Curvature

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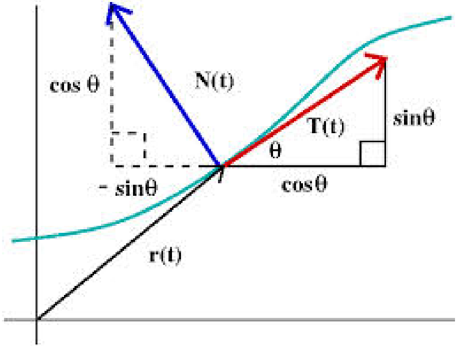
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Radius of curvature is given by

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The center of curvature (h, k) is

$$h = x - \frac{y'(1 + (y')^2)}{y''}, \quad k = y - \frac{(1 + (y')^2)}{y''}.$$

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Q(3): Find the curvature of the parabola given by $y = x - \frac{1}{4}x^2$ at $x = 2$. Sketch the circle of the curvature at $(2,1)$.

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Curvature:

$$\mathcal{K} = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = a_N \frac{1}{\|r'(t)\|^2}$$

Example (1): The position vector of a moving object at time t is

$$r(t) = 3ti + t^3 j + 3t^2 k,$$

find the tangential and normal component of acceleration and curvature at time t.

Example (2): Find the tangential and normal component of acceleration and curvature at time $t = 1$ of a particle moving along the curve C which is given by

$$r(t) = 2ti + 7tj + 3tk$$