Mechanical Springs

ME 305 Mechanical Engineering Design 2

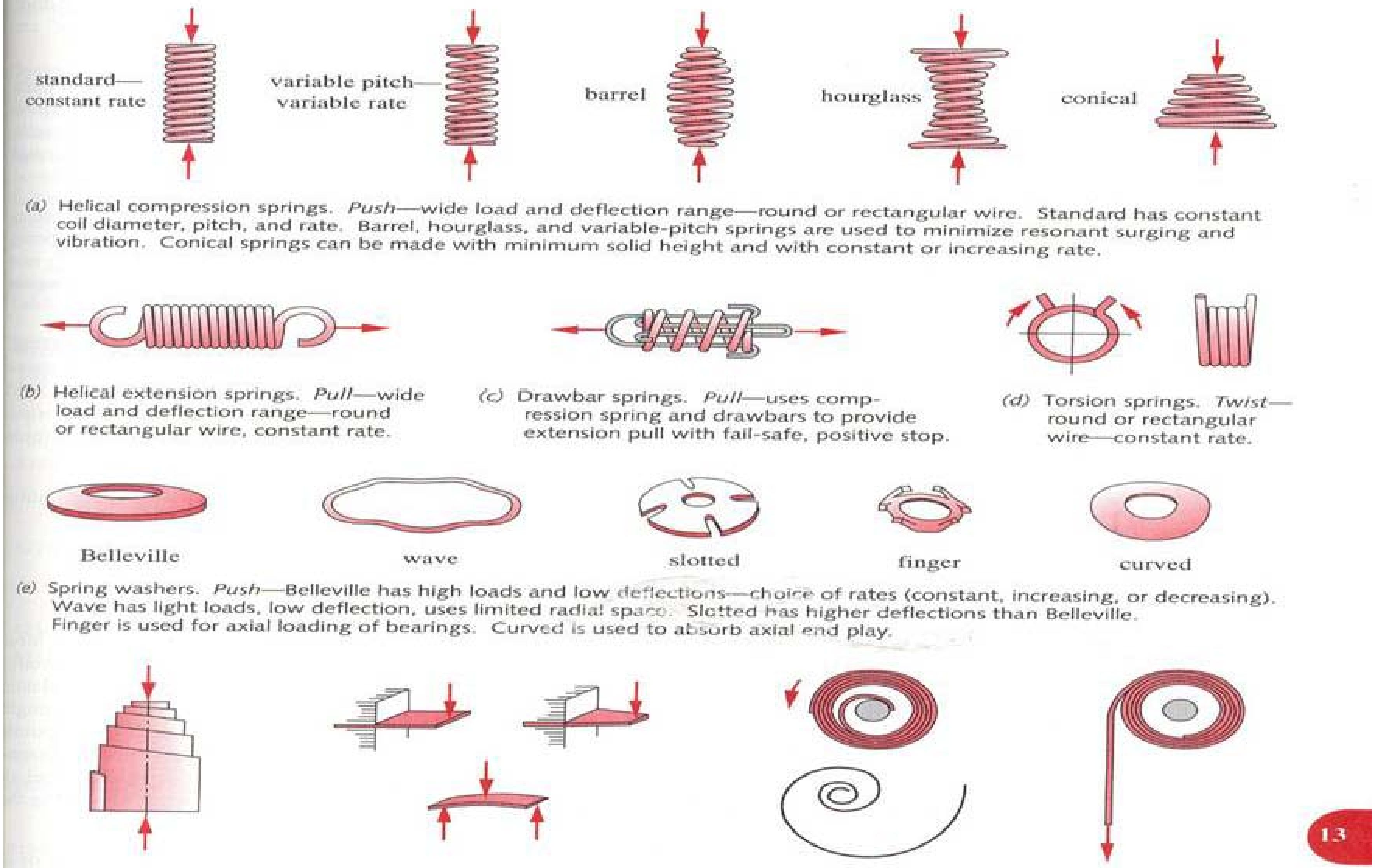
Topics to be Covered

- Stresses in Helical Springs
- The Curvature Effect
- Deflection of Helical Springs
- Compression Springs
- Stability
- Spring Materials
- Helical Compression Spring Design for static Service
- Critical Frequency of Helical Springs
- Fatigue Loading of Helical Compression Spring
- Helical Compression Spring Design for Fatigue Loading
- Extension Springs
- Helical Coil Torsion Springs

Mechanical Springs

- Produce a pull, a push, or a twist (torque) force when displaced.
- Store or absorb energy.
- May be made of **round or rectangular wire** bent into a suitable form such as a coil, or made of **flat stock** loaded as a beam.

Types of Spring



(a) Helical compression springs. *Push*—wide load and deflection range—round or rectangular wire. Standard has constant coil diameter, pitch, and rate. Barrel, hourglass, and variable-pitch springs are used to minimize resonant surging and vibration. Conical springs can be made with minimum solid height and with constant or increasing rate.

(b) Helical extension springs. *Pull*—wide load and deflection range—round or rectangular wire, constant rate.

(c) Drawbar springs. *Pull*—uses compression spring and drawbars to provide extension pull with fail-safe, positive stop.

(d) Torsion springs. *Twist*—round or rectangular wire—constant rate.

(e) Spring washers. *Push*—Belleville has high loads and low deflections—choice of rates (constant, increasing, or decreasing). Wave has light loads, low deflection, uses limited radial space. Slotted has higher deflections than Belleville. Finger is used for axial loading of bearings. Curved is used to absorb axial end play.

(f) Volute spring. *Push*—may have an inherently high friction damping.

(g) Beam springs. *Push* or *Pull*—wide load but low deflection range—rectangular or shaped cantilever, or simply supported.

(h) Power or motor springs. *Twist*—exerts torque over many turns. Shown in, and removed from, retainer.

(i) Constant Force. *Pull*—long deflection at low or zero rate.

FIGURE 13-2

Spring Configurations (Adapted from: *Design Handbook: Engineering Guide to Spring Design*, 1987, Associated Spring, Barnes Group Inc., 10 Main St., Bristol Conn., with permission)

Push Function

- **Push function** is provided by **helical compression springs**, spring washers, volute springs, and beam springs. These are shown in the previous page.
- Helical Compression Springs: Used in **applications involving large deflections**, such as **shock absorbers** in automobiles or to **hold** batteries in consumer products. Used in **valve-return springs** in engine, die springs, etc.
- Conical Springs: Spring rate is nonlinear. By varying coil pitch, a nearly constant spring rate can be obtained. Advantage is the ability to close to a height as small as one wire diameter if the coils nest.

Push Function – Cont'd

- Barrel/Hourglass/Variable Pitch Springs: Can be thought of a two conical springs back to back, also having a nonlinear spring rate. Barrel, hourglass, and variable pitch springs are used to minimize resonant surging and vibration.
- Spring Washers: Used for **small deflections** associated with motion along a bolt or other guide. Used to **load something axially**, such as to take up endplay on a bearing.
- Volute Springs: Can be used for damping and also to resist buckling. Very expensive. Shear cutter for trimming, and has significant friction and hysteresis (significant energy loss)
- Beam Springs: Can be used to push or pull. Examples are diving boards. Spring rate and stress distribution can be controlled with changes in beam width or depth along its length. Loads can be high but deflections are limited.

Pull Function

- **Pull function** is provided by **helical extension** springs and **constant force** springs.

- Helical Extension Springs: Capable of large deflection. Used in door closers and counterbalances, automobile wiper blades, children's car seats and car hoods. Hooks more highly stressed than **coils** and **usually fail first**. When hook fails this spring becomes unsafe.

Twisting Function

- **Twisting function** is provided by **helical torsion** springs and **spiral** springs (coils in the same plane).
- Helical Torsion Springs: Used for garage-door counter-balancers and counterbalancing of such things as doors which rotate about a horizontal edge. Clothespins, mousetraps and finger exercisers are examples.
- Spiral Springs:
- • *Hairsprings* are used in instruments and mechanical clocks and watches. One of their **characteristics is low hysteresis** (small energy loss).
- • *Brush Springs*: Hold motor and generator brushes against their commutators.

Twisting Function – Cont'd

- Motor, Clock or Power Springs: Used to **supply rotational energy** and used in **windup clocks and mechanical toys**.
- Prestressed Power Springs: Has large energy storage capacity. Used in seatbelt retractors.
- Constant-Torque Spring Motor: Used to provide level torque.
- Drawbar Springs: **Unlike helical extension** spring, it will **support the load safely when it fails.**

Spring Rate

- **Every spring** configuration has a spring rate, k, defined as slope of its force-deflection curve.
- If slope is constant, it is a linear spring, and $$k = \frac{F}{y}$$

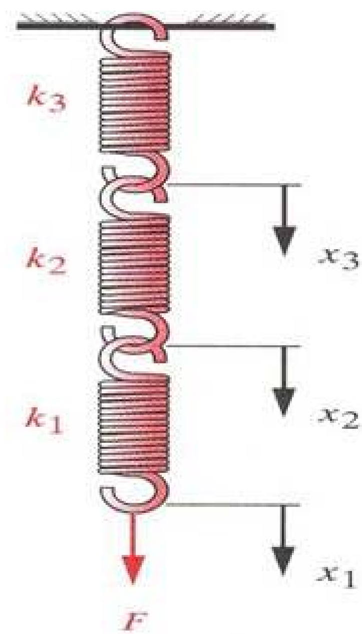
Where: F is applied force, and y is deflection.

- When spring rate varies with deflection, it is called a nonlinear spring.
- We often want a linear spring to control loading.
- Many spring configurations have constant spring rates and **few have zero rates** (constant force).

Spring Rate – Cont'd

- When **multiple springs** are combined, resulting spring rate depends on whether they are combined in series or parallel.
- Springs in series have same force passing through them, as each contributes to total deflection .

$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots + \frac{1}{k_n}$$

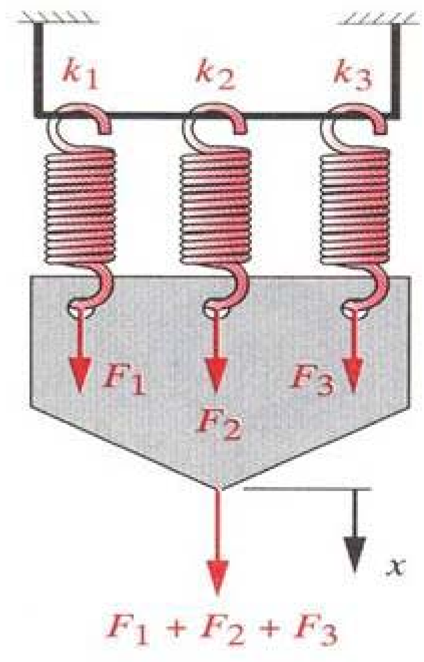


(a) Series

Spring Rate – Cont'd

- Springs in parallel have same deflection, and total force is split among them . For springs in parallel, individual spring rates add directly.

$$k_{total} = k_1 + k_2 + k_3 + \ldots + k_n$$



(b) Parallel

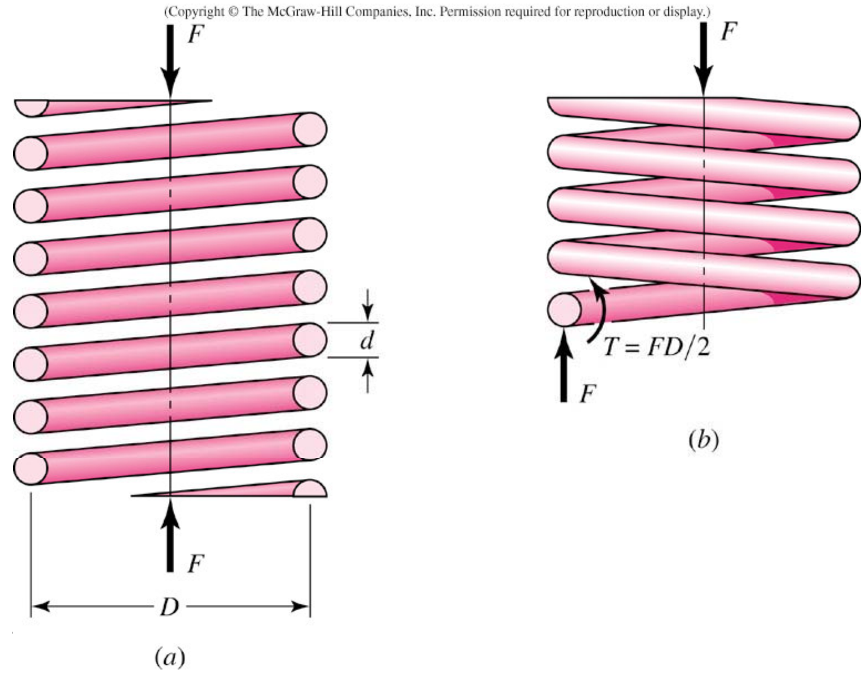
10.1 Stresses in Helical Springs

$$\tau_{\max} = \frac{Tr}{J} + \frac{F}{A}$$

- Replace $T = FD/2$, $r = d/2$, $J = \pi d^4/32$, $A = \pi d^2/4$, $\tau_{\max} = \tau$:

$$\tau = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2} \qquad (10\text{-}1)$$

- A round-wire helical compression spring is loaded by the axial force F.



Define the spring index as: $C = \dfrac{D}{d}$

- The mean coil diameter D
- The wire diameter d.

For most springs the range for C is: $6 \le C \le 12$.

$$\tau = K_S \frac{8FD}{\pi d^3}$$

$$K_S = \frac{2C+1}{2C}$$

Where K_S is a shear-stress correction factor

10.2 The Curvature Effect

- $\tau = K_S \dfrac{8FD}{\pi d^3}$ is based on the wire being straight.
- In fatigue load: It is important to include the curvature stress.
- To include the curvature effect, the factor K_S needs to be modified.
- The curvature of the wire increases the stress on the inside of the spring but decreases it only slightly on the outside.
- In static load: these stresses can normally be neglected because it will be relieved by local yielding with first application of a load.
- Wahl factor: $$K_W = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$
- Bergstrasser factor: $$K_B = \frac{4C+2}{4C-3}$$

- The curvature correction factor can now be obtained by canceling out the effect of the direct shear form K_B, thus

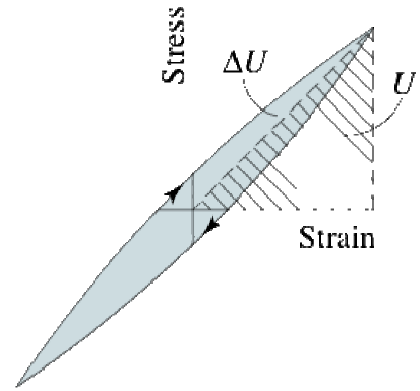
$$K_C = \frac{K_B}{K_S} = \frac{2C(4C+2)}{(4C-3)(2C+1)} \qquad (10\text{-}7)$$

- To predict the largest shear stress we will use the equation:

$$\tau = K_B \frac{8FD}{\pi d^3}$$

10.3 Deflection of Helical Springs

- Using the strain energy method to include both the torsional and shear components, thus



$$U = \frac{T^2 l}{2GJ} + \frac{F^2 l}{2AG}$$

- Substituting $T = FD/2$, $l = \pi DN$, $J = \pi d^4/32$,

$A = \pi d^2/4$, in the previous equation we have:

$$U = \frac{4F^2 D^3 N}{d^4 G} + \frac{2F^2 DN}{d^2 G}$$

Where $N = N_a$ = number of active coils

- The total deflection y can now be calculated by:

$$y = \frac{\partial U}{\partial F} = \frac{8FD^3N}{d^4G} + \frac{4FDN}{d^2G}$$

- Since, $C = D/d$, the deflection y becomes:

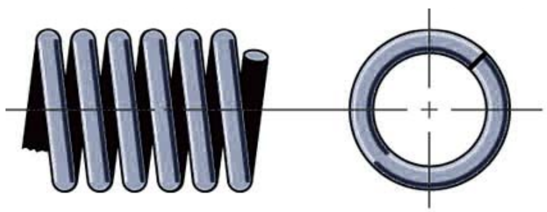
$$y \cong \frac{8FD^3N}{d^4G} \qquad \Longrightarrow \qquad (10\text{-}8)$$

- The spring rate can be calculated by

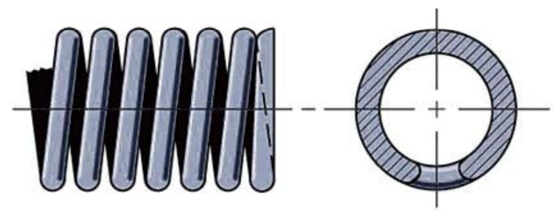
$$k = \frac{F}{y} = \frac{d^4G}{8D^3N} \qquad \Longrightarrow \qquad (10\text{-}9)$$

10.4 Compression Springs

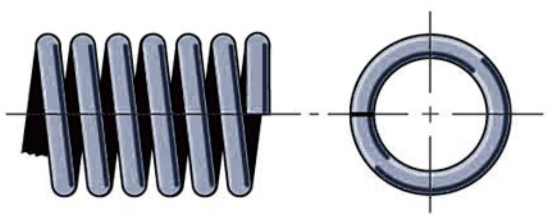
- There are four standard types of ends in helical compression springs. They are plain end, squared end, plain-ground end, and squared-ground end.



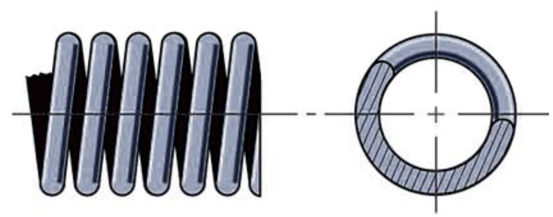
(*a*) Plain end, right hand



(*c*) Squared and ground end, left hand



(*b*) Squared or closed end, right hand



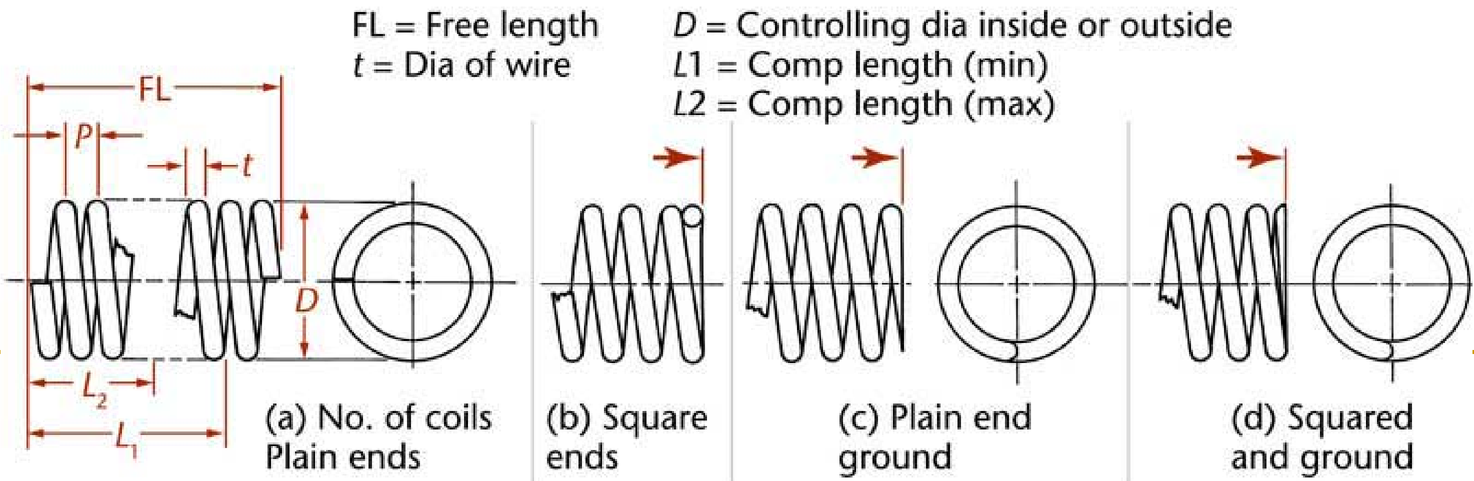
(*d*) Plain end, ground, left hand

- A spring with plain ends has a noninterrupted helicoids; the ends are the same as if a long spring had been cut into sections.

- A spring with plain ends that are squared or closed is obtained by deforming the ends to a zero-degree helix angle.

- Springs should always be both squared and ground for important applications, because a better transfer of the load is obtained.

- A spring with squared and ground ends compressed between rigid plates can be considered to have fixed ends.



- The type of end used affects the number of active coils N_a and the solid height of the spring.

- Square ends effectively decrease the number of total coils N_t by approximately two: $N_t = N_a+2$

•Forys gives an expression for calculating the solid length of squared and ground ends

$$L_s=(N_t - a)d$$

Where a varies, with an average of 0.75 which means in this case that the entry dN_t in table 10-1 may be overstated.
The way to check these variations is to take a spring and count the wire diameters in the solid stack

- Table 10-1 shows how the type of end used affects the number of coils and the spring length.

	Types of Springs Ends			
Term	**Plain**	**Plain and ground**	**Squared or Closed**	**Squared and Ground**
End coils, N_e	0	1	2	2
Total coils, N_t	N_a	N_a+1	N_a+2	N_a+2
Free length, L_0	pN_a+d	$p(N_a+1)$	pN_a+3d	pN_a+2d
Solid length, L_s	$d(N_t+1)$	dN_t	$d(N_t+1)$	dN_t
Pitch, p	$(L_0-d)/ N_a$	$L_0/ (N_a+1)$	$(L_0-3d)/ N_a$	$(L_0-2d)/ N_a$

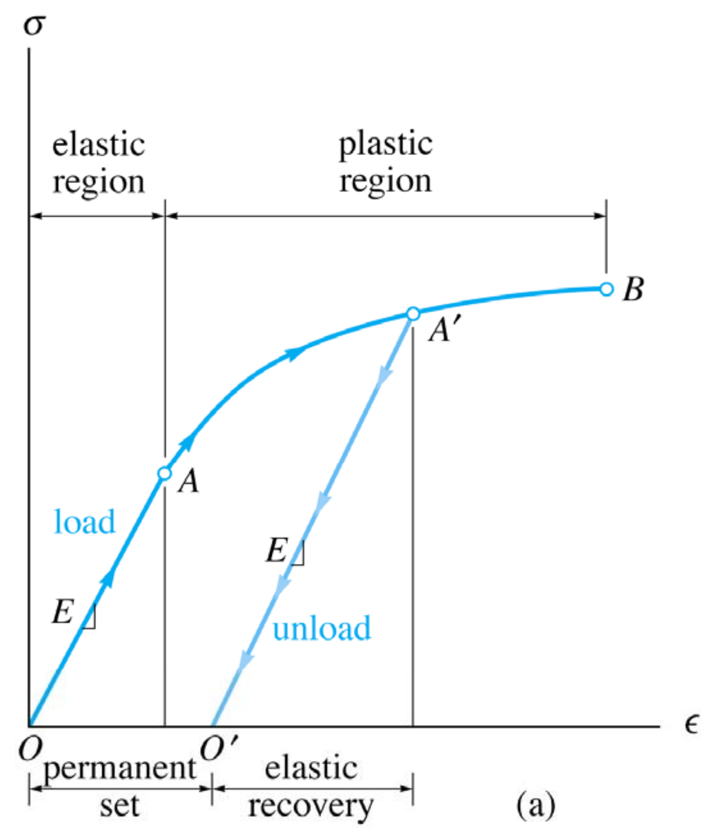
Set removal or presetting:

Set removal or *presetting* is a process used in the manufacture of compression springs to induce useful residual stresses. It is done by making the spring longer than needed and then compressing it to its solid height. This operation *sets* the spring to the required final free length and, since the torsional yield strength has been exceeded, induces residual stresses opposite in direction to those induced in service. Springs to be preset should be designed so that 10 to 30 percent of the initial free length is removed during the operation. If the stress at the solid height is greater than 1.3 times the torsional yield strength, distortion may occur. If this stress is much less than 1.1 times, it is difficult to control the resulting free length.

Set removal increases the strength of the spring and so is especially useful when the spring is used for energy-storage purposes. However, set removal should not be used when springs are subject to fatigue.

Set removal or presetting:

- A process used to induce useful residual stresses.
- It is done by making the spring longer than needed and then compressing it to its solid height L_s.

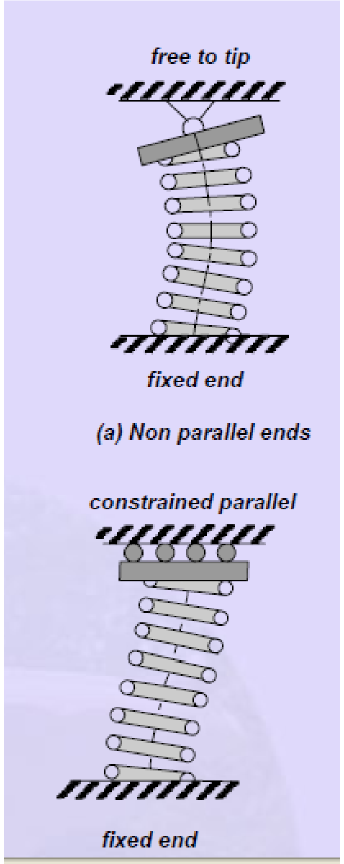


- This operation sets the spring to the required final free length L_0 and, since the torsional yield strength has been exceeded, it induces residual stresses opposite in the direction to those induced in service.
- Set removal increases the strength of the spring and so is especially useful when the spring is used for energy-storage purposes. But, set removal should not be used when springs are subjects to fatigue.

10.5 Stability

- Compression coil springs may buckle when the deflection becomes too large.
- The critical deflection is given by the equation:

$$y_{cr} = L_0 C_1' \left[1 - \left(1 - \frac{C_2'}{\lambda_{eff}^2} \right)^{1/2} \right] \qquad (10\text{-}10)$$



Where

y_{cr} is the deflection corresponding to the onset of instability

λ_{eff} is the effective slenderness ratio and is given by:

$$\lambda_{eff} = \frac{\alpha L_0}{D} \qquad (10\text{-}11)$$

α the end condition constant

- $C_1^{'}$ & $C_2^{'}$ are elastic constants defined by the equations:

$$C_1^{'} = \frac{E}{2(E-G)}; C_2^{'} = \frac{2\pi^2 (E-G)}{2G+E}$$

- The end condition constant α depends upon how the ends of the spring are support.

End condition	Constant α
Spring supported between flat parallel surfaces (fixed ends)	0.5
One end supported by flat surface perpendicular to spring axis (fixed); other end pivoted (hinged)	0.707
Both ends pivoted (hinged)	1
One end clamped; other end free	2

- Absolute stability occurs when the term C_2' / λ_{eff}^2 in equation (10-10) is greater than unity. Thus, for stability we have:

$$L_0 < \frac{\pi D}{\alpha}\left[\frac{2(E-G)}{2G+E}\right]^{1/2} \quad (10\text{-}12)$$

- For steels, equation 10-12 becomes:

$L_0 < 2.63D/\alpha$

- For squared and ground ends

$\alpha = 0.5$ and $L_0 < 5.26D$

Spring Materials

- Limited **number of materials and alloys** are suitable for use as springs.
- **Ideal spring material** would have **high ultimate strength, high yield point, and low modulus of elasticity** in order to **provide maximum energy storage** (area under elastic portion of stress-strain curve).
- For **dynamically loaded** springs, **fatigue strength properties** of material are of **primary importance**.

Spring Materials – Cont'd

- High strength and high yield points are attainable from medium-to high-carbon and alloy steels, and these are most common spring materials, despite their high modulus of elasticity. Few stainless-steel alloys are suitable for springs, as are beryllium copper and phosphor bronze, among copper alloys.
- Springs are manufactured by hot-or cold-working process. Winding springs induces residual stresses through bending. These stresses are relieved through mild heat treatment.

Table 10–3

High-Carbon and Alloy Spring Steels

Source: From Harold C. R. Carlson, "Selection and Application of Spring Materials," *Mechanical Engineering*, vol. 78, 1956, pp. 331–334.

Name of Material	Similar Specifications	Description
Music wire, 0.80–0.95C	UNS G10850 AISI 1085 ASTM A228-51	This is the best, toughest, and most widely used of all spring materials for small springs. It has the highest tensile strength and can withstand higher stresses under repeated loading than any other spring material. Available in diameters 0.12 to 3 mm (0.005 to 0.125 in). Do not use above 120°C (250°F) or at subzero temperatures.
Oil-tempered wire, 0.60–0.70C	UNS G10650 AISI 1065 ASTM 229-41	This general-purpose spring steel is used for many types of coil springs where the cost of music wire is prohibitive and in sizes larger than available in music wire. Not for shock or impact loading. Available in diameters 3 to 12 mm (0.125 to 0.5000 in), but larger and smaller sizes may be obtained. Not for use above 180°C (350°F) or at subzero temperatures.
Hard-drawn wire, 0.60–0.70C	UNS G10660 AISI 1066 ASTM A227-47	This is the cheapest general-purpose spring steel and should be used only where life, accuracy, and deflection are not too important. Available in diameters 0.8 to 12 mm (0.031 to 0.500 in). Not for use above 120°C (250°F) or at subzero temperatures.
Chrome-vanadium	UNS G61500 AISI 6150 ASTM 231-41	This is the most popular alloy spring steel for conditions involving higher stresses than can be used with the high-carbon steels and for use where fatigue resistance and long endurance are needed. Also good for shock and impact loads. Widely used for aircraft-engine valve springs and for temperatures to 220°C (425°F). Available in annealed or pretempered sizes 0.8 to 12 mm (0.031 to 0.500 in) in diameter.
Chrome-silicon	UNS G92540 AISI 9254	This alloy is an excellent material for highly stressed springs that require long life and are subjected to shock loading. Rockwell hardnesses of C50 to C53 are quite common, and the material may be used up to 250°C (475°F). Available from 0.8 to 12 mm (0.031 to 0.500 in) in diameter.

Spring Wire

- It is common to use round wire as spring material. It is available in selection of alloys and wide range of sizes.
- Rectangular wire is available only in limited sizes.

Tensile Strength

- <u>Tensile Strength</u>
- Relationship between wire size and tensile strength is shown below. Cold-drawn process used in reducing wire diameter is responsible for hardening and strengthening material at expense of much of its ductility.

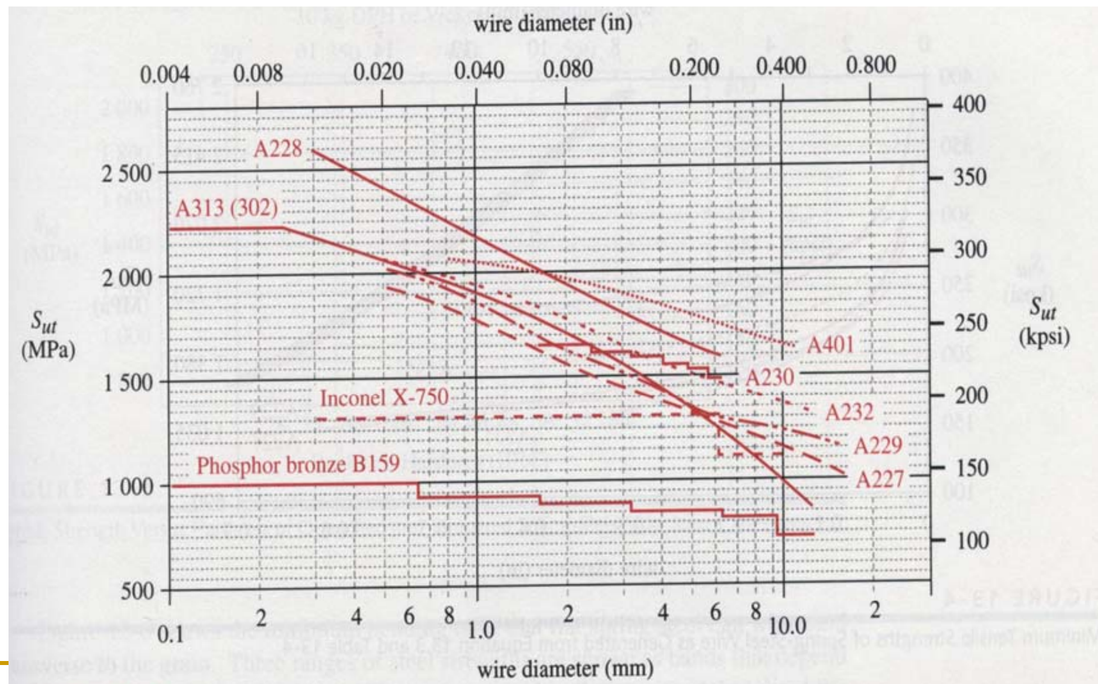


FIGURE 13-3

Minimum Tensile Strengths of Spring Wire—Identified by ASTM Number—See Table 13-1 Source: *Design Handbook: Engineering Guide to Spring Design*, 1987, Associated Spring, Barnes Group Inc., Bristol, Conn.

Tensile Strength

- **Empirical equation resulting** from **fitting exponential function** through **data for five of materials**, is:

$$S_{ut} = \frac{A}{d^m}$$

- Where **A and m** are defined in Table 10-4 for these wires materials.

Table 10–4

Constants A and m of $S_{ut} = A/d^m$ for Estimating Minimum Tensile Strength of Common Spring Wires

Source: From *Design Handbook,* 1987, p. 19. Courtesy of Associated Spring.

Material	ASTM No.	Exponent m	Diameter, in	A, kpsi · in^m	Diameter, mm	A, MPa · mm^m	Relative Cost of wire
Music wire*	A228	0.145	0.004–0.256	201	0.10–6.5	2211	2.6
OQ&T wire†	A229	0.187	0.020–0.500	147	0.5–12.7	1855	1.3
Hard-drawn wire‡	A227	0.190	0.028–0.500	140	0.7–12.7	1783	1.0
Chrome-vanadium wire§	A232	0.168	0.032–0.437	169	0.8–11.1	2005	3.1
Chrome-silicon wire‖	A401	0.108	0.063–0.375	202	1.6–9.5	1974	4.0
302 Stainless wire#	A313	0.146	0.013–0.10	169	0.3–2.5	1867	7.6–11
		0.263	0.10–0.20	128	2.5–5	2065	
		0.478	0.20–0.40	90	5–10	2911	
Phosphor-bronze wire**	B159	0	0.004–0.022	145	0.1–0.6	1000	8.0
		0.028	0.022–0.075	121	0.6–2	913	
		0.064	0.075–0.30	110	2–7.5	932	

Shear Strength

- A very rough estimate of the torsional yield strength can be obtained by assuming that the tensile yield strength is between 60 and 90 percent of the tensile strength.

$$0.6S_{ut} < S_y < 0.9S_{ut}$$

- Distortion-energy theory can be applied to obtain torsional yield strength: $S_{ys} = 0.577S_y$.
- Therefore, for steel:

$$0.577(0.6)S_{ut} < S_{ys} < 0.577(0.9)S_{ut}$$

$$0.35S_{ut} < S_{ys} < 0.52S_{ut}$$

- For wires listed in table 10-5, the maximum allowable shear stress in a spring can be seen in column 3.
- Joerres uses the maximum allowable torsional stress for static applications and it is given in table 10-6
- Samonov shows that $S_{sy} = \tau_{all} = 0.56 S_{ut}$ for high-tensile spring steels, which is close to the value given by Joerres for hardened alloy steels.

Table 10–5

Mechanical Properties of Some Spring Wires

Material	Elastic Limit, Percent of S_{ut} Tension	Torsion	Diameter d, in	E Mpsi	E GPa	G Mpsi	G GPa
Music wire A228	65–75	45–60	<0.032	29.5	203.4	12.0	82.7
			0.033–0.063	29.0	200	11.85	81.7
			0.064–0.125	28.5	196.5	11.75	81.0
			>0.125	28.0	193	11.6	80.0
HD spring A227	60–70	45–55	<0.032	28.8	198.6	11.7	80.7
			0.033–0.063	28.7	197.9	11.6	80.0
			0.064–0.125	28.6	197.2	11.5	79.3
			>0.125	28.5	196.5	11.4	78.6
Oil tempered A239	85–90	45–50		28.5	196.5	11.2	77.2
Valve spring A230	85–90	50–60		29.5	203.4	11.2	77.2
Chrome-vanadium A231	88–93	65–75		29.5	203.4	11.2	77.2
A232	88–93			29.5	203.4	11.2	77.2
Chrome-silicon A401	85–93	65–75		29.5	203.4	11.2	77.2
Stainless steel							
A313*	65–75	45–55		28	193	10	69.0
17-7PH	75–80	55–60		29.5	208.4	11	75.8
414	65–70	42–55		29	200	11.2	77.2
420	65–75	45–55		29	200	11.2	77.2
431	72–76	50–55		30	206	11.5	79.3

Table 10–6

Maximum Allowable Torsional Stresses for Helical Compression Springs in Static Applications

Source: Robert E. Joerres, "Springs," Chap. 6 in Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds.), *Standard Handbook of Machine Design*, 3rd ed., McGraw-Hill, New York, 2004.

Material	Maximum Percent of Tensile Strength: Before Set Removed (includes K_W or K_B)	After Set Removed (includes K_s)
Music wire and cold-drawn carbon steel	45	60–70
Hardened and tempered carbon and low-alloy steel	50	65–75
Austenitic stainless steels	35	55–65
Nonferrous alloys	35	55–65

Example

- A helical compression spring is made of hard drawn carbon A227 steel has a wire diameter of 0.94 mm. The outside diameter of the spring is 11.11mm. The ends are squared and there are 12.5 total turns.

a) Estimate the torsional yield strength of the wire.
b) Estimate the static load corresponding to the yield strength.
c) Estimate the scale of the spring (the spring constant).
d) Estimate the deflection that would be caused by the load in part b
e) Estimate the solid length of the spring
f) What length should the spring be to ensure that when it is compressed solid and then released, there will be no permanent change in the free length?
g) Given the length found in part (f), is buckling a possibility?
h) What is the pitch of the body coil?

Solution

a. The torsional yield strength of the wire:

$d = 0.94$mm

From table 10-4, A=1783MPa.mm^m and $m = 0.19$

From equation 10-14:

S_{ut} = A/d^m = 1783/(0.94)$^{0.19}$ =1804 MPa

From table 10-6: $S_{sy} = 0.50S_{ut}$ = 902 MPa

b. The static load corresponding to the yield strength:

The main spring coil diameter is $D = D_{out}$ - d = 11.11-0.94=10.17mm

Thus, the spring index is: $C = D/d = 10.17/0.94 = 10.82$

Therefore, from equation 10-3 replacing K_s and τ with K_B and S_{sy} respectively, Thus:

$$\tau = S_{sy} = K_B \frac{8FD}{\pi d^3}$$

$$\Rightarrow F = \frac{\pi d^3 S_{sy}}{8K_B D} \quad ; \quad K_B = \frac{4C+2}{4C-3} = \frac{4(10.82)+2}{4(10.82)-3} = 1.124$$

$$\Rightarrow F = 25.734N$$

c. The scale/rate of the spring:

$$k = \frac{d^4 G}{8D^3 N_a}$$

From Table 10-5, d = 0.94 mm = 0.037in $\Rightarrow$ G = 80GPa

N_a for both ends squared:

$N_t = N_a + 2 \quad \Rightarrow \quad N_a$ = 12.5-2=10.5 turns

$$k = \frac{d^4 G}{8D^3 N_a} = 707\,N/m$$

d. The deflection that would be caused by the load in part (b):

$$y = \frac{F}{k} = \frac{25.734}{707} = 0.0364m = 36.4mm$$

e. The solid length of the spring, from table 10-1:

$$L_s = d(N_t + 1) = 0.94(12.5 + 1) = 12.69mm$$

f. The free length of the spring suppose to have to be sure when it is compressed and then released, there will be no permanent change in its free length:

$$L_0 = L_s + y = 12.69 + 36.4 = 49.09mm$$

g. To Check if buckling is possible:

For steel and squared ground:

$L_0 < 5.26D \quad \Rightarrow \quad L_0 < 53.94\text{mm},$

since $L_0 = 49.9\text{mm}$, therefore: there will be no buckling.

h. The pitch of the body coil is:

$$p = \frac{(L_0 - 3d)}{N_a} = \frac{(49.09 - 3(0.94))}{10.5} = 4.41$$

10.7 Helical Compression Spring Design for Static Service

- The design of a new spring involves the following considerations:
  - Space into which the spring must fit and operate
  - Values of working forces and deflections
  - Accuracy and reliability needed
  - Tolerances and permissible variations in specifications
  - Environmental conditions such as temperature and presence of a corrosive atmosphere
  - Cost and quantities needed
- Designers use these factors to select a material and specify suitable values for wire size, the number of turns, the diameter and free length, the type of ends, and the spring rate needed to satisfy the working force-deflection requirements.

- Some important limits:
  - The range index is:

$$4 \le C \le 12 \qquad (10\text{-}18)$$

Lower indexes being more difficult to form (because of the danger of surface cracking).

Higher indexes tending to tangle (knot) often enough to require individual packing.

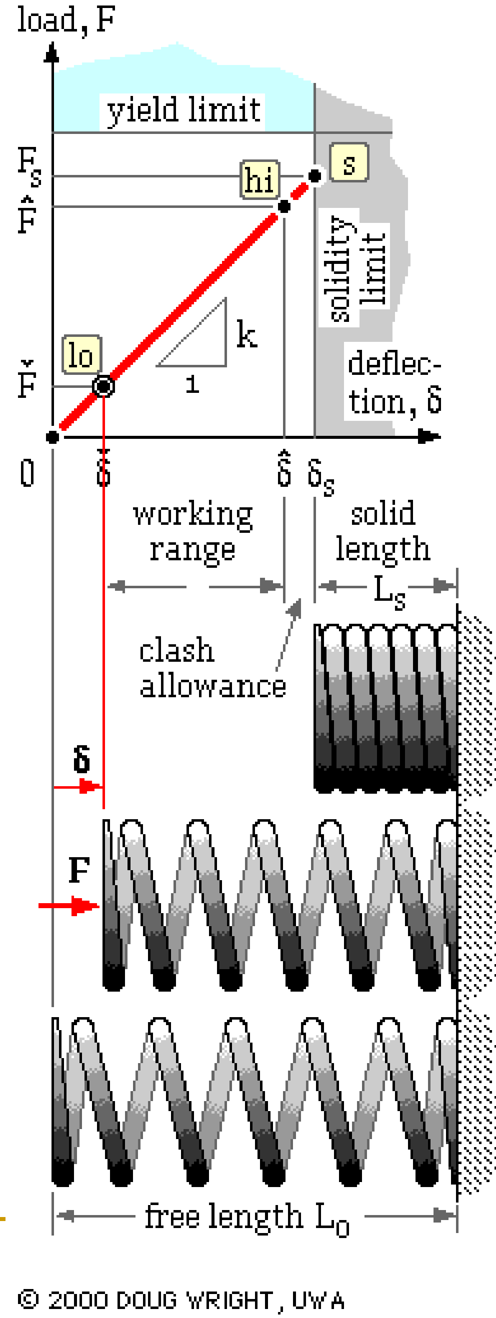
  - The recommended range of active turns is:

$$3 \le N_a \le 15 \qquad (10\text{-}19)$$

  - The designer confines (limits) the spring's operating point to the central 75% of the curve between no load $F = 0$ and closure (end) $F = F_S$, thus

The maximum operating force $F_{\max} \le \frac{7}{8} F_S$

Maximum Operating Force



Defining the fractional overrun to closure as ξ, where

$$F_S = (1+\xi)F_{max} \qquad \frac{F_S}{1+\xi} \le \frac{7}{8} F_S \qquad (10\text{-}17)$$

$$\xi \ge \frac{1}{7} == 0.15$$

- The factor of safety at closure (solid height) : $n_s \ge 1.2$

- Figure of merit:

  The cost of wire from which the spring is wound (coiled) helps in making the decision for the optimal spring design.

  Formula:

$$fom = -(RMC)\frac{\gamma \pi^2 d^2 N_t D}{4} \qquad (10\text{-}22)$$

Where *RMC* is the relative material cost.

- Spring design is an open-ended process with many decisions to be made, and many possible solution paths as well as solutions.
- One possible approach for design spring coil is:
  - Make a priori decisions, with hard-drawn steel wire the first choice (relative material cost =1)
  - Choose a wire size d
  - generate a column of parameters:

    d, D, OD or ID, N_a, L_s, L_o, $(L_o)_{\mathrm{cr}}$, n_s, and fom

  - By incrementing wire sizes available, a table of parameters will be generated. Then, the design recommendation conditions are applied to choose the good design.

- After wire sizes are eliminated, the spring design with the highest figure of merit will be chosen.

- This will give the optimal design despite the presence of a discrete design variable *d* and aggregation of equality and inequality constraints.

STATIC SPRING DESIGN

Choose d

Over-a-rod — As-wound or set

$$D = d_{rod} + d + \text{allow}$$

Free

As-wound

$$S_{sy} = \text{const}(A)/d^{m\,\dagger}$$

$$C = \frac{2\alpha - \beta}{4\beta} + \sqrt{\left(\frac{2\alpha - \beta}{4\beta}\right)^2 - \frac{3\alpha}{4\beta}}$$

$$\alpha = \frac{S_{sy}}{n_s} \qquad \beta = \frac{8(1+\xi)F_{max}}{\pi d^2}$$

$$D = Cd$$

Set removed

$$S_{sy} = 0.65A/d^m$$

$$D = \frac{S_{sy}\pi d^3}{8n_s(1+\xi)F_{max}}$$

In-a-hole — As-wound or set

$$D = d_{hole} - d - \text{allow}$$

$$C = D/d$$

$$K_B = (4C + 2)/(4C - 3)$$

$$\tau_s = K_B 8(1+\xi)F_{max}D/(\pi d^3)$$

$$n_s = S_{sy}/\tau_s$$

$$\text{OD} = D + d$$

$$\text{ID} = D - d$$

$$N_a = Gd^4 y_{max}/(8D^3 F_{max})$$

N_t: Table 10–1

L_s: Table 10–1

L_O: Table 10–1

$$(L_O)_{cr} = 2.63D/\alpha$$

$$\text{fom} = -(\text{rel. cost})\,\gamma\pi^2 d^2 N_t D/4$$

set-removed springs, operating over a rod, or in a hole free of rod or hole. In as-wound springs the controlling equation must be solved for the spring index as follows. From Eq. (10–3) with $\tau = S_{sy}/n_s$, $C = D/d$, K_B from Eq. (10–6), and Eq. (10–17),

$$\frac{S_{sy}}{n_s} = K_B \frac{8F_s D}{\pi d^3} = \frac{4C+2}{4C-3}\left[\frac{8(1+\xi)\,F_{\max}C}{\pi d^2}\right] \tag{a}$$

Let

$$\alpha = \frac{S_{sy}}{n_s} \tag{b}$$

$$\beta = \frac{8\,(1+\xi)\,F_{\max}}{\pi d^2} \tag{c}$$

Substituting Eqs. (b) and (c) into (a) and simplifying yields a quadratic equation in C. The larger of the two solutions will yield the spring index

$$C = \frac{2\alpha-\beta}{4\beta} + \sqrt{\left(\frac{2\alpha-\beta}{4\beta}\right)^2 - \frac{3\alpha}{4\beta}} \tag{10–23}$$

EXAMPLE 10–2 A music wire helical compression spring is needed to support an 89-N load after being compressed 50.8 mm. Because of assembly considerations the solid height cannot exceed 25.4 mm and the free length cannot be more than 101.6 mm. Design the spring.

Solution The a priori decisions are

- Music wire, A228; from Table 10–4, $A = 2211$ MPa·mm^m; $m = 0.145$; from Table 10–5, $E = 196.5$ GPa, $G = 81$ GPa (expecting $d > 1.61$ mm)
- Ends squared and ground
- Function: $F_{max} = 89$ N, $y_{max} = 50.8$ mm
- Safety: use design factor at solid height of $(n_s)_d = 1.2$
- Robust linearity: $\xi = 0.15$
- Use as-wound spring (cheaper), $S_{sy} = 0.45 S_{ut}$ from Table 10–6
- Decision variable: $d = 2.03$ mm, music wire gage #30, Table A–26. From Fig. 10–3 and Table 10–6,

$$S_{sy} = 0.45 \frac{2211}{2.03^{0.145}} = 897.9 \text{ MPa}$$

From Fig. 10–3 or Eq. (10–23)

$$\alpha = \frac{S_{sy}}{n_s} = \frac{897.9}{1.2} = 748.3 \text{ MPa}$$

$$\beta = \frac{8(1+\xi)F_{max}}{\pi d^2} = \frac{8(1+0.15)89}{\pi(2.03^2)} = 63.2 \text{ MPa}$$

$$C = \frac{2(748.3) - 63.2}{4(63.2)} + \sqrt{\left[\frac{2(748.3) - 63.2}{4(63.2)}\right]^2 - \frac{3(748.3)}{4(63.2)}} = 10.5$$

Continuing with Fig. 10–3:

$$D = Cd = 10.5(2.03) = 21.3 \text{ mm}$$

$$K_B = \frac{4(10.5)+2}{4(10.5)-3} = 1.128$$

$$\tau_s = 1.128 \frac{8(1+0.15)89(21.3)}{\pi(2.03)^3} = 748 \text{ MPa}$$

$$n_s = \frac{897.9}{748} = 1.2$$

$$\text{OD} = 21.3 + 2.03 = 23.3 \text{ mm}$$

$$N_a = \frac{2.03^4(81\,000)50.8}{8(21.3)^3 89} = 10.16 \text{ turns}$$

$$N_t = 10.16 + 2 = 12.16 \text{ total turns}$$

$$L_s = 2.03(12.16) = 24.3 \text{ mm}$$

$$L_0 = 24.3 + (1 + 0.15)50.8 = 82.7 \text{ mm}$$

$$(L)_{cr} = 2.63(21.3/0.5) = 112 \text{ mm}$$

$$\text{fom} = -2.6\frac{\pi^2(2.03)^2 12.16(21.3)}{4(25.4)^3} = -0.417$$

Repeat the above for other wire diameters and form a table (easily accomplished with a spreadsheet program):

d:	1.6	1.7	1.8	1.9	2.03	2.1	2.3	2.4
D	9.9	12.2	14.6	17.5	21.3	25.2	31.1	36.0
C	6.2	7.2	8.1	9.2	10.5	12.0	13.5	15.0
OD	11.5	13.8	16.4	19.3	23.3	27.8	32.8	38.4
N_a	39.1	26.9	19.3	14.2	10.2	7.3	5.4	4.1
L_s	65.2	48.8	38.1	30.7	24.3	19.9	16.8	14.6
L_0	123.8	107.3	96.6	89.1	82.7	78.3	75.2	73
$(L_0)_{cr}$	52.1	63.7	76.9	91.5	112	160.5	161.0	189.9
n_s	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
fom	−0.41	−0.40	−0.40	−0.404	−0.42	−0.44	−0.47	−0.51

Now examine the table and perform the adequacy assessment. The constraint $3 \leq N_a \leq 15$ rules out wire diameters less than 1.9 mm. The spring index constraint $4 \leq C \leq 12$ rules out diameters larger than 2.1 mm. The $L_s \leq 1$ constraint rules out diameters less than 0.203 mm. The $L_0 \leq 4$ constraint rules out diameters less than 1.8 mm. The buckling criterion rules out free lengths longer than $(L_0)_{cr}$, which rules out diameters less than 1.9 mm. The factor of safety n_s is exactly 1.20 because the mathematics forced it. Had the spring been in a hole or over a rod, the helix diameter would be chosen without reference to $(n_s)_d$. The result is that there are only two springs in the feasible domain, one with a wire diameter of 0.203 mm and the other with a wire diameter of 2.1 mm. The figure of merit decides and the decision is the design with 0.203 mm wire diameter.

Example

- An A228 wire helical compression spring is needed to support a 20-Ibf (100 N) load after being compressed 2 in (50 mm). Because of assembly considerations the solid height cannot exceed 1 in (25 mm) and the free length cannot be more than 4 in (100 mm). Design the spring.

Solution

- For A228 wire helical spring: From Table, A=201 kpsi-in^m and m=0.145
- From Table 10-5, E = 28.5Mpsi, G =11.75Mpsi (choosing d > 0.064 in)
- Ends squared and ground
- Function: F_{max}= 20 Ibf, y_{max}= 2 in
- Safety: use design factor at solid height of $(n_s)d$ =1.2
- Select robust linearity: ξ = 0.15
- Use as wound spring, S_{sy}= 0.45S_{ut} from table 10-6

- Decision variable: d =0.08 in, from figure 10-3 and table 10-6:

$$S_{sy} = 0.45\frac{A}{d^m} = 130.46kpsi$$

$$\alpha = \frac{S_{sy}}{n_s} = 108.713kpsi$$

$$\beta = \frac{8(1+\xi)F_{\max}}{\pi d^2} = 9.15kpsi$$

$$C = \frac{2\alpha - \beta}{4\beta} + \left(\left(\frac{2\alpha - \beta}{4\beta}\right)^2 - \frac{3\alpha}{4\beta}\right)^{1/2} = 10.53$$

$$D = Cd = 0.8424$$

$$K_B = (4C+2)/(4C-3) = 1.128$$

$$\tau_s = K_B 8(1+\xi)F_{\max} D/(\pi d^3) = 108.7kpsi$$

$$n_s = S_{sy}/\tau_s = 1.2$$

$$OD = D + d = 0.923in$$

$$N_a = Gd^4 y_{\max}/(8D^3 F_{\max}) = 10.05turns$$

$$N_t = 10.05 + 2 = 12.05turns$$

$$L_s = dN_t = 0.964in$$

$$L_o = L_s + y_s = L_s + (1+\xi)y_{\max} = 3.264in$$

$$\left(L_o\right)_{cr} = 2.63D/\alpha = 4.43in$$

$$fom = -2.6\pi^2 d^2 N_t D/4 = -0.417$$

- Repeat the above analysis for other diameters and form a table to select the best spring design:

d	**0.063**	**0.067**	**0.071**	**0.075**	**0.08**	**0.085**	**0.09**	**0.095**
D	0.391	0.479	0.578	0.688	0.843	1.017	1.211	1.427
C	6.205	7.153	8.143	9.178	10.53	11.96	13.46	15.02
OD	0.454	0.546	0.649	0.763	0.923	1.102	1.301	1.522
N_a	39.1	26.9	19.3	14.2	10.1	7.3	5.4	4.1
L_s	2.587	1.936	1.513	1.219	0.964	0.790	0.668	0.581
L_o	4.887	4.236	3.813	3.519	3.264	3.090	2.968	2.881
$(L_o)_{cr}$	2.06	2.52	3.04	3.62	4.43	5.35	6.37	7.51
n_s	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
fom	-0.409	-0.399	-0.398	-0.404	-0.417	-0.438	-0.467	-0.505

- Examine the table and perform the adequacy assessment.
  - The constraint $3 \le N_a \le 15$ cancel wire diameters less than 0.08 in
  - The constraint $4 \le C \le 12$ cancel diameters larger than 0.085 in.
  - The constraint $L_s < 1$ in cancel diameters less than 0.080 in
  - The constraint $L_o < 4$ in cancel diameters less than 0.071 in
  - The buckling criterion cancel free length longer than $(L_o)_{cr}$, which cancel diameters less than0.075in.
  - The result is that there are only two springs in the feasible domain.
  - The figure of merit decides that the wire diameter is 0.08 in.

d	0.063	0.067	0.071	0.075	0.08	0.085	0.09	0.095
D	0.391	0.479	0.578	0.688	0.843	1.017	1.211	1.427
C	6.205	7.153	8.143	9.178	10.53	11.96	13.46	15.02
OD	0.454	0.546	0.649	0.763	0.923	1.102	1.301	1.522
N_a	39.1	26.9	19.3	14.2	10.1	7.3	5.4	4.1
L_s	2.587	1.936	1.513	1.219	0.964	0.790	0.668	0.581
L_o	4.887	4.236	3.813	3.519	3.264	3.090	2.968	2.881
$(L_o)_{cr}$	2.06	2.52	3.04	3.62	4.43	5.35	6.37	7.51
n_s	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
fom	-0.409	-0.399	-0.398	-0.404	-0.417	-0.438	-0.467	-0.505

10.8 Critical Frequency of Helical Springs

- The natural frequency of the spring should not be close to the frequency of the applied force; otherwise resonance may occur, resulting in damaging stresses.

- The governing equation for the translational vibration of a spring is the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{W}{kgl^2}\frac{\partial^2 u}{\partial t^2} \qquad (10\text{-}24)$$

Where

k = spring rate ; g = acceleration due to gravity

l = length of spring; W= weight of spring

x = coordinate along length of spring

u = motion of any particle at distance x

- Equation (10-24) has a harmonic solution and it depends on:
    - Given physical properties
    - End conditions of the spring
- The natural frequencies for a spring placed between two flat and parallel plates:

$$\omega = n\pi\sqrt{\frac{kg}{W}} \qquad n = 1,2,3,\ldots$$

- Since $\omega = 2\pi f$, thus $f = \frac{1}{2}\sqrt{\frac{kg}{W}} \qquad n = 1,2,3,\ldots$
- If n =1, it is called the fundamental frequency and it is equal to:

$$f = \frac{1}{2}\sqrt{\frac{kg}{W}} \tag{10-25}$$

- For spring that has one end against a flat plate and other end free, the frequency is

$$f = \frac{1}{4}\sqrt{\frac{kg}{W}} \qquad (10\text{-}26)$$

- W can be calculated as:

$$W = AL\gamma = \frac{\pi d^2}{4}\left(\pi D N_a\right)\gamma = \frac{1}{4}\pi^2 d^2 D N_a \gamma$$

Where γ is the specific weight (weight per unit volume)

- To avoid resonance with the harmonic it is required that the fundamental critical frequency is 15~20 ≥ the frequency of the force or motion of the spring.

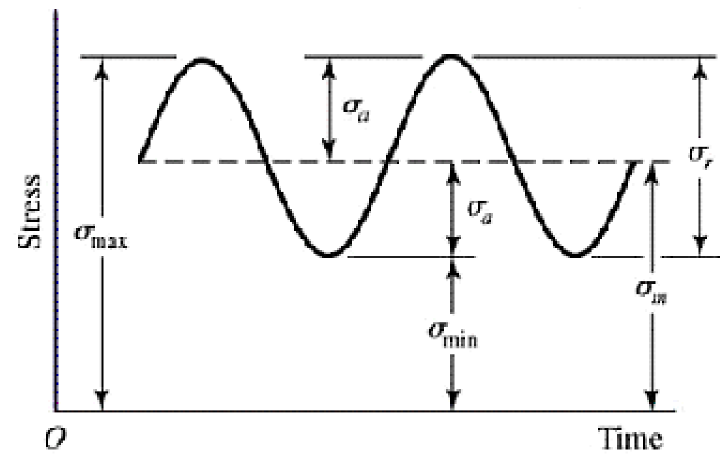
$$\frac{f}{f_{op}} \geq 20$$

10.9 Fatigue Loading of Helical Compression Springs

- Helical springs are never used as both compression and extension springs. This is because they are usually assembled with a preload so that the working load is additional. Thus, the spring application fall under the condition of **fluctuating loads.**
- Thus,

$$F_a = \frac{F_{\max} - F_{\min}}{2}$$

$$F_m = \frac{F_{\max} + F_{\min}}{2} = F_{\min} + F_a$$



- Therefore, the amplitude and midrange shear stresses respectively can be written as:

$$\tau_a = K_B \frac{8F_a D}{\pi d^3}; \tau_m = K_B \frac{8F_m D}{\pi d^3}$$

- Endurance limits for **infinite life** were found to be for unpeened and peened springs (Torsional endurance strength components)

Unpeened:	S_{sa}= 35.0 kpsi (241MPa)	S_{sm}= 55.0kpsi (379MPa)
Peened:	S_{sa}= 57.5 kpsi (398MPa)	S_{sm}= 77.5kpsi (534MPa)

- Then S_{se} can be calculated using Goodman theory: Torsional endurance strength :

$$S_{se} = \frac{S_{sa}}{1-\left(S_{sm}/S_{su}\right)}$$

where S_{su}=0.67S_{ut} *(S_{su} is the ultimate torsional strength)*

EXAMPLE 10–4 An as-wound helical compression spring, made of music wire, has a wire size of 2.3 mm, an outside coil diameter of 14 mm, a free length of 98 mm, 21 active coils, and both ends squared and ground. The spring is unpeened. This spring is to be assembled with a preload of 22 N and will operate with a maximum load of 156 N during use.

(*a*) Estimate the factor of safety guarding against fatigue failure using a torsional Gerber fatigue failure criterion with Zimmerli data.

(*b*) Repeat part (*a*) using the Sines torsional fatigue criterion (steady stress component has no effect), with Zimmerli data.

(*c*) Repeat using a torsional Goodman failure criterion with Zimmerli data.

(*d*) Estimate the critical frequency of the spring.

Solution The mean coil diameter is $D = 14 - 2.3 = 11.7$ mm. The spring index is $C = D/d = 11.7/2.3 = 5.09$. Then

$$K_B = \frac{4C+2}{4C-3} = \frac{4(5.09)+2}{4(5.09)-3} = 1.288$$

From Eqs. (10–31),

$$F_a = \frac{156-22}{2} = 67\text{ N} \qquad F_m = \frac{156+22}{2} = 89\text{ N}$$

The alternating shear-stress component is found from Eq. (10–32) to be

$$\tau_a = K_B\frac{8F_aD}{\pi d^3} = (1.288)\frac{8(67)11.7}{\pi(2.3)^3} = 211.3\text{ MPa}$$

Equation (10–33) gives the midrange shear-stress component

$$\tau_m = K_B\frac{8F_mD}{\pi d^3} = (1.288)\frac{8(89)11.7}{\pi(2.3)^3} = 280.7\text{ MPa}$$

From Table 10–4 we find $A = 2211\text{ MPa}\cdot\text{mm}^m$ and $m = 0.145$. The ultimate tensile strength is estimated from Eq. (10–14) as

$$S_{ut} = \frac{A}{d^m} = \frac{2211}{2.3^{0.145}} = 1959\text{ MPa}$$

Also the shearing ultimate strength is estimated from

$$S_{su} = 0.67S_{ut} = 0.67(1959) = 1312\text{ MPa}$$

The load-line slope $r = \tau_a/\tau_m = 211.3/280.7 = 0.75$.

(*a*) The Gerber ordinate intercept for the Zimmerli data, Eq. (10–28), is

$$S_{se} = \frac{S_{sa}}{1 - (S_{sm}/S_{su})^2} = \frac{241}{1-(379/1312)^2} = 263 \text{ MPa}$$

The amplitude component of strength S_{sa}, from Table 6–7, p. 307, is

$$S_{sa} = \frac{r^2 S_{su}^2}{2S_{se}}\left[-1 + \sqrt{1 + \left(\frac{2S_{se}}{rS_{su}}\right)^2}\right]$$

$$= \frac{0.75^2 1312^2}{2(263)}\left\{-1 + \sqrt{1 + \left[\frac{2(263)}{0.75(1312)}\right]^2}\right\} = 246.5 \text{ MPa}$$

and the fatigue factor of safety n_f is given by

Answer
$$n_f = \frac{S_{sa}}{\tau_a} = \frac{246.5}{211.3} = 1.17$$

(*b*) The Sines failure criterion ignores S_{sm} so that, for the Zimmerli data with S_{sa} = 241 MPa,

Answer
$$n_f = \frac{S_{sa}}{\tau_a} = \frac{241}{211.3} = 1.14$$

(*c*) The ordinate intercept S_{se} for the Goodman failure criterion with the Zimmerli data is

$$S_{se} = \frac{S_{sa}}{1 - (S_{sm}/S_{su})} = \frac{241}{1-(379/1312)} = 338.9 \text{ MPa}$$

The amplitude component of the strength S_{sa} for the Goodman criterion, from Table 6–6, p. 307, is

$$S_{sa} = \frac{rS_{se}S_{su}}{rS_{su} + S_{se}} = \frac{0.75(338.9)1312}{0.75(1312) + 338.9} = 252 \text{ MPa}$$

The fatigue factor of safety is given by

Answer

$$n_f = \frac{S_{sa}}{\tau_a} = \frac{252}{211.3} = 1.19$$

(*d*) Using Eq. (10–9) and Table 10–5, we estimate the spring rate as

$$k = \frac{d^4 G}{8D^3 N_a} = \frac{2.3^4(81\,000)}{8(11.7)^3 21} = 8.4 \text{ N/mm}$$

From Eq. (10–27) we estimate the spring weight as

$$W = \frac{\pi^2(2.3^2)11.7(21)82 \times 10^{-6}}{4} = 0.26 \text{ N}$$

and from Eq. (10–25) the frequency of the fundamental wave is

Answer

$$f_n = \frac{1}{2}\left[\frac{8400(9.81)}{0.26}\right]^{1/2} = 281 \text{ Hz}$$

If the operating or exciting frequency is more than $281/20 = 14.1$ Hz, the spring may have to be redesigned.

EXAMPLE 10–5 A music wire helical compression spring with infinite life is needed to resist a dynamic load that varies from 20 to 80 N at 5 Hz while the end deflection varies from 12 to 50 mm. Because of assembly considerations, the solid height cannot exceed 25 mm and the free length cannot be more than 100 mm. The springmaker has the following wire sizes in stock: 1.7, 1.8, 2.0, 2.15, 2.3, 2.4, 2.6, and 2.8 mm.

Solution The a priori decisions are:

- Material and condition: for music wire, $A = 2211$ MPa·mm^m, $m = 0.145$, $G = 81$ GPa; relative cost is 2.6
- Surface treatment: unpeened
- End treatment: squared and ground
- Robust linearity: $\xi = 0.15$
- Set: use in as-wound condition
- Fatigue-safe: $n_f = 1.5$ using the Sines-Zimmerli fatigue-failure criterion
- Function: $F_{min} = 20$ N, $F_{max} = 80$ N, $y_{min} = 12$ mm, $y_{max} = 50$ mm, spring operates free (no rod or hole)
- Decision variable: wire size d

The figure of merit will be the volume of wire to wind the spring, Eq. (10–22). The design strategy will be to set wire size d, build a table, inspect the table, and choose the satisfactory spring with the highest figure of merit.

Solution Set $d = 2.8$ mm. Then

$$F_a = \frac{80-20}{2} = 30\text{ N} \qquad F_m = \frac{80+20}{2} = 50\text{ N}$$

$$k = \frac{F_{max}}{y_{max}} = \frac{80}{50} = 1.6\text{ N/mm}$$

$$S_{ut} = \frac{2211}{2.8^{0.145}} = 1901\text{ MPa}$$

$$S_{su} = 0.67(1901) = 1276\text{ MPa}$$

$$S_{sy} = 0.45(1901) = 857\text{ MPa}$$

From Eq. (10–28), with the Sines criterion, $S_{se} = S_{sa} = 241$ MPa. Equation (10–23) can be used to determine C with S_{sa}, n_f, and F_a in place of S_{sy}, n_s, and $(1+\xi)F_{max}$, respectively. Thus,

$$\alpha = \frac{S_{sa}}{n_f} = \frac{241}{1.5} = 161\text{ MPa}$$

$$\beta = \frac{8F_a}{\pi d^2} = \frac{8(30)}{\pi(2.8^2)} = 9.7\text{ MPa}$$

$$C = \frac{2(161) - 9.7}{4(9.7)} + \sqrt{\left[\frac{2(161) - 9.7}{4(9.7)}\right]^2 - \frac{3(161)}{4(9.7)}} = 15.3$$

$$D = Cd = 15.3(2.8) = 42.8\text{ mm}$$

$$F_s = (1+\xi)F_{max} = (1+0.15)80 = 92\text{N}$$

$$N_a = \frac{d^4G}{8D^3k} = \frac{2.8^4(81\,000)}{8(42.8)^3 1.6} = 4.96\text{ turns}$$

$$N_t = N_a + 2 = 4.96 + 2 = 6.96\text{ turns}$$

$$L_s = dN_t = 2.8\,(6.96) = 19.5\text{ mm}$$

$$L_0 = L_s + \frac{F_s}{k} = 19.5 + 92/1.6 = 77\text{ mm}$$

$$\text{ID} = 42.8 - 2.8 = 40\text{ mm}$$

$$\text{OD} = 42.8 + 2.8 = 45.6\text{ mm}$$

$$y_s = L_0 - L_s = 77 - 19.5 = 57.5\text{ mm}$$

$$(L_0)_{cr} < \frac{2.63D}{\alpha} = 2.63\frac{(42.8)}{0.5} = 225\text{ mm}$$

$$K_B = \frac{4(15.3) + 2}{4(15.3) - 3} = 1.09$$

$$W = \frac{\pi^2 d^2 D N_a \gamma}{4} = \frac{\pi^2 2.8^2(42.8)4.96(82 \times 10^{-6})}{4} = 0.34\text{ N}$$

$$f_n = 0.5\sqrt{\frac{9.81k}{W}} = 0.5\sqrt{\frac{9.81(1600)}{0.34}} = 107\text{ Hz}$$

$$\tau_a = K_B\frac{8F_aD}{\pi d^3} = 1.09\frac{8(30)42.8}{\pi 2.8^3} = 162\text{ MPa}$$

$$\tau_m = \tau_a\frac{F_m}{F_a} = 162(50)/30 = 270\text{ MPa}$$

$$\tau_s = \tau_a\frac{F_s}{F_a} = 162(92)/30 = 497\text{ MPa}$$

$$n_f = \frac{S_{sa}}{\tau_a} = 241/162 = 1.5$$

$$n_s = \frac{S_{sy}}{\tau_s} = 857/497 = 1.7$$

$$\text{fom} = -(\text{relative material cost})\pi^2 d^2 N_t D/4$$

$$= -2.6\pi^2(2.8)^2 6.96(42.8)/[4(25.4)^3] = -1.01$$

Inspection of the results shows that all conditions are satisfied except for $4 \le C \le 12$. Repeat the process using the other available wire sizes and develop the following table:

d:	1.7	1.8	2.0	2.15	2.3	2.4	2.6	2.8
D	9.5	10.6	15.4	18.4	22.5	26.4	34.8	42.8
ID	[illegible]	7.3	11.9	15.0	18.6	22.6	32.1	40
OD	9.9	10.9	16.1	19.1	22.2	27.5	37.1	45.6
C	5.6	5.9	7.7	8.7	9.8	11.0	13.1	15.3
N_a	105.2	84.7	37.0	25.21	17.61	12.75	7.15	4.96
L_s	94.3	61.5	81.5	59.9	45.8	36.1	24.3	19.5
L_0	[illegible]	354.1	45.6	[illegible]	106.0	95.2	82.3	77
$(L_0)_{cr}$	47.6	47.6	92.8	90.7	110.1	151.8	182.8	225
n_f	1.5	1.5	1.5	1.5	1.5	1.5	1.5	1.5
n_s	1.82	1.81	1.78	1.77	1.75	1.74	1.71	1.7
f_n	86.7	88.9	96.0	98.8	101.0	102.8	105.6	107
fom	1.17	1.12	0.98	0.95	0.93	0.93	0.96	1.01

The problem-specific inequality constraints are

$$L_s \le 25\text{ mm}$$

$$L_0 \le 100\text{ mm}$$

$$f_n \ge 5(20) = 100\text{ Hz}$$

The general constraints are

$$3 \le N_a \le 15$$

$$4 \le C \le 12$$

$$(L_0)_{cr} > L_0$$

We see that none of the diameters satisfy the given constraints. The 2.6-mm-diameter wire is the closest to satisfying all requirements. The value of $C = 13.1$ is not a serious deviation and can be tolerated. However, the tight constraint on L_s needs to be addressed. If the assembly conditions can be relaxed to accept a solid height of 24.3 mm, we have a solution. If not, the only other possibility is to use the 2.8-mm diameter and accept a value $C = 15.3$, individually package the springs, and possibly reconsider supporting the spring in service.

Example

- A helical compression spring, made of A228 wire, has a wire size of 0.092 in (2.3 mm), an outside diameter of 0.5625 in (14.6 mm), a free length of 4.125 in (35.9 mm), 21 active coils, and both ends squared and ground. The spring is to be assembled with a preload of 5 lb (22.27 N) and will operate to a maximum load of 35 lb (160 N) during use. Knowing that the spring is unpeened type.

1. Find the factor of safety guarding against a fatigue failure.
2. Find the critical operating frequency

Solution

- Given $OD = 0.5625$ in, $d = 0.092$ in, $N_a = 21$
- A228 spring type material, $G = 11.75$ Gpsi
- $F_{\max} = 35$ lb, $F_{\min} = 5$ lb
- Both ends squared and ground, and unpeened.

1. The fatigue factor of safety:

$$D = OD - d = 0.4705 in; C = D/d = 5.11$$

$$K_B = (4C+2)/(4C-3) = 1.287$$

$$F_a = (35-5)/2 = 15 lb; F_m = (35+5)/2 = 20 lb$$

$$\tau_a = K_B \frac{8F_a D}{\pi d^3} = 29.7 kpsi; \tau_m = K_B \frac{8F_m D}{\pi d^3} = 33.8 kpsi$$

- From table 10-4, A = 201 kpsi-in^m, m = 0.145, therefore

$$S_{ut} = \frac{A}{d^m} = 284.1kpsi$$

$$S_{su} = 0.67S_{ut} = 190.347kpsi$$

- For unpeened spring:

$$S_{sa} = 35kpsi \qquad S_{sm} = 55kpsi$$

$$S_{se} = \frac{S_{sa}}{1-\left(S_{sm}/S_{su}\right)} = 49.22kpsi$$

- The factor of safety guarding against failure to be

$$\frac{\tau_a}{S_{se}} + \frac{\tau_m}{S_{su}} = \frac{1}{n_f} \Rightarrow n_f = \frac{S_{se}S_{su}}{\tau_a S_{su} + \tau_m S_{se}} = 1.28$$

2. The critical frequency:

$$k = \frac{d^4 G}{8D^3 N_a} = 48.1 lb/in$$

$$W = \frac{1}{4}\pi^2 d^2 D N_a \gamma ; \quad \gamma = 0.284 lb/in^3$$

$$\therefore W = 0.0586 lbf$$

$$f = \frac{1}{2}\sqrt{\frac{kg}{W}} = \frac{1}{2}\sqrt{\frac{48.1(386)}{0.586}} = 281 Hz$$

- Check for operating frequency:
- For good design

$$\frac{f}{f_{op}} \geq 20 \Rightarrow f_{op} \leq \frac{f}{20} \Rightarrow f_{op} \leq 14.1$$

10.10 Helical Compression Spring Design for Fatigue Loading

- Example

  A helical compression spring, made of A228 wire, with infinite life is needed to resist a dynamic load that varies from 5 to 20 lbf at 5 Hz while the end deflection varies from 0.5 to 2 in. Because of assembly considerations, the solid height cannot exceed 1.2 in and the free length cannot be more than 4 in. The springmaker has the following wire sizes in stock: 0.069, 0.071, 0.080, 0.085, 0.090, 0.095, 0.105, and 0.112 in.

- Solution

  From table 10-4 for A228: A=201 kpsi.in^m, m = 0.145,

  G =11.75Mpsi, relative cost of wire = 2.6

  Surface treatment: unpeended

  End treatment: squared and ground

  Select robust linearity: $\xi = 0.15$ and f_{op} = 5Hz

  Fatigue safety: n_f =1.5 using the Sines-Zimmerli fatigue-failure criterion

  Use as wound spring, $S_{sy} = 0.45 S_{ut}$ from table 10-6

  $F_{\min}$ = 5lbf, $F_{\max}$ = 20lbf, $y_{\min}$ = 0.5 in, $y_{\max}$ = 2 in, spring operates free (no rod or hole)

  Decision variable: wire size d

- The design strategy will be to set wire size *d, build a table, inspect the table, and choose* the satisfactory spring with the highest figure of merit.

STATIC SPRING DESIGN

Choose d

Over-a-rod

As-wound or set

$D = d_{rod} + d + \text{allow}$

Free

As-wound

$S_{sy} = \text{const}(A)/d^{m\dagger}$

$$C = \frac{2\alpha - \beta}{4\beta} + \sqrt{\left(\frac{2\alpha - \beta}{4\beta}\right)^2 - \frac{3\alpha}{4\beta}}$$

$$\alpha = \frac{S_{sy}}{n_s} \qquad \beta = \frac{8(1+\xi)F_{max}}{\pi d^2}$$

$D = Cd$

Set removed

$S_{sy} = 0.65A/d^m$

$$D = \frac{S_{sy}\pi d^3}{8n_s(1+\xi)F_{max}}$$

In-a-hole

As-wound or set

$D = d_{hole} - d - \text{allow}$

$C = D/d$

$K_B = (4C+2)/(4C-3)$

$\tau_s = K_B 8(1+\xi)F_{max}D/(\pi d^3)$

$n_s = S_{sy}/\tau_s$

$\text{OD} = D + d$

$\text{ID} = D - d$

$N_a = Gd^4 y_{max}/(8D^3 F_{max})$

N_t: Table 10–1

L_s: Table 10–1

L_O: Table 10–1

$(L_O)_{cr} = 2.63D/\alpha$

$\text{fom} = -(\text{rel. cost})\gamma\pi^2 d^2 N_t D/4$

Replace S_{sy} by S_{se}, ns by n_f and $(1+\xi)F_{max}$ by F_a

nf =Ssa/ τa

Design analysis based on d = 0.112 in

$$F_a = \frac{20-5}{2} = 7.5\text{lbf} \qquad F_m = \frac{20+5}{2} = 12.5\text{lbf}$$

$$k = \frac{F_{\max}}{y_{\max}} = \frac{20}{2} = 10\text{lbf/in}$$

$$S_{ut} = \frac{201}{0.112^{0.145}} = 276.1\text{kpsi}$$

$$S_{su} = 0.67 S_{ut} = 185.0\text{kpsi}$$

$$S_{sy} = 0.45 S_{ut} = 124.2\text{kpsi}$$

- For Unpeened spring, from equation (10-28): S_{sa}= 35.0 kpsi
  S_{sm}= 55.0kpsi
- In Sines failure criterion, the terms S_{sm} *is* ignored, thus

$$S_{se} = \frac{S_{sa}}{1-\left(S_{sm}/S_{su}\right)} = \frac{35}{1-0} = 35\text{kpsi}$$

- To find C, we replace S_{sy} by S_{se} , ns by n_f, and $(1+\xi)F_{\max}$ by F_a , thus:

$$\alpha = \frac{S_{se}}{n_f} = 23.333\text{kpsi}$$

$$\beta = \frac{8F_a}{\pi d^2} = 1.523\text{kpsi}$$

$$C = \frac{2\alpha - \beta}{4\beta} + \left(\left(\frac{2\alpha - \beta}{4\beta} \right)^2 - \frac{3\alpha}{4\beta} \right)^{1/2} = 14.005$$

$$D = Cd = 1.569\text{in}$$

$$F_s = (1 + \xi)F_{\max} = 23\text{lbf}$$

$$N_a = Gd^4 / (8D^3 k) = 5.98\text{turns}$$

$$N_t = 5.98 + 2 = 7.98\text{turns}$$

$$L_s = dN_t = 0.894\text{in}$$

$$L_o = L_s + y_s = L_s + \left(F_s / k\right) = 3.194\text{in}$$

$$ID = D - d = 1.457\text{in}$$

$$OD = D + d = 1.681\text{in}$$

$$y_s = L_o - L_s = 2.3\text{in}$$

$$\left(L_o\right)_{cr} = 2.63D / \alpha = 8.253\text{in}$$

$$W = \frac{\pi^2 d^2 D N_a \gamma}{4} = 0.0825\text{lbf}$$

$$f_n = 0.5\sqrt{\frac{386k}{W}} = 108\text{Hz}$$

$$K_B = (4C+2)/(4C-3) = 1.094$$

$$\tau_a = K_B \frac{8F_a D}{\pi d^3} = 23.334\text{kpsi}$$

$$\tau_m = K_B \frac{8F_m D}{\pi d^3} = 38.89\text{kpsi}$$

$$\tau_s = K_B \frac{8F_s D}{\pi d^3} = 71.56\text{kpsi}$$

$$n_f = S_{sa} / \tau_a = 1.5$$

$$n_s = S_{sy} / \tau_s = 1.74$$

$$fom = -2.6\pi^2 d^2 N_t D / 4 = -1.01$$

Repeat the above analysis for other wire diameters and form a table to select the best spring design:

- Knowing that we have two types of constrains
- General Constraints:

  The constraint $3 \le N_a \le 15$ cancel wire diameters less than 0.105 in

  The constraint $4 \le C \le 12$ cancel diameters larger than 0.105 in.
- Problem Constraints:

  The constraint $L_s \le 1.2$ in cancel diameters less than 0.1050 in

  The constraint $L_o \le 4$ in cancel diameters less than 0.095 in

  $f_n \ge 20 f_{op} \Rightarrow f_n \ge 100\text{Hz}$ cancel diameters less than 0.090 in

  The buckling criterion cancel free length longer than $(L_o)_{cr}$, which cancel diameters less than0.075in.

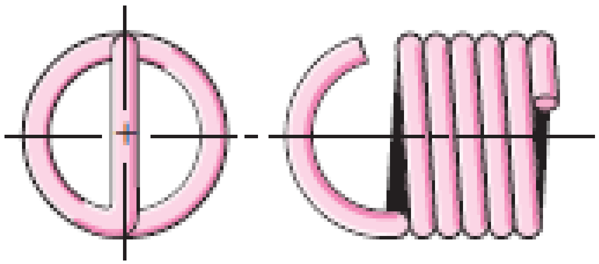
Repeat the above analysis for other wire diameters and form a table to select the best spring design:

d	**0.069**	**0.071**	**0.080**	**0.085**	**0.090**	**0.095**	**0.105**	**0.112**
D	0.297	0.332	0.512	0.632	0.767	0.919	1.274	1.569
ID	0.228	0.261	0.432	0.547	0.677	0.824	1.169	1.457
OD	0.366	0.403	0.592	0.717	0.857	1.014	1.379	1.681
C	4.33	4.67	6.40	7.44	8.53	9.67	12.14	14.00
N_a	127.2	102.4	44.8	30.5	21.3	15.4	8.63	6.0
L_s	8.916	7.414	3.74	2.75	2.10	1.655	1.116	0.895
L_o	11.216	9.714	6.040	5.05	4.40	3.955	3.416	3.195
$(L_o)_{cr}$	1.562	1.744	2.964	3.325	4.036	4.833	6.703	8.25
n_f	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50
n_s	1.86	1.85	1.82	1.81	1.79	1.78	1.75	1.74
f_n	87.5	89.7	96.9	99.7	101.9	103.8	106.6	108
fom	-1.17	-1.12	-0.983	-0.948	-0.930	-0.927	-0.958	-1.01

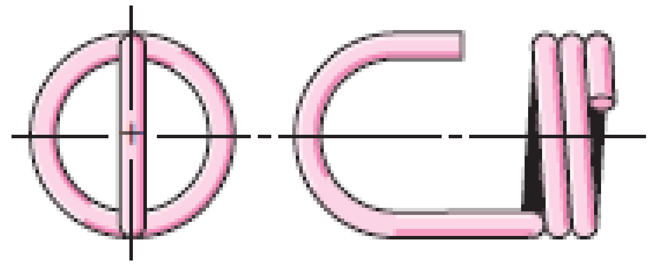
10.11 Extension Springs

- Most of the preceding discussion of compression springs applied equally to helical extension springs
- The natural frequency of a helical extension spring with both ends fixed against axial deflection is the same as that for a helical spring in compression
- However:
  - In extension spring, the coils are usually close wound so that there is an initial tension or so termed preload F. Therefore, no deflection occurs until the initial tension built into the spring is overcome. That is: the applied load F becomes larger than initial tension ($F > F_i$)
  - The load transfer can be done with: a threaded plug or a swivel hook

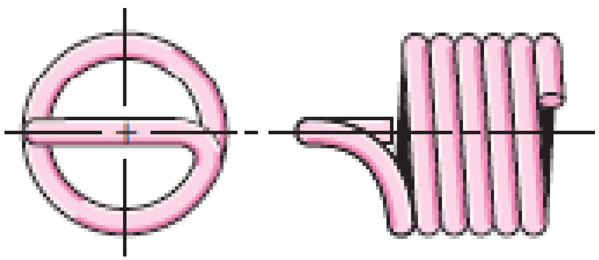
Types of ends used on extension springs



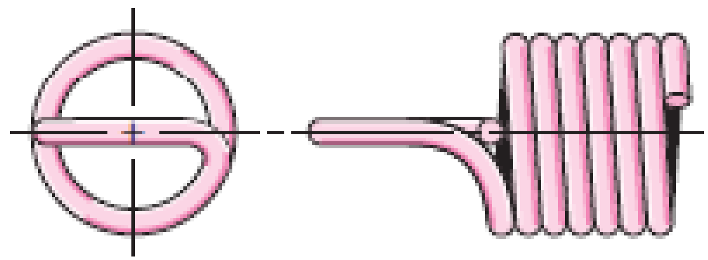
(*a*) Machine half loop–open



(*b*) Raised hook

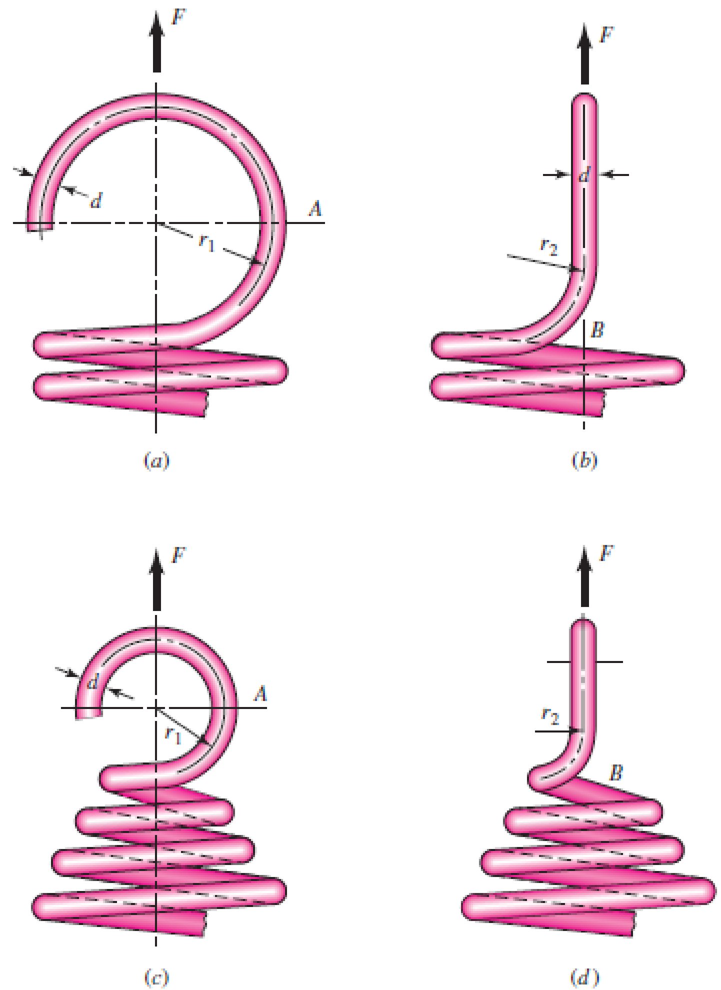


(*c*) Short twisted loop



(*d*) Full twisted loop

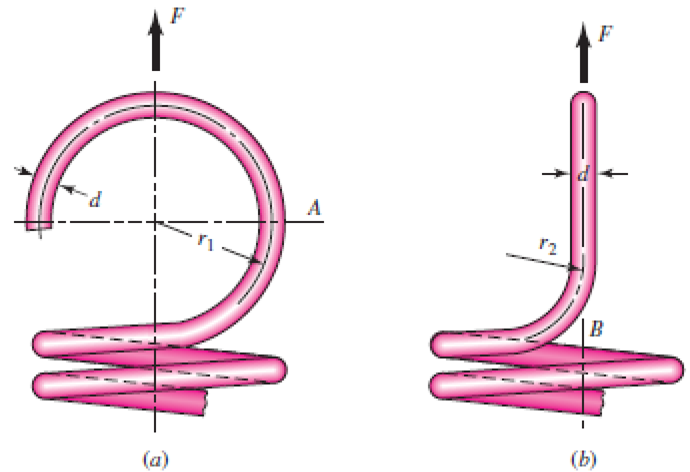
- In designing a spring with a hook end, bending and torsion in the hook must be included in the analysis.
- The figure shows two common used method of designing the end.
- *c* and *d* show an improved design due to a reduced coil diameter.



Note: Radius r_1 is in the plane of the end coil for curved beam bending stress. Radius r_2 is at a right angle to the end coil for torsional shear stress.

- The maximum tensile stress at A, due to bending and axial loading, is given by:

$$\sigma_A = F\left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2}\right]$$

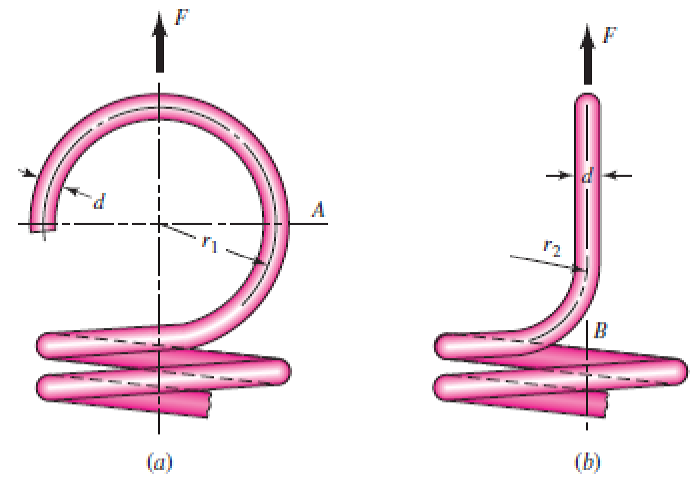


- Where $(K)_A$ is a bending stress correction factor for curvature, given by:

$$(K)_A = \frac{4C_1^2 - C_1 - 1}{4C_1(C_1 - 1)} \qquad C_1 = \frac{2r_1}{d}$$

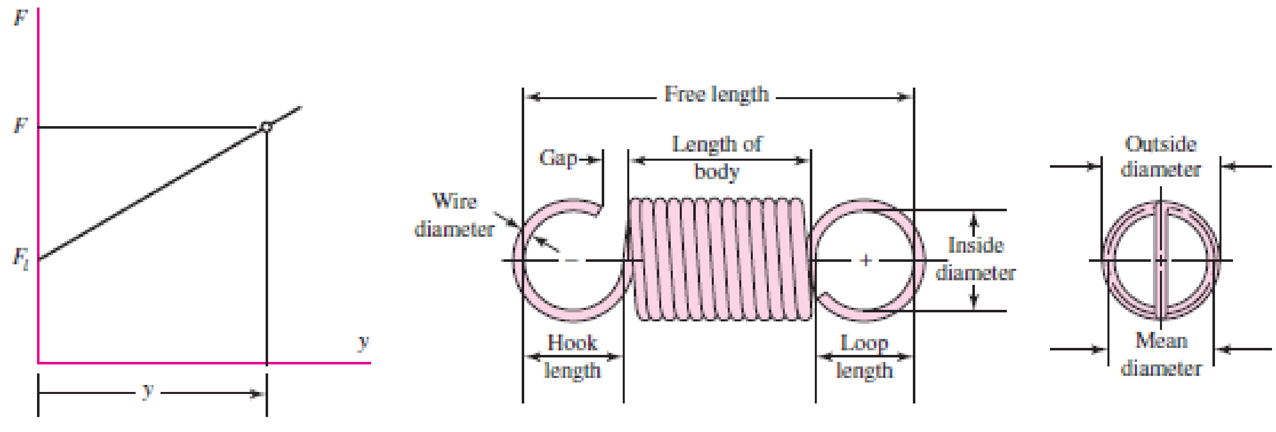
- The maximum torsional stress at point B is given by:

$$\tau_B = (K)_B \frac{8FD}{\pi d^3}$$



- Where $(K)_B$ is the stress correction factor for curvature and it is given by:

$$(K)_B = \frac{4C_2 - 1}{4C_2 - 4} \qquad C_2 = \frac{2r_2}{d}$$



(a) (b)

- If coils in contact with one another, the springs is known as close-wound.
- The load-deflection relation can be written as:

  $F = F_i + ky$

  Where k is the spring rate

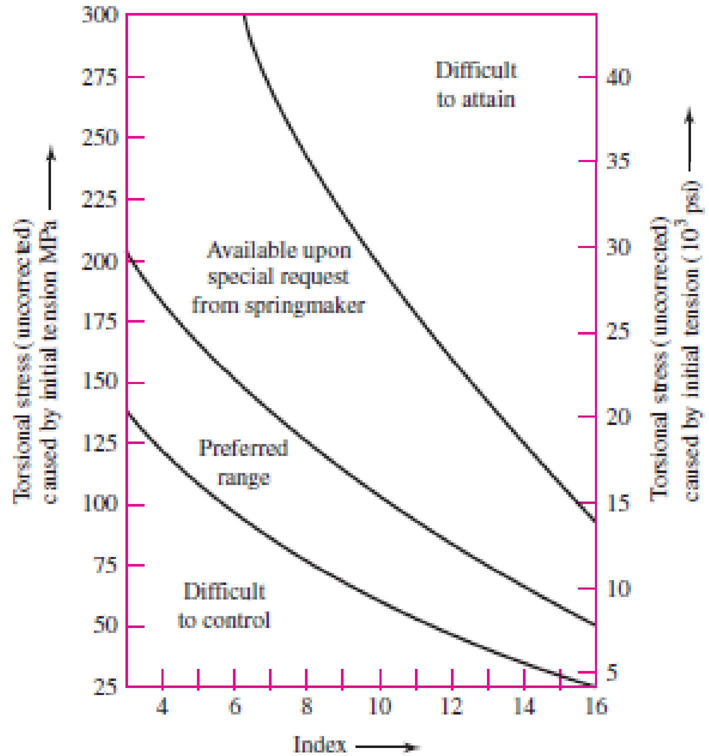
- The free length L_o of a spring measured inside the end loops or hooks can be expressed as

$$L_o = 2(D-d) + (N_b+1)d = (2C-1+N_b)d$$

D mean coil diameter; N_b number of body coils; C the spring index

- The equivalent number of active helical turns N_a for the spring rate k is:

  $N_a = N_b + G/E$
- Where G and E are the shear the tensile modulus of elasticity
- The amount of initial tension that a springmaker can routinely incorporate is as shown in the figure c



(c)

- The preferred range can be expressed in terms of the uncorrected torsional stress τ_i as:

$$\tau_i = \frac{33500}{\exp(0.105C)} \pm 1000\left(4 - \frac{C-3}{6.5}\right) psi \qquad (10\text{-}41)$$

- The maximum allowable corrected stresses (i.e.; using K_W or K_B) for static applications of extension springs are given in table 10-7

Percent of Tensile Strength

Materials	In Torsion body	In Torsion end	In Bending end
Patented, cold-drawn or hardened and tempered carbon and low-alloy steels	45-50	40	75
Austenitic stainless steel and nonferrous alloys	35	30	55

Table 10–7

Maximum Allowable Stresses (K_W or K_B corrected) for Helical Extension Springs in Static Applications

Source: From *Design Handbook,* 1987, p. 52. Courtesy of Associated Spring.

	Percent of Tensile Strength		
	In Torsion		In Bending
Materials	**Body**	**End**	**End**
Patented, cold-drawn or hardened and tempered carbon and low-alloy steels	45–50	40	75
Austenitic stainless steel and nonferrous alloys	35	30	55

This information is based on the following conditions: set not removed and low temperature heat treatment applied. For springs that require high initial tension, use the same percent of tensile strength as for end.

EXAMPLE 10–6 A hard-drawn steel wire extension spring has a wire diameter of 0.9 mm, an outside coil diameter of 6.3 mm, hook radii of $r_1 = 2.7$ mm and $r_2 = 2.3$ mm, and an initial tension of 5 N. The number of body turns is 12.17. From the given information:

(*a*) Determine the physical parameters of the spring.

(*b*) Check the initial preload stress conditions.

(*c*) Find the factors of safety under a static 23 N load.

Solution (*a*)

$$D = \mathrm{OD} - d = 6.3 - 0.9 = 5.4 \text{ mm}$$

$$C = \frac{D}{d} = \frac{5.4}{0.9} = 6.0$$

$$K_B = \frac{4C + 2}{4C - 3} = 1.24$$

Eq. (10–40) and Table 10–5:

$$N_a = N_b + G/E = 12.17 + 79/198 = 12.57 \text{ turns}$$

Eq. (10–9): $$k = \frac{d^4 G}{8D^3 N_a} = \frac{0.9^4(79\,000)}{8(5.4^3)12.57} = 3.27 \text{ N/mm}$$

Eq. (10–39): $$L_0 = (2C - 1 + N_b)d = [2(6.0) - 1 + 12.17]\,0.9 = 20.9 \text{ mm}$$

The deflection under the service load is

$$y_{\max} = \frac{F_{\max} - F_i}{k} = \frac{23 - 5}{3.27} = 5.5 \text{ mm}$$

where the spring length becomes $L = L_0 + y = 20.9 + 5.5 = 26.4$ mm
(b) The uncorrected initial stress is given by Eq. (10–2) without the correction factor. That is,

$$(\tau_i)_{\text{uncorr}} = \frac{8F_i D}{\pi d^3} = \frac{8(5)(5.4)}{\pi(0.9^3)} = 94.3 \text{ MPa}$$

The preferred range is given by Eq. (10–41) and for this case is

$$(\tau_i)_{\text{pref}} = \frac{231}{\exp(0.105C)} \pm 6.9\left(4 - \frac{C-3}{6.5}\right)$$

$$= \frac{231}{\exp[0.105(6.0)]} \pm 6.9\left(4 - \frac{6.0-3}{6.5}\right)$$

$$= 123 \pm 24.4 = 147.4,\ 98.6 \text{ MPa}$$

Answer Thus, the initial tension of 94.3 MPa is in the preferred range.
(c) For hard-drawn wire, Table 10–4 gives $m = 0.190$ and $A = 1783$ MPa $\cdot$ mm^m. From Eq. (10–14)

$$S_{ut} = \frac{A}{d^m} = \frac{1783}{0.9^{0.190}} = 1819 \text{ MPa}$$

For torsional shear in the main body of the spring, from Table 10–7,

$$S_{sy} = 0.45S_{ut} = 0.45(1819) = 818.6 \text{ MPa}$$

The shear stress under the service load is

$$\tau_{\max} = \frac{8K_B F_{\max} D}{\pi d^3} = \frac{8(1.24)23(5.4)}{\pi(0.9^3)} = 538 \text{ MPa}$$

Thus, the factor of safety is

Answer

$$n = \frac{S_{sy}}{\tau_{\max}} = \frac{818.6}{538} = 1.52$$

For the end-hook bending at A,

$$C_1 = 2r_1/d = 2(2.7)/0.9 = 6$$

From Eq. (10–35)

$$(K)_A = \frac{4C_1^2 - C_1 - 1}{4C_1(C_1 - 1)} = \frac{4(6^2) - 6 - 1}{4(6)(6 - 1)} = 1.14$$

From Eq. (10–34)

$$\sigma_A = F_{\max}\left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2}\right]$$

$$= 23\left[1.14 \frac{16(5.4)}{\pi(0.9^3)} + \frac{4}{\pi(0.9^2)}\right] = 1025.8 \text{ MPa}$$

The yield strength, from Table 10–7, is given by

$$S_y = 0.75 S_{ut} = 0.75(1819) = 1364.3 \text{ MPa}$$

The factor of safety for end-hook bending at A is then

Answer

$$n_A = \frac{S_y}{\sigma_A} = \frac{1364.3}{1025.8} = 1.33$$

For the end-hook in torsion at B, from Eq. (10–37)

$$C_2 = 2r_2/d = 2(2.3)/0.9 = 5.1$$

$$(K)_B = \frac{4C_2 - 1}{4C_2 - 4} = \frac{4(5.1) - 1}{4(5.1) - 4} = 1.18$$

and the corresponding stress, given by Eq. (10–36), is

$$\tau_B = (K)_B \frac{8 F_{\max} D}{\pi d^3} = 1.18 \frac{8(23)5.4}{\pi(0.9^3)} = 511.9 \text{ MPa}$$

Using Table 10–7 for yield strength, the factor of safety for end-hook torsion at B is

Answer

$$n_B = \frac{(S_{sy})_B}{\tau_B} = \frac{0.4(1819)}{511.9} = 1.42$$

Yield due to bending of the end hook will occur first.

EXAMPLE 10–7 The helical coil extension spring of Ex. 10–6 is subjected to a dynamic loading from 6.5 to 20 N. Estimate the factors of safety using the Gerber failure criterion for (*a*) coil fatigue, (*b*) coil yielding, (*c*) end-hook bending fatigue at point A of Fig. 10–7*a*, and (*d*) end-hook torsional fatigue at point B of Fig. 10–7*b*.

Solution A number of quantities are the same as in Ex. 10–6: $d = 0.9$ mm, $S_{ut} = 1819$ MPa, $D = 5.4$ mm, $r_1 = 2.7$ mm, $C = 6$, $K_B = 1.24$, $(K)_A = 1.14$, $(K)_B = 1.18$, $N_b = 12.17$ turns, $L_0 = 20.9$ mm, $k = 3.27$ N/mm, $F_i = 5$ N, and $(\tau_i)_{\text{uncorr}} = 94.3$ MPa. Then

$$F_a = (F_{\max} - F_{\min})/2 = (20 - 6.5)/2 = 6.75 \text{ N}$$

$$F_m = (F_{\max} + F_{\min})/2 = (20 + 6.5)/2 = 13.25 \text{ N}$$

The strengths from Ex. 10–6 include $S_{ut} = 1819$ MPa, $S_y = 1364.3$ MPa, and $S_{sy} =$ 818.6 MPa. The ultimate shear strength is estimated from Eq. (10–30) as

$$S_{su} = 0.67S_{ut} = 0.67(1819) = 1218.7 \text{ MPa}$$

(*a*) Body-coil fatigue:

$$\tau_a = \frac{8K_B F_a D}{\pi d^3} = \frac{8(1.24)6.75(5.4)}{\pi(0.9^3)} = 157.9 \text{ MPa}$$

$$\tau_m = \frac{F_m}{F_a}\tau_a = \frac{13.25}{6.75}157.9 = 310 \text{ MPa}$$

Using the Zimmerli data of Eq. (10–28) gives

$$S_{se} = \frac{S_{sa}}{1 - \left(\dfrac{S_{sm}}{S_{su}}\right)^2} = \frac{241}{1 - \left(\dfrac{379}{1218.7}\right)^2} = 266.8 \text{ MPa}$$

From Table 6–7, p. 307, the Gerber fatigue criterion for shear is

Answer

$$(n_f)_{\text{body}} = \frac{1}{2}\left(\frac{S_{su}}{\tau_m}\right)^2 \frac{\tau_a}{S_{se}}\left[-1+\sqrt{1+\left(2\frac{\tau_m}{S_{su}}\frac{S_{se}}{\tau_a}\right)^2}\right]$$

$$= \frac{1}{2}\left(\frac{1218.7}{310}\right)^2 \frac{157.9}{266.8}\left[-1+\sqrt{1+\left(2\frac{310}{1218.7}\frac{266.8}{157.9}\right)^2}\right] = 1.46$$

(*b*) The load-line for the coil body begins at $S_{sm} = \tau_i$ and has a slope $r = \tau_a/(\tau_m - \tau_i)$. It can be shown that the intersection with the yield line is given by $(S_{sa})_y = [r/(r+1)](S_{sy} - \tau_i)$. Consequently, $\tau_i = (F_i/F_a)\tau_a = (5/6.75)157.9 = 117$ MPa, $r = 157.9/(310 - 117) = 0.82$, and

$$(S_{sa})_y = \frac{0.82}{0.82+1}(818.6 - 117) = 316 \text{ MPa}$$

Thus,

Answer

$$(n_y)_{\text{body}} = \frac{(S_{sa})_y}{\tau_a} = \frac{316}{157.9} = 2.0$$

(*c*) End-hook bending fatigue: using Eqs. (10–34) and (10–35) gives

$$\sigma_a = F_a\left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2}\right]$$

$$= 6.75\left[1.14\frac{16(5.4)}{\pi(0.9^3)} + \frac{4}{\pi(0.9^2)}\right] = 301 \text{ MPa}$$

$$\sigma_m = \frac{F_m}{F_a}\sigma_a = \frac{13.25}{6.75}301 = 590.9 \text{ MPa}$$

To estimate the tensile endurance limit using the distortion-energy theory,

$$S_e = S_{se}/0.577 = 266.8/0.577 = 462.4 \text{ MPa}$$

Using the Gerber criterion for tension gives

Using the Gerber criterion for tension gives

Answer

$$(n_f)_A = \frac{1}{2}\left(\frac{S_{ut}}{\sigma_m}\right)^2 \frac{\sigma_a}{S_e}\left[-1+\sqrt{1+\left(2\frac{\sigma_m}{S_{ut}}\frac{S_e}{\sigma_a}\right)^2}\,\right]$$

$$= \frac{1}{2}\left(\frac{1819}{590.9}\right)^2 \frac{301}{462.4}\left[-1+\sqrt{1+\left(2\,\frac{590.9}{1819}\,\frac{462.4}{301}\right)^2}\,\right] = 1.27$$

(d) End-hook torsional fatigue: from Eq. (10–36)

$$(\tau_a)_B = (K)_B\,\frac{8F_aD}{\pi d^3} = 1.18\,\frac{8(6.75)(5.4)}{\pi(0.9^3)} = 150.2 \text{ MPa}$$

$$(\tau_m)_B = \frac{F_m}{F_a}(\tau_a)_B = \frac{13.25}{6.75}\,150.2 = 294.8 \text{ MPa}$$

Then, again using the Gerber criterion, we obtain

Answer

$$(n_f)_B = \frac{1}{2}\left(\frac{S_{su}}{\tau_m}\right)^2 \frac{\tau_a}{S_{se}}\left[-1+\sqrt{1+\left(2\frac{\tau_m}{S_{su}}\frac{S_{se}}{\tau_a}\right)^2}\,\right]$$

$$= \frac{1}{2}\left(\frac{1218.7}{294.8}\right)^2 \frac{150.2}{266.8}\left[-1+\sqrt{1+\left(2\,\frac{294.8}{1218.7}\,\frac{266.8}{150.2}\right)^2}\,\right] = 1.53$$

Example

- A hard drawn steel wire extension spring has a wire diameter of 0.035 in, an outside coil diameter of 0.248 in, hook radii of r_1= 0.106 in and r_2 = 0.089in, and an initial tension of 1.19 lbf. The number of body turns is 12.17. From the given information:
  - Determine the physical parameters of the spring (D, C, K_B, N_a, k, L_o, y_{max})
  - Check the initial preload stress conditions
  - Find the factors of safety under a static 5.25 lbf load.

Solution

- $d = 0.035$ in, $OD = 0.248$ in, $r_1 = 0.106$ in, $r_2 = 0.089$ in, $N_b = 12.17$; $F_i = 1.19$ lbf, $F_{max} = 5.25$
- HD steel $\Rightarrow$ Form table 10-5, with $0.033 < d < 0.063 \Rightarrow E = 28.7$Mpsi, G=11.6Mpsi
- From table 10-4, A =140kpsi-in^m, $m = 0.19$
- The physical parameters

$$D = OD - d = 0.248 - 0.035 = 0.213$$

$$C = D / d = 6.086; K_B = \frac{4C+2}{4C-3} = 1.234$$

$$N_a = N_b + G / E = 12.17 + 11.6 / 28.7 = 12.57 \text{turns}$$

$$k = \frac{d^4 G}{8D^3 N_a} = 17.76 \text{lbf/in}$$

$$L_o = (2C - 1 + N_b) d = 0.817 \text{in}$$

- The deflection under the service load is:

$$y_{\max} = \frac{\left(F_{\max} - F_i\right)}{k} = \frac{5.25 - 1.19}{17.76} = 0.229\text{in}$$

- Therefore the maximum spring length is:

$$L = L_o + \mathrm{y}_{\max} = 0.817 + 0.229 = 1.046 \text{ in}$$

- Initial preload stress condition:
- The uncorrected initial stress is given by equation (10-3) without the correction factor:

$$\left(\tau_i\right)_{uncorr} = \frac{8F_iD}{\pi d^3} = 15.1\text{kpsi}$$

- The preferred range is given by equation (10-41):

$$\left(\tau_i\right)_{pref} = \frac{33500}{\exp(0.105C)} \pm 1000\left(4 - \frac{C-3}{6.5}\right) = 17681 \pm 3525 = 14.2\text{kpsi} \quad or \quad 21.2\text{kpsi}$$

Thus, the initial tension of 15.1 kpsi is in the preferred range.

The factor of safety under static load

- We need to check three positions:
  - The shear stress under the service load
  - The bending at the end hook which is represented by point A
  - The torsion at the end hook which is represented by point B

1. For the shear stress under the service load

$$n = \frac{S_{sy}}{\tau_{\max}} \quad ; \quad S_{sy} = 0.45 S_{ut} \quad \& \quad \tau_{\max} = K_B \frac{8 F_{\max} D}{\pi d^3}$$

$$S_{ut} = \frac{A}{d^m} = \frac{140}{0.035^{0.19}} = 264.7\text{kpsi}$$

$$\tau_{\max} = 1.234 \frac{8(5.25)(0.213)}{\pi\left(0.035^3\right)} = 82\text{kpsi}$$

$$n = \frac{0.45(264.7)}{82} = 1.45$$

2. The bending at the end hook which is represented by point A

$$n_A = \frac{S_y}{\sigma_A} \quad ; \quad S_y = 0.75 S_{ut} = 0.75(264.7) = 198.5\text{kpsi}$$

$$\sigma_A = F_{\max}\left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2}\right] \quad ; \quad (K)_A = \frac{4C_1^2 - C_1 - 1}{4C_1(C_1 - 1)} \qquad C_1 = 2r_1 / d = 6.057$$

$$(K)_A = 1.14 \Rightarrow \sigma_A = 156.9\text{kpsi}$$

$$n_A = \frac{198.5}{156.9} = 1.27$$

3. The torsion at the end hook which is represented by point B

$$n_B = \frac{S_{sy}}{\tau_B} \quad ; \quad S_{sy} = 0.4S_{ut} = 0.4(264.7) = 105.88\text{kpsi}$$

$$\tau_B = (K)_B \frac{8F_{\max}D}{\pi d^3} \quad ; \quad (K)_B = \frac{4C_2 - 1}{4C_2 - 4} \qquad C_2 = 2r_2 / d = 5.086$$

$$(K)_B = 1.18 \Rightarrow \tau_B = 78.4\text{kpsi}$$

$$n_B = \frac{105.88}{78.4} = 1.35$$

Note that S_{sy}=0.4 Sut from table 10-7 under torsion for the end part.

- From all three calculations, the yield will first occur due to the bending of the end hook.

Fatigue Example

- The helical extension spring of the pervious example is subjected to a dynamic loading from 1.5 lbf to 5 lbf. Estimate the factors of safety using Goodman failure criterion for
  - The n_f coil fatigue for the body spring
  - The n_y coil yield for the body spring
  - The $(n_f)_A$ end hook bending fatigue at point A
  - The $(n_f)_B$ end hook torsion fatigue at point B

- Solution
- $d = 0.035$ in, $D = 0.213$ in, $r_1 = 0.106$ in, $r_2 = 0.089$ in, $N_b = 12.17$
- $F_i = 1.19$ lbf, $F_{\text{min}}=1.5$, $F_{\text{max}}=5$

- From the pervious example we have:

  $C=6.086$, $L_o=0.817$, $k = 17.76$ lbf/in

  $K_B=1.234$, $(K)_A=1.14$, $(K)_B=1.18$, $(\tau_i)_{\text{uncorr}} = 15.1$ kpsi

  $S_{ut} = 264.7$kpsi, $S_{su} = 0.67S_{ut} = 177.3$ kpsi, $S_y = 198.5$ kpsi, $S_{sy} = 119.1$ kpsi (shear in body)

- The fatigue in the body coil:

$$F_a = (5-1.5)/2 = 1.75\text{lbf}, F_m = (5+1.5)/2 = 3.25\text{lbf}$$

$$\tau_a = K_B \frac{8F_a D}{\pi d^3} = 27.3\text{kpsi}, \tau_m = K_B \frac{8F_m D}{\pi d^3} = 50.7\text{kpsi}$$

- For unpeened spring:

$$S_{sa} = 35\text{kpsi} \qquad S_{sm} = 55\text{kpsi}$$

$$S_{se} = \frac{S_{sa}}{1-(S_{sm}/S_{su})} = 50.74\text{kpsi}$$

The factor of safety guarding against failure to be

$$\frac{\tau_a}{S_{se}} + \frac{\tau_m}{S_{su}} = \frac{1}{n_f} \Rightarrow n_f = \frac{S_{se}S_{su}}{\tau_a S_{su} + \tau_m S_{se}} = 1.214$$

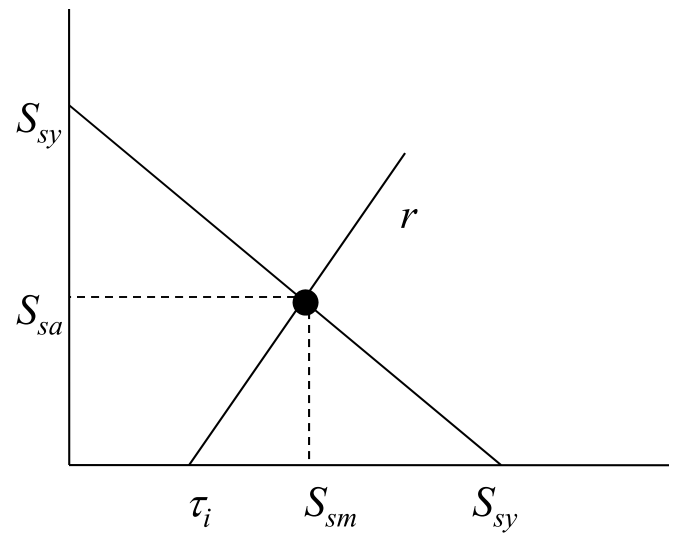
- The coil yield

$$n_y = \frac{S_{sa}}{\tau_a}$$

- To find the values of S_{sa}:
- The load line:

$$r = \frac{\tau_a}{\tau_m - \tau_i} \qquad (1)$$

$$\frac{S_{sa}}{S_{sy}} + \frac{S_{sm}}{S_{sy}} = 1 \qquad (2)$$



- The intersection between 1 & 2 gives:

$$S_{sa} = \frac{r}{r+1}\left(S_{sy} - \tau_i\right)$$

- Therefore:

$$\frac{\tau_i}{F_i} = \frac{\tau_a}{F_a} \Rightarrow \tau_i = \left(\frac{F_i}{F_a}\right)\tau_a = \left(\frac{1.19}{1.75}\right)27.3 = 18.6\text{kpsi}$$

$$r = \frac{27.3}{50.7 - 18.6} = 0.85$$

$$S_{sa} = 46.2\text{kpsi}$$

$$Thus:$$

$$n_y = \frac{46.2}{27.3} = 1.69$$

- The end hook bending fatigue at point A

$$n_f = 1/\left(\sigma_a / S_e + \sigma_m / S_{ut}\right) \quad ;$$

$$\sigma_a = F_a\left[\left(K\right)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2}\right] = 52.3\text{kpsi} \quad ;$$

$$\sigma_m = \left(\frac{F_m}{F_a}\right)\sigma_a = 97.1\text{kpsi}$$

$$S_e = \frac{S_{se}}{0.577} = 67.1\text{kpsi}$$

$$\Rightarrow \left(n_f\right)_A = 0.87$$

- The end hook torsion fatigue at point B

$$n_f = 1/\left(\tau_{sa}/S_{se} + \tau_{sm}/S_{su}\right)$$

$$\left(\tau_a\right)_B = \left(K\right)_B \frac{8F_a D}{\pi d^3} = 26.1\text{kpsi}$$

$$\left(\tau_m\right)_B = \left(K\right)_B \frac{8F_m D}{\pi d^3} = 48.5\text{kpsi}$$

$$\Rightarrow \left(n_f\right)_B = 1.27$$

Given A229 (OQ&T spring steel), squared and ground-ended helical compression spring, $d = 3.4$ mm, $OD = 50.8$ mm, $L_o = 74{,}6$ mm, $N_t = 5.25$.
Is the spring solid safe? If not, what is the largest free length to which it can be wound using $n_s = 1.2$?

$T_s < S_{sy}$

$T_s = K_B \, (\, 8 \, F_s \, D/(\pi \; d^3)$

$F_s = k \, y_s$

Table 10–4

Constants A and m of $S_{ut} = A/d^m$ for Estimating Minimum Tensile Strength of Common Spring Wires

Source: From *Design Handbook,* 1987, p. 19. Courtesy of Associated Spring.

Material	ASTM No.	Exponent m	Diameter, in	A, kpsi · in^m	Diameter, mm	A, MPa · mm^m	Relative Cost of wire
Music wire*	A228	0.145	0.004–0.256	201	0.10–6.5	2211	2.6
OQ&T wire†	A229	0.187	0.020–0.500	147	0.5–12.7	1855	1.3
Hard-drawn wire‡	A227	0.190	0.028–0.500	140	0.7–12.7	1783	1.0
Chrome-vanadium wire§	A232	0.168	0.032–0.437	169	0.8–11.1	2005	3.1
Chrome-silicon wire‖	A401	0.108	0.063–0.375	202	1.6–9.5	1974	4.0
302 Stainless wire#	A313	0.146	0.013–0.10	169	0.3–2.5	1867	7.6–11
		0.263	0.10–0.20	128	2.5–5	2065	
		0.478	0.20–0.40	90	5–10	2911	
Phosphor-bronze wire**	B159	0	0.004–0.022	145	0.1–0.6	1000	8.0
		0.028	0.022–0.075	121	0.6–2	913	
		0.064	0.075–0.30	110	2–7.5	932	

Table 10–5

Mechanical Properties of Some Spring Wires

Material	Elastic Limit, Percent of S_{ut} Tension	Elastic Limit, Percent of S_{ut} Torsion	Diameter *d*, in	*E* Mpsi	*E* GPa	*G* Mpsi	*G* GPa
Music wire A228	65–75	45–60	<0.032	29.5	203.4	12.0	82.7
			0.033–0.063	29.0	200	11.85	81.7
			0.064–0.125	28.5	196.5	11.75	81.0
			>0.125	28.0	193	11.6	80.0
HD spring A227	60–70	45–55	<0.032	28.8	198.6	11.7	80.7
			0.033–0.063	28.7	197.9	11.6	80.0
			0.064–0.125	28.6	197.2	11.5	79.3
			>0.125	28.5	196.5	11.4	78.6
Oil tempered A239	85–90	45–50		28.5	196.5	11.2	77.2
Valve spring A230	85–90	50–60		29.5	203.4	11.2	77.2
Chrome-vanadium A231	88–93	65–75		29.5	203.4	11.2	77.2
A232	88–93			29.5	203.4	11.2	77.2
Chrome-silicon A401	85–93	65–75		29.5	203.4	11.2	77.2
Stainless steel							
A313*	65–75	45–55		28	193	10	69.0
17-7PH	75–80	55–60		29.5	208.4	11	75.8
414	65–70	42–55		29	200	11.2	77.2
420	65–75	45–55		29	200	11.2	77.2
431	72–76	50–55		30	206	11.5	79.3

Table 10–6

Maximum Allowable Torsional Stresses for Helical Compression Springs in Static Applications

Source: Robert E. Joerres, "Springs," Chap. 6 in Joseph E. Shigley, Charles R. Mischke, and Thomas H. Brown, Jr. (eds.), *Standard Handbook of Machine Design*, 3rd ed., McGraw-Hill, New York, 2004.

	Maximum Percent of Tensile Strength	
Material	**Before Set Removed (includes K_W or K_B)**	**After Set Removed (includes K_s)**
Music wire and cold-drawn carbon steel	45	60–70
Hardened and tempered carbon and low-alloy steel	50	65–75
Austenitic stainless steels	35	55–65
Nonferrous alloys	35	55–65