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# Computation of Greeks in LIBOR models driven by time-inhomogeneous Lévy processes 

Ernst Eberlein ${ }^{\text {a }}$, M’hamed Eddahbi ${ }^{\text {b }}$ and S. M. Lalaoui Ben Cherif ${ }^{\text {c }}$<br>${ }^{\text {a }}$ Department of Mathematical Stochastics, University of Freiburg, Freiburg im Breisgau, Germany; ${ }^{\text {b }}$ Faculty of Sciences and Techniques, Department of Mathematics, Cadi Ayyad University, Marrakech, Morocco; ${ }^{\text {cFaculty }}$ of Sciences Semlalia, Department of Mathematics, Cadi Ayyad University, Marrakech, Morocco


#### Abstract

The aim of this article is to compute Greeks, i.e. price sensitivities in the framework of the Lévy LIBOR model. Two approaches are discussed. The first approach is based on the integration-by-parts formula, which lies at the core of the application of the Malliavin calculus to finance. The second approach consists of using Fourierbased methods for pricing derivatives. We illustrate the result by applying the formula to a caplet price where the jump part of the driving process of the underlying model is given by a time-inhomogeneous Gamma process and alternatively by a Variance Gamma process.


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## 1. Introduction

When a financial institution sells a derivative product to a customer, it is usually interested to preserve the initial margin and consequently tries to avoid or at least to reduce the market risk by hedging the position. In some cases, one can make use of a static hedging strategy, that means hedge-and-forget. Unfortunately for most derivatives, there is no static hedge, instead one has to hedge dynamically. For this, traders need to know the exposure of their positions to changes of parameters which go into the valuation formula, called the sensitivities or Greeks. Note that the possibility to hedge a position is also crucial for the classical pricing approach via the cost of hedging. From the mathematical point of view, Greeks are partial derivatives of the pricing functional with respect to specific parameters. Traditionally, they are estimated by means of a finite difference approximation. Two kinds of errors are produced this way: one which comes from the approximation of the derivative by a finite difference and another one which results from the numerical computation of the expectation which represents the pricing functional. To eliminate the former source of error, Fournié et al. (1999) adopted a new approach which allows to transform the differential operator into a random weight. The result is an expectation operator applied to the product of the pay-off function with a random weight function.

In the context of equity models, this approach has been successfully used not only for diffusion-driven approaches, but also for models which allow jumps. El-Khatib and Privault (2004) study a model which is driven by a discontinuous process with Poisson jump times and random jump sizes. Although European options do not satisfy the regularity conditions required in their approach, they show that Asian options can be considered due to the smoothing effect of the integral over time. Davis and Johansson (2006) investigate a jump diffusion setting with a deterministic jump amplitude. An additional separability assumption has to be imposed. Bayazit and Nolder (2013) consider exponential Lévy models driven by Variance Gamma and by Normal Inverse Gaussian Processes. They obtain approximations of various sensitivities after replacing the underlying continuous time driving process by an approximating sequence of random variables.

A rather different approach to obtain sensitivities in exponential Lévy models exploits the Fourier-based valuation formulas for derivatives. For a systematic investigation under which conditions Fourier-based methods apply, see Eberlein, Glau, and Papapantoleon (2010). This reference contains also a section where sensitivities are derived. A more recent paper along these lines is De Olivera and Mordecki (2014). As far as hedging issues in exponential Lévy models are concerned, we recommend the review by Tankov (2011).

It is the purpose of the current paper to study sensitivities in a different area namely for fixed income markets. We focus on the celebrated LIBOR market model (LMM). In the framework of the classical diffusion version of the LMM as introduced by Brace, Gatarek, and Musiela (1997) (see also Miltersen, Sandmann, and Sondermann (1997) as well as Jamshidian (1997)), Glasserman and Zhao (1999) investigate the estimation of option price sensitivities based on Monte Carlo simulation of the forward LIBOR rates. In spite of the complexity of the LMM which is well above that of equity models, they succeed to extract conditions under which the discrete time estimator gives unbiased derivative estimates for the simulated process and also under which it converges to the correct continuous time limit.

The diffusion-based LIBOR market model offers a high degree of analytical tractability. However, due to the rigidity of the normal distribution, it has limitations as far as its calibration to price data is concerned. Among other shortcomings, it cannot reproduce the phenomenon of changing volatility smiles along the maturity axis. In order to gain more flexibility in a first step, one can replace the driving Brownian motion by a (time-homogeneous) Lévy process such as a generalized hyperbolic Lévy process or a process from a suitable subclass. This improves the model performance already considerably but the shape of the volatility surface produced by cap and floor prices is typically too sophisticated in order to be matched sufficiently well by a model which is driven by a time-homogeneous process. A more accurate calibration of the model across different strikes and maturities and thus a reduction of model risk can be achieved by using the larger class of time-inhomogeneous Lévy processes (see, e.g., Eberlein and Kluge (2005) and Eberlein and Özkan (2005)). In practical applications, a mild form of time-inhomogeneity turns out to be sufficient. Changing the process parameters twice along the time axis produces already satisfactory results (Eberlein and Kluge (2005)). This corresponds to choosing one Lévy parameter set for short maturities, one for the middle range and another one for the long maturities.

In order to take these aspects into account, we focus in the following on the computation of Greeks in the framework of the Lévy LIBOR model (LLM) which was introduced in Eberlein and Özkan (2005). Note also that time-inhomogeneity of the driving processes develops naturally during the measure changes along the tenor structure. Therefore, one does not give up simplifying structural properties by starting already with the wider class of time-inhomogeneous processes. We discuss the computation of Greeks in both approaches mentioned earlier which come from totally different mathematical fields. The first approach is based on the integration-by-parts formula, which lies at the core of the application of the Malliavin calculus to finance as developed in Fournié et al. (1999), León et al. (2002), Petrou (2008) and Yablonski (2008). The second approach consists of using Fourier-based methods for pricing derivatives. For a recent survey of Fourier-based methods, see Eberlein (2014). We illustrate the result by applying the formulas to the pricing of caplets where the jump part of the driving process of the underlying model is given by a time-inhomogeneous Gamma process and alternatively by a Variance Gamma process. For the ease of reading in an appendix, the relevant definitions and results of the Malliavin calculus are presented on the level of generality which is required here.

## 2. The LLM

In the classical Heath-Jarrow-Morton theory, instantaneous forward rates represent the basic quantity in interest rate term structure modelling. These rates are an infinitesimal quantity given by

$$
\begin{equation*}
f(t, T)=-\frac{\partial}{\partial T} \ln B(t, T) \tag{1}
\end{equation*}
$$

where $B(t, T)$ is the price at time $t \in[0, T]$ of a default-free zero coupon bond which matures at time T. $f(t, T)$ is however not observable in the market. What can be observed instead are forward LIBOR rates $L(t, T)$ which are the discretely compounded, annualized interest rates which can be earned from investment during a future interval starting at $T$ and ending at $T+\delta$ considered at the time point $t<T$

$$
\begin{equation*}
L(t, T)=\frac{1}{\delta}\left(\frac{B(t, T)}{B(t, T+\delta)}-1\right) \tag{2}
\end{equation*}
$$

We recall that the forward price process for two maturities $T$ and $T+\delta$ is defined as the quotient of the corresponding discount factors

$$
F(t, T, T+\delta)=\frac{B(t, T)}{B(t, T+\delta)}
$$

The LLM was developed by Eberlein and Özkan (2005) following the seminal articles on LIBOR market model driven by Brownian motion by Miltersen, Sandmann, and Sondermann (1997) and Brace, Gatarek, and Musiela (1997). We repeat in this section some facts as presented in Eberlein (2014). The model is developed via a backward induction and driven by a time-inhomogeneous Lévy process $L^{T^{*}}$ defined on a complete stochastic basis $\left(\Omega, \mathcal{F}=\mathcal{F}_{T^{*}}, \mathbb{F}, \mathbb{P}_{T^{*}}\right)$ where $\mathbb{P}_{T^{*}}$ should be regarded as the
forward martingale measure for the settlement date $T^{*}$ and the filtration $\mathbb{F}=$ $\left(\mathcal{F}_{t}\right)_{t \in\left[0, T^{*}\right]}$ satisfies the usual conditions. $L^{T^{*}}$ is given by

$$
\begin{equation*}
L_{t}^{T^{*}}=\int_{0}^{t} b_{s}^{T^{*}} d s+\int_{0}^{t} c_{s}^{\frac{1}{2}} d W_{s}^{T^{*}}+\int_{0}^{t} \int_{\mathbb{R}} x\left(\mu^{T^{*}}-v^{T^{*}}\right)(d s, d x) \tag{3}
\end{equation*}
$$

with characteristics

$$
B_{t}^{T^{*}}=\int_{0}^{t} b_{s}^{T^{*}} d s, \quad C_{t}=\int_{0}^{t} c_{s} d s, \quad v^{T^{*}}(d s, d x)=F_{s}^{T^{*}}(d x) d s
$$

Here $T^{*}$ denotes the end point of a tenor structure $0<T_{1}<\ldots<T_{n-1}<T_{n}=T^{*}$ where the corresponding contract runs from $T_{1}$ to $T_{n},\left(W_{t}^{T^{*}}\right)_{t \geq 0}$ is a $\mathbb{P}_{T^{*}}$ standard Brownian motion, $\mu^{T^{*}}$ the random measure of jumps of $L^{T^{*}}$ and $v^{T^{*}}$ is the $\mathbb{P}_{T^{*}}$ compensator of $\mu^{T^{*}}$. We assume $\delta=T_{k+1}-T_{k}$ for $1 \leq k \leq n-1$ to be independent of $k$ and the usual integrability assumptions.

Assumption 2.1. The drift term $b_{s}^{T^{*}} \in \mathbb{R}$, the volatility coefficients $c_{s}$ and the Lévy measure $F_{s}^{T^{*}}$ satisfy

$$
\exists \sigma>0 ; \quad \forall s \in\left[0, T^{*}\right]: \quad c_{s}>\sigma
$$

and

$$
\int_{0}^{T^{*}}\left(\left|b_{s}^{T^{*}}\right|+\left|c_{s}\right|+\int_{\mathbb{R}}\left(|x|^{2} \wedge 1\right) F_{s}^{T^{*}}(d x)\right) d s<\infty
$$

Assumption 2.2 (EM). There exists a constant $M>1$ such that

$$
\int_{|x|>1} \exp (u x) F_{s}^{T^{*}}(d x)<\infty \quad \forall u \in[-M, M], \quad \forall s \in\left[0, T^{*}\right]
$$

Two ingredients are needed for the LLM.
Assumption 2.3 ( $\mathbb{L} \mathbb{R} .1$ ) For any maturity $T_{k}$ there is a deterministic function $\lambda\left(., T_{k}\right):\left[0, T^{*}\right] \mapsto \mathbb{R}$ which represents the volatility of the forward LIBOR rate process $L\left(., T_{k}\right)$. These functions satisfy

- $\sum_{k=1}^{n-1} \lambda\left(s, T_{k}\right) \leq M^{\prime} ; \forall s \in\left[0, T^{*}\right]$, for some $M^{\prime}<\frac{M}{2}$, where $M$ is the constant from Assumption (EM),
- $\lambda\left(s, T_{k}\right)>0$ for all $s \in\left[0, T_{k}\left[\right.\right.$ and $\lambda\left(s, T_{k}\right)=0$ for $s \geq T_{k}$, for any maturity $T_{k}$.

Assumption 2.4 ( $\mathbb{L} \mathbb{R}$.2). The initial term structure $B\left(0, T_{k}\right)$ for $1 \leq k \leq n$ is strictly positive and strictly decreasing in $k$.

The backward induction starts by setting the most distant LIBOR rate $L\left(t, T_{n-1}\right)$ under the forward martingale measure $\mathbb{P}_{T^{*}}$ as

$$
\begin{equation*}
L\left(t, T_{n-1}\right)=L\left(0, T_{n-1}\right) \exp \left(\int_{0}^{t} \lambda\left(s, T_{n-1}\right) d L_{s}^{T^{*}}\right) \tag{4}
\end{equation*}
$$

Expressed as a differential we get the $\mathbb{P}_{T^{*}}$ dynamics

$$
\begin{aligned}
d L\left(t, T_{n-1}\right)= & L\left(t-, T_{n-1}\right)\left(c_{t}^{\frac{1}{2}} \lambda\left(t, T_{n-1}\right) d W_{t}^{T^{*}}+\int_{\mathbb{R}}\left(e^{\lambda\left(t, T_{n-1}\right) x}-1\right)\left(\mu^{T^{*}}-v^{T^{*}}\right)(d t, d x)\right. \\
& +\left(\lambda\left(t, T_{n-1}\right) b_{t}^{T^{*}}+\frac{1}{2} c_{t} \lambda^{2}\left(t, T_{n-1}\right)\right) d t \\
& \left.+\int_{\mathbb{R}}\left(e^{x \lambda\left(t, T_{n-1}\right)}-1-x \lambda\left(t, T_{n-1}\right)\right) v^{T^{*}}(d t, d x)\right)
\end{aligned}
$$

One forces this process to become a $\mathbb{P}_{T^{*}}$ martingale by choosing $b^{T^{*}}$ such that

$$
\begin{align*}
\int_{0}^{t} \lambda\left(s, T_{n-1}\right) b_{s}^{T^{*}} d s= & -\frac{1}{2} \int_{0}^{t} c_{s} \lambda^{2}\left(s, T_{n-1}\right) d s \\
& -\int_{0}^{t} \int_{\mathbb{R}}\left(e^{x \lambda\left(s, T_{n-1}\right)}-1-x \lambda\left(s, T_{n-1}\right)\right) v^{T^{*}}(d s, d x) \tag{5}
\end{align*}
$$

Define

$$
\begin{aligned}
& \ell\left(t-, T_{n-1}\right)=\frac{\delta L\left(t-, T_{n-1}\right)}{1+\delta L\left(t-, T_{n-1}\right)} \\
& \alpha\left(t, T_{n-1}\right)=\ell\left(t-, T_{n-1}\right) \lambda\left(t, T_{n-1}\right)
\end{aligned}
$$

and

$$
\beta\left(t, x, T_{n-1}\right)=\ell\left(t-, T_{n-1}\right)\left(e^{x \lambda\left(t, T_{n-1}\right)}-1\right)+1
$$

Then the forward process $F\left(., T_{n-1}, T^{*}\right)$ is given as a stochastic exponential

$$
F\left(t, T_{n-1}, T^{*}\right)=F\left(0, T_{n-1}, T^{*}\right) \mathcal{E}_{t}(Z)
$$

with

$$
Z_{t}=\int_{0}^{t} c_{s}^{\frac{1}{2}} \alpha\left(s, T_{n-1}\right) d W_{s}^{T^{*}}+\int_{0}^{t} \int_{\mathbb{R}}\left(\beta\left(s, x, T_{n-1}\right)-1\right)\left(\mu^{T^{*}}-v^{T^{*}}\right)(d s, d x)
$$

and is consequently a $\mathbb{P}_{T^{*}}$ martingale. We use this forward process as a density process and define the forward measure $\mathbb{P}_{T_{n-1}}$ by setting

$$
\frac{d \mathbb{P}_{T_{n-1}}}{d \mathbb{P}_{T^{*}}}=\frac{F\left(T_{n-1}, T_{n-1}, T^{*}\right)}{F\left(0, T_{n-1}, T^{*}\right)}=\mathcal{E}_{T_{n-1}}(Z) .
$$

By the semimartingale version of Girsanov's theorem (see Jacod and Shiryaev (1987))

$$
W_{t}^{T_{n-1}}:=W_{t}^{T^{*}}-\int_{0}^{t} c_{s}^{\frac{1}{2}} \alpha\left(s, T_{n-1}\right) d s
$$

is a $\mathbb{P}_{T_{n-1}}$ standard Brownian motion and

$$
v^{T_{n-1}}(d t, d x):=\beta\left(t, x, T_{n-1}\right) v^{T^{*}}(d t, d x)
$$

is the $\mathbb{P}_{T_{n-1}}$ compensator of $\mu^{T^{*}}$. We take $\mathbb{P}_{T_{n-1}}$ as the new underlying probability measure and define $L\left(t, T_{n-2}\right)$ in the same way as $L\left(t, T_{n-1}\right)$ defined in Equation (4). Continuing this way one gets forward LIBOR rates $L\left(t, T_{k}\right)$ and forward measures $\mathbb{P}_{T_{k+1}}$ such that for $k \in\{1, \ldots, n-1\}$ and $0 \leq t \leq T_{k}$

$$
\begin{equation*}
L\left(t, T_{k}\right)=L\left(0, T_{k}\right) \exp \left(\int_{0}^{t} \lambda\left(s, T_{k}\right) d L_{s}^{T_{k+1}}\right) \tag{6}
\end{equation*}
$$

and the successive densities are given by the recursive relation

$$
\frac{d \mathbb{P}_{T_{k}}}{d \mathbb{P}_{T_{k+1}}}=\frac{1+\delta L\left(T_{k}, T_{k}\right)}{1+\delta L\left(0, T_{k}\right)}
$$

The driving process has the form

$$
\begin{equation*}
L_{t}^{T_{k+1}}=\int_{0}^{t} b_{s}^{T_{k+1}} d s+\int_{0}^{t} c_{s}^{\frac{1}{2}} d W_{s}^{T_{k+1}}+\int_{0}^{t} \int_{\mathbb{R}} x\left(\mu^{T^{*}}-v^{T_{k+1}}\right)(d s, d x) \tag{7}
\end{equation*}
$$

where $\left(W_{t}^{T_{k+1}}\right)_{t \geq 0}$ is a $\mathbb{P}_{T_{k+1}}$ standard Brownian motion with

$$
\left\{\begin{array}{l}
W_{t}^{T_{k}}=W_{t}^{T_{k+1}}-\int_{0}^{t} c_{s}^{\frac{1}{2}} \alpha\left(s, T_{k}\right) d s \quad k \in\{2, \ldots, n-1\},  \tag{8}\\
W_{t}^{T_{n}}=W_{t}^{T^{*}}
\end{array}\right.
$$

and $v^{T_{k+1}}(d s, d x)=F_{s}^{T_{k+1}}(d x) d s$ is the $\mathbb{P}_{T_{k+1}}$ compensator of $\mu^{T^{*}}$ with

$$
\left\{\begin{array}{l}
F_{s}^{T_{k}}(d x)=\beta\left(s, x, T_{k}\right) F^{T_{k+1}}(d x)  \tag{9}\\
F_{s}^{T_{n}}(d x)=F_{s}^{T^{*}}(d x),
\end{array} \quad k \in\{2, \ldots, n-1\}\right.
$$

such that

$$
\begin{aligned}
& \ell\left(t-, T_{k}\right)=\frac{\delta L\left(t-, T_{k}\right)}{1+\delta L\left(t-, T_{k}\right)}, \\
& \alpha\left(t, T_{k}\right)=\ell\left(t-, T_{k}\right) \lambda\left(t, T_{k}\right),
\end{aligned}
$$

and

$$
\beta\left(t, x, T_{k}\right)=\ell\left(t-, T_{k}\right)\left(e^{x \lambda\left(t, T_{k}\right)}-1\right)+1
$$

We conclude that for all $k \in\{2, \ldots, n-1\}$

$$
\begin{equation*}
W_{t}^{T_{k}}=W_{t}^{T^{*}}-\int_{0}^{t} c_{s}^{\frac{1}{2}} \sum_{j=k}^{n-1} \alpha\left(s, T_{j}\right) d s \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{s}^{T_{k}}(d x)=\prod_{j=k}^{n-1} \beta\left(s, x, T_{j}\right) F_{s}^{T^{*}}(d x) \tag{11}
\end{equation*}
$$

The drift term $b^{T_{k+1}}$ is chosen in such a way that the process $L\left(., T_{k}\right)$ becomes a martingale under the forward measure $\mathbb{P}_{T_{k+1}}$

$$
\begin{align*}
\int_{0}^{t} \lambda\left(s, T_{k}\right) b_{s}^{T_{k+1}} d s= & -\frac{1}{2} \int_{0}^{t} c_{s} \lambda^{2}\left(s, T_{k}\right) d s \\
& -\int_{0}^{t} \int_{\mathbb{R}}\left(e^{x \lambda\left(s, T_{k}\right)}-1-x \lambda\left(s, T_{k}\right)\right) v^{T_{k+1}}(d s, d x) \tag{12}
\end{align*}
$$

We propose the following choice for the functions $b^{T_{k+1}}$ for all $k \in\{1, \ldots, n-1\}$

$$
\begin{cases}b_{s}^{T_{k+1}}=-\frac{1}{2} \lambda\left(s, T_{k}\right) c_{s}-\int_{\mathbb{R}}\left(\frac{e^{x \lambda\left(s, T_{k}\right)}-1}{\lambda\left(s, T_{k}\right)}-x\right) F_{s}^{T_{k+1}}(d x) & \left(0 \leq s<T_{k}\right)  \tag{13}\\ b_{s}^{T_{k+1}}=0 & \left(s \geq T_{k}\right)\end{cases}
$$

The driving process $L^{T_{k+1}}$ becomes

$$
\begin{align*}
L_{t}^{T_{k+1}}= & -\int_{0}^{t}\left(\frac{1}{2} \lambda\left(s, T_{k}\right) c_{s}+\int_{\mathbb{R}}\left(\frac{e^{x \lambda\left(s, T_{k}\right)}-1}{\lambda\left(s, T_{k}\right)}-x\right) F_{s}^{T_{k+1}}(d x)\right) d s+\int_{0}^{t} c_{s}^{\frac{1}{2}} d W_{s}^{T_{k+1}} \\
& +\int_{0}^{t} \int_{\mathbb{R}} x\left(\mu^{T^{*}}-v^{T_{k+1}}\right)(d s, d x) \tag{14}
\end{align*}
$$

Since $L\left(t, T_{k}\right)$ is a $\mathbb{P}_{T_{k+1}}$ martingale $1+\delta L\left(t, T_{k}\right)$ is a $\mathbb{P}_{T_{k+1}}$ martingale as well, which is up to the constant $\left(1+\delta L\left(0, T_{k}\right)\right)^{-1}$ the density process

$$
\left.\frac{d \mathbb{P}_{T_{k}}}{d \mathbb{P}_{T_{k+1}}}\right|_{\mathcal{F}_{t}}=\frac{1+\delta L\left(t, T_{k}\right)}{1+\delta L\left(0, T_{k}\right)} \quad\left(0 \leq t<T_{k}\right)
$$

By iterating this, we get

$$
\begin{aligned}
\frac{d \mathbb{P}_{T_{k+1}}}{d \mathbb{P}_{T^{*}}} & =\prod_{j=k+1}^{n-1} \frac{1+\delta L\left(T_{k+1}, T_{j}\right)}{1+\delta L\left(0, T_{j}\right)} \\
& =\frac{B\left(0, T^{*}\right)}{B\left(0, T_{k+1}\right)} \prod_{j=k+1}^{n-1}\left(1+\delta L\left(T_{k+1}, T_{j}\right)\right) .
\end{aligned}
$$

Applying iteratively Proposition III.3.8 of Jacod and Shiryaev (1987) - which is a very fundamental result for interest rate modelling - we see that its restriction to $\mathcal{F}_{t}$

$$
\begin{equation*}
\frac{\left.\left.\left.d \mathbb{P}_{T_{k+1}}\right|_{\mathbb{P}_{T^{*}}}=\frac{B\left(0, T^{*}\right)}{B\left(0, T_{k+1}\right)} \prod_{j=k+1}^{n-1}\left(1+\delta L\left(t, T_{j}\right)\right)\right) \text { (0<t<t} T_{k+1}\right)}{n-1} \tag{15}
\end{equation*}
$$

is a $\mathbb{P}_{T^{*}}$ martingale.
As has already been pointed out in Eberlein (2014) as a consequence of representations of the type (Equation (15)) of arbitrary quotients $\frac{B\left(t, T_{j}\right)}{B\left(t, T_{k}\right)}$ as products of quotients with successive maturities $T_{k}$ and $T_{k+1}$, Proposition III.3.8 of Jacod and Shiryaev (1987) guarantees also that properly discounted zero coupon bond prices $\frac{B\left(t, T_{j}\right)}{B\left(t, T_{k}\right)}$ are $\mathbb{P}_{T_{k}}$ martingales. This means that the LIBOR approach as developed above creates an arbitrage-free model.

With respect to numerical aspects and the application of the Malliavin calculus in Section 3, it is important to note that already with the first measure change one looses the property that the driving processes $L^{T_{k+1}}$ are time-inhomogeneous Lévy processes. This is because the coefficients $\alpha\left(s, T_{k}\right)$ and $\beta\left(s, x, T_{k}\right)$ contain the random quantity $L\left(s-, T_{k}\right)$. The simplest approach to preserve this property is to replace $\ell\left(s-, T_{k}\right)$ by its deterministic starting value. Henceforth, we will make this assumption in the following sections.

Assumption 2.5 (Frozen drift approximation). For each $k$ and all $s$ belonging to [ $0, T_{k}$ ], we assume the approximation

$$
\ell\left(s-, T_{k}\right) \simeq \ell\left(0, T_{k}\right)=\frac{\delta L\left(0, T_{k}\right)}{1+\delta L\left(0, T_{k}\right)} .
$$

## 3. Sensitivity analysis

Following Eberlein and Kluge (2005) and Eberlein (2014), we shall consider valuation formulas for standard interest rate derivatives such as caps, floors and swaptions in the LLM. These formulas are computationally efficient. Since floor prices can be derived from the corresponding put-call-parity relation, we concentrate on caps. A cap consists of a sequence of caplets which are call options on LIBOR rates. The pay-off of a caplet with strike rate $K$ and maturity $T_{k}$ is

$$
\delta\left(L\left(T_{k}, T_{k}\right)-K\right)^{+},
$$

where the payment is made at time point $T_{k+1}$. Its time-0-price, denoted by $\widetilde{C p l t}_{0}\left(T_{k}, K, \delta\right)$, is given by

$$
\begin{equation*}
\widetilde{C p l t}\left(T_{k}, K, \delta\right)=B\left(0, T_{k+1}\right) \delta \mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[\left(L\left(T_{k}, T_{k}\right)-K\right)^{+}\right] \tag{16}
\end{equation*}
$$

### 3.1. Greeks computed by the Malliavin approach

In this part, we present an application of the Malliavin calculus to the computation of Greeks within the LLM. We refer to the literature, for example, Di Nunno, Øksendal, and Proske (2009) and Nualart (2006) for details on the theoretical aspects of Malliavin calculus, but we mainly follow Fournié et al. (1999) and Yablonski (2008) (see the Appendix) for the presentation of the mathematical results used in the sequel. The forward LIBOR rates $L\left(t, T_{k}\right)$ under the forward measures $\mathbb{P}_{T_{k+1}}$ can be written as stochastic exponentials. Expressed as a differential, we get the $\mathbb{P}_{T_{k+1}}$ dynamics in the form

$$
\begin{equation*}
d L\left(t, T_{k}\right)=L\left(t-, T_{k}\right)\left(c_{t}^{\frac{1}{2}} \lambda\left(t, T_{k}\right) d W_{t}^{T_{k+1}}+\int_{\mathbb{R}}\left(e^{\lambda\left(t, T_{k}\right) x}-1\right)\left(\mu^{T^{*}}-v^{T_{k+1}}\right)(d t, d x)\right) \tag{17}
\end{equation*}
$$

As in the classical Malliavin calculus, we are able to associate the solution of Equation (17) with the process $Y\left(t, T_{k}\right):=\frac{\partial L\left(t, T_{k}\right)}{\partial L\left(0, T_{k}\right)}$, called the first variation process of $L\left(t, T_{k}\right)$. By applying Proposition A.10, the following proposition provides an expression for the Malliavin derivative operator $D_{r, 0}$ when applied to the LIBOR rates $L\left(t, T_{k}\right)$. This expression is simpler than the original one which follows from Theorem A.9.

Proposition 3.1. Let $L\left(t, T_{k}\right)_{t \in\left[0, T^{*}\right]}$ be the solution of Equation (17). Then $L\left(t, T_{k}\right) \in \mathbb{D}^{1,2}$, which is the domain of the derivative operator in $L^{2}(\Omega)$. The Malliavin derivative satisfies the following equation:

$$
\begin{equation*}
D_{r, 0} L\left(t, T_{k}\right)=Y\left(t, T_{k}\right) Y\left(r-, T_{k}\right)^{-1} L\left(r-, T_{k}\right) c_{r}^{\frac{1}{2}} \lambda\left(r, T_{k}\right) \mathbf{1}_{\{r \leq t\}} \text { a.e.. } \tag{18}
\end{equation*}
$$

### 3.1.1. Variation in the initial condition

In this section, we provide an expression for the Delta, the partial derivative of the expectation $\operatorname{Cplt}_{0}\left(T_{k}, K, \delta\right)$ with respect to the initial condition $L\left(0, T_{k}\right)$ given by

$$
\frac{\Delta\left(L\left(0, T_{k}\right)\right)=\partial \widetilde{C p l t}_{0}\left(T_{k}, K, \delta\right)}{\partial L\left(0, T_{k}\right)}
$$

For a convenient representation of this expectation, we introduce two processes which turn out to be $\mathbb{P}_{T^{*}}$ martingales (see Equation (15))

$$
\begin{equation*}
M_{t}^{1}=\prod_{j=k+1}^{n-1}\left(1+\delta L\left(t, T_{j}\right)\right) \frac{L\left(t, T_{k}\right)}{K} \quad\left(0 \leq t<T_{k}\right) \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{t}^{2}=\prod_{j=k+1}^{n-1}\left(1+\delta L\left(t, T_{j}\right)\right) \quad\left(0 \leq t<T_{k+1}\right) \tag{20}
\end{equation*}
$$

As mentioned in Fournié et al. (1999), let us define the set

$$
\widetilde{T}_{k}=\left\{h_{k} \in L^{2}\left(\left[0, T_{k}\right]\right): \int_{0}^{T_{k}} h_{k}(u) d u=1\right\} .
$$

For all $t \in\left[0, T^{*}\right]$, define

$$
\begin{equation*}
f^{k}(t)=\sum_{j=k+1}^{n-1} \ell\left(0, T_{j}\right) \lambda\left(t, T_{j}\right) \tag{21}
\end{equation*}
$$

Proposition 3.2. For all functions $h_{k} \in \widetilde{T}_{k}$, we have

$$
\begin{align*}
\Delta\left(L\left(0, T_{k}\right)\right)= & \frac{\delta K B\left(0, T^{*}\right)}{L\left(0, T_{k}\right)} \mathbb{E}_{\mathbb{P}_{T^{*}}}\left[\left(M_{T_{k}}^{1}-M_{T_{k}}^{2}\right)^{+}\right. \\
& \left.\times\left(\int_{0}^{T_{k}} \frac{h_{k}(u)}{c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d W_{u}^{T^{*}}-\int_{0}^{T_{k}} \frac{h_{k}(u) f^{k}(u)}{\lambda\left(u, T_{k}\right)} d u\right)\right] . \tag{22}
\end{align*}
$$

Proof. Following the results of Fournié et al. (1999) and Petrou (2008) concerning the application of Malliavin calculus to finance, we consider now a more general pay-off of the form $H\left(L\left(T_{k}, T_{k}\right)\right)$ such that $H: \mathbb{R} \rightarrow \mathbb{R}$ is a locally integrable function satisfying

$$
\mathbb{E}_{\mathbb{P}_{k+1}}\left[H\left(L\left(T_{k}, T_{k}\right)\right)^{2}\right]<\infty .
$$

First, assume that $H$ is a continuously differentiable function with compact support. Then we can differentiate inside the expectation ${ }^{1}$ and we get

$$
\begin{aligned}
\Delta_{H}\left(L\left(0, T_{k}\right)\right) & :=\frac{\partial \mathbb{E}_{\mathbb{P}_{k+1}}\left[H\left(L\left(T_{k}, T_{k}\right)\right)\right]}{\partial L\left(0, T_{k}\right)} \\
& =\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[H^{\prime}\left(L\left(T_{k}, T_{k}\right)\right) \frac{\partial L\left(T_{k}, T_{k}\right)}{\partial L\left(0, T_{k}\right)}\right] \\
& =\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[H^{\prime}\left(L\left(T_{k}, T_{k}\right)\right) Y\left(T_{k}, T_{k}\right)\right]
\end{aligned}
$$

For any $h_{k} \in \widetilde{T}_{k}$ and using Proposition 3.1, we find

$$
Y\left(T_{k}, T_{k}\right)=\int_{0}^{T_{k}} h_{k}(u) D_{u, 0} L\left(T_{k}, T_{k}\right) \frac{Y\left(u-, T_{k}\right)}{L\left(u-, T_{k}\right) c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d u
$$

From the chain rule Proposition A. 5 we find

$$
\begin{aligned}
\Delta_{H}\left(L\left(0, T_{k}\right)\right) & =\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[\int_{0}^{T_{k}} H^{\prime}\left(L\left(T_{k}, T_{k}\right)\right) h_{k}(u) D_{u, 0} L\left(T_{k}, T_{k}\right) \frac{Y\left(u-, T_{k}\right)}{L\left(u-, T_{k}\right) c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d u\right] \\
& =\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[\int_{0}^{T_{k}} D_{u, 0} H\left(L\left(T_{k}, T_{k}\right)\right) h_{k}(u) \frac{Y\left(u-, T_{k}\right)}{L\left(u-, T_{k}\right) c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d u\right] \\
& =\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[\int_{0}^{T_{k}} \int_{\mathbb{R}} D_{u, x} H\left(L\left(T_{k}, T_{k}\right)\right) h_{k}(u) \frac{Y\left(u-, T_{k}\right)}{L\left(u-, T_{k}\right) c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d u \delta_{0}(d x)\right] .
\end{aligned}
$$

By the definition of the Skorohod integral (see Section A.5), we reach

$$
\Delta_{H}\left(L\left(0, T_{k}\right)\right)=\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[H\left(L\left(T_{k}, T_{k}\right)\right) \delta\left(h_{k}(.) \frac{Y\left(., T_{k}\right)}{L\left(., T_{k}\right) c^{\frac{1}{2}} \lambda\left(., T_{k}\right)} \delta_{0}(x)\right)\right]
$$

However, $\left(h_{k}(u) \frac{Y\left(u-, T_{k}\right)}{L\left(u-, T_{k} \frac{c_{u}^{1}}{\frac{1}{2}} \lambda\left(u, T_{k}\right)\right.}\right)_{0 \leq u \leq T_{k}}$ is a predictable process, thus the Skorohod integral coincides with the Itô stochastic integral and we get

$$
\begin{equation*}
\Delta_{H}\left(L\left(0, T_{k}\right)\right)=\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[H\left(L\left(T_{k}, T_{k}\right)\right) \int_{0}^{T_{k}} h_{k}(u) \frac{Y\left(u-, T_{k}\right)}{L\left(u-, T_{k}\right) c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d W_{u}^{T_{k+1}}\right] . \tag{23}
\end{equation*}
$$

By Lemma 12.28, p. 208, in Di Nunno, Øksendal, and Proske (2009), the result (Equation (23)) holds for any locally integrable function $H$ such that

$$
\mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[H\left(L\left(T_{k}, T_{k}\right)\right)^{2}\right]<\infty .
$$

In particular, if one takes $H\left(L\left(T_{k}, T_{k}\right)\right)=B\left(0, T_{k+1}\right) \delta\left(L\left(T_{k}, T_{k}\right)-K\right)^{+}$, we can express the derivatives of the expectation $\widetilde{\operatorname{Cplt}}\left(T_{k}, K, \delta\right)$ with respect to the initial condition $L\left(0, T_{k}\right)$ in the form of a weighted expectation as follows:

$$
\Delta\left(L\left(0, T_{k}\right)\right)=\delta B\left(0, T_{k+1}\right) \mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[\left(L\left(T_{k}, T_{k}\right)-K\right)^{+} \int_{0}^{T_{k}} \frac{h_{k}(u) Y\left(u-, T_{k}\right)}{c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right) L\left(u-, T_{k}\right)} d W_{u}^{T_{k+1}}\right]
$$

We show easily that $Y\left(u-, T_{k}\right)=\frac{L\left(u-, T_{k}\right)}{L\left(0, T_{k}\right)}$ and we reach

$$
\Delta\left(L\left(0, T_{k}\right)\right)=\frac{\delta B\left(0, T_{k+1}\right)}{L\left(0, T_{k}\right)} \mathbb{E}_{\mathbb{P}_{T_{k+1}}}\left[\left(L\left(T_{k}, T_{k}\right)-K\right)^{+} \int_{0}^{T_{k}} \frac{h_{k}(u)}{c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d W_{u}^{T_{k+1}}\right]
$$

In accordance with Equation (8) and taking the frozen drift approximation (Assumption 2.5) into consideration, we can write for all $k \in\{1, \ldots, n-2\}$

$$
W_{t}^{T_{k+1}}=W_{t}^{T^{*}}-\int_{0}^{t} c_{s}^{\frac{1}{2}} f^{k}(s) d s
$$

One can easily show that

$$
\begin{array}{r}
K\left(M_{T_{k}}^{1}-M_{T_{k}}^{2}\right)^{+}=\left(L\left(T_{k}, T_{k}\right)-K\right)^{+} \prod_{j=k+1}^{n-1}  \tag{24}\\
\left(1+\delta L\left(T_{k}, T_{j}\right)\right) .
\end{array}
$$

By making a measure change using the fact that

$$
\left.\frac{d \mathbb{P}_{T_{k+1}}}{d \mathbb{P}_{T^{*}}}\right|_{\mathcal{F}_{T_{k}}}=\frac{B\left(0, T^{*}\right)}{B\left(0, T_{k+1}\right)} \prod_{j=k+1}^{n-1}\left(1+\delta L\left(T_{k}, T_{j}\right)\right)
$$

we end up with

$$
\begin{aligned}
\Delta\left(L\left(0, T_{k}\right)\right)= & \frac{\delta K B\left(0, T^{*}\right)}{L\left(0, T_{k}\right)} \mathbb{E}_{\mathbb{P}_{T^{*}}}\left[\left(M_{T_{k}}^{1}-M_{T_{k}}^{2}\right)^{+} \int_{0}^{T_{k}} \frac{h_{k}(u)}{c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d W_{u}^{T_{k+1}}\right] \\
= & \frac{\delta K B\left(0, T^{*}\right)}{L\left(0, T_{k}\right)} \mathbb{E}_{\mathbb{T}_{T^{*}}}\left[\left(M_{T_{k}}^{1}-M_{T_{k}}^{2}\right)^{+}\right. \\
& \left.\times\left(\int_{0}^{T_{k}} \frac{h_{k}(u)}{c_{u}^{\frac{1}{2}} \lambda\left(u, T_{k}\right)} d W_{u}^{T^{*}}-\int_{0}^{T_{k}} \frac{h_{k}(u) f^{k}(u)}{\lambda\left(u, T_{k}\right)} d u\right)\right] .
\end{aligned}
$$

Remark 3.3. The function $h_{k}$ used in this formula allows a lot of flexibility. An obvious choice which simplifies the formula considerably is $h_{k}(u)=\frac{\lambda\left(u, T_{k}\right)}{\int_{0}^{T_{k}} \lambda\left(u, T_{k}\right) d u}$.

### 3.2. Greeks computed by the approximative Fourier-based valuation method

Making use of Equation (24), we can write the time-0-price $\widetilde{C p l t_{0}}\left(T_{k}, K, \delta\right)$ as

$$
\begin{equation*}
\widetilde{\operatorname{Cplt}_{0}}\left(T_{k}, K, \delta\right)=\delta K B\left(0, T^{*}\right) \mathbb{E}_{\mathbb{P}_{T^{*}}}\left[\left(M_{T_{k}}^{1}-M_{T_{k}}^{2}\right)^{+}\right] \tag{25}
\end{equation*}
$$

Substituting $L\left(t, T_{j}\right)$ in Equations (19) and (20) by its explicit form (Equation (6)) and using the fact that $L^{T_{j+1}}$ and $L^{T^{*}}$ differ only by a drift term, we get the representation

$$
\begin{align*}
M_{t}^{1}= & \prod_{j=k+1}^{n-1}\left[1+\delta L\left(0, T_{j}\right) \exp \left(\int_{0}^{t} \lambda\left(s, T_{j}\right) d L_{s}^{T^{*}}+d\left(t, T_{j}\right)\right)\right] \\
& \times \frac{L\left(0, T_{k}\right)}{K} \exp \left(\int_{0}^{t} \lambda\left(s, T_{k}\right) d L_{s}^{T^{*}}+d\left(t, T_{k}\right)\right) \tag{26}
\end{align*}
$$

and

$$
\begin{equation*}
M_{t}^{2}=\prod_{j=k+1}^{n-1}\left(1+\delta L\left(0, T_{j}\right) \exp \left(\int_{0}^{t} \lambda\left(s, T_{j}\right) d L_{s}^{T^{*}}+d\left(t, T_{j}\right)\right)\right) \tag{27}
\end{equation*}
$$

For small values $|x|$ and $\varepsilon>0$, we use now the approximative relation

$$
\begin{equation*}
1+\varepsilon \exp (x) \approx(1+\varepsilon) \exp \left(\frac{\varepsilon}{1+\varepsilon} x\right) \tag{28}
\end{equation*}
$$

As a consequence, we can approximate

$$
1+\delta L\left(0, T_{j}\right) \exp \left(\int_{0}^{t} \lambda\left(s, T_{j}\right) d L_{s}^{T^{*}}+d\left(t, T_{j}\right)\right)
$$

by

$$
\left(1+\delta L\left(0, T_{j}\right)\right) \exp \left(\int_{0}^{t} \ell\left(0, T_{j}\right) \lambda\left(s, T_{j}\right) d L_{s}^{T^{*}}+\ell\left(0, T_{j}\right) d\left(t, T_{j}\right)\right)
$$

$\left(M_{t}^{1}\right)_{0 \leq t \leq T_{k}}$ and $\left(M_{t}^{2}\right)_{0 \leq t \leq T_{k+1}}$ are $\mathbb{P}_{T^{*}}$ martingales. We replace now the factors in Equations (26) and (27) by the approximating terms and determine appropriate exponential compensators $D^{1}$ and $D^{2}$ such that we get again $\mathbb{P}_{T^{*}}$ martingales. The resulting processes $\left(\widetilde{M}_{t}^{1}\right)_{0 \leq t \leq T_{k}}$ and $\left(\widetilde{M}_{t}^{2}\right)_{0 \leq t \leq T_{k+1}}$ can explicitly be written in the form

$$
\widetilde{M}_{t}^{1}=\frac{L\left(0, T_{k}\right)}{K} \frac{B\left(0, T_{k+1}\right)}{B\left(0, T^{*}\right)} \exp \left(\int_{0}^{t} f^{k}(s) d L_{s}^{T^{*}}+\int_{0}^{t} \lambda\left(s, T_{k}\right) d L_{s}^{T^{*}}+D_{t}^{1}\right)
$$

and

$$
\widetilde{M}_{t}^{2}=\frac{B\left(0, T_{k+1}\right)}{B\left(0, T^{*}\right)} \exp \left(\int_{0}^{t} f^{k}(s) d L_{s}^{T^{*}}+D_{t}^{2}\right)
$$

where

$$
D_{t}^{1}=\ln \left(\mathbb{E}_{\mathbb{P}_{T^{*}}}\left[\exp \left(\int_{0}^{t} f^{k}(s) d L_{s}^{T^{*}}+\int_{0}^{t} \lambda\left(s, T_{k}\right) d L_{s}^{T^{*}}\right)\right]^{-1}\right)
$$

and

$$
D_{t}^{2}=\ln \left(\mathbb{E}_{\mathbb{P}_{T^{*}}}\left[\exp \left(\int_{0}^{t} f^{k}(s) d L_{s}^{T^{*}}\right)\right]^{-1}\right)
$$

With the new processes we get an approximative formula for the caplet price (Equation (25))

$$
\begin{equation*}
\widetilde{\operatorname{Cpl}} t_{0}\left(T_{k}, K, \delta\right) \approx \delta K B\left(0, T^{*}\right) \mathbb{E}_{\mathbb{P}_{T^{*}}}\left[\left(\widetilde{M}_{T_{k}}^{1}-\widetilde{M}_{T_{k}}^{2}\right)^{+}\right] \tag{29}
\end{equation*}
$$

Define

$$
\begin{equation*}
\widetilde{\operatorname{Cplt}_{0}}\left(T_{k}, K, \delta\right):=\delta K B\left(0, T^{*}\right) \mathbb{E}_{\mathbb{P}_{T^{*}}}\left[\left(\widetilde{M}_{T_{k}}^{1}-\widetilde{M}_{T_{k}}^{2}\right)^{+}\right], \tag{30}
\end{equation*}
$$

and

$$
\widetilde{\Delta}\left(L\left(0, T_{k}\right)\right):=\frac{\partial \widetilde{C_{p l t}}\left(T_{k}, K, \delta\right)}{\partial L\left(0, T_{k}\right)}
$$

Then

$$
\begin{align*}
\widetilde{C p l t}_{0}\left(T_{k}, K, \delta\right) & \left.\approx{\widetilde{\operatorname{Cpl}} t_{0}}^{( } T_{k}, K, \delta\right)  \tag{31}\\
\Delta\left(L\left(0, T_{k}\right)\right) & \approx \widetilde{\Delta}\left(L\left(0, T_{k}\right)\right) \tag{32}
\end{align*}
$$

Since $\widetilde{M}^{1}$ and $\widetilde{M}^{2}$ are $\mathbb{P}_{T^{*}}$ martingales, we can introduce a $\widetilde{\mathbb{P}}_{T_{k+1}}$ forward measure on ( $\Omega, \mathcal{F}_{T_{k+1}}$ ) by setting

$$
\frac{d \widetilde{\mathbb{P}}_{T_{k+1}}}{d \mathbb{P}_{T^{*}}}=\frac{\widetilde{M}_{T_{k+1}}^{2}}{\widetilde{M}_{0}^{2}}=\exp \left(\int_{0}^{T_{k+1}} f^{k}(s) d L_{s}^{T^{*}}+D_{T_{k+1}}^{2}\right)
$$

By the semimartingale version of Girsanov's theorem (see Jacod and Shiryaev (1987))

$$
\begin{equation*}
\widetilde{W}_{t}^{T_{k+1}}:=W_{t}^{T^{*}}-\int_{0}^{t} c_{s}^{\frac{1}{2}} f^{k}(s) d s \tag{33}
\end{equation*}
$$

is a $\widetilde{\mathbb{P}}_{T_{k+1}}$ standard Brownian motion and

$$
\begin{equation*}
\widetilde{v}^{T_{k+1}}(d t, d x):=\exp \left(f^{k}(s) x\right) v^{T^{*}}(d t, d x) \tag{34}
\end{equation*}
$$

is the $\widetilde{\mathbb{P}}_{T_{k+1}}$ compensator of $\mu^{T^{*}}$. Expressing Equation (30) in terms of the new measure, we get

$$
\begin{equation*}
\widetilde{\operatorname{Cplt}_{0}}\left(T_{k}, K, \delta\right)=\delta K B\left(0, T_{k+1}\right) \mathbb{E}_{\tilde{\mathbb{P}}_{T_{k+1}}}\left[\left(\exp \left(X_{T_{k}}\right)-1\right)^{+}\right], \tag{35}
\end{equation*}
$$

where $X$ is defined as the process

$$
X_{t}=\ln \frac{\widetilde{M}_{t}^{1}}{\widetilde{M}_{t}^{2}}=\ln \left(\frac{L\left(0, T_{k}\right)}{K}\right)+\int_{0}^{t} \lambda\left(s, T_{k}\right) d L_{s}^{T^{*}}+D_{t}^{1}-D_{t}^{2}
$$

Proposition 3.4. Suppose $R \in(1,1+\varepsilon)$ such that the moment-generating function of $X_{T_{k}}$ with respect to $\widetilde{\mathbb{P}}_{T_{k+1}}$ is finite, then

$$
\begin{align*}
\tilde{\Delta}\left(L\left(0, T_{k}\right)\right)= & \frac{\delta}{2 \pi} B\left(0, T_{k+1}\right) \int_{\mathbb{R}}\left\langle( \frac { L ( 0 , T _ { k } ) } { K } ) ^ { - 1 + R + i u } \operatorname { e x p } \left\{\int _ { 0 } ^ { T _ { k } } \left(\int _ { \mathbb { R } } \left[ e^{x f^{k}(s)}\left(e^{x(R+i u) \lambda\left(s, T_{k}\right)}-1\right)\right.\right.\right.\right. \\
& \left.-(R+i u) e^{x f^{k}(s)}\left(e^{x \lambda\left(s, T_{k}\right)}-1\right)\right] F_{s}^{T^{*}}(d x) \\
& \left.\left.\left.+\frac{1}{2} c_{s}\left(\lambda\left(s, T_{k}\right)\right)^{2}(R+i u)(-1+R+i u)\right) d s\right\} \frac{1}{-1+R+i u}\right\rangle d u \tag{36}
\end{align*}
$$

Proof. Using the Fourier-based valuation approach (for details, see Theorem 2.7 in Eberlein, Glau, and Papapantoleon (2010)), we get the following explicit integral representation for Equation (35). Suppose $R \in(1,1+\varepsilon)$ such that the momentgenerating function of $X_{T_{k}}$ with respect to $\widetilde{\mathbb{P}}_{T_{k+1}}$ is finite at $R$, i.e. $\widetilde{M}_{X_{T_{k}}}(R)<\infty$, then

$$
\widetilde{C p l t}_{0}\left(T_{k}, K, \delta\right)=\frac{K \delta}{2 \pi} B\left(0, T_{k+1}\right) \int_{\mathbb{R}} \widetilde{M}_{X_{T k}}(R+i u) \frac{1}{(-R-i u)(1-R-i u)} d u
$$

where the moment-generating function $\widetilde{M}_{X_{T_{k}}}$ is given by (for details, we refer to Eberlein and Kluge (2005))

$$
\begin{align*}
\widetilde{M}_{X_{T_{k}}}(z)= & \left(\frac{L\left(0, T_{k}\right)}{K}\right)^{z} \\
& \times \exp \left(\int _ { 0 } ^ { T _ { k } } \left[\theta_{s}\left(f^{k}(s)+z \lambda\left(s, T_{k}\right)\right)-z \theta_{s}\left(f^{k}(s)+\lambda\left(s, T_{k}\right)\right)\right.\right.  \tag{37}\\
& \left.\left.+(z-1) \theta_{s}\left(f^{k}(s)\right)\right] d s\right)
\end{align*}
$$

for all $z \in \mathbb{C}$ with $\operatorname{Re}(z)=R$, with cumulant function

$$
\begin{equation*}
\theta_{s}(z)=z b_{s}^{T^{*}}+\frac{1}{2} c_{s} z^{2}+\int_{\mathbb{R}}\left(e^{z x}-1-z x\right) F_{s}^{T^{*}}(d x) \tag{38}
\end{equation*}
$$

and

$$
\begin{equation*}
f^{k}(s)=\sum_{j=k+1}^{n-1} \frac{\delta L\left(0, T_{j}\right)}{1+\delta L\left(0, T_{j}\right)} \lambda\left(s, T_{j}\right) \tag{39}
\end{equation*}
$$

Taking into account the choice of the drift coefficient in Equation (13), the cumulant function $\theta_{s}$ and the moment-generating function $\widetilde{M}_{X_{T_{k}}}$, respectively, become

$$
\theta_{s}(z)=z \int_{\mathbb{R}}\left(\frac{e^{x z}-1}{z}-\frac{e^{x \lambda\left(s, T_{n-1}\right)}-1}{\lambda\left(s, T_{n-1}\right)}\right) F_{s}^{T^{*}}(d x)+\frac{1}{2} c_{s} z\left(z-\lambda\left(s, T_{n-1}\right)\right),
$$

$$
\begin{gather*}
\tilde{M}_{X_{T_{k}}}(z)=\left(\frac{L\left(0, T_{k}\right)}{K}\right)^{z} \exp \left\{\int _ { 0 } ^ { T _ { k } } \left(\int _ { \mathbb { R } } \left[e^{x f^{k}(s)}\left(e^{x z \lambda\left(s, T_{k}\right)}-1\right)\right.\right.\right.  \tag{40}\\
\left.-z e^{x f^{k}(s)}\left(e^{x \lambda\left(s, T_{k}\right)}-1\right)\right] F_{s}^{T^{*}}(d x) \\
\left.\left.\quad+\frac{1}{2} c_{s}\left(\lambda\left(s, T_{k}\right)\right)^{2} z(z-1)\right) d s\right\} .
\end{gather*}
$$

Hence, the approximative valuation formula can be written as

$$
\begin{aligned}
\widetilde{C p l} l t_{0}\left(T_{k}, K, \delta\right)= & \frac{K \delta}{2 \pi} B\left(0, T_{k+1}\right) \int_{\mathbb{R}}\left\langle( \frac { L ( 0 , T _ { k } ) } { K } ) ^ { R + i u } \operatorname { e x p } \left\{\int _ { 0 } ^ { T _ { k } } \left(\int _ { \mathbb { R } } \left[ e^{x f^{k}(s)}\left(e^{x(R+i u) \lambda\left(s, T_{k}\right)}-1\right)\right.\right.\right.\right. \\
& \left.-(R+i u) e^{x f^{k}(s)}\left(e^{x \lambda\left(s, T_{k}\right)}-1\right)\right] F_{s}^{T^{*}}(d x) \\
& \left.\left.+\frac{1}{2} c_{s}\left(\lambda\left(s, T_{k}\right)\right)^{2}(R+i u)(-1+R+i u)\right) d s\right\} \\
& \left.\times \frac{1}{(-R-i u)(1-R-i u)}\right\rangle d u .
\end{aligned}
$$

We conclude that the Delta computed using the approximative Fourier-based valuation method is given by

$$
\begin{align*}
\tilde{\Delta}\left(L\left(0, T_{k}\right)\right)= & \frac{\delta}{2 \pi} B\left(0, T_{k+1}\right) \int_{\mathbb{R}}\left\langle( \frac { L ( 0 , T _ { k } ) } { K } ) ^ { - 1 + R + i u } \operatorname { e x p } \left\{\int _ { 0 } ^ { T _ { k } } \left(\int _ { \mathbb { R } } \left[ e^{x f^{k}(s)}\left(e^{x(R+i u) \lambda\left(s, T_{k}\right)}-1\right)\right.\right.\right.\right. \\
& \left.-(R+i u) e^{x f^{k}(s)}\left(e^{x \lambda\left(s, T_{k}\right)}-1\right)\right] F_{s}^{T^{*}}(d x) \\
& \left.\left.\left.+\frac{1}{2} c_{s}\left(\lambda\left(s, T_{k}\right)\right)^{2}(R+i u)(-1+R+i u)\right) d s\right\} \frac{1}{-1+R+i u}\right\rangle d u . \tag{41}
\end{align*}
$$

Example 3.5 (Variance Gamma (VG) component) We suppose that the jump component of the driving process $L^{T^{*}}$ given in Equation (3) is described by a Variance Gamma process with the Lévy density $v$ given by

$$
v(d x)=F_{V G}(x) d x=\frac{1}{\eta|x|} \exp \left(\frac{\theta}{\sigma^{2}} x-\frac{1}{\sigma} \sqrt{\frac{2}{\eta}+\frac{\theta^{2}}{\sigma^{2}}}|x|\right) d x,
$$

where $(\theta, \sigma, \eta)$ are the parameters such that $\theta \in \mathbb{R}, \sigma>0$ and $\eta>0$. A more convenient parametrization is achieved by setting $B=\frac{\theta}{\sigma^{2}}$ and $C=\frac{1}{\sigma} \sqrt{\frac{2}{\eta}+\frac{\theta^{2}}{\sigma^{2}}}$. Then

$$
\begin{equation*}
F_{V G}(x)=\frac{\exp (B x-C|x|)}{\eta|x|} \tag{42}
\end{equation*}
$$

Using Frullani's integral (see, for details, Ostrowski (1949)), we can show that, if $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ such that $\mathcal{R e}(\alpha)>0, \quad \mathcal{R e}(\beta)>0$ and $\frac{\beta}{\alpha} \in \mathbb{C} \backslash \mathbb{R}^{-}$where $\left.\left.\mathbb{R}^{-}=\right]-\infty ; 0\right]$, then

$$
\begin{equation*}
I_{(\alpha, \beta)}:=\int_{0}^{+\infty} \frac{e^{-\alpha x}-e^{-\beta x}}{x} d x=\log \left(\frac{\beta}{\alpha}\right) \tag{43}
\end{equation*}
$$

where $\log$ is the principal value of the logarithm. Exploiting this formula and setting

$$
\begin{gather*}
\alpha_{k}(s, z)=-\left(z \lambda\left(s, T_{k}\right)+f^{k}(s)+B-C\right)  \tag{44}\\
\beta_{k}(s)=-\left(f^{k}(s)+B-C\right) \tag{45}
\end{gather*}
$$

a tedious computation shows that the moment-generating function becomes

$$
\begin{aligned}
\widetilde{M}_{X_{T_{k}}}(z)= & \left(\frac{L\left(0, T_{k}\right)}{K}\right)^{z} \times \exp \left(\frac{1}{\eta} \int_{0}^{T_{k}} \log \left(\frac{\beta_{k}(s)\left(2 C-\beta_{k}(s)\right)}{\alpha_{k}(s, z)\left(2 C-\alpha_{k}(s, z)\right)}\right) d s\right) \\
& \times \exp \left(-\frac{z}{\eta} \int_{0}^{T_{k}} \log \left(\frac{\beta_{k}(s)\left(2 C-\beta_{k}(s)\right)}{\left(\beta_{k}(s)-\lambda\left(s, T_{k}\right)\right)\left(2 C-\beta_{k}(s)+\lambda\left(s, T_{k}\right)\right)}\right) d s\right) \\
& \times \exp \left(\int_{0}^{T_{k}} \frac{c_{s}}{2} \lambda^{2}\left(s, T_{k}\right) z(z-1) d s\right)
\end{aligned}
$$

Plugging this in the approximative valuation formula, we get

$$
\begin{aligned}
\widetilde{C p l} l t_{0}\left(T_{k}, K, \delta\right)= & \frac{K \delta}{2 \pi} B\left(0, T_{k+1}\right) \int_{\mathbb{R}}\left(\frac{L\left(0, T_{k}\right)}{K}\right)^{R+i u} \\
& \times \exp \left(\frac{1}{\eta} \int_{0}^{T_{k}} \log \left(\frac{\beta_{k}(s)\left(2 C-\beta_{k}(s)\right)}{\alpha_{k}(s, R+i u)\left(2 C-\alpha_{k}(s, R+i u)\right)}\right) d s\right) \\
& \times \exp \left(-\frac{R+i u}{\eta} \int_{0}^{T_{k}} \log \left(\frac{\beta_{k}(s)\left(2 C-\beta_{k}(s)\right)}{\left(\beta_{k}(s)-\lambda\left(s, T_{k}\right)\right)\left(2 C-\beta_{k}(s)+\lambda\left(s, T_{k}\right)\right)}\right) d s\right) \\
& \times \exp \left(\int_{0}^{T_{k}} \frac{c_{s}}{2} \lambda^{2}\left(s, T_{k}\right)(R+i u)(-1+R+i u) d s\right) \frac{1}{(-R-i u)(1-R-i u)} d u .
\end{aligned}
$$

The approximate Delta is therefore given by

$$
\begin{aligned}
\tilde{\Delta}\left(L\left(0, T_{k}\right)\right)= & \frac{\delta}{2 \pi} B\left(0, T_{k+1}\right) \int_{\mathbb{R}}\left(\frac{L\left(0, T_{k}\right)}{K}\right)^{-1+R+i u} \\
& \times \exp \left(\frac{1}{\eta} \int_{0}^{T_{k}} \log \left(\frac{\beta_{k}(s)\left(2 C-\beta_{k}(s)\right)}{\alpha_{k}(s, R+i u)\left(2 C-\alpha_{k}(s, R+i u)\right)}\right) d s\right) \\
& \times \exp \left(-\frac{R+i u}{\eta} \int_{0}^{T_{k}} \log \left(\frac{\beta_{k}(s)\left(2 C-\beta_{k}(s)\right)}{\left(\beta_{k}(s)-\lambda\left(s, T_{k}\right)\right)\left(2 C-\beta_{k}(s)+\lambda\left(s, T_{k}\right)\right)}\right) d s\right) \\
& \times \exp \left(\int_{0}^{T_{k}} \frac{c_{s}}{2} \lambda^{2}\left(s, T_{k}\right)(R+i u)(-1+R+i u) d s\right) \frac{1}{-1+R+i u} d u .
\end{aligned}
$$

For the approximate Gamma, one gets the form

$$
\begin{aligned}
\widetilde{\Gamma}\left(L\left(0, T_{k}\right)\right):= & \frac{\partial^{2} \widetilde{C p l} t_{0}\left(T_{k}, K, \delta\right)}{\partial^{2} L\left(0, T_{k}\right)}=\frac{\delta}{2 \pi K} B\left(0, T_{k+1}\right) \int_{\mathbb{R}}\left(\frac{L\left(0, T_{k}\right)}{K}\right)^{-2+R+i u} \\
& \times \exp \left(\frac{1}{\eta} \int_{0}^{T_{k}} \log \left(\frac{\beta_{k}(s)\left(2 C-\beta_{k}(s)\right)}{\alpha_{k}(s, R+i u)\left(2 C-\alpha_{k}(s, R+i u)\right)}\right) d s\right) \\
& \times \exp \left(-\frac{R+i u}{\eta} \int_{0}^{T_{k}} \log \left(\frac{\beta_{k}(s)\left(2 C-\beta_{k}(s)\right)}{\left(\beta_{k}(s)-\lambda\left(s, T_{k}\right)\right)\left(2 C-\beta_{k}(s)+\lambda\left(s, T_{k}\right)\right)}\right) d s\right) \\
& \times \exp \left(\int_{0}^{T_{k}} \frac{c_{s}}{2} \lambda^{2}\left(s, T_{k}\right)(R+i u)(-1+R+i u) d s\right) d u .
\end{aligned}
$$

Example 3.6. (Non-homogeneous Gamma (IGP) component) We suppose that the jump component of the driving process $L^{T^{*}}$ given in Equation (3) is described by an inhomogeneous Gamma process (IGP), which is introduced by Berman (1981) as follows

Definition 3.7. Let $A(t)$ be a non-decreasing function from $\mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$and $B>0$. A Gamma process with shape function $A$ and scale parameter $B$ is a stochastic process $\left(L_{t}\right)_{t \geq 0}$ on $\mathbb{R}^{+}$such that
(1) $L_{0}=0$,
(2) Independent increments: for every increasing sequence of times $t_{0}, \ldots, t_{n}$, the random variables $L_{t_{0}}, L_{t_{1}}-L_{t_{0}}, \ldots, L_{t_{n}}-L_{t_{n-1}}$ are independent,
(3) For $0 \leq s<t$, the distribution of the random variable $L_{t}-L_{s}$ is the Gamma distribution $\Gamma(A(t)-A(s), B)$.

Remark 3.8. If the shape function $A$ is differentiable, we can write the shape function $A$ in the form

$$
A(t)=\int_{0}^{t} \dot{A}(s) d s+A(0)
$$

for all $t \in \mathbb{R}^{+}$where $\dot{A}$ denotes the derivative of $A$. In this case, the Lévy measure of the Gamma process $L$ is given by

$$
F_{s}^{G}(d x)=\dot{A}(s) \frac{e^{-B x}}{x} \mathbf{1}_{x>0} d x
$$

The approximate Delta is now

$$
\begin{aligned}
\widetilde{\Delta}\left(L\left(0, T_{k}\right)\right)= & \frac{\partial \widetilde{\operatorname{Cpl} t_{0}}\left(T_{k}, K, \delta\right)}{\partial L\left(0, T_{k}\right)} \\
= & \frac{\delta}{2 \pi} B\left(0, T_{k+1}\right) \int_{\mathbb{R}}\left\langle\left(\frac{L\left(0, T_{k}\right)}{K}\right)^{-1+R+i u}\right. \\
& \times \exp \left\{\int _ { 0 } ^ { T _ { k } } \left(\dot{A}(s) \log \left(\frac{-f^{k}(s)+B}{-(R+i u) \lambda\left(s, T_{k}\right)-f^{k}(s)+B}\right)\right.\right. \\
& -\dot{A}(s)(R+i u) \log \left(\frac{-f^{k}(s)+B}{-\lambda\left(s, T_{k}\right)-f^{k}(s)+B}\right) \\
& \left.\left.+\frac{1}{2} c_{s}\left(\lambda\left(s, T_{k}\right)\right)^{2}(R+i u)(-1+R+i u)\right) d s\right\} \\
& \left.\times \frac{1}{-1+R+i u}\right\rangle d u
\end{aligned}
$$

whereas the approximate Gamma is

$$
\begin{aligned}
\widetilde{\Gamma}\left(L\left(0, T_{k}\right)\right)= & \frac{\partial^{2} \widetilde{C p l t}_{0}\left(T_{k}, K, \delta\right)}{\partial^{2} L\left(0, T_{k}\right)} \\
= & \frac{\delta}{2 \pi K} B\left(0, T_{k+1}\right) \iint_{\mathbb{R}}\left\langle\left(\frac{L\left(0, T_{k}\right)}{K}\right)^{-2+R+i u}\right. \\
& \times \exp \left\{\int _ { 0 } ^ { T _ { k } } \left(\dot{A}(s) \log \left(\frac{-f^{k}(s)+B}{-(R+i u) \lambda\left(s, T_{k}\right)-f^{k}(s)+B}\right)\right.\right. \\
& -\dot{A}(s)(R+i u) \log \left(\frac{-f^{k}(s)+B}{-\lambda\left(s, T_{k}\right)-f^{k}(s)+B}\right) \\
& \left.\left.\left.+\frac{1}{2} c_{s}\left(\lambda\left(s, T_{k}\right)\right)^{2}(R+i u)(-1+R+i u)\right) d s\right\}\right\rangle d u .
\end{aligned}
$$

## Note

1. See Fournié et al. (1999) for details

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## Appendix Malliavin calculus for time-inhomogeneous Lévy processes

For the ease of reading, we present in this appendix the Malliavin derivative $D$ as well as the Skorohod integral $\delta$ for the class of processes which we use. Yablonski (2008) studied these notions in a slightly more general context, namely for processes with conditionally independent increments. Conditioning is not relevant in our case. The $\sigma$ field $\mathcal{H}$ considered by Yablonski is trivial for time-inhomogeneous Lévy processes.
A. 1 Isonormal Lévy process (ILP)

Let $\mu$ and $v$ be $\sigma$ finite measures without atoms on the measurable spaces $(\mathrm{T}, \mathcal{A})$ and $\left(\mathrm{T} \times \mathrm{X}_{0}, \mathcal{B}\right)$, respectively. Define a new measure

$$
\begin{equation*}
\pi(d t, d z):=\mu(d t) \delta_{\Theta}(d z)+v(d t, d z) \tag{A1}
\end{equation*}
$$

on a measurable space $(\mathrm{T} \times \mathrm{X}, \mathcal{G})$, where $\mathrm{X}=\mathrm{X}_{0} \cup\{\Theta\}, \mathcal{G}=\sigma(\mathcal{A} \times\{\Theta\}, \mathcal{B})$ and $\delta_{\Theta}(d z)$ is the measure which gives mass one to the point $\Theta$. Consider the Hilbert space $H=L^{2}(T \times X, \mathcal{G}, \pi)$ and assume that this space is separable.
Definition A.1. We say that a stochastic process $\mathrm{L}=\{\mathrm{L}(h), h \in \mathrm{H}\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ is an isonormal Lévy process (or Lévy process on $H$ ) if the following conditions are satisfied
(1) The mapping $h \rightarrow L(h)$ is linear;
(2) $\mathrm{E}\left[e^{i x L(h)}\right]=\exp (\Psi(x, h))$, where

$$
\Psi(x, h)=\int_{\mathrm{T} \times \mathrm{X}}\left(\left(e^{i x h(t, z)}-1-i x h(t, z)\right) \mathbf{1}_{\mathrm{X}_{0}}(z)-\frac{1}{2} x^{2} h^{2}(t, z) \mathbf{1}_{\Theta}(z)\right) \pi(d t, d z) .
$$

Remark A.2. It is easy to show that $\mathrm{E}[L(h)]=0$ and $\mathrm{E}[L(h) L(g)]=\langle h ; g\rangle_{H}$ for all $h, g \in H$.

## A. 2 The derivative operator

In this section, we introduce the derivative operator $D$ which coincides with the classical Malliavin derivative in the Gaussian case (see, e.g., Nualart (2006)) and with the difference operator defined in Nualart and Vives (1990) and Picard (1996) in the Poisson case.

Let $\mathcal{S}$ denote the class of smooth random variables, that is the class of random variables $\xi$ of the form

$$
\begin{equation*}
\xi=f\left(L\left(h_{1}\right), \ldots, L\left(h_{n}\right)\right), \tag{A2}
\end{equation*}
$$

where $f$ belongs to $\mathrm{C}_{b}^{\infty}\left(\mathbb{R}^{n}\right), h_{1}, \ldots, h_{n}$ are in $H$, and $n \geq 1$. The set $\mathcal{S}$ is dense in $L^{p}(\Omega)$ for any $p \geq 1$. We refer for the following definition to Yablonski (2008).

Definition A.3. The stochastic derivative of a smooth random variable of the form (Equation (A2)) is the $H$-valued random variable $D \xi=\left\{D_{t, x} \xi,(t, x) \in T \times X\right\}$ given by

$$
\begin{align*}
D_{t, x} \xi= & \sum_{k=1}^{n} \frac{\partial f}{\partial y_{k}}\left(L\left(h_{1}\right), \ldots, L\left(h_{n}\right)\right) h_{k}(t, x) \mathbf{1}_{\Theta}(x)  \tag{A3}\\
& +\left(f\left(L\left(h_{1}\right)+h_{1}(t, x), \ldots, L\left(h_{n}\right)+h_{n}(t, x)\right)\right. \\
& \left.-f\left(L\left(h_{1}\right), \ldots, L\left(h_{n}\right)\right)\right) \mathbf{1}_{X_{0}}(x)
\end{align*}
$$

We will consider $D \xi$ as an element of $L^{2}(T \times X \times \Omega) \cong L^{2}(\Omega ; H)$; namely $D \xi$ is a random process indexed by the parameter space $T \times X$.
(1) If the measure $v$ is zero or $h_{k}(t, x)=0, k=1, \ldots, n$ when $x \neq \Theta$ then $D \xi$ coincides with the Malliavin derivative (see, e.g., Nualart (2006)).
(2) If the measure $\mu$ is zero or $h_{k}(t, x)=0, k=1, \ldots, n$ when $x=\Theta$ then $D \xi$ coincides with the difference operator (see, e.g., Picard (1996)).

## A. 3 Integration-by-parts formula

Theorem A.4. Suppose that $\xi$ and $\eta$ are smooth random variables and $h \in H$. Then
(1) $\mathrm{E}[\xi L(h)]=\mathrm{E}\left[\langle D \xi ; h\rangle_{H}\right]$,
(2) $\mathrm{E}[\xi \eta L(h)]=\mathrm{E}\left[\eta\langle D \xi ; h\rangle_{H}\right]+\mathrm{E}\left[\xi\langle D \eta ; h\rangle_{H}\right]+\mathrm{E}\left[\left\langle D \eta ; h \mathbf{1}_{X_{0}} D \xi\right\rangle_{H}\right]$.

As a consequence of the above theorem we obtain the following result

- The expression of the derivative $D \xi$ given in Equation (A3) does not depend on the particular representation of $\xi$ in Equation (A2).
- The operator $D$ is closable as an operator from $L^{2}(\Omega)$ to $L^{2}(\Omega ; H)$.


## A. 4 The chain rule

We will denote the closure of $D$ again by $D$ and its domain in $L^{2}(\Omega)$ by $\mathbb{D}^{1,2}$.
Proposition A. 5 (see Yablonski (2008), Proposition 4.8) Suppose $F=\left(F_{1}, F_{2}, \ldots, F_{n}\right)$ is a random vector whose components belong to the space $\mathbb{D}^{1,2}$. Let $\phi \in \mathcal{C}^{1}\left(\mathbb{R}^{n}\right)$ be a function with bounded partial derivatives such that $\phi(F) \in \mathrm{L}^{2}(\Omega)$. Then $\phi(F) \in \mathbb{D}^{1,2}$ and

$$
D_{t, x} \phi(F)= \begin{cases}\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_{i}}(F) D_{t, \Theta} F_{i} & x=\Theta \\ \phi\left(F_{1}+D_{t, x} F_{1}, \ldots, F_{n}+D_{t, x} F_{n}\right)-\phi\left(F_{1}, \ldots, F_{n}\right) & x \neq \Theta\end{cases}
$$

## A. 5 The Skorohod integral

In this section, we consider the adjoint of the operator $D$ which coincides with the Skorohod integral (see Skorokhod (1976)) in the Gaussian case and with the extended stochastic integral introduced by Kabanov (see Kabanov (1975)) in the pure jump Lévy case. See also Benth et al. (2003) and Lokka and Benth (2004). Consequently, it can be considered as a generalization of the stochastic integral.

We recall that the derivative operator $D$ is a closed and unbounded operator defined on the dense subset $\mathbb{D}^{1,2}$ of $\mathrm{L}^{2}(\Omega)$ with values in $\mathrm{L}^{2}(\Omega ; H)$.

We denote by $\delta$ the adjoint of the operator $D$ and we call it the Skorohod integral. The operator $\delta$ is defined on a subset $\operatorname{Dom} \delta$ of $\mathrm{L}^{2}(\Omega ; H)$ with values in $\mathrm{L}^{2}(\Omega)$, where Dom $\delta$ is the set of processes $u \in \mathrm{~L}^{2}(\Omega ; H)$ such that

$$
\left|\mathrm{E}\left[\int_{T \times X} D_{t, z} F u(t, z) \pi(d t, d z)\right]\right| \leq c\|F\|_{\mathrm{L}^{2}(\Omega)}
$$

for all $F \in \mathbb{D}^{1,2}$ and where $c$ is some constant depending on $u$. If $u \in \operatorname{Dom} \delta$, then $\delta(u)$ is the element of $\mathrm{L}^{2}(\Omega)$ such that

$$
\begin{equation*}
\mathrm{E}[F \delta(u)]=\mathrm{E}\left[\int_{T \times X} D_{t, z} F u(t, z) \pi(d t, d z)\right] \tag{A4}
\end{equation*}
$$

for any $F \in \mathbb{D}^{1,2}$. Let us note that $\delta$ is a closed and unbounded operator on Dom $\delta$.
A. 6 Commutativity relationship between the derivative and divergence operators

Let $\mathbb{L}^{1,2}$ denote the class of processes $u \in \mathrm{~L}^{2}(T \times X \times \Omega)$ such that $u(t, x) \in \mathbb{D}^{1,2}$ for almost all $(t, x)$, and such that there exists a measurable version of the multi-process $D_{t, x} u(s, y)$ satisfying

$$
\mathrm{E}\left[\int_{T \times X} \int_{T \times X}\left(D_{t, x} u(s, y)\right)^{2} \pi(d t, d x) \pi(d s, d y)\right]<\infty
$$

Proposition A.6. Suppose that $u \in \mathbb{L}^{1,2}$ such that for almost all $(t, z) \in T \times X$ the two-parameter process $\left(D_{t, z} u(s, y)\right)_{(s, y) \in T \times X}$ is Skorohod integrable, and such that there exists a version of the process $\left(\delta\left(D_{t, z} u(., .)\right)\right)_{(t, z) \in T \times X}$ which belongs to $\mathrm{L}^{2}(T \times X \times \Omega)$. Then $\delta(u) \in \mathbb{D}^{1,2}$ and we have

$$
\begin{equation*}
D_{t, z} \delta(u)=u(t, z)+\delta\left(D_{t, z} u(., .)\right) \tag{A5}
\end{equation*}
$$

A. 7 The Itô stochastic integral as a particular case of the Skorohod integral

Let $W=\left\{W_{t}, 0 \leq t \leq T\right\}$ be a $d$-dimensional standard Brownian motion, $\tilde{N}$ a compensated Poisson random measure on $[0, T] \times \mathbb{R}^{d} \backslash\{0\}$ with compensator $v(d t, d x)=\beta_{t}(d x) d t$, where $\left(\beta_{t}\right)_{t \in[0, T]}$ is a family of Lévy measures satisfying $\int_{T}^{0}\left(\int_{\mathbb{R}^{d}}\left(|x|^{2} \wedge 1\right) \beta_{t}(d x)\right) d t<\infty$. For each $t \in$ $[0, T]$ denote by $\mathcal{F}_{t}$ the $\sigma$ algebra generated by the random variables

$$
\left\{W_{s}^{i}, \tilde{N}((0, s] \times A) ; 0 \leq s \leq t, i=1, \ldots, d, A \in \mathcal{B}\left(\mathbb{R}^{d} \backslash\{0\}\right), \sup _{0 \leq s \leq t} \beta_{s}(A)<\infty\right\}
$$

and the null sets of $\mathcal{F}$. We denote by $L_{p}^{2}$ the subset of $L^{2}(\Omega ; H)$ formed by the $\left(\mathcal{F}_{t}\right)$ predictable processes.
Proposition A. $7 L_{p}^{2} \subset D o m \delta$, and the restriction of the operator $\delta$ to the space $L_{p}^{2}$ coincides with the usual stochastic integral, that is

$$
\begin{equation*}
\delta(u)=\sum_{i=1}^{d} \int_{0}^{T} u^{i}(t, 0) d W_{t}^{i}+\int_{0}^{T} \int_{\mathbb{R}^{d} \backslash\{0\}} u(t, x) \tilde{N}(d t, d x) \tag{A6}
\end{equation*}
$$

## A. 8 Regularity of solutions of SDEs driven by time-inhomogeneous Lévy processes

We focus on a class of models in which the price of the underlying asset is given by the following stochastic differential equation (see Di Nunno, Øksendal, and Proske (2009) and Petrou (2008) for details)

$$
\begin{gather*}
d S_{t}=b\left(t, S_{t-}\right) d t+\sigma\left(t, S_{t-}\right) d W_{t}+\int_{\mathbb{R}_{0}} \varphi\left(t, S_{t-}, z\right) \tilde{N}(d t, d z),  \tag{A7}\\
S_{0}=x
\end{gather*}
$$

where $x \in \mathbb{R}^{d},\left\{W_{t}, 0 \leq t \leq T\right\}$ is a $m$-dimensional standard Brownian motion, $\tilde{N}$ is a compensated Poisson random measure on $[0, T] \times \mathbb{R}_{0}$ with compensator $v_{t}(d z) d t$. The coefficients $b: \mathbb{R}^{+} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \sigma: \mathbb{R}^{+} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d} \times \mathbb{R}^{m}$ and $\varphi: \mathbb{R}^{+} \times \mathbb{R}^{d} \times \mathbb{R} \rightarrow \mathbb{R}^{d} \times \mathbb{R}$ are continuously differentiable with bounded derivatives and the family of positive measures $\left(v_{t}\right)_{t \in[0, T]}$ satisfies $\int_{T}^{0}\left(\int_{\mathbb{R}_{0}}\left(\|z\|^{2} \wedge 1\right) v_{t}(\mathrm{~d} z)\right) \mathrm{d} t<\infty$ and $v_{t}(\{0\})=0$. The coefficients are assumed to satisfy the following linear growth condition

$$
\|b(t, x)\|^{2}+\|\sigma(t, x)\|^{2}+\int_{\mathbb{R}_{0}}\|\varphi(t, x, z)\|^{2} v_{t}(\mathrm{~d} z) \leq C\left(1+\|x\|^{2}\right)
$$

for all $t \in[0, T], x \in \mathbb{R}^{d}$ where $C$ is a positive constant.
Furthermore we suppose that there exists a function $\rho: \mathbb{R} \rightarrow \mathbb{R}$ with

$$
\begin{equation*}
\sup _{0 \leq t \leq T} \int_{\mathbb{R}_{0}}|\rho(z)|^{2} v_{t}(\mathrm{~d} z)<\infty \tag{A8}
\end{equation*}
$$

and a positive constant $K$ such that

$$
\begin{equation*}
\|\varphi(t, x, z)-\varphi(t, y, z)\| \leq K|\rho(z)|\|x-y\| \tag{A9}
\end{equation*}
$$

for all $t \in[0, T], x, y \in \mathbb{R}^{d}$ and $z \in \mathbb{R}_{0}$. Similarly to the homogeneous case, see more details on page 334 in Di Nunno, Øksendal, and Proske (2009), we have the following lemma

Lemma A.8. Under the above conditions, there exists a unique solution $\left(S_{t}\right)_{t \in[0, T]}$ for Equation (A7). Moreover, there exists a positive constant $C_{0}$ such that

$$
\mathrm{E}\left[\sup _{0 \leq t \leq T}\left|S_{t}\right|^{2}\right]<C_{0}
$$

In the sequel, we provide a theorem which proves that under specific conditions the solution of a stochastic differential equation belongs to the domain $\mathbb{D}^{1,2}$. For details, we refer to Eddahbi and Lalaoui Ben Cherif (2016) and Di Nunno, Øksendal, and Proske (2009).

Theorem A.9. Let $\left(S_{t}\right)_{t \in[0, T]}$ be the solution of Equation (A7) and assume that the condition (Equation (A8)) is satisfied. Then $S_{t} \in \mathbb{D}^{1,2}$ for all $t \in[0, T]$ and we have
(1) The derivative $D_{r, 0} S_{t}$ satisfies the following linear equation

$$
\begin{aligned}
D_{r, 0} S_{t}= & \int_{r}^{t} \frac{\partial b}{\partial x}\left(u, S_{u-}\right) D_{r, 0} S_{u-} d u+\sigma\left(r, S_{r-}\right)+\int_{r}^{t} \frac{\partial \sigma}{\partial x}\left(u, S_{u-}\right) D_{r, 0} S_{u-} d W_{u} \\
& +\int_{r}^{t} \int_{\mathbb{R}_{0}} \frac{\partial \varphi}{\partial x}\left(u, S_{u-}, y\right) D_{r, 0} S_{u-} \tilde{N}(d u, d y)
\end{aligned}
$$

for $0 \leq r \leq t$ a.e. and $D_{r, 0} S_{t}=0$ a.e. otherwise.
(2) For all $z \in \mathbb{R}_{0}$ the derivative $D_{r, z} S_{t}$ satisfies the following linear equation

$$
\begin{aligned}
D_{r, z} S_{t}= & \int_{r}^{t} D_{r, z} b\left(u, S_{u-}\right) d u+\int_{r}^{t} D_{r, z} \sigma\left(u, S_{u-}\right) d W_{u}+\varphi\left(r, S_{r-}, z\right) \\
& +\int_{r}^{t} \int_{\mathbb{R}_{0}} D_{r, z} \varphi\left(u, S_{u-}, y\right) \tilde{N}(d u, d y)
\end{aligned}
$$

for $0 \leq r \leq t$ a.e. and $D_{r, z} S_{t}=0$ a.e. otherwise.
Having in mind the applications in finance, we will also provide a specific expression for the Wiener directional derivative of the solution. As in the classical Malliavin calculus, we are able to associate the solution of Equation (A7) with the first variation process $Y_{t}:=\frac{\partial S_{t}}{\partial x}$. We reach the following proposition which provides a simpler expression for $D_{r, 0} S_{t}$. For details, we refer to Eddahbi and Lalaoui Ben Cherif (2016) and Di Nunno, Øksendal, and Proske (2009).

Proposition A.10. Let $\left(S_{t}\right)_{t \in[0, T]}$ be the solution of Equation (A7). Then the derivative satisfies the following equation

$$
\begin{equation*}
D_{r, 0} S_{t}=Y_{t} Y_{r-}^{-1} \sigma\left(r, S_{r-}\right) \mathbf{1}_{\{r \leq t\}} \text { a.e.. } \tag{A10}
\end{equation*}
$$

