King Saud University

College of Science

Statistics and Operations Research Department

# Stat 319

# Theory of Statistics (2)

# Exercises

**References:**

1. Introduction to Mathematical Statistics, Sixth Edition, by R. Hogg, J. McKean, and A. Craig, Prentice Hall.
2. Introduction to Theory of Statistics, A. Mood, F. Graybill and B. Boes, McGrow-Hill.
3. Introduction to Mathematical Statistics, Fourth Edition, by Hogg and Craig, Macmillan Publishing Co., Inc.
4. Statistical Inference, second Edition, by G. Casella and R. Berger, Duxbury.

**By**

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# Confidence Intervals

1. Let $\bar{X}$ be the mean of a random sample from the exponential distribution, $Exp(\theta)$.
   (a) Show that $\bar{X}$ is an unbiased point estimator of $\theta$.
   (b) Using the MGF technique determine the distribution of $\bar{X}$.
   (c) Use (b) to show that $Y = 2n\bar{X}/\theta$ has a $\chi^2$ distribution with 2n degrees of freedom.
   (d) Based on Part (c), find a 95% confidence interval for $\theta$ if $n = 10$. Hint: Find c and d such that $P\left(c < \frac{2n\bar{X}}{\theta} < d\right) = 0.95$ and solve the inequalities for $\theta$.
2. Let $X_1, \dots, X_n$ be a random sample from the $\Gamma(2, \theta)$ distribution, where $\theta$ is unknown. Let $Y = \sum_{i=1}^{n} X_i$.
   (a) Find the distribution of Y and determine c so that cY is an unbiased estimator of $\theta$.
   (b) If $n = 5$, show that $P\left(9.59 < \frac{2Y}{\theta} < 34.2\right) = 0.95$.
   (c) Using Part (b), show that if y is the value of Y once the sample is drawn, then the interval $\left(\frac{2y}{34.2}, \frac{2y}{9.59}\right)$ is a 95% confidence interval for $\theta$.
   (d) Suppose the sample results in the values,
   44.8079 1.5215 12.1929 12.5734 43.2305
   Based on these data, obtain the point estimate of $\theta$ as determined in Part (a) and the computed 95% confidence interval in Part (c). What does the confidence interval mean?
3. Let the observed value of the mean $\bar{X}$ of a random sample of size 20 from a distribution that $N(\mu, 80)$ be 81.2. Find a 95% confidence interval for $\mu$.
4. Let $\bar{X}$ be the mean of a random sample of size n from a distribution that is $N(\mu, 9)$. Find n such that $P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90$, approximately.
5. Let a random sample of size 17 from the normal distribution $N(\mu, \sigma^2)$ yield $\bar{x} = 4.7$ and $s^2 = 5.76$. Determine a 90% confidence interval for $\mu$.
6. Let $\bar{X}$ denote the mean of a random sample of size n from a distribution that has mean $\mu$ and variance $\sigma^2 = 10$. Find n so that the probability is approximately 0.954 that the random interval $\left(\bar{X} - \frac{1}{2} < \mu < \bar{X} + \frac{1}{2}\right)$ includes $\mu$.
7. Let Y be $Bin(300, p)$. If the observed value of Y is $y = 75$, find an approximate 90% confidence interval for $p$.
8. Let $\bar{x}$ be the observed mean of a random sample of size n from a distribution having mean $\mu$ and known variance $\sigma^2$. Find n so that $\bar{x} - \frac{\sigma}{4}$ to $\bar{x} + \frac{\sigma}{4}$ is an approximate 95% confidence interval for $\mu$.
9. Let $X_1, \dots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both parameters $\mu$ and $\sigma^2$ are unknown. A confidence interval for $\sigma^2$ can be found as follows:
   We know that $(n-1)S^2/\sigma^2$ is a random variable with a $\chi^2_{(n-1)}$ distribution. Thus, we can find constants a and b so that $P((n-1)S^2/\sigma^2 < b) = 0.975$ and $P(a < (n-1)S^2/\sigma^2 < b) = 0.95$.
   (a) Show that this second probability statement can be written as
   $P((n-1)S^2/b < \sigma^2 < (n-1)S^2/a) = 0.95$.

(b) If $n = 9$ and $s^2 = 7.93$, find a 95% confidence interval for $\sigma^2$.

10. Let $X_1, \dots, X_n$ be a random sample from gamma distributions with known parameter $\alpha = 3$ and unknown $\beta > 0$. Discuss the construction of a confidence interval for $\beta$.

    Hint: What is the distribution of $2\sum_{i=1}^{n} \frac{X_i}{\beta}$? Follow the procedure outlined in Exercise 9.
11. When 100 tacks were thrown on a table, 60 of them landed point up. Obtain a 95% confidence interval for the probability that a tack of this type will land point up. Assume independence.
12. Let two independent random samples, each of size 10, from two normal distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ yield $\bar{x} = 4.8$, $s_1^2 = 8.64$, $\bar{y} = 5.6, s_2^2 = 7.88$. Find a 95% confidence interval for $\mu_1 - \mu_2$.
13. Let two independent random variables, $Y_1$ and $Y_2$, with binomial distribution that have parameters $n_1 = n_2 = 100, p_1$ and $p_2$, respectively, be observed to be equal to $y_1 = 50$ and $y_2 = 40$. Determine an approximate 90% confidence interval for $p_1 - p_2$.
14. Let $\bar{X}$ and $\bar{Y}$ be the means of two independent random samples, each of size n, from the respective distributions $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, where the common variance is known. Find n such that $P\left(\bar{X} - \bar{Y} - \frac{\sigma}{5} < \mu_1 - \mu_2 < \bar{X} - \bar{Y} + \frac{\sigma}{5}\right) = 0.90$.
15. If 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, 7.5 are the observed values of a random sample of size 9 from a distribution that is $N(8, \sigma^2)$, construct a 90% confidence interval for $\sigma^2$.
16. Let $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ be two independent random samples from the respective normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, where the four parameters are unknown. To construct a confidence interval for the ratio, $\frac{\sigma_1^2}{\sigma_2^2}$, of the variances, from the quotient of the two independent chi-square variables, each divided by its degrees of freedom, namely $F = \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2}$, where $S_1^2$ and $S_2^2$ are the respective sample variances.
    (a) What kind of distribution does F have?
    (b) From the appropriate table, a and b can be found so that $P(F < b) = 0.975$ and $P(a < F < b) = 0.95$.
    (c) Rewrite the second probability statement as $P\left(a\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < b\frac{S_1^2}{S_2^2}\right) = 0.95$. The observed values, $s_1^2$ and $s_2^2$, can be inserted in these inequalities to provide a 95% confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$.
17. Let two independent random samples of sizes $n = 16$, $m = 10$, taken from two independent normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, yield $\bar{x} = 3.6$, $s_1^2 = 4.14$, $\bar{y} = 13.6, s_2^2 = 7.26$. Fin a 90% confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$ when $\mu_1$ and $\mu_2$ are unknown.
18. Find a 90 percent confidence interval for the mean of a normal distribution with $\sigma = 3$ given the sample (3.3, -0.3, -0.6, -0.9). What would be the confidence interval if $\sigma$ were unknown?
19. The breaking strengths in pounds of five specimens of manila rope of diameter $\frac{3}{16}$ inch were found to be 660, 460, 540, 580 and 550.

(a) Estimate the mean breaking strength by a 95% confidence interval assuming normality.
(b) Estimate $\sigma^2$ by a 90% confidence interval; also $\sigma$.

20. A sample was drawn from each of five populations assumed to be normal with the same variance. The values of $(n-1)S^2 = \sum(X_i - \bar{X})$ and n, the sample size, were

| | | | | | |
|---|---|---|---|---|---|
| $S^2$: | 40 | 30 | 20 | 42 | 50 |
| $n$: | 6 | 4 | 3 | 7 | 8 |

Find 98% confidence interval for the common variance.

21. To test two promising new lines of hybrid corn under normal farming conditions, a seed company selected eight farms at random in Iowa and planted both lines in experimental plots on each farm. The yields (converted to bushels per acre) for the eight locations were

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Line A: | 86 | 87 | 56 | 93 | 84 | 93 | 75 | 79 |
| Line B: | 80 | 79 | 58 | 91 | 77 | 82 | 74 | 66 |

Assuming that the two yields are jointly normally distributed, estimate the difference between the mean yields by a 95% confidence interval.

22. Let $\bar{X}$ denote the mean of a random sample of size 25 from a gamma distribution with $\alpha = 4$ and $\beta > 0$. Use the Central Limit Theorem to find an approximate 0.954 confidence interval for $\mu$, the mean of the gamma distribution. Hint: Use the random variable $\frac{(\bar{X}-4\beta)}{\sqrt{4\beta/25}} = \frac{5\bar{X}}{2\beta} - 10$.

23. Let $S_1^2$ and $S_2^2$ denote, respectively, the variances of random samples, of sizes n and m, from two independent distributions that are $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$.
(a) Derive $(1-\alpha)100\%$ confidence interval for the common unknown variance $\sigma^2$.
(b) Find 99% confidence interval for $\sigma^2$ if $n = 12, s_1^2 = 4.74, m = 15, s_2^2 = 5.66$.