# CHAPTER OBJECTIVES

- Use various methods to determine the deflection and slope at specific pts on beams and shafts:

  1. Integration method
  2. Discontinuity functions
  3. Method of superposition
 4. Moment-area method

12.1 THE ELASTIC CURVE

- It is useful to sketch the deflected shape of the loaded beam, to "visualize" computed results and partially check the results.

- The deflection diagram of the longitudinal axis that passes through the centroid of each x-sectional area of the beam is called the elastic curve.
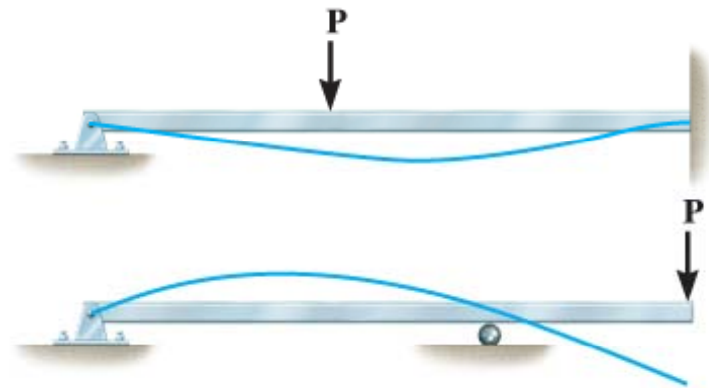
12.1 THE ELASTIC CURVE

- Draw the moment diagram for the beam first before creating the elastic curve.

- Use beam convention as shown and established in chapter 6.1.

+M ⟶ ⟵ +M
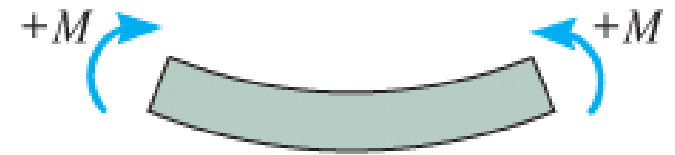
Positive internal moment
concave upwards

(a)

−M ⟶ ⟵ −M

Negative internal moment
concave downwards

(b)

12.1 THE ELASTIC CURVE

- For example, due to roller and pin supports at B and D, displacements at B and D is zero.

- For region of -ve moment AC, elastic curve concave downwards.

- Within region of +ve moment CD, elastic curve concave upwards.

- At pt C, there is an inflection pt where curve changes from concave up to concave down (zero moment).
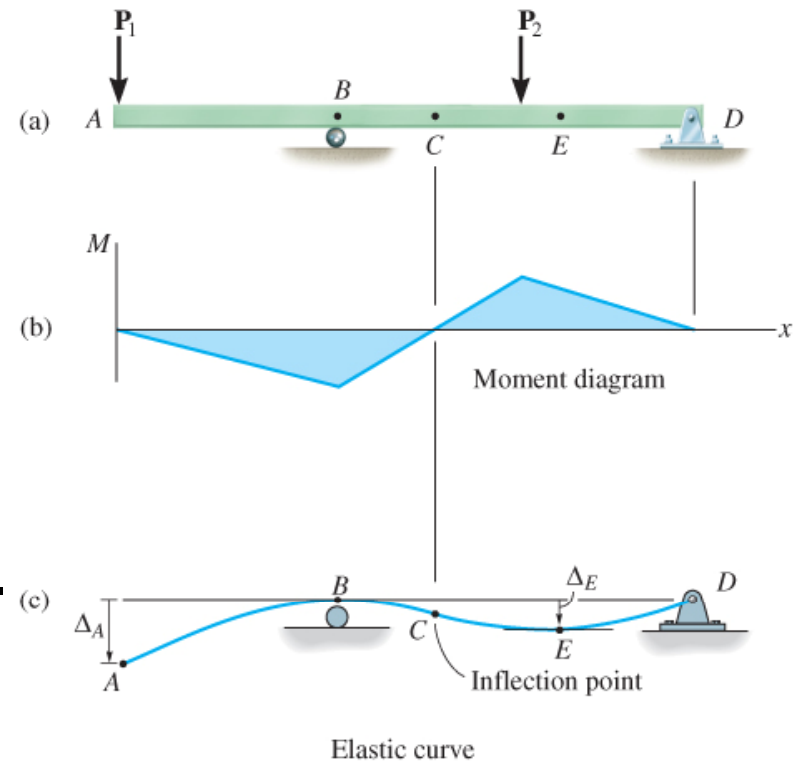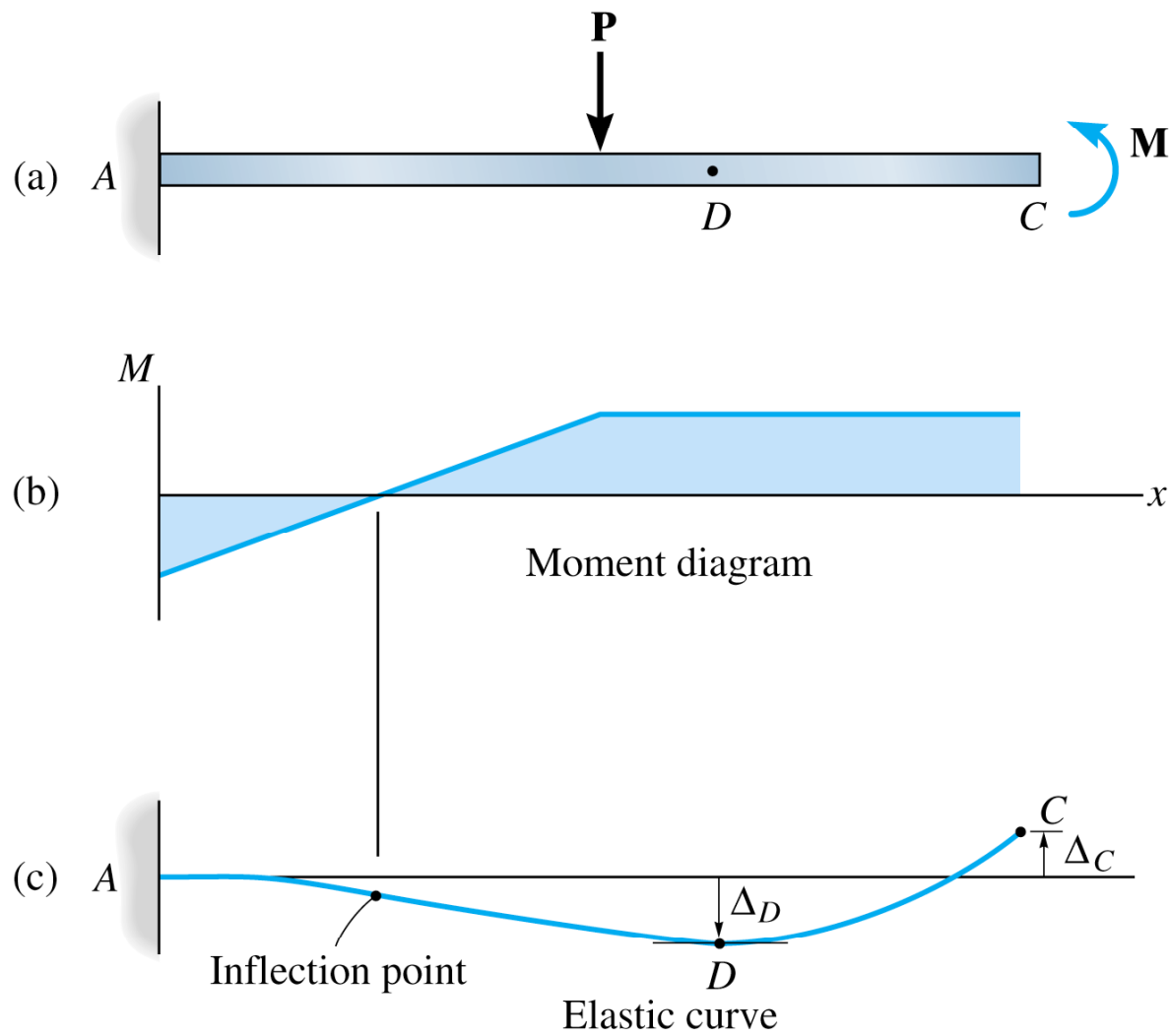
Moment diagram

Elastic curve
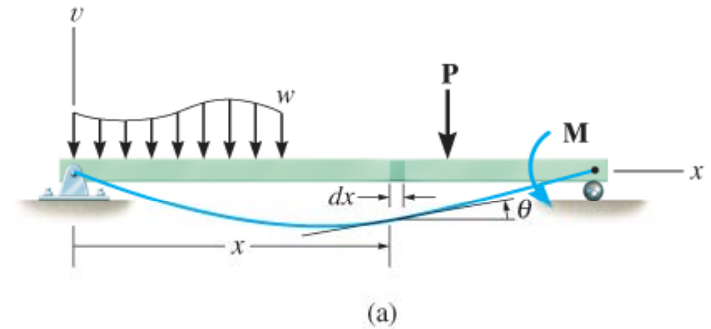
12.1 THE ELASTIC CURVE

(a)

(b) Moment diagram

(c) Inflection point
Elastic curve

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

(a)

- x axis extends +ve to the right, along longitudinal axis of beam.

- A differential element of undeformed width dx is located.

- v axis extends +ve upwards from x axis. It measures the displacement of the centroid on x-sectional area of element.

- A "localized" y coordinate is specified for the position of a fiber in the element.

- It is measured +ve upward from the neutral axis.

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

Before deformation   After deformation

(b)

- Limit analysis to the case of initially straight beam elastically deformed by loads applied perpendicular to beam's x axis and lying in the x-v plane of symmetry for beam's x-sectional area.

- Internal moment **M** deforms element such that angle between x-sections is $d\theta$.

- Arc dx is a part of the elastic curve that intersects the neutral axis for each x-section.

- Radius of curvature for this arc defined as the distance ρ, measured from center of curvature O' to dx.
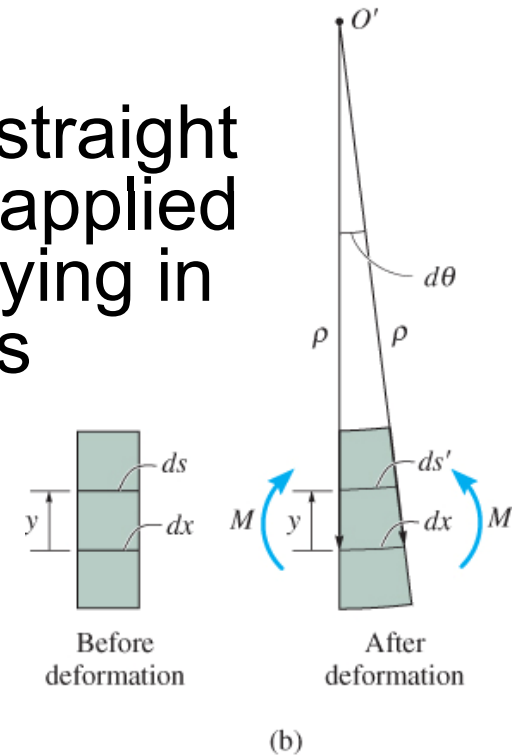
12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- Strain in arc ds, at position y from neutral axis, is

$$\varepsilon = \frac{ds'-ds}{ds}$$

But $ds = dx = \rho d\theta$ and $ds' = (\rho - y)d\theta$

$$\varepsilon = \frac{[(\rho - y)d\theta - \rho d\theta_s]}{\rho d\theta} \text{ or}$$

$$\frac{1}{\rho} = -\frac{\varepsilon}{y} \qquad (12\text{-}1)$$

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- If material is homogeneous and shows linear-elastic behavior, Hooke's law applies. Since flexure formula also applies, we combing the equations to get

☐ $\varepsilon = \sigma/E$, $\sigma = -My/I$ → $$\frac{1}{\rho} = \frac{M}{EI} \qquad (12\text{-}2)$$

ρ = radius of curvature at a specific pt on elastic curve ($1/\rho$ is referred to as the curvature).

M = internal moment in beam at pt where is to be determined.
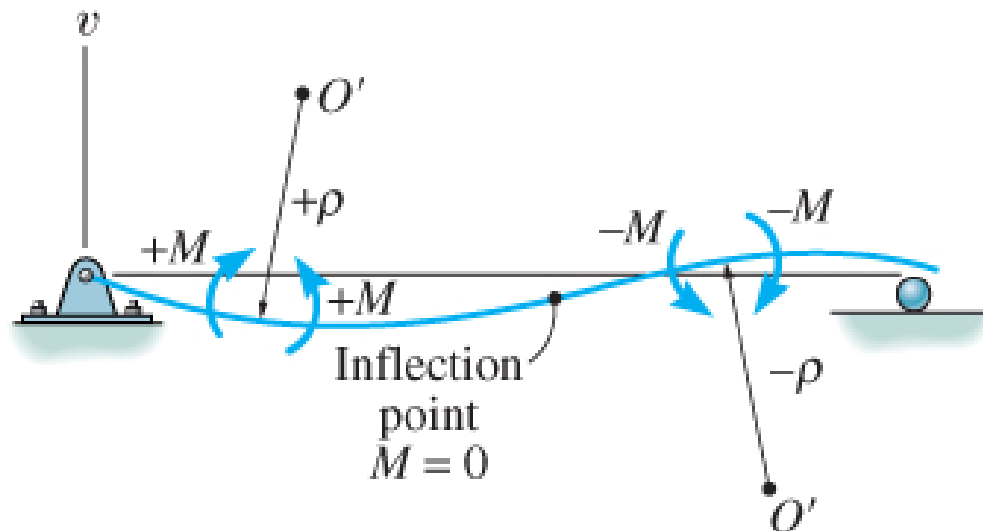
E = material's modulus of elasticity.

I = beam's moment of inertia computed about neutral axis.

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- EI is the flexural rigidity and is always positive.
- Sign for ρ depends on the direction of the moment.
- As shown, when M is +ve, ρ extends above the beam. When M is –ve, ρ extends below the beam.

12.1 THE ELASTIC CURVE

Moment-Curvature Relationship

- Using flexure formula, $\sigma = -My/I$, curvature is also

$$\frac{1}{\rho} = -\frac{\sigma}{Ey} \qquad (12\text{-}3)$$

- Eqns 12-2 and 12-3 valid for either small or large radii of curvature.

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Let's represent the curvature in terms of v and x.

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1 + \left(dv/dx\right)^2\right]^{3/2}}$$

- Substitute into Eqn 12-2

$$\frac{d^2v/dx^2}{\left[1 + \left(dv/dx\right)^2\right]^{3/2}} = \frac{M}{EI} \qquad (12\text{-}4)$$

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Most engineering codes specify limitations on deflections for tolerance or aesthetic purposes.

- Slope of elastic curve determined from dv/dx is very small and its square will be negligible compared with unity.

- Therefore, by approximation $1/\rho = d^2v/dx^2$, Eqn 12-4 rewritten as

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI} \qquad (12\text{-}5)$$

- Differentiate each side w.r.t. x and substitute $V = dM/dx$, we get

$$\frac{d}{dx}\left(EI\frac{d^2\upsilon}{dx^2}\right) = V(x) \qquad (12\text{-}6)$$

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Differentiating again, using $-w = dV/dx$ yields

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\upsilon}{dx^2}\right) = -w(x) \quad (12\text{-}7)$$

- Flexural rigidity is constant along beam, thus

$$EI\frac{d^4\upsilon}{dx^4} = -w(x) \quad (12\text{-}8)$$

$$EI\frac{d^3\upsilon}{dx^3} = V(x) \quad (12\text{-}9)$$

$$EI\frac{d^2\upsilon}{dx^2} = M(x) \quad (12\text{-}10)$$
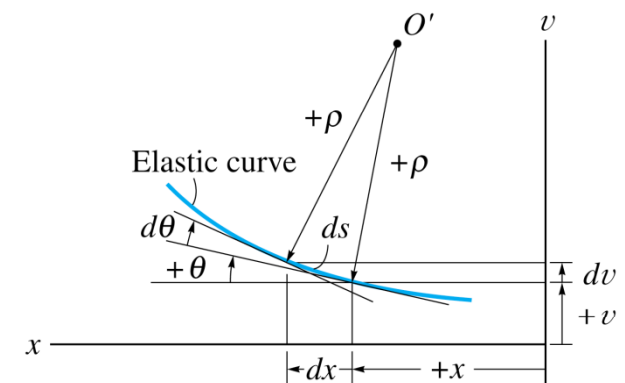
12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Generally, it is easier to determine the internal moment M as a function of x, integrate twice, and evaluate only two integration constants.

- For convenience in writing each moment expression, the origin for each x coordinate can be selected arbitrarily.

Sign convention and coordinates

- Use the proper signs for M, V and w.

Positive sign convention
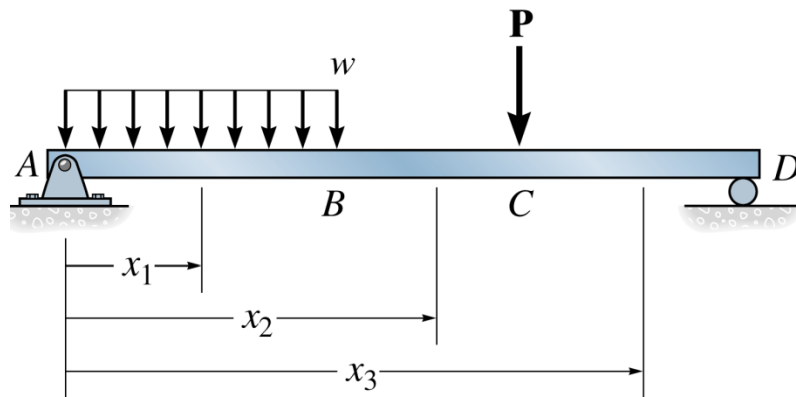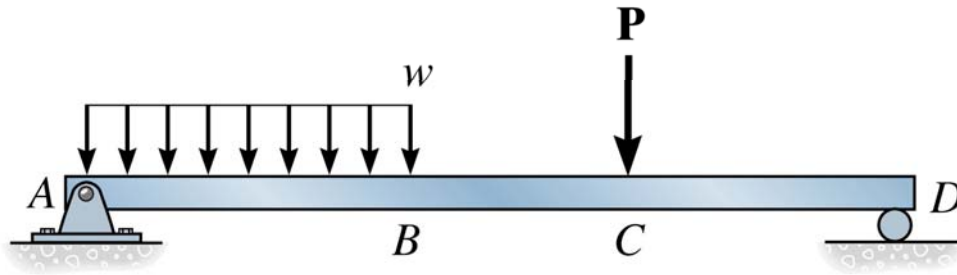
(a)
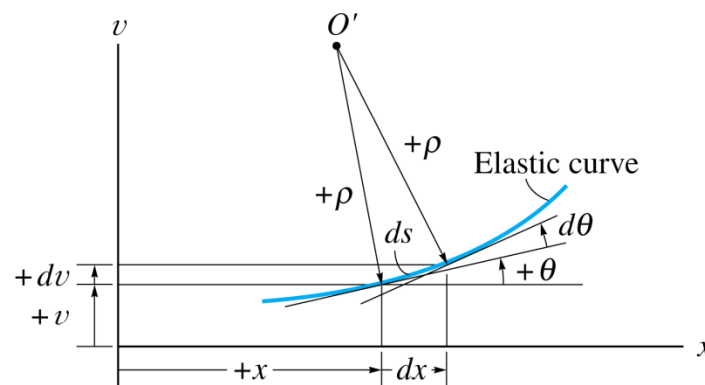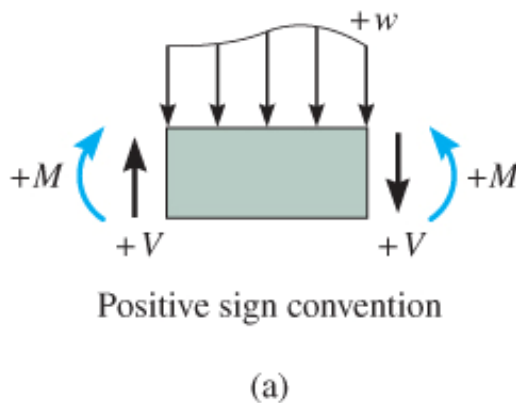
Positive sign convention

Positive sign convention

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

- Positive deflection v is upward.

- Positive slope angle θ will be measured counterclockwise when x is positive to the right.

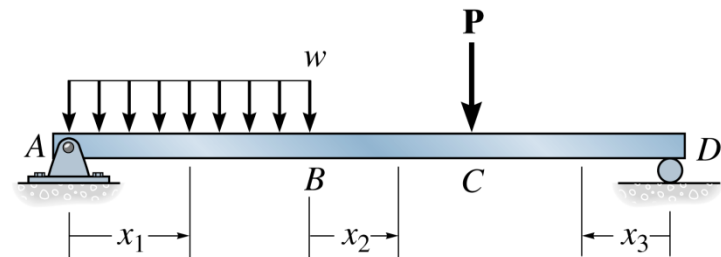- Positive slope angle θ will be measured clockwise when x is positive to the left.

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Boundary and continuity conditions

- Possible boundary conditions are shown here.

Boundary and continuity conditions

- If a single x coordinate cannot be used to express the eqn for beam's slope or elastic curve, then continuity conditions must be used to evaluate some of the integration constants.
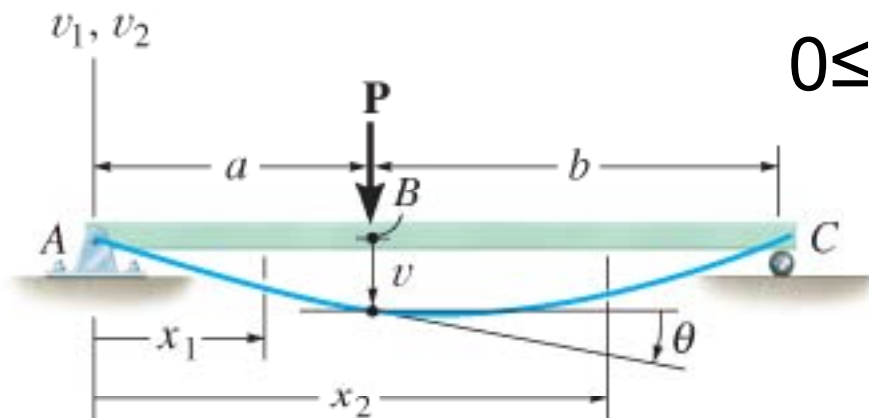
$$0 \leq x_1 \leq a \ , \ a \leq x_2 \leq (a+b)$$

$$\theta_1 (a) = \theta_2 (a)$$

$$v_1 (a) = v_2 (a)$$

$$0 \leq x_1 \leq a \ , \ 0 \leq x_2 \leq b$$

$$\theta_1 (a) = - \theta_2 (b)$$

$$v_1 (a) = v_2 (b)$$
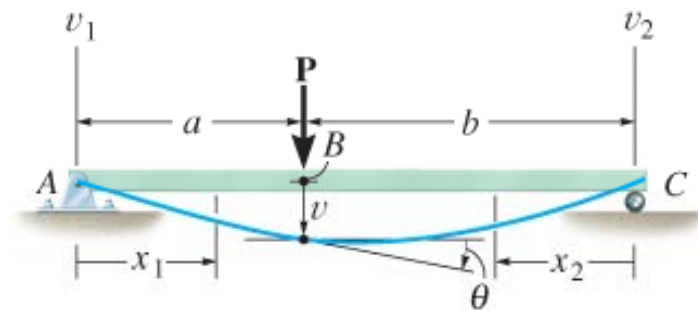
(b)

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Elastic curve

- Draw an exaggerated view of the beam's elastic curve.

- Recall that zero slope and zero displacement occur at all fixed supports, and zero displacement occurs at all pin and roller supports.

- Establish the x and v coordinate axes.

- The x axis must be parallel to the undeflected beam and can have an origin at any pt along the beam, with +ve direction either to the right or to the left.

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Elastic curve

- If several discontinuous loads are present, establish x coordinates that are valid for each region of the beam between the discontinuties.

- Choose these coordinates so that they will simplify subsequent algrebraic work.

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Load or moment function

- For each region in which there is an x coordinate, express that loading w or the internal moment M as a function of x.

- In particular, always assume that M acts in the +ve direction when applying the eqn of moment equilibrium to determine $M = f(x)$.

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Slope and elastic curve

- Provided EI is constant, apply either the load eqn $EI\, d^4 v/dx^4 = -w(x)$, which requires four integrations to get $v = v(x)$, or the moment eqns $EI\, d^2 v/dx^2 = M(x)$, which requires only two integrations. For each integration, we include a constant of integration.

- Constants are evaluated using boundary conditions for the supports and the continuity conditions that apply to slope and displacement at pts where two functions meet.

12.2 SLOPE AND DISPLACEMENT BY INTEGRATION

Procedure for analysis

Slope and elastic curve

- Once constants are evaluated and substituted back into slope and deflection eqns, slope and displacement at specific pts on elastic curve can be determined.

- The numerical values obtained is checked graphically by comparing them with sketch of the elastic curve.

- Realize that +ve values for slope are counterclockwise if the x axis extends +ve to the right, and clockwise if the x axis extends +ve to the left. For both cases, +ve displacement is upwards.

EXAMPLE 12.1

Cantilevered beam shown is subjected to a vertical load **P** at its end. Determine the eqn of the elastic curve. EI is constant.

(a)
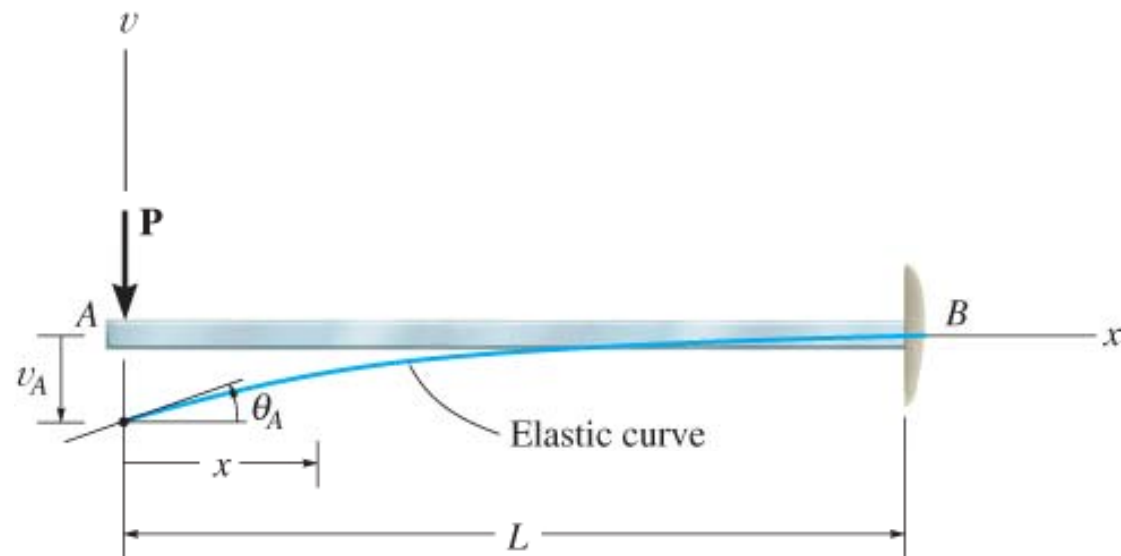
EXAMPLE 12.1 (SOLN)

Elastic curve: Load tends to deflect the beam. By inspection, the internal moment can be represented throughout the beam using a single x coordinate.

(a)
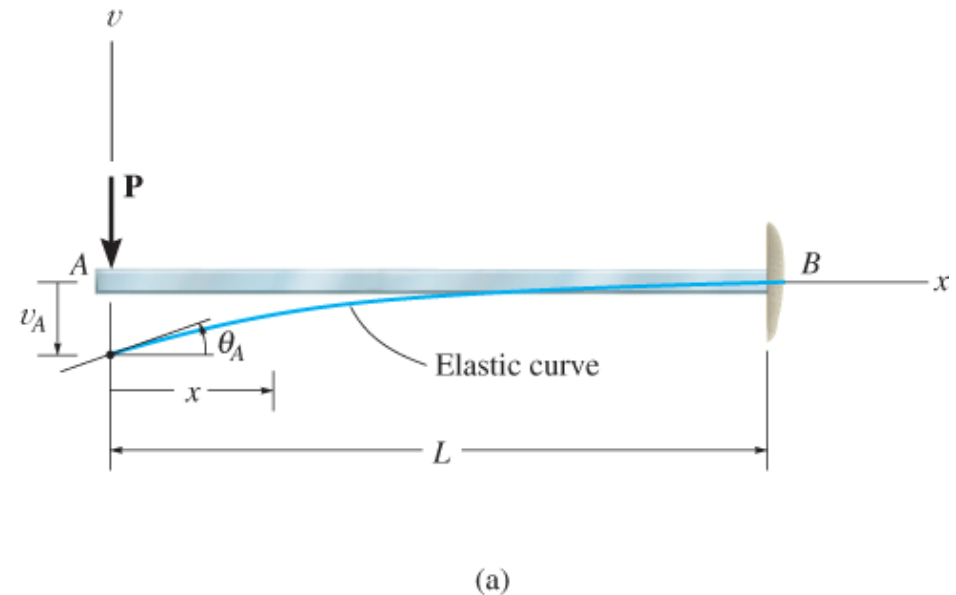
Moment function: From free-body diagram, with **M** acting in the +ve direction, we have

$$M = -Px$$
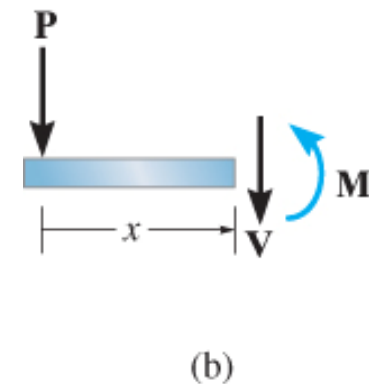
(b)

EXAMPLE 12.1 (SOLN)

Slope and elastic curve:

Applying Eqn 12-10 and integrating twice yields

$$EI \frac{d^2 \upsilon}{dx^2} = -Px \qquad (1)$$

$$EI \frac{d \upsilon}{dx} = -\frac{Px^2}{2} + C_1 \qquad (2)$$

$$EI\upsilon = -\frac{Px^3}{6} + C_1 x + C_2 \qquad (3)$$

EXAMPLE 12.1 (SOLN)

Slope and elastic curve:

Using boundary conditions $dv/dx = 0$ at $x = L$, and $v = 0$ at $x = L$, Eqn (2) and (3) becomes

$$0 = -\frac{PL^2}{2} + C_1$$

$$0 = -\frac{PL^3}{6} + C_1L + C_2$$

EXAMPLE 12.1 (SOLN)

Slope and elastic curve:

Thus, $C_1 = PL^2/2$ and $C_2 = PL^3/3$. Substituting these results into Eqns (2) and (3) with $\theta = dv/dx$, we get

$$\theta = -\frac{P}{2EI}\left(L^2 - x^2\right)$$

$$\upsilon = \frac{P}{6EI}\left(-x^3 + 3L^2 x - 2L^3\right)$$

Maximum slope and displacement occur at A ($x = 0$),

$$\theta_A = \frac{PL^2}{2EI} \qquad \qquad \upsilon_A = -\frac{PL^3}{3EI}$$

EXAMPLE 12.1 (SOLN)

Slope and elastic curve:

Positive result for θ_A indicates counterclockwise rotation and negative result for A indicates that v_A is downward.

Consider beam to have a length of 5 m, support load P = 30 kN and made of A-36 steel having E_{st} = 200 GPa.

EXAMPLE 12.1 (SOLN)

Slope and elastic curve:

Using methods in chapter 11.3, assuming allowable normal stress is equal to yield stress σ_{allow} = 250 MPa, then a W310×39 would be adequate (I = 84.8(10^6) mm^4).

From Eqns (4) and (5),

$$\theta_A = \frac{PL^2}{2EI}$$

$$\upsilon_A = -\frac{PL^3}{3EI}$$

EXAMPLE 12.1 (SOLN)

Slope and elastic curve:

From Eqns (4) and (5),

$$\theta_A = \frac{30 \text{ kN}\left(10^3 \text{ N/kN}\right) \times \left[5 \text{ m}\left(10^3 \text{ mm/m}\right)^2\right]^2}{2\left[200\left(10^3\right) \text{N/mm}^2\right]\left(84.8\left(10^6\right) \text{mm}^4\right)} = 0.0221 \text{ rad}$$

$$\upsilon_A = -\frac{30 \text{ kN}\left(10^3 \text{ N/kN}\right) \times \left[5 \text{ m}\left(10^3 \text{ mm/m}\right)^2\right]^3}{3\left[200\left(10^3\right) \text{N/mm}^2\right]\left(84.8\left(10^6\right) \text{mm}^4\right)} = -73.7 \text{ mm}$$

EXAMPLE 12.1 (SOLN)

Slope and elastic curve:

Since $\theta^2_A = (dv/dx)^2 = 0.000488 << 1$, this justifies the use of Eqn 12-10 than the more exact 12-4.

EXAMPLE 12.1 (SOLN)

SOLUTION 2

Using Eqn 12-8 to solve the problem. Here $w(x) = 0$ for $0 \leq x \leq L$, so that upon integrating once, we get the form of Eqn 12-19

$$EI\frac{d^4\upsilon}{dx^4} = 0$$

$$EI\frac{d^3\upsilon}{dx^3} = C'_1 = V$$

EXAMPLE 12.1 (SOLN)

Solution II

Shear constant C'_1 can be evaluated at $x = 0$, since $V_A = -P$. Thus, $C'_1 = -P$. Integrating again yields the form of Eqn 12-10,

$$EI \frac{d^3 \upsilon}{dx^3} = -P$$

$$EI \frac{d^2 \upsilon}{dx^2} = -Px + C'_2 = M$$
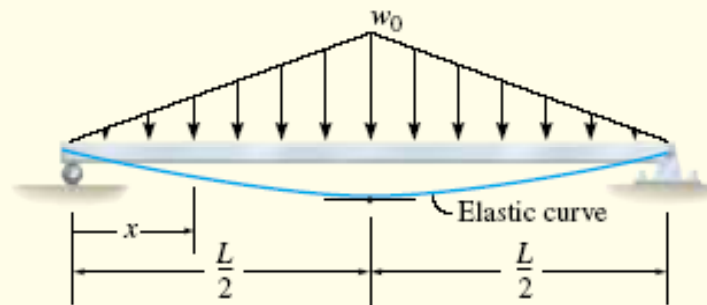
Here, $M = 0$ at $x = 0$, so $C'_2 = 0$, and as a result, we obtain Eqn 1 and solution proceeds as before.

E X A M P L E   **12.2**

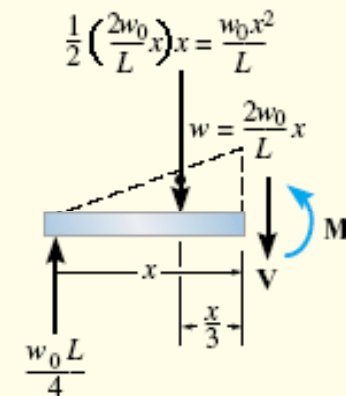The simply supported beam shown in Fig. 12–11a supports the triangular distributed loading. Determine its maximum deflection. EI is constant.

(a)

$$\frac{1}{2}\left(\frac{2w_0}{L}x\right)x = \frac{w_0 x^2}{L}$$

$$w = \frac{2w_0}{L}x$$

(b)

Solution I

Elastic Curve. Due to symmetry, only one x coordinate is needed for the solution, in this case $0 \leq x \leq L/2$. The beam deflects as shown in Fig. 12–11a. Notice that maximum deflection occurs at the center since the slope is zero at this point.

Moment Function. The distributed load acts downward, and therefore, it is positive according to our sign convention. A free-body diagram of the segment on the left is shown in Fig. 12–11b. The equation for the distributed loading is

$$w = \frac{2w_0}{L}x \qquad (1)$$

Hence,

$$\zeta+ \Sigma M_{NA} = 0; \quad M + \frac{w_0 x^2}{L}\left(\frac{x}{3}\right) - \frac{w_0 L}{4}(x) = 0$$

$$M = -\frac{w_0 x^3}{3L} + \frac{w_0 L}{4}x$$

Slope and Elastic Curve.

Using Eq. 12–10 and integrating twice, we have

$$EI \frac{d^2v}{dx^2} = M = -\frac{w_0}{3L}x^3 + \frac{w_0 L}{4}x \tag{2}$$

$$EI \frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0 L}{8}x^2 + C_1$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0 L}{24}x^3 + C_1 x + C_2$$

The constants of integration are obtained by applying the boundary condition $v = 0$ at $x = 0$ and the symmetry condition that $dv/dx = 0$ at $x = L/2$. This leads to

$$C_1 = -\frac{5w_0 L^3}{192} \qquad C_2 = 0$$

Hence,

$$EI \frac{dv}{dx} = -\frac{w_0}{12L}x^4 + \frac{w_0 L}{8}x^2 - \frac{5w_0 L^3}{192}$$

$$EIv = -\frac{w_0}{60L}x^5 + \frac{w_0 L}{24}x^3 - \frac{5w_0 L^3}{192}x$$

Determining the maximum deflection at $x = L/2$, we have

$$v_{max} = -\frac{w_0 L^4}{120EI} \qquad \text{\textit{Ans.}}$$

Solution II

E X A M P L E 12.3

The simply supported beam shown in Fig. 12–12a is subjected to the concentrated force **P**. Determine the maximum deflection of the beam. EI is constant.

(a)

(b)
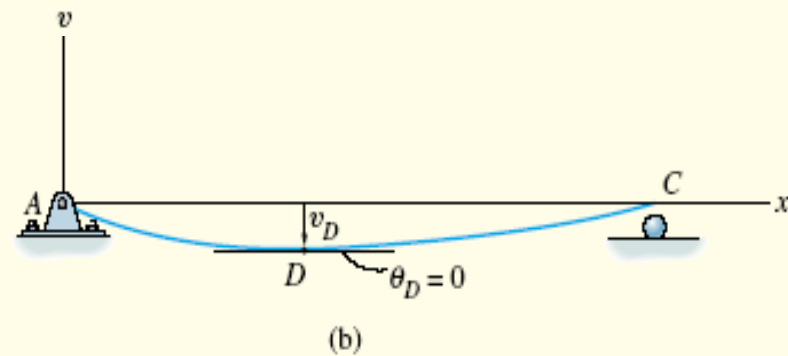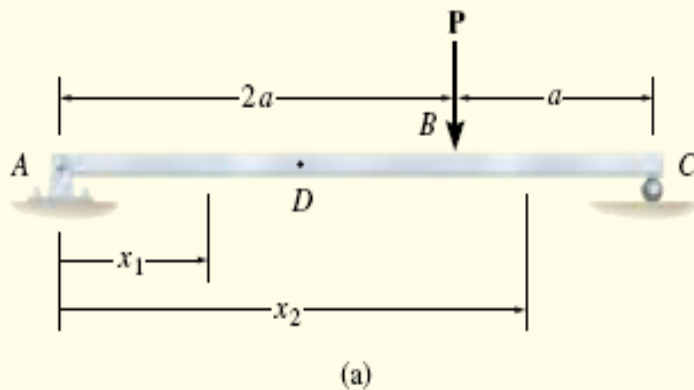
Elastic Curve. The beam deflects as shown in Fig. 12–12b. Two coordinates must be used, since the moment becomes discontinuous at P. Here we will take x_1 and x_2, having the *same origin* at A, so that $0 \leq x_1 < 2a$ and $2a < x_2 \leq 3a$.

Moment Function. From the free-body diagrams shown in Fig. 12–12c,

$$M_1 = \frac{P}{3}x_1$$

$$M_2 = \frac{P}{3}x_2 - P(x_2 - 2a) = \frac{2P}{3}(3a - x_2)$$

Slope and Elastic Curve. Applying Eq. 12–10 for M_1 and integrating twice yields

$$EI\frac{d^2v_1}{dx_1^2} = \frac{P}{3}x_1$$

$$EI\frac{dv_1}{dx^1} = \frac{P}{6}x_1^2 + C_1 \tag{1}$$

$$EIv_1 = \frac{P}{18}x_1^3 + C_1x_1 + C_2 \tag{2}$$

Likewise for M_2,

$$EI\frac{d^2v_2}{dx_2^2} = \frac{2P}{3}(3a - x_2)$$

$$EI\frac{dv_2}{dx_2} = \frac{2P}{3}\left(3ax_2 - \frac{x_2^2}{2}\right) + C_3 \tag{3}$$

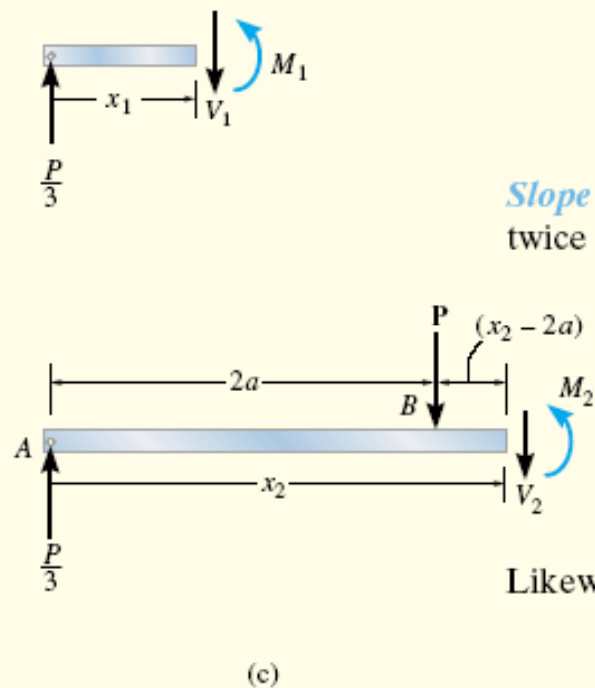$$EIv_2 = \frac{2P}{3}\left(\frac{3}{2}ax_2^2 - \frac{x_2^3}{6}\right) + C_3x_2 + C_4 \tag{4}$$

(c)

Fig. 12–12

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The four constants are evaluated using *two* boundary conditions, namely, $x_1 = 0$, $v_1 = 0$ and $x_2 = 3a$, $v_2 = 0$. Also, *two* continuity conditions must be applied at B, that is, $dv_1/dx_1 = dv_2/dx_2$ at $x_1 = x_2 = 2a$ and $v_1 = v_2$ at $x_1 = x_2 = 2a$. Substitution as specified results in the following four equations:

$v_1 = 0$ at $x_1 = 0$; $\qquad\qquad 0 = 0 + 0 + C_2$

$v_2 = 0$ at $x_2 = 3a$; $\qquad 0 = \dfrac{2P}{3}\left(\dfrac{3}{2}a(3a)^2 - \dfrac{(3a)^3}{6}\right) + C_3(3a) + C_4$

$\dfrac{dv_1(2a)}{dx_1} = \dfrac{dv_2(2a)}{dx_2}$; $\qquad \dfrac{P}{6}(2a)^2 + C_1 = \dfrac{2P}{3}\left(3a(2a) - \dfrac{(2a)^2}{2}\right) + C_3$

$v_1(2a) = v_2(2a)$; $\qquad \dfrac{P}{18}(2a)^3 + C_1(2a) + C_2 = \dfrac{2P}{3}\left(\dfrac{3}{2}a(2a)^2 - \dfrac{(2a)^3}{6}\right) + C_3(2a) + C_4$

Solving these equations, we get

$$C_1 = -\dfrac{4}{9}Pa^2 \qquad C_2 = 0$$

$$C_3 = -\dfrac{22}{9}Pa^2 \qquad C_4 = \dfrac{4}{3}Pa^3$$

Thus, Eqs. 1–4 become

$$\frac{dv_1}{dx_1} = \frac{P}{6EI}x_1{}^2 - \frac{4Pa^2}{9EI} \tag{5}$$

$$v_1 = \frac{P}{18EI}x_1{}^3 - \frac{4Pa^2}{9EI}x_1 \tag{6}$$

$$\frac{dv_2}{dx_2} = \frac{2Pa}{EI}x_2 - \frac{P}{3EI}x_2{}^2 - \frac{22Pa^2}{9EI} \tag{7}$$

$$v_2 = \frac{Pa}{EI}x_2{}^2 - \frac{P}{9EI}x_2{}^3 - \frac{22Pa^2}{9EI}x_2 + \frac{4Pa^3}{3EI} \tag{8}$$

By inspection of the elastic curve, Fig. 12–12b, the maximum deflection occurs at D, somewhere within region AB. Here the slope must be zero. From Eq. 5,

$$\frac{1}{6}x_1{}^2 - \frac{4}{9}a^2 = 0$$

$$x_1 = 1.633a$$

Substituting into Eq. 6,

$$v_{max} = -0.484\frac{Pa^3}{EI} \qquad\qquad Ans.$$

The negative sign indicates that the deflection is downward.

EXAMPLE 12.4

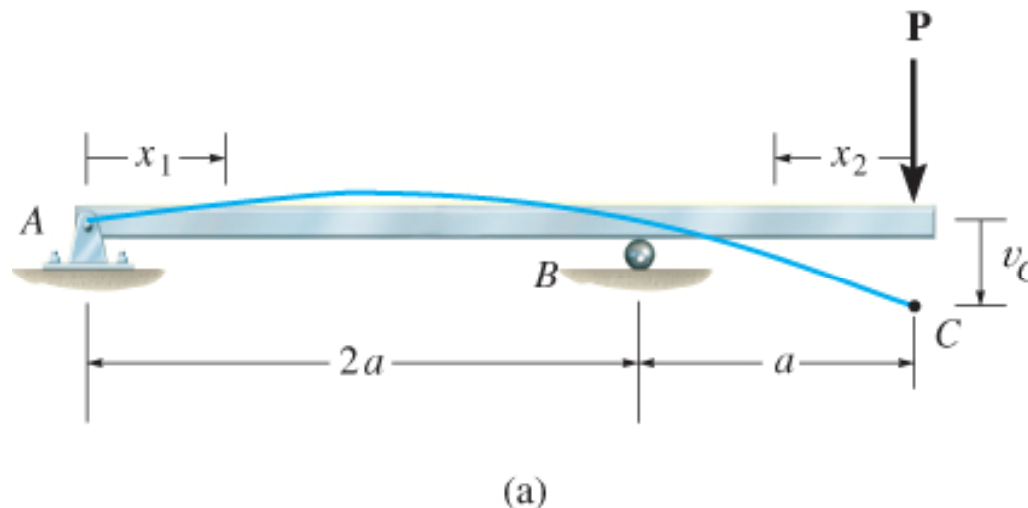Beam is subjected to load P at its end. Determine the displacement at C. EI is a constant.

(a)
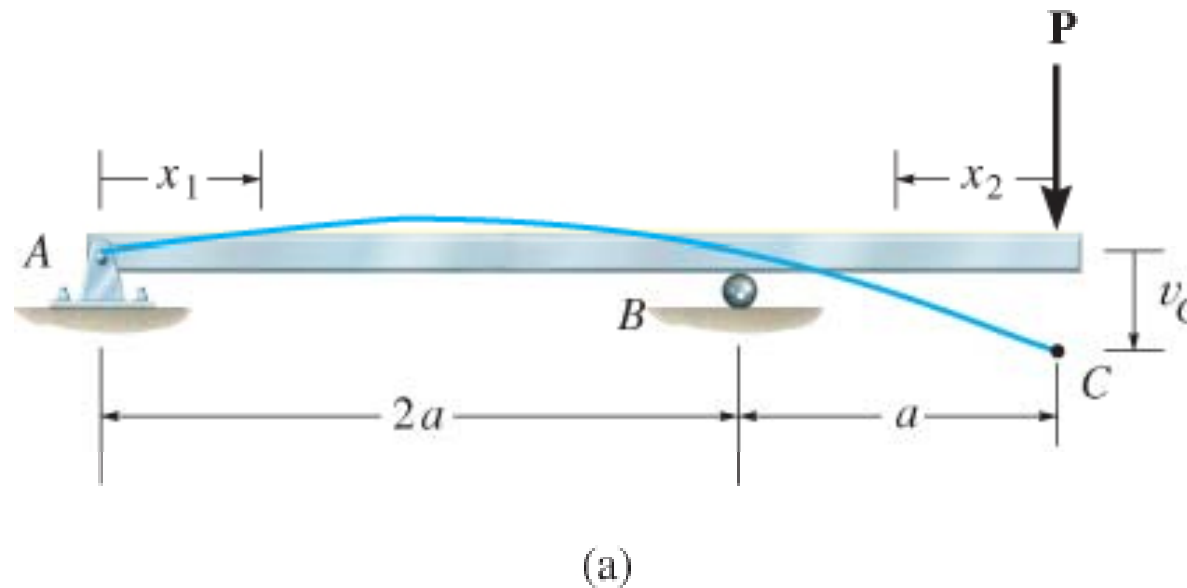
EXAMPLE 12.4 (SOLN)

Elastic curve

Beam deflects into shape shown. Due to loading, two x coordinates will be considered, $0 \leq x_1 \leq 2a$ and $0 \leq x_2 \leq a$, where x_2 is directed to the left from C since internal moment is easy to formulate.

(a)

EXAMPLE 12.4 (SOLN)

Moment functions

Using free-body diagrams, we have

$$M_1 = -\frac{P}{2}x_1 \qquad M_2 = -Px_2$$

(b)
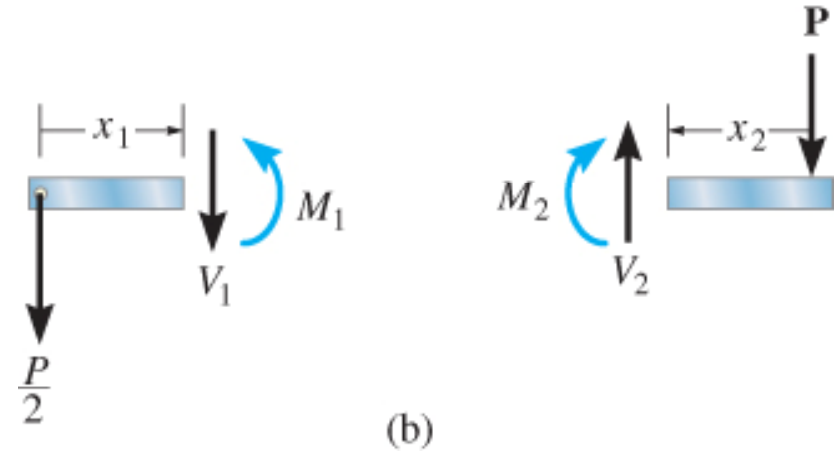
Slope and Elastic curve: Applying Eqn 10-12,

for $0 \le x_1 \le 2a$

$$EI\frac{d^2\upsilon_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{d\upsilon_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EI\upsilon_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

EXAMPLE 12.4 (SOLN)

Slope and Elastic curve:

Applying Eqn 10-12,

$$\text{for } 0 \le x_2 \le a \qquad EI = \frac{d^2 \upsilon_2}{dx_2{}^2} = -Px_2$$

$$EI \frac{d\upsilon_2}{dx_2} = -\frac{P}{2}x_2{}^2 + C_3 \qquad (3)$$

$$EI\upsilon_2 = -\frac{P}{6}x_2{}^3 + C_3 x_2 + C_4 \qquad (4)$$

EXAMPLE 12.4 (SOLN)

Slope and Elastic curve:

The four constants of integration determined using three boundary conditions, $v_1 = 0$ at $x_1 = 0$, $v_1 = 0$ at $x_1 = 2a$, and $v_2 = 0$ at $x_2 = a$ and a discontinuity eqn. Here, continuity of slope at roller requires $dv_1/dx_1 = -dv_2/dx_2$ at $x_1 = 2a$ and $x_2 = a$.

$$v_1 = 0 \ at \ x_1 = 0; \qquad 0 = 0 + 0 + C2$$

$$v_1 = 0 \ at \ x_1 = 2a; \quad 0 = -\frac{P}{12}(2a)^2 + C_1(2a) + C_2$$

EXAMPLE 12.4 (SOLN)

Slope and Elastic curve:

$$\upsilon_2 = 0 \text{ at } x_2 = a; \quad 0 = -\frac{P}{6}a^3 + C_3 a + C_4$$

$$\frac{d\upsilon_1(2a)}{dx_1} = -\frac{d\upsilon_2(a)}{dx_2}; \quad -\frac{P}{4}(2a)^2 + C_1 = -\left(-\frac{P}{2}(a)^2 + C_3\right)$$

Solving, we obtain

$$C_1 = \frac{Pa^2}{3} \qquad C_2 = 0 \qquad C_3 = \frac{7}{6}Pa_2 \qquad C_4 = -Pa^3$$

EXAMPLE 12.4 (SOLN)

Slope and Elastic curve:

Substituting C_3 and C_4 into Eqn (4) gives

$$\upsilon_2 = -\frac{P}{6EI}x_2{}^3 + \frac{7Pa^2}{6EI}x_2 - \frac{Pa^3}{EI}$$

Displacement at C is determined by setting $x_2 = 0$,

$$\upsilon_C = -\frac{Pa^3}{EI}$$

EXAMPLE 12.4 (SOLN)

- If several different loadings act on the beam the method of integration becomes more tedious to apply, because separate loadings or moment functions must be written for each region of the beam.
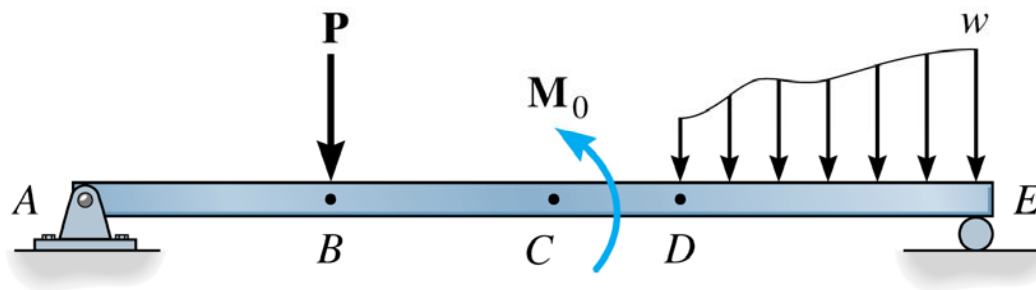
8 constants of integration

*12.3 DISCONTINUITY FUNCTIONS

- A simplified method for finding the eqn of the elastic curve for a multiply loaded beam using a single expression, formulated from the loading on the beam , $w = w(x)$, or the beam's internal moment, $M = M(x)$ is discussed below.

Discontinuity functions

Macaulay functions

- Such functions can be used to describe distributed loadings, written generally as

$$\langle x - a \rangle^n = \begin{cases} 0 & for\ x < a \\ (x-a)^n & for\ x \geq a \end{cases}$$

$$n \geq 0$$

$$(12\text{-}11)$$

*12.3 DISCONTINUITY FUNCTIONS

Discontinuity functions
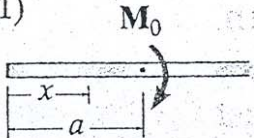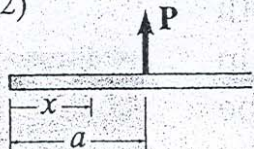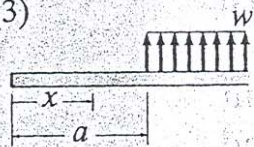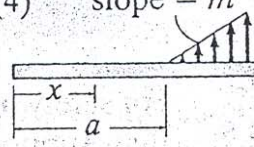
Macaulay functions

- x represents the coordinate position of a pt along the beam

- a is the location on the beam where a "discontinuity" occurs, or the pt where a distributed loading begins.

- The functions describe both uniform load and triangular load.

*12.3 DISCONTINUITY FUNCTIONS

Discontinuity functions
Macaulay functions

	TABLE 12–2		
Loading	**Loading Function** $w = w(x)$	**Shear** $V = \int w(x)dx$	**Moment** $M = \int V dx$
(1) M_0	$w = M_0\langle x-a\rangle^{-2}$	$V = M_0\langle x-a\rangle^{-1}$	$M = M_0\langle x-a\rangle^{0}$
(2) P	$w = P\langle x-a\rangle^{-1}$	$V = P\langle x-a\rangle^{0}$	$M = P\langle x-a\rangle^{1}$
(3) w_0	$w = w_0\langle x-a\rangle^{0}$	$V = w_0\langle x-a\rangle^{1}$	$M = \dfrac{w_0}{2}\langle x-a\rangle^{2}$
(4) slope = m	$w = m\langle x-a\rangle^{1}$	$V = \dfrac{m}{2}\langle x-a\rangle^{2}$	$M = \dfrac{m}{6}\langle x-a\rangle^{3}$

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Macaulay Functions. For purposes of beam or shaft deflection, Macaulay functions, named after the mathematician W. H. Macaulay, can be used to describe *distributed loadings*. These functions can be written in general form as

$$\langle x - a \rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x - a)^n & \text{for } x \geq a \end{cases} \quad (12\text{--}11)$$
$$n \geq 0$$

Here x represents the coordinate position of a point along the beam, and a is the location on the beam where a "discontinuity" occurs, namely the point where a distributed loading *begins*. Note that the Macaulay function $\langle x - a \rangle^n$ is written with angle brackets to distinguish it from the ordinary function $(x - a)^n$, written with parentheses. As stated by the equation, only when $x \geq a$ is $\langle x - a \rangle^n = (x - a)^n$, otherwise it is zero. Furthermore, these functions are valid only for exponential values $n \geq 0$. Integration of Macaulay functions follows the same rules as for ordinary functions, i.e.,

$$\int \langle x - a \rangle^n \, dx = \frac{\langle x - a \rangle^{n+1}}{n + 1} + C \qquad \qquad \int \langle x - a \rangle^n dx = \langle x - a \rangle^{n+1}, n = -1, -2$$

Note how the Macaulay functions describe both the *uniform load* $w_0 \, (n = 0)$ and *triangular load* $(n = 1)$, shown in Table 12–2, items 3 and 4. This type of description can, of course, be extended to distributed loadings having other forms. Also, it is possible to use superposition with

Singularity Functions. These functions are only used to describe the point location of concentrated forces or couple moments acting on a beam or shaft. Specifically, a concentrated force **P** can be considered as a special case of a distributed loading, where the intensity of the loading is $w = P/\epsilon$ such that its length is ϵ, where $\epsilon \to 0$, Fig. 12–15. The area under this loading diagram is equivalent to P, *positive upward*, and so we will use the singularity function

$$w = P\langle x - a \rangle^{-1} = \begin{cases} 0 & \text{for } x \neq a \\ P & \text{for } x = a \end{cases} \qquad (12\text{–}13)$$

to describe the force P. Here $n = -1$ so that the units for w are force per length, as it should be. Furthermore, the function takes on the value of **P** only at the point $x = a$ where the load occurs, otherwise it is zero.

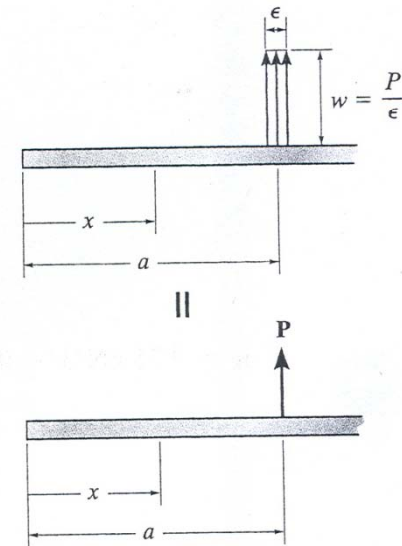Fig. 12–15

Singularity functions

In a similar manner, a couple moment M_0, considered *positive clockwise*, is a limit as $\epsilon \to 0$ of two distributed loadings as shown in Fig. 12–16. Here the following function describes its value.

$$w = M_0\langle x - a \rangle^{-2} = \begin{cases} 0 & \text{for } x \neq a \\ M_0 & \text{for } x = a \end{cases} \qquad (12\text{–}14)$$

The exponent $n = -2$, in order to ensure that the units of w, force per length, are maintained.

$$w = \frac{P}{\epsilon} = \frac{M_0}{\epsilon^2}$$

$$w = \frac{P}{\epsilon} = \frac{M_0}{\epsilon^2}$$
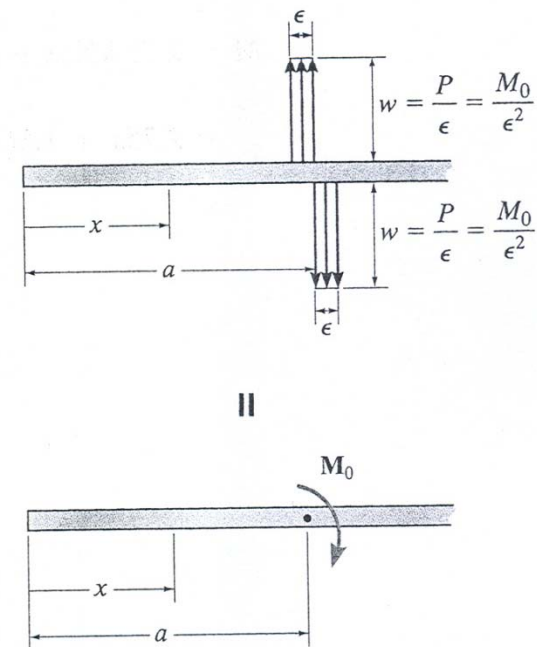
$$=$$

Fig. 12–16

Application of Eqs. 12–11 through 12–15 provides a rather direct means for expressing the loading or the internal moment in a beam as a function of x. When doing so, close attention must be paid to the signs of the external loadings. As stated above, and as shown in Table 12–2, *concentrated forces and distributed loads are positive upward, and couple moments are positive clockwise.* If this sign convention is followed, then the internal shear and moment are in accordance with the beam sign convention established in Sec. 6.1.

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(a)

(b)
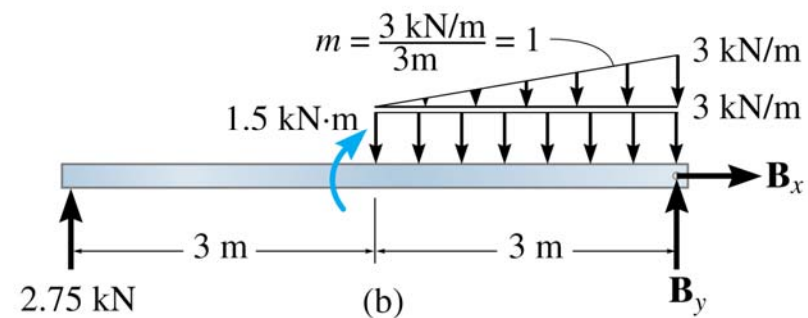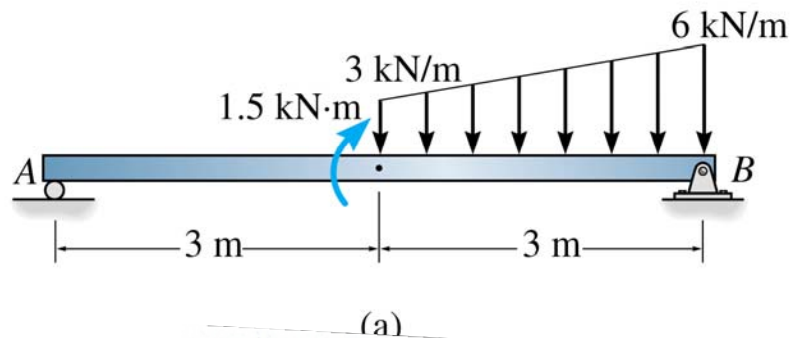
$$w = 2.75 \text{ kN}\langle x - 0 \rangle^{-1} + 1.5 \text{ kN} \cdot \text{m}\langle x - 3 \text{ m}\rangle^{-2} - 3 \text{ kN/m}\langle x - 3 \text{ m}\rangle^0 - 1 \text{ kN/m}^2\langle x - 3 \text{ m}\rangle^1$$

The reactive force at B is not included here since x is never greater than 6 m, and furthermore, this value is of no consequence in calculating the slope or deflection. We can determine the moment expression directly from Table 12–2, rather than integrating this expression twice. In either case,

$$M = 2.75 \text{ kN}\langle x - 0 \rangle^1 + 1.5 \text{ kN} \cdot \text{m}\langle x - 3 \text{ m}\rangle^0 - \frac{3 \text{ kN/m}}{2}\langle x - 3 \text{ m}\rangle^2 - \frac{1 \text{ kN/m}^2}{6}\langle x - 3 \text{ m}\rangle^3$$

$$= 2.75x + 1.5\langle x - 3 \rangle^0 - 1.5\langle x - 3 \rangle^2 - \frac{1}{6}\langle x - 3 \rangle^3$$

The deflection of the beam can now be determined after this equation is integrated two successive times and the constants of integration are evaluated using the boundary conditions of zero displacement at A and B.

*12.3 DISCONTINUITY FUNCTIONS

Procedure for analysis

Elastic curve

- Sketch the beam's elastic curve and identify the boundary conditions at the supports.

- Zero displacement occurs at all pin and roller supports, and zero slope and zero displacement occurs at fixed supports.

- Establish the x axis so that it extends to the right and has its origin at the beam's left end.

Load or moment function

- Calculate the support reactions and then use the discontinuity functions in Table 12-2 to express either the loading w or the internal moment M as a function of x.

*12.3 DISCONTINUITY FUNCTIONS

Procedure for analysis

Load or moment function

- Calculate the support reactions and then use the discontinuity functions in Table 12-2 to express either the loading w or the internal moment M as a function of x.

- Make sure to follow the sign convention for each loading as it applies for this equation.

- Note that the distributed loadings must extend all the way to the beam's right end to be valid. If this does not occur, use the method of superposition.

*12.3 DISCONTINUITY FUNCTIONS

Procedure for analysis

Slope and elastic curve

- Substitute w into $EI\ d^4v/dx^4 = -w(x)$ or M into the moment curvature relation $EI\ d^2v/dx^2 = M$, and integrate to obtain the eqns for the beam's slope and deflection.

- Evaluate the constants of integration using the boundary conditions, and substitute these constants into the slope and deflection eqns to obtain the final results.

- When the slope and deflection eqns are evaluated at any pt on the beam, a +ve slope is counterclockwise, and a +ve displacement is upward.

EXAMPLE 12.5
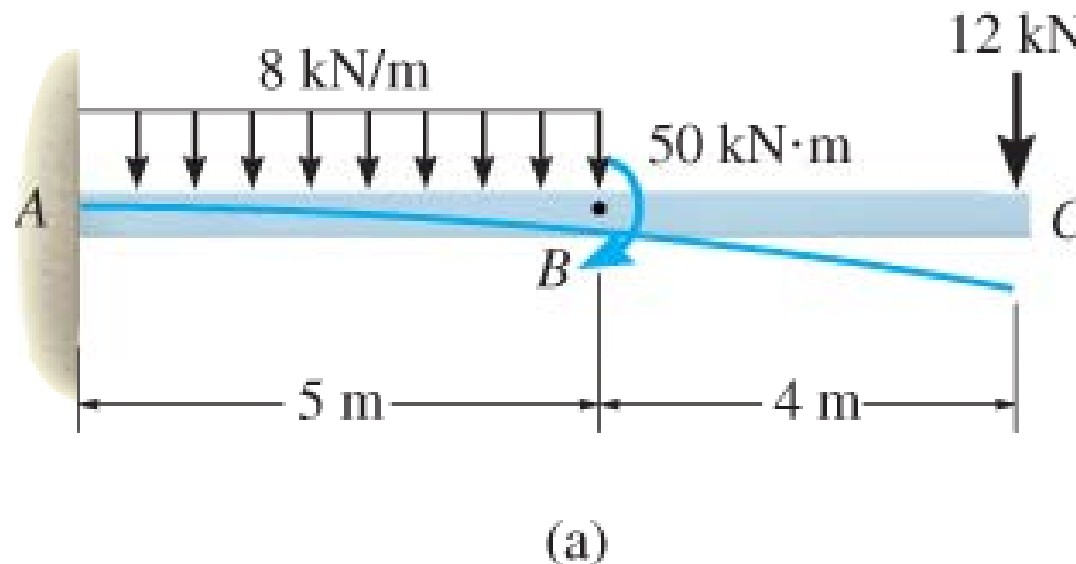
Determine the eqn of the elastic curve for the cantilevered beam shown. EI is constant.

(a)
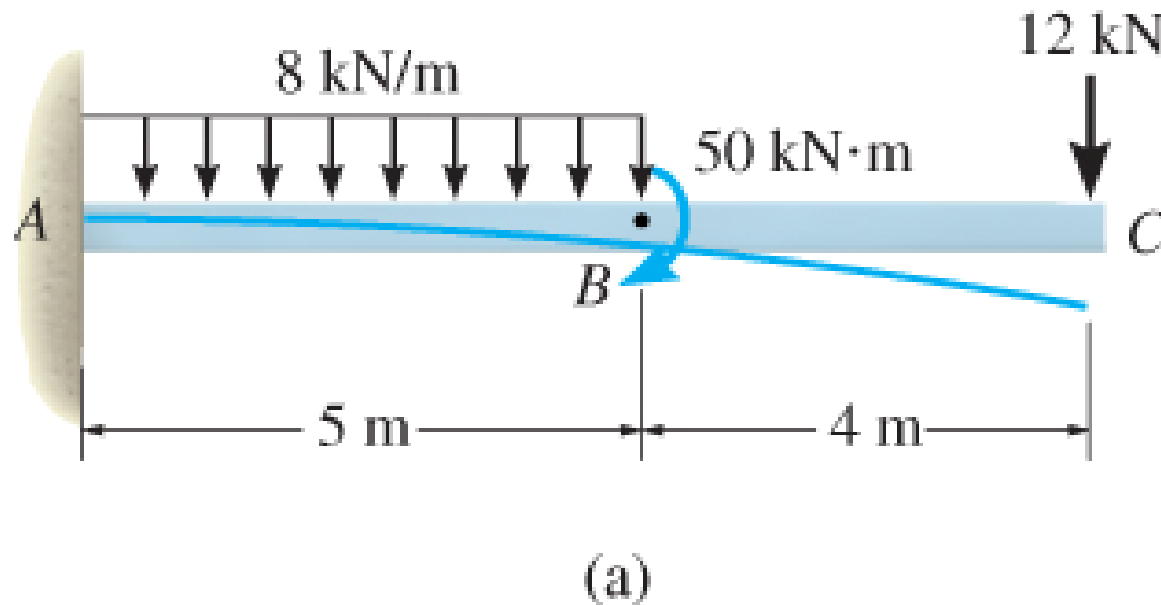
EXAMPLE 12.5 (SOLN)

Elastic curve

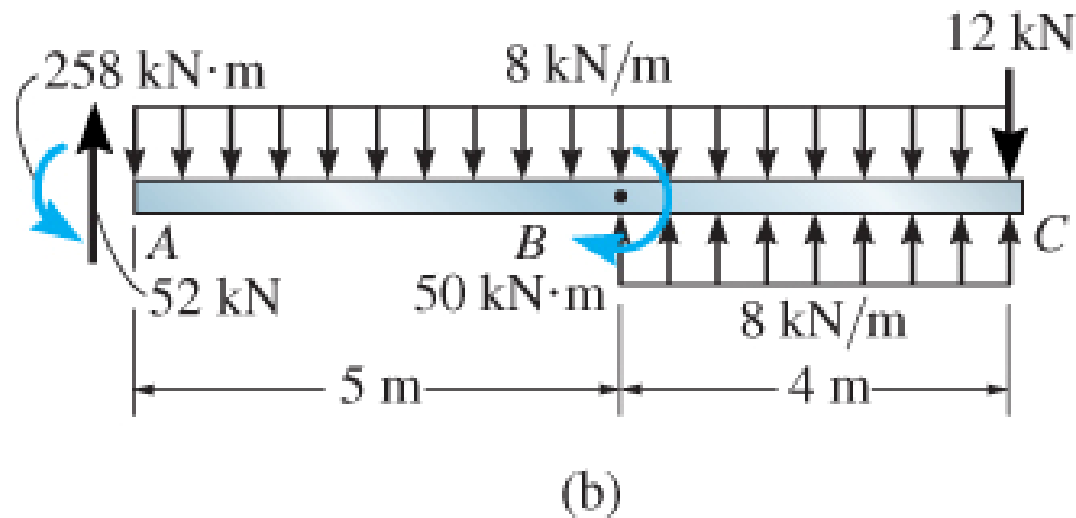The loads cause the beam to deflect as shown. The boundary conditions require zero slope and displacement at A.

(a)

EXAMPLE 12.5 (SOLN)

Loading functions

Support reactions shown on free-body diagram. Since distributed loading does not extend to C as required, use superposition of loadings to represent same effect.

(b)

EXAMPLE 12.5 (SOLN)

Loading functions

Therefore, $w = 52 \text{ kN}\langle x - 0 \rangle^{-1} - 258 \text{ kN} \cdot \text{m}\langle x - 0 \rangle^{-2} - 8KN/m\langle x - 0 \rangle^{0}$

$$+ 50 \text{ kN} \cdot \text{m}\langle x - 5 \text{ m}\rangle^{-2} + 8 \text{ kN}/\text{m}\langle x - 5 \text{ m}\rangle^{0}$$

The 12-kN load is not included, since x cannot be greater than 9 m. Because $dV/dx = -w(x)$, then by integrating, neglect constant of integration since reactions are included in load function, we have

$$V = 52\langle x - 0 \rangle^{0} - 258\langle x - 0 \rangle^{-1} - 8\langle x - 0 \rangle^{1}$$

$$+ 50\langle x - 5 \rangle^{-1} + 8\langle x - 5 \rangle^{1}$$

EXAMPLE 12.5 (SOLN)

Loading functions

Furthermore, $dM/dx = V$, so integrating again yields

$$M = -258\langle x-0\rangle^0 + 52\langle x-0\rangle^1 - \frac{1}{2}(8)\langle x-0\rangle^2 + 50\langle x-5\rangle^0 + \frac{1}{2}(8)\langle x-5\rangle^2$$

$$= \left(-258 + 52x - 4x^2 + 4\langle x-5\rangle^2 + 50\langle x-5\rangle^0\right) kN\cdot m$$

The same result can be obtained directly from Table 12-2.

EXAMPLE 12.5 (SOLN)

Slope and elastic curve

Applying Eqn 12-10 and integrating twice, we have

$$EI\frac{d^2\upsilon}{dx^2} = -258 + 52x - 4x^2 + 50\langle x-5 \rangle^0 + 4\langle x-5 \rangle^2$$

$$EI\frac{d\upsilon}{dx} = -258x + 26x^2 - \frac{4}{3}x^3 + 50\langle x-5 \rangle^1 + \frac{4}{3}\langle x-5 \rangle^3 + C_1$$

$$EI\upsilon = -129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4 + 25\langle x-5 \rangle^2$$

$$+ \frac{1}{3}\langle x-5 \rangle^4 + C_1x + C_2$$

EXAMPLE 12.5 (SOLN)

Slope and elastic curve
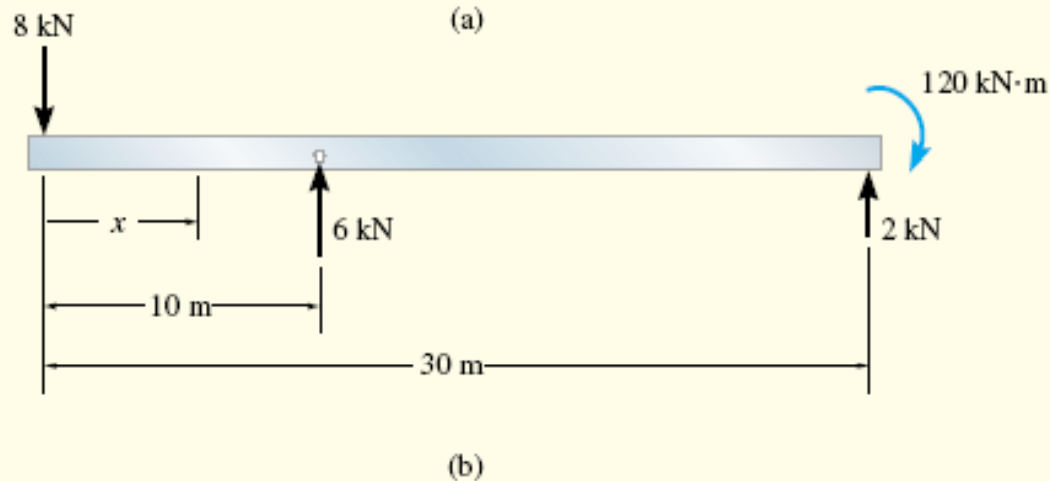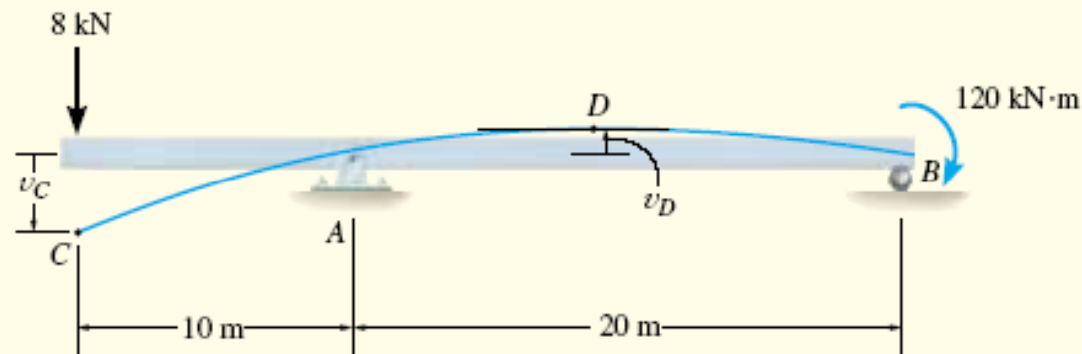
Since $dv/dx = 0$ at $x = 0$, $C_1 = 0$; and $v = 0$ at $x = 0$, so $C_2 = 0$. Thus

$$v = \frac{1}{EI}\left(-129x^2 + \frac{26}{3}x^3 - \frac{1}{3}x^4\right.$$

$$\left. +25\langle x-5\rangle^2 + \frac{1}{3}\langle x-5\rangle^4\right) \text{ m}$$

E X A M P L E 12.6

Determine the maximum deflection of the beam shown in Fig. 12–20a. EI is constant.

(a)

(b)

Loading Function. The reactions have been calculated and are shown on the free-body diagram in Fig. 12–20b. The loading function for the beam can be written as

$$w = 8 \text{ kN}\langle x - 0\rangle^{-1} - 6 \text{ kN}\langle x - 10 \text{ m}\rangle^{-1}$$

The couple moment and force at B are not included here, since they are located at the right end of the beam, and x cannot be greater than 30 m. Applying $dV/dx = -w(x)$, we get

$$V = -8\langle x - 0\rangle^0 + 6\langle x - 10\rangle^0$$

In a similar manner, $dM/dx = V$ yields

$$M = -8\langle x - 0\rangle^1 + 6\langle x - 10\rangle^1$$
$$= (-8x + 6\langle x - 10\rangle^1) \text{ kN} \cdot \text{m}$$

Slope and Elastic Curve. Integrating twice yields

$$EI\frac{d^2v}{dx^2} = -8x + 6\langle x - 10\rangle^1$$

$$EI\frac{dv}{dx} = -4x^2 + 3\langle x - 10\rangle^2 + C_1$$

$$EIv = -\frac{4}{3}x^3 + \langle x - 10\rangle^3 + C_1 x + C_2$$

From Eq. 1, the boundary condition $v = 0$ at $x = 10$ m and $v = 0$ at $x = 30$ m gives

$$0 = -1333 + (10 - 10)^3 + C_1(10) + C_2$$

$$0 = -36\,000 + (30 - 10)^3 + C_1(30) + C_2$$

Solving these equations simultaneously for C_1 and C_2, we get $C_1 = 1333$ and $C_2 = -12\,000$. Thus,

$$EI\frac{dv}{dx} = -4x^2 + 3(x - 10)^2 + 1333 \tag{2}$$

$$EIv = -\frac{4}{3}x^3 + (x - 10)^3 + 1333x - 12\,000 \tag{3}$$

From Fig. 12-20a, maximum displacement may occur either at C, or at D, where the slope $dv/dx = 0$. To obtain the displacement of C, set $x = 0$ in Eq. 3. We get

$$v_C = -\frac{12\,000 \text{ kN} \cdot \text{m}^3}{EI}$$

To locate point D, use Eq. 2 with $x > 10$ m and $dv/dx = 0$.

$$0 = -4x_D^2 + 3(x_D - 10)^2 + 1333$$
$$x_D^2 + 60x_D - 1633 = 0$$

Solving for the positive root,

$$x_D = 20.3 \text{ m}$$

Hence, from Eq. 3,

$$EIv_D = -\frac{4}{3}(20.3)^3 + (20.3 - 10)^3 + 1333(20.3) - 12\,000$$

$$v_D = \frac{5000 \text{ kN} \cdot \text{m}^3}{EI} \qquad \textit{Ans.}$$

Comparing this value with v_C, we see that $v_{max} = v_C$.