Chapter 7

Digital Control Systems

1 Introduction

- In this chapter, we introduce analysis and design of stability, steady-state error, and transient response for computer-controlled systems.

- Transfer functions, representing compensators built with analog components, are now replaced with a digital computer that performs calculations that emulate the physical compensator.

- The digital computer (or micro-Controller, microprocessor) receives the error or only the reference signal and performs calculations (program) in order to provide an output near or equals the desired signal.

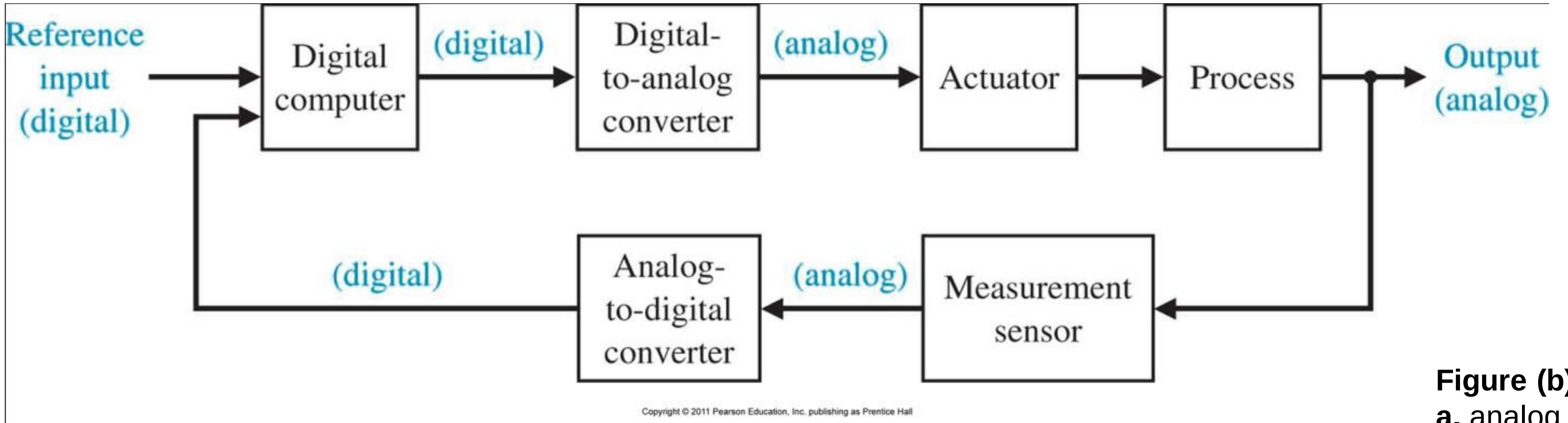


Figure (a) block diagram of a single-loop digital control system.

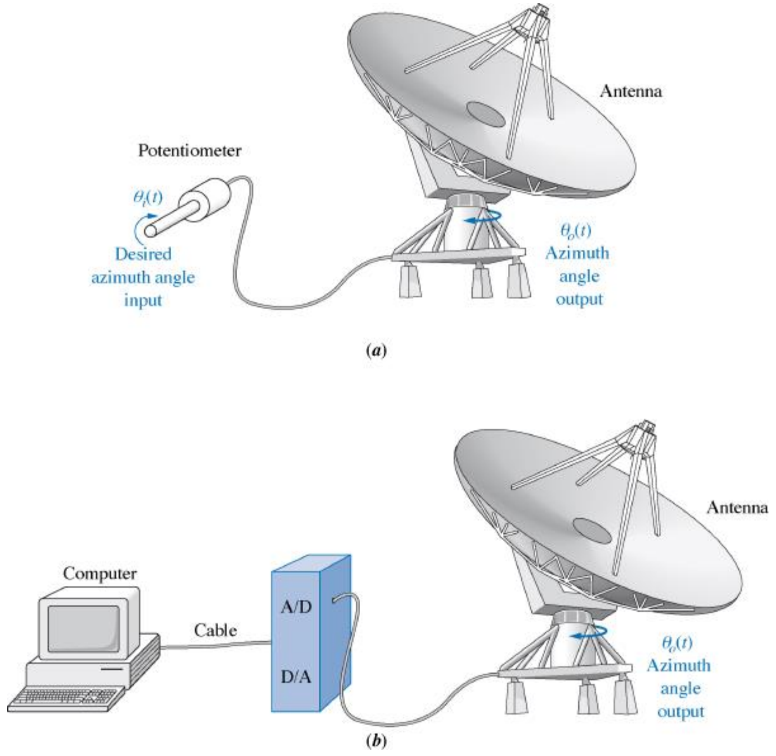


Figure (b) Conversion of antenna azimuth position control system from: **a.** analog control to **b.** digital control

1.1 Advantages of Digital Computers

- The use of digital computers in the loop yields the following advantages

  1. **Reduced cost.**
  A single digital computer can replace numerous analog controllers with a subsequent reduction in cost, analog controllers implied numerous adjustments and resulting hardware.

  2. **Flexibility in response to design changes**.
  Any changes or modifications that are required in the future can be implemented with simple software changes rather than expensive hardware modifications.

  3. **Noise immunity.**
  Digital systems exhibit more noise immunity than analog systems.

- The computer replaces the cascade compensator and uses digital signals Fig. (a)

- A device that converts analog signals to digital signals is called an analog-to-digital (A/D) converter.

- a device that converts digital signals to analog signals is called a digital-to analog (D/A) converter

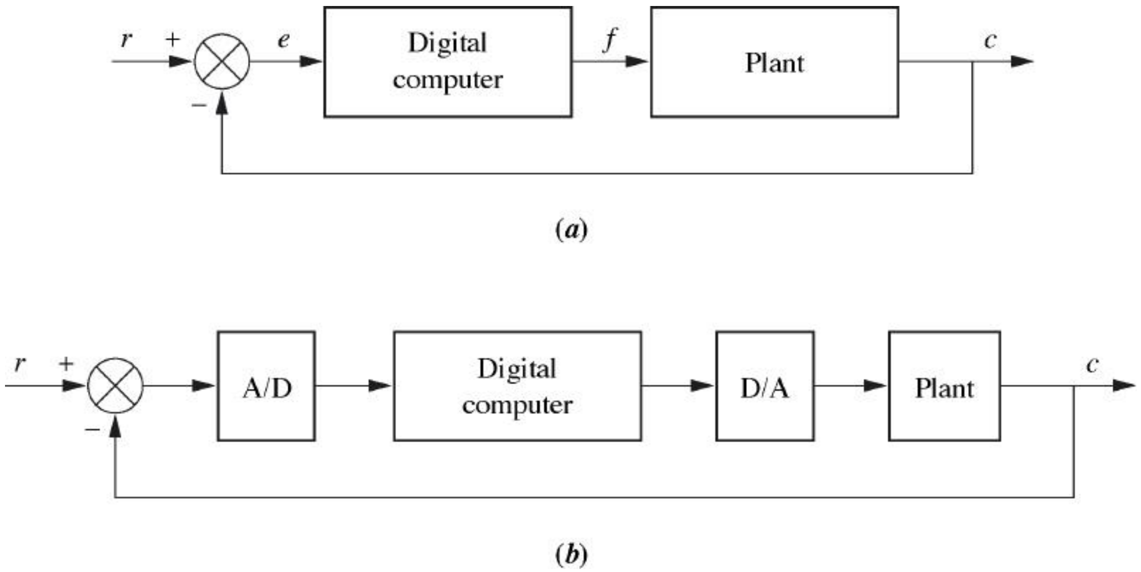


Figure (a) a. Placement of the digital computer within the loop;
b. detailed block diagram showing placement of A/D and D/A converters

1.2 Digital-to-Analog Conversion (DAC)

- Digital-to-analog conversion is simple and effectively instantaneous.
- Properly weighted voltages are summed together to yield the analog output.
- if the binary number is 110_2, the center and bottom switches are on, and the analog output is 6 volts Fig(a).
- the switches are electronic and are set by the input binary code.

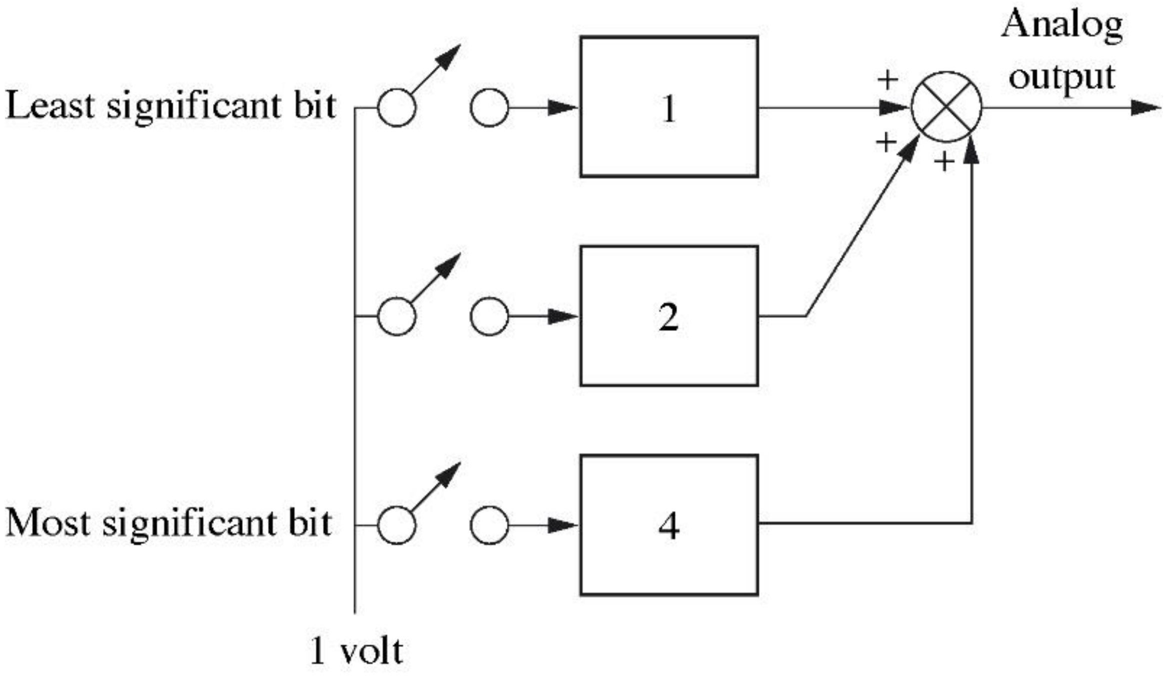


Figure (a) Digital-to-analog converter

1.3 Analog-to-Digital Conversion (ADC)

- The analog signal is first converted to a sampled signal and then converted to a sequence of binary numbers, the digital signal.
- ADC is a two-step process (sampling and quantization) with delay between the input analog voltage and the output digital word (not instantaneous).
- The sampling rate must be at least twice the bandwidth of the signal (Nyquist sampling rate),

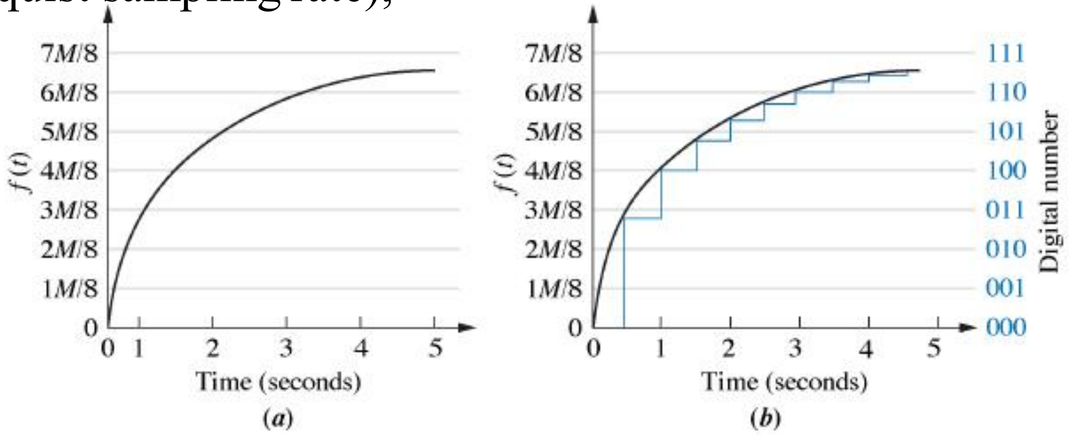
- The analog signal is sampled and held at periodic intervals by a ***zero-order-holder*** (Z-O-H) (staircase approximation to the analog signal.)
- The ADC converts the sample to a digital number.

- The analog signal's voltage is divided into discrete levels $\frac{M}{2^B}$, and each level is assigned a digital number. M is the maximal value of the analog input signal and B is the length of the binary bits.

- In Fig (a) the analog signal is divided into eight levels 2^3. A three-bit digital number (B=3) can represent each of the eight.

- The ***quantization error*** ***e*** is due to the quantization process that rounds off the analog voltage to the next higher or lower level.

$$|e| \le \frac{1}{2}\left(\frac{M}{2^B}\right) = \frac{M}{2^{B+1}}$$



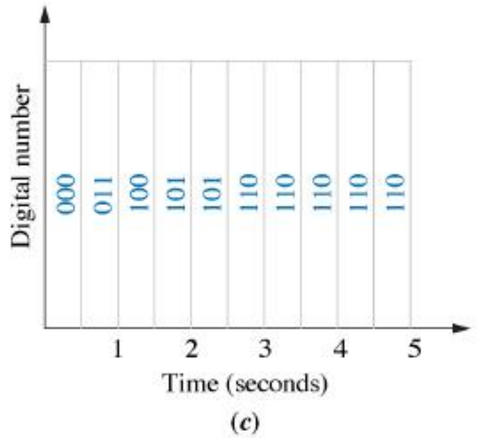


Figure (a) Steps in analog-to-digital conversion:
a. analog signal;
b. analog signal after sample-and-hold;
c. conversion of samples to digital numbers

2 Modeling the Digital Computer

2.1 Modeling the Sampler

- The objective is to derive a mathematical model (transfer function of a subsystem) for the digital computer as represented by a sampler and zero-order hold.
- When signals are sampled, the Laplace transform can be replaced by another related transform called ***the z-transform***.

- The sampling model is a switch turning on and off at a uniform sampling rate. Fig(a)
- Sampling can also be considered to be a product of the time waveform to be sampled $f(t)$ and a sampling function $s(t)$. Fig(b)

- The time equation of the sampled waveform $f_{TW}^{*}(t)$

integer

period of the pulse train

pulse width

$$f_{TW}^{*}(t) = f(t) \cdot s(t) = f(t) \cdot \sum_{k=-\infty}^{+\infty} [u(t - kT) - u(t - kT - T_W)]$$

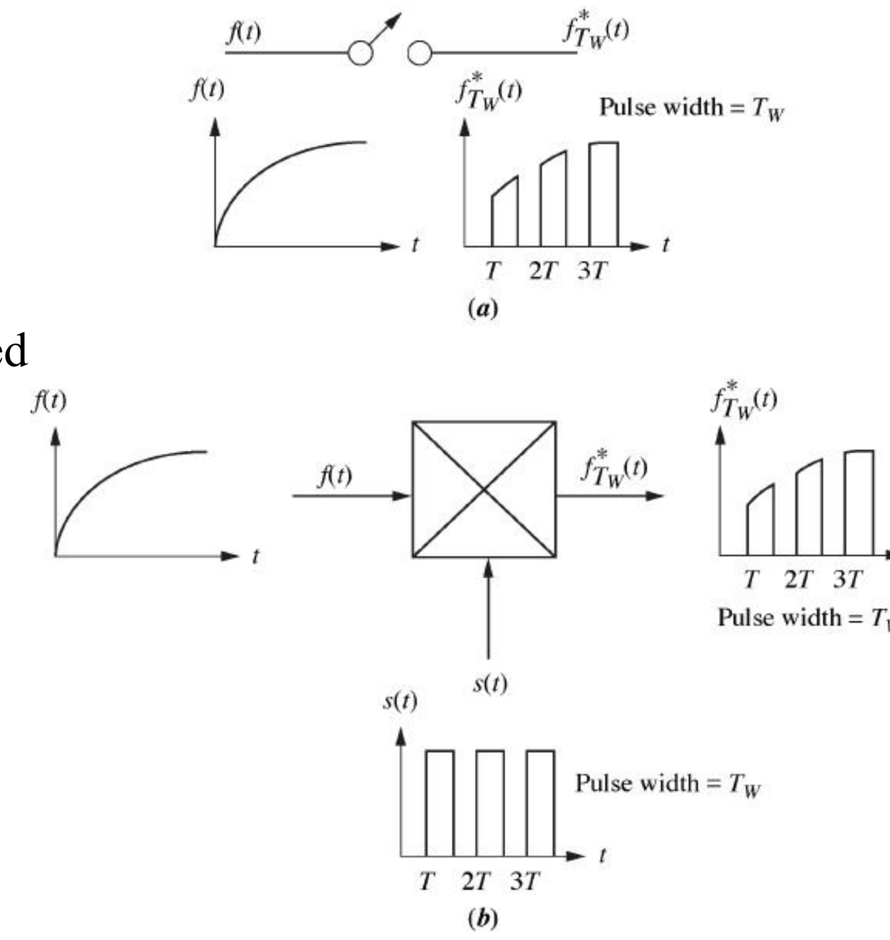


FIGURE (1) Two views of uniform-rate sampling:
a. switch opening and closing;
b. product of time waveform and sampling waveform

Using the Laplace transform

$$F_{Tw}^{*}(s) = \sum_{k=-\infty}^{+\infty} f(kT)\left[\frac{e^{-kTs}}{s} - \frac{e^{-kTs-T_w s}}{s}\right] = \sum_{k=-\infty}^{+\infty} f(kT)\left[\frac{1-e^{-T_w s}}{s}\right] e^{-kTs} = T_w \cdot f^{*}(t)$$

Unit Step Function

$$u(t-a) = 0, t < a$$
$$\quad 1, t \geq a$$

$$L\{u(t-a)\} = \int_0^{\infty} e^{-st} u(t-a)dt$$
$$= \int_0^{a} e^{-st}(0)dt + \int_a^{\infty} e^{-st}(1)dt$$
$$= \left[\frac{e^{-st}}{-s}\right]_a^{\infty} = \frac{1}{s}e^{-as}$$

if $a = 0$

$$L\{u(t)\} = \frac{1}{s}$$

Replacing $e^{-T_w s}$ with its series expansion, we obtain

$$F_{Tw}^{*}(s) = \sum_{k=-\infty}^{+\infty} f(kT)\left[\frac{1-\{1-T_w s+\frac{(T_w s)^2}{2!}-\cdots\}}{s}\right] e^{-kTs}$$

For small T_w

$$F_{Tw}^{*}(s) = \sum_{k=-\infty}^{+\infty} f(kT)\left[\frac{T_w s}{s}\right] e^{-kTs} \quad = \sum_{k=-\infty}^{+\infty} f(kT) T_w e^{-kTs}$$

Laplace inverse transform

$$[\ LIT\left(e^{-kTs}\right) = \delta(t-kT)]$$

Finally, converting back to the time domain, we have

Dirac Delta function

$$f_{Tw}^{*}(t) = T_w \sum_{k=-\infty}^{+\infty} f(kT)\delta(t-kT) = T_w \cdot f^{*}(t)$$

Hold

ideal sampler

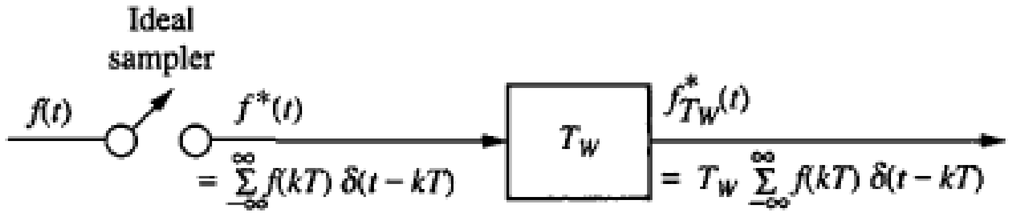
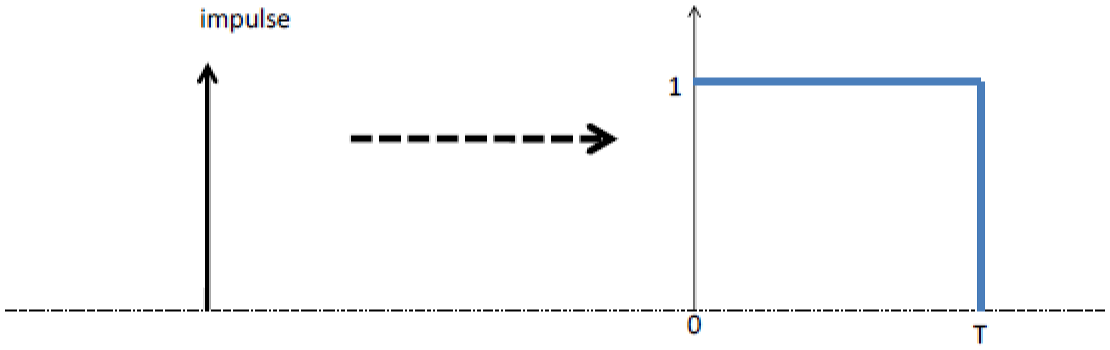


Figure: Model of sampling with a uniform rectangular pulse train

2.2 Modeling the zero-order hold

- The final step in modeling the digital computer is modeling the zero-order hold, it follows the sampler and holds the last sampled value of *f(t)*.
- Using an impulse input at zero time, the output is a step that starts at $t = 0$ and ends at $t = T$



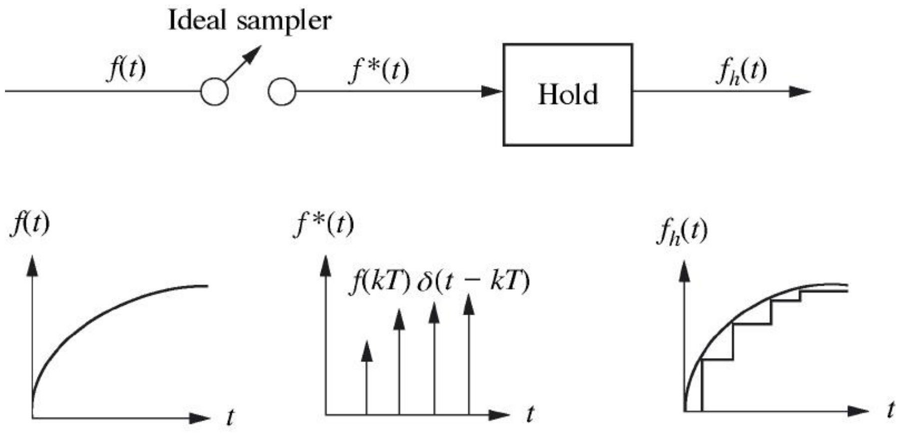


FIGURE (1) Ideal sampling and the zero-order hold

The output is: $y(t) = h(t) = u(t) - u(t-T)$ → Laplace transform is:

$$L\{u(t) - u(t-T)\} = \frac{1}{s} - \frac{1}{s}\, e^{-Ts} = \frac{1 - e^{Ts}}{s}$$

$$\boxed{ZOH(s) = G_h(s) = \frac{1 - e^{Ts}}{s}}$$

- The digital computer is modeled by cascading two elements: (1) an ideal sampler and (2) a zero-order hold.

3 The z-Transform

- The goal is to develop a transform that contains the information of sampling from which sampled-data systems can be modeled with transfer functions, analyzed and designed

- Laplace transform of a sampled signal $f^*(t)$ is:

$$f^*(t) = \sum_{k=-\infty}^{+\infty} f(kT)\cdot\delta(t-kT) \quad \Rightarrow \quad F^*(s) = \int \sum_{k=-\infty}^{+\infty} f(kT)\cdot\delta(t-kT)\, e^{-st} dt$$

$$F^*(s) = \sum_{k=-\infty}^{+\infty} f(kT)\cdot\int \delta(t-kT)e^{-st} dt = \sum_{k=-\infty}^{+\infty} f(kT)\cdot e^{-skT}$$

- It's nonlinear. To overcome the nonlinearity problem, we transform S-domain to another domain where the operator is linear: Z- domain by setting $z = e^{sT}$, The z-transform is defined as:

$$\boxed{F(z) = \sum_{k=-\infty}^{+\infty} f(kT)\cdot z^{-k}}$$

Example 1

Find the z-transform of a sampled unit ramp.

SOLUTION

- For a unit ramp, $f(kT) = kT$. The ideal sampled step:

$$f^*(\mathrm{t}) = \sum_{k=0}^{+\infty} k\,T\,\delta(t-kT) \xrightarrow{\text{Z transform}} F(\mathrm{z}) = \sum_{k=0}^{+\infty} f(kT)z^{-k} \Rightarrow F(\mathrm{z}) = \sum_{k=0}^{+\infty} kTz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots)$$

Multiplying by Z

$$\Rightarrow z\,F(\mathrm{z}) = T(1 + 2z^{-1} + 3z^{-2} + \cdots) \Rightarrow z\,F(\mathrm{z}) - F(\mathrm{z}) = T(1 + z^{-1} + z^{-2} + \cdots)$$

$$\Rightarrow (z-1)F(z) = T(1 + z^{-1} + z^{-2} + \cdots)$$

We have the sum of geometric series ($\mathrm{r} < 1$):

$$\left.\begin{aligned} s &= 1 + r^1 + r^2 + r^3 \ldots + r^N \\ \mathrm{r}s &= r + r^2 + r^3 + r^4 \ldots + r^N + r^{N+1} \end{aligned}\right\} \Rightarrow s - rs = 1 - r^{N+1} \Rightarrow s = \frac{1 - r^{N+1}}{1 - r} \Rightarrow s = \frac{1}{1-r}\ for\ N \to \infty$$

Then, $1 + z^{-1} + z^{-2} + z^{-3} \ldots = \dfrac{1}{1 - z^{-1}} \Rightarrow F(z) = T\dfrac{z}{(z-1)^2}$

- Hint: use $\sum_{i=0}^{N} a^{i} = \frac{1-a^{N+1}}{1-a}$

- Z-transform of unit impulse: $\delta(z) = \sum_{k=0}^{\infty} \delta(k) z^{-k} = z^{0} = 1$

- Z-Transform of the unit-step: $U(z) = \sum_{k=0}^{\infty} u(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}}$

- Z-transform of the exponential function: $f(t) = r^{t} \Rightarrow F(z) = \sum_{k=0}^{\infty} r^{kT} z^{-k} = \sum_{k=0}^{\infty} (r^{T} z^{-1})^{k} = \frac{1}{1-r^{T} z^{-1}} = \frac{z}{z-r^{T}}$
- $If\ r = e^{-a}\ then\ F(z) = \frac{z}{z-e^{-aT}}$

- For general sinusoidal form: $f(k) = r^{k} sin(k\theta) \Rightarrow F(z) = \frac{z\left(z-rcos(\theta)\right)}{z^{2}-2rcos(\theta)z+r^{2}}$

3.1 Partial table of z- and s-transforms

TABLE 13.1 Partial table of z- and s-transforms

	$f(t)$	$F(s)$	$F(z)$	$f(kT)$
1.	$u(t)$	$\frac{1}{s}$	$\frac{z}{z-1}$	$u(kT)$
2.	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$	kT
3.	t^n	$\frac{n!}{s^{n+1}}$	$\lim_{a\to 0}(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$	e^{-akT}
5.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n e^{-akT}$
6.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$	$\sin \omega kT$
7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$	$\cos \omega kT$
8.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$	$e^{-akT}\sin \omega kT$
9.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2 - ze^{-aT}\cos \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$	$e^{-akT}\cos \omega kT$

3.2 Some Properties

	Theorem	Name
1.	$z\{af(t)\} = aF(z)$	Linearity theorem
2.	$z\{f_1(t) + f_2(t)\} = F_1(z) + F_2(z)$	Linearity theorem
3.	$z\{e^{-aT}f(t)\} = F(e^{aT}z)$	Complex differentiation
4.	$z\{f(t-nT)\} = z^{-n}F(z)$	Real translation
5.	$z\{tf(t)\} = -Tz\dfrac{dF(z)}{dz}$	Complex differentiation
6.	$f(0) = \lim_{z\to\infty} F(z)$	Initial value theorem
7.	$f(\infty) = \lim_{z\to 1}(1 - z^{-1})F(z)$	Final value theorem

3.3 The Inverse z-Transform

- Three methods for finding the inverse z-transform (the sampled time function from its z-transform) will be described:

    1. partial-fraction expansion.
    2. Residue.
    3. the power series method.

- Since the z-transform came from the sampled waveform, the inverse z-transform will yield only the values of the time function at the sampling instants.

3.4 Inverse z-Transforms via Partial-Fraction Expansion

- The Laplace transform consists of a partial fraction that yields a sum of terms leading to exponentials, that is, $A/(s+a)$.

- Knowing that: $\frac{z}{z-a} \rightarrow a^{kT}$ or $\frac{z}{z-e^{-bT}} \rightarrow \frac{1}{s+b} \rightarrow e^{-b\,k\,T}$

$$F(z) = \frac{N(z)}{D(z)} \Longrightarrow \frac{F(z)}{z} = \frac{A}{z-a} + \frac{B}{z-b} + \cdots \Longrightarrow F(z) = \frac{A\,z}{z-a} + \frac{B\,z}{z-b} + \cdots$$

- The inverse is:

$$f(kT) = A\,a^k + B\,b^k + \cdots$$

Example

Given the function F(z) find the sampled time function.

$$F(z) = \frac{0.5\,z}{(z-0.5)(z-0.7)}$$

SOLUTION

- First we divide F(z) by z and then we perform a partial-fraction expansion

$$\frac{F(z)}{z} = \frac{0.5}{(z-0.5)(z-0.7)} = \frac{A}{z-0.5} + \frac{B}{z-0.7} = \frac{-2.5}{z-0.5} + \frac{2.5}{z-0.7}$$

- Next, multiply through by *z*. $F(z) = \frac{0.5\,z}{(z-0.5)(z-0.7)} = \frac{-2.5\,z}{z-0.5} + \frac{2.5\,z}{z-0.7}$

- Thus the inverse z-transform $f(kT) = -2.5(0.5)^k + 2.5(0.7)^k$

- The ideal sampled time function is: $f^*(t) = \sum_{k=0}^{\infty} f(kT)\cdot\delta(t-Kt) = \sum_{k=0}^{\infty}\left(-2.5(0.5)^k + 2.5(0.7)^k\right)\cdot\delta(t-Kt)$

- Plot by hand the function $f^*(t)$ for k = 0, 1, 2 ..

4 Transfer Functions of a discrete system

- The objective is to find transfer functions of sampled-data systems. Consider the continuous system shown in Figure (a) the output is conceptually sampled in synchronization with the input by a phantom sampler.

- The sampled input, $r^*(t)$, to the system of Figure(c) is: (convolution theorem)

$$r^*(t) = \sum_{k=0}^{\infty} r(nT)\delta(t-nT)$$

← sampled input

sampled input, system

$$c(t) = \sum_{n=0}^{\infty} r(nT)g(t-nT)$$

→ Time out put of G(s), discrete convolution

Change t by kT

$$C(z) = \sum_{k=0}^{\infty} c(kT)z^{-k} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} r(nT)g[(k-n)T]z^{-k}$$

← C(t) for t = kT

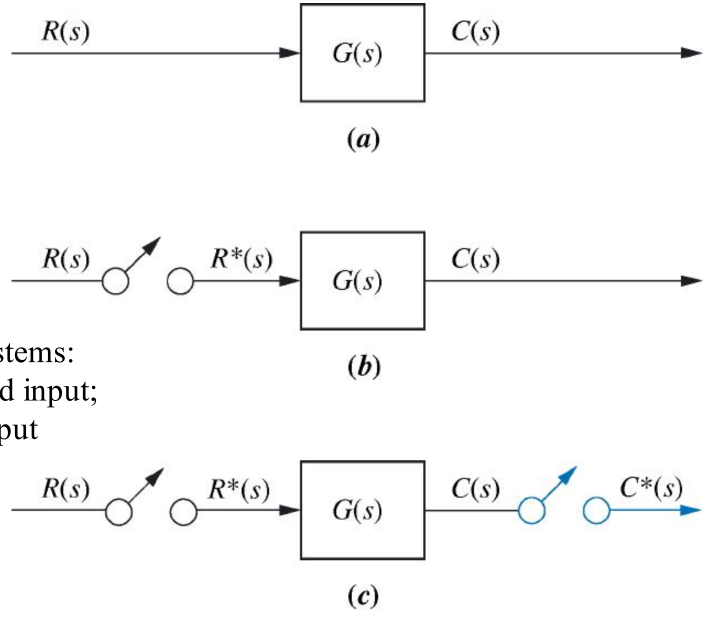


Figure Sampled-data systems:
a. continuous; b. sampled input;
c. sampled input and output

Note: Phantom sampler is shown in color.

- let m = k-n→ k = m + n

$$C(z) = \sum_{n=0}^{\infty}\sum_{m+n=0}^{\infty} r(nT)\ g(mT)\ z^{-(m+n)}$$

$$C(z) = \sum_{m=0}^{\infty} g(mT)\ z^{-m} \sum_{n=0}^{\infty} r(nT)\ z^{-n} = G(z)R(z)$$

m+n changed to m because m+n=0 yields to m<0 for n>0 and g(mT) = 0 for m<0.

- The transform of the sampled output is the product of the transforms of the sampled input and the pulse transfer function of the system.

Example: Converting G1(s) in Cascade with z.o.h. to G(z)

Given a z.o.h. in cascade with $G_1(s) = \frac{s+2}{s+1}$ or $G(s) = \frac{1-e^{-Ts}}{s}\frac{s+2}{s+1}$

find the sampled-data transfer function, *G(z)*, if the sampling time, *T*, is 0.5 second.

SOLUTION

- Knowing that $Z^{-1} = e^{-sT}$ $\Rightarrow$ $G(s) = (1-e^{-Ts})\frac{G_1(s)}{s}$ $\Rightarrow$ $G(z) = (1-z^{-1})Z\left[\frac{G_1(s)}{s}\right] = \frac{z-1}{z}Z\left[\frac{G_1(s)}{s}\right]$

- the impulse response (inverse Laplace transform) of $\frac{G_1(s)}{s}$

$$G_2(s) = \frac{G_1(s)}{s} = \frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{2}{s} - \frac{1}{s+1}$$

- Taking the inverse Laplace transform $g_2(t) = 2 - e^{-t}$ $\Rightarrow$ $g_2(kT) = 2 - e^{-kT}$ $\xrightarrow{\text{From Table}}$ $G_2(z) = \frac{2z}{z-1} - \frac{z}{z-e^{-T}}$

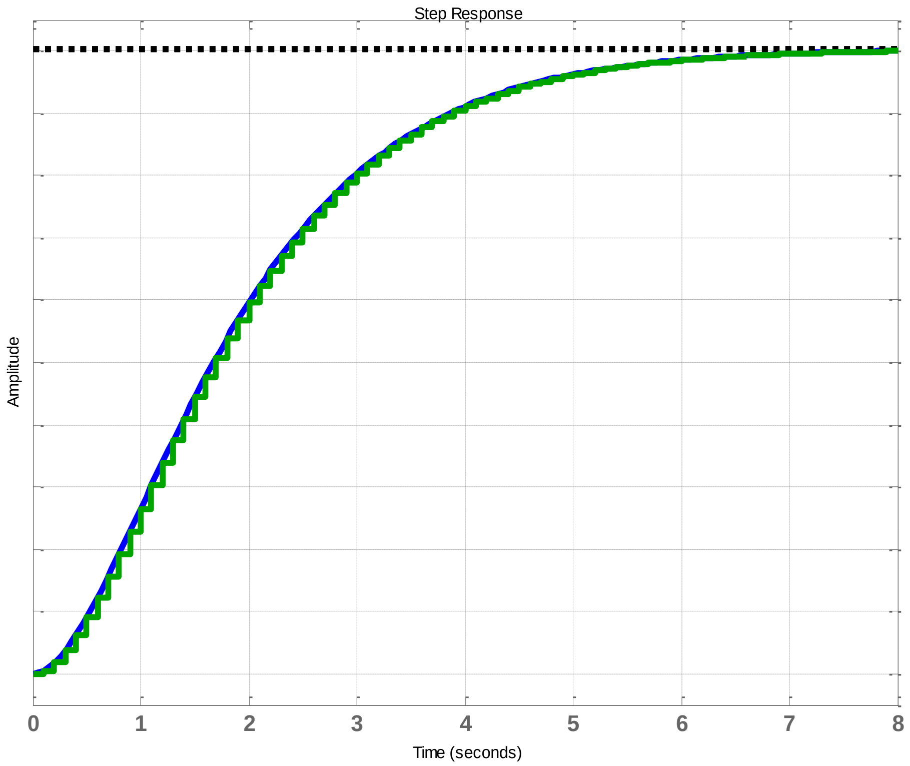
- Substituting $T = 0.5s$ $G_2(z) = Z\left[\frac{G_1(s)}{s}\right] = \frac{2z}{z-1} - \frac{z}{z-0.607} = \frac{z^2 - 0.213z}{(z-1)(z-0.607)}$

$$\Rightarrow \boxed{G(z) = \frac{z-1}{z}Z\left[\frac{G_1(s)}{s}\right] = \frac{z-0.213}{z-0.607}}$$

Available Commands for Continuous/Discrete Conversion

The commands c2d, d2c, and d2d perform continuous to discrete, discrete to continuous, and discrete to discrete (resampling) conversions, respectively.

```
sysd = c2d(sysc,Ts)  % Discretization w/ sample period Ts
sysc = d2c(sysd)     % Equivalent continuous-time model
sysd1= d2d(sysd,Ts)  % Resampling at the period Ts
```




```
>>  sys=tf(1,[1 2 1])

Transfer function:
     1
-------------
s^2 + 2 s + 1

>> Ts=0.1; sysd=c2d(sys,Ts)

Transfer function:
0.004679 z + 0.004377
---------------------
z^2 - 1.81 z + 0.8187

Sampling time (seconds): 0.1
>> step(sys,sysd)
```

MATLAB Code :

Continuous/Discrete system

```
>> T=1;Num=1;Den=[1 0 0];
>> sysc=tf(Num,Den);
>> sysd=c2d(sysc,T,'zoh')
Transfer function:
 0.5 z + 0.5
-------------
z^2 - 2 z + 1
Sampling time (seconds): 1
```

5 Block Diagram Reduction

- Our objective here is to be able to find the closed-loop sampled-data transfer function of an arrangement of subsystems that have a computer in the loop. When manipulating block diagrams for sampled-data systems, the rule is:

$$Z\{G_1(s)G_2(s)\} \neq G_1(z)G_2(z) \qquad \{G_1(s)G_2(s)\}^* \neq G_1(s)^*\, G_2(s)^* \qquad \{G_1(s)G_2(s)^*\}^* = G_1(s)^*\, G_2(s)^*$$

Sampled data systems

Their z-transforms

$$C(z) = R(z)G(z)$$

no sampler between $G_1(s)$ and $G_2(s)$

single transfer function $G_1(s)G_2(s)$=$G_{12}\ (s)$

$$C(z) = R(z)G_{12}(z)$$

cascaded of two subsystems each like in (a)

$$C(z) = R(z)\ G_1(z)\ G_2(z)$$

the continuous signal entering the sampler is $R(s)G_1(s)$

$$RG_1(z)\text{=Z}\{R(s)G_1(s)\}$$

$$C(z) = RG_1(z)\ G_2(z)$$

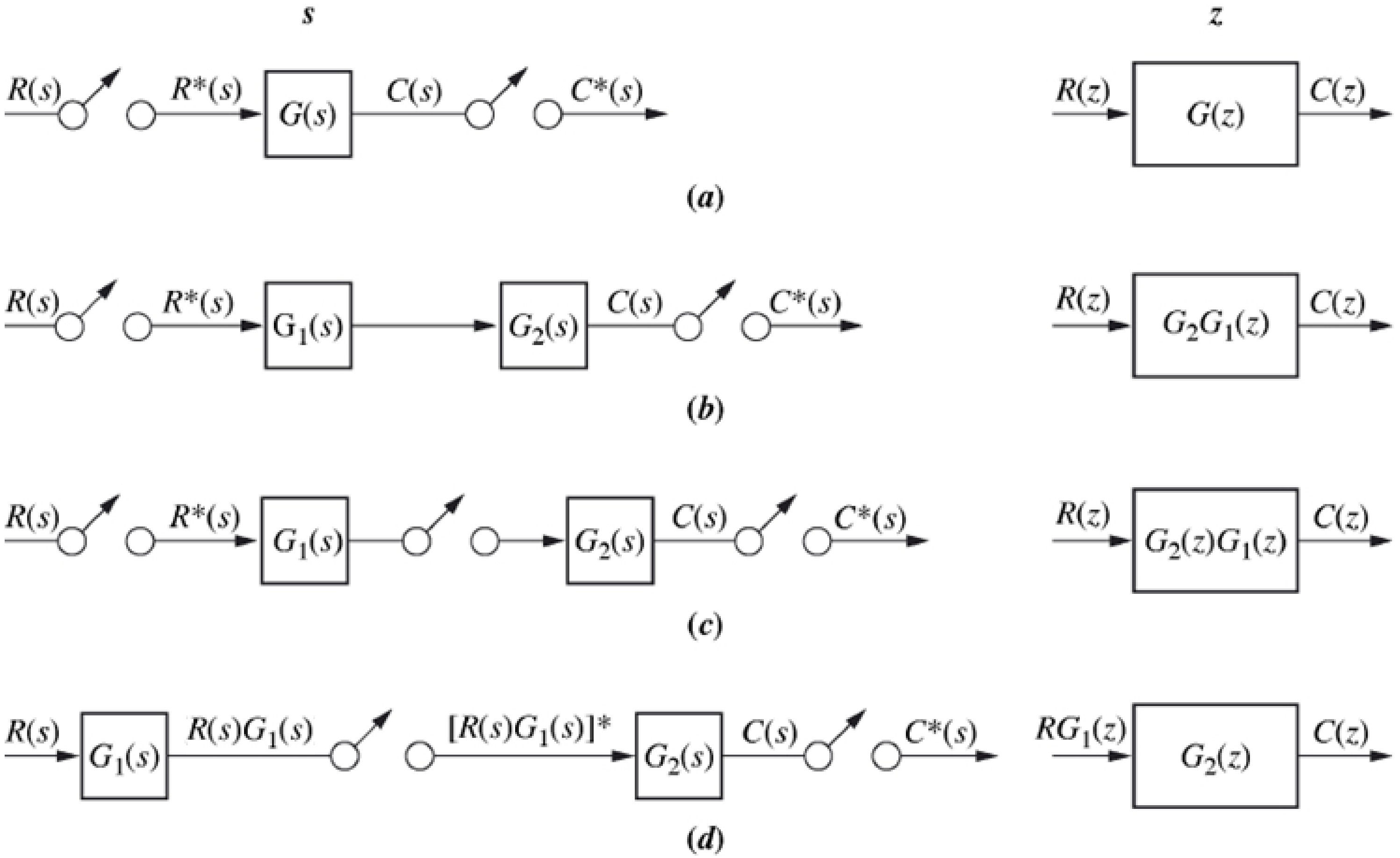
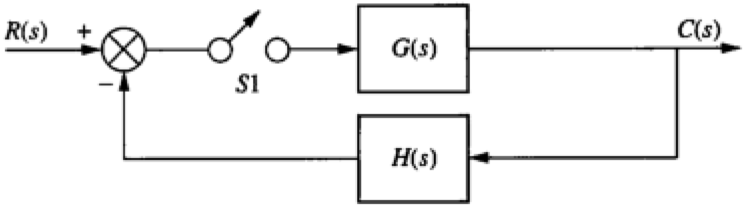


Figure Sampled-data systems and their z-transforms

Example



Find the z-transform of the system shown in Figure

SOLUTION

- The objective is to reduce the block diagram of Figure (a) and reducing it to the one shown in Figure (f).

1. place a phantom sampler at the output of any subsystem that has a sampled input (is not an input to other subsystem)

2. add phantom samplers S2 and S3 at the input to a summing junction whose output is sampled (synchronized samplers).

3. move sampler S1 and G(s) to the right past the pickoff point (to yield a sampler at the input of G(s)H(s))

   $G(s)H(s)$ with samplers S1 and S3 becomes $GH(z)$

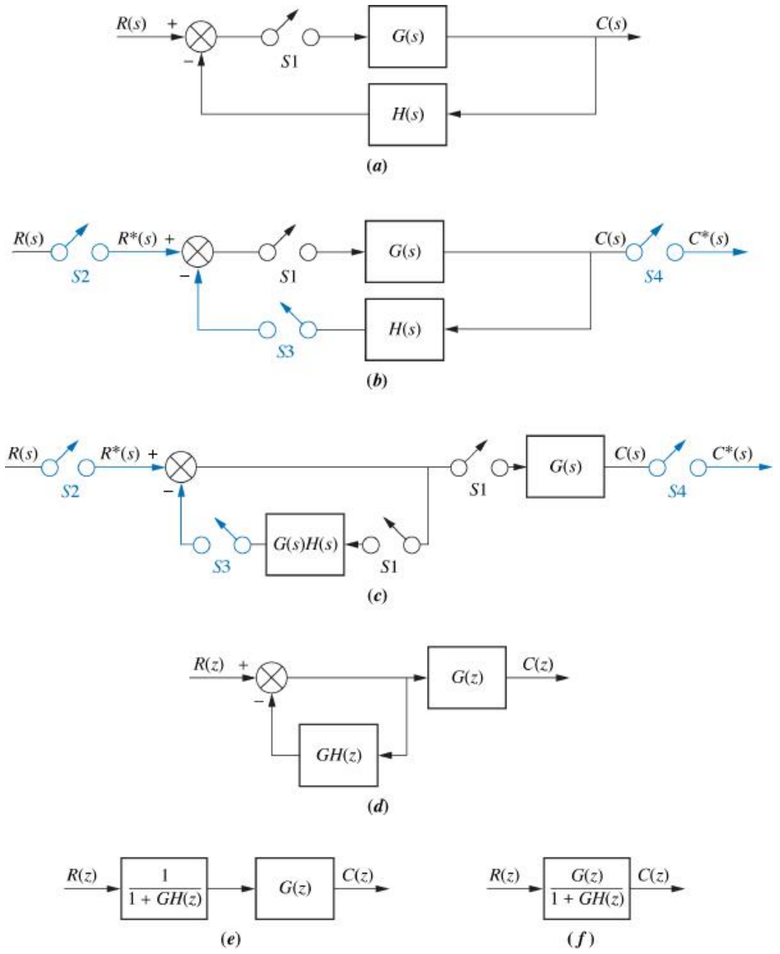
   $G(s)$ with samplers S1 and S4 becomes $G(z)$

   Converting $R^{*}(s)$ to $R(z)$ and $C^{*}(s)$ to $C(z)$

   Now we have the system represented totally in the z-domain

4. using the feedback formula, we obtain the first block (Fig(e))

5. multiplication of the cascaded sampled-data systems yields the final result (Fig(f))



Note: Phantom samplers are shown in color.

FIGURE Steps in block diagram reduction of a sampled-data system

6 Stability

- The stability of digital system can be analyzed in Z-Domain or in S-Domain.

- Changes in sampling rate not only change the nature of the response from over-damped to underdamped, but also can turn a stable system into an unstable one.

6.1 Digital System stability via Z-Plane

- In the S-plane, the region of stability is the left half-plane.

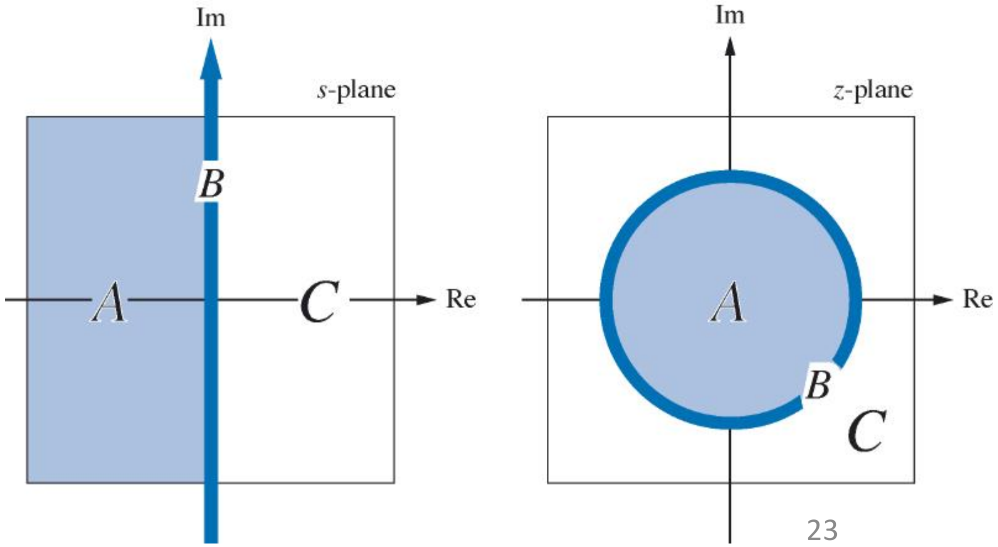
- If the transfer function, *G(s),* is transformed into a sampled-data transfer function, *G(z),* the region of stability on the z-plane can be evaluated from $Z = e^{Ts}$.

- Letting $s = \alpha + j\omega$ we obtain: $Z = e^{Ts} = e^{T(\alpha+j\omega)} = e^{\alpha T}e^{j\omega T} = e^{\alpha T}(\cos\omega T + j sin\omega T) = e^{\alpha T}\angle\omega T$

- From the above equation, we can deduce that the stable domain that corresponds to $\alpha<0$, lies inside the unity circle, the $j\omega$ ($\alpha=0$) axis lies on the unity circle, and the unstable domain $\alpha>0$ lies outside the unity circle.

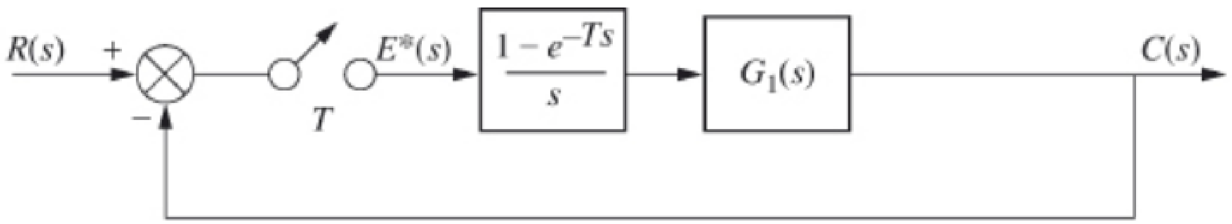
- Thus, a digital system is stable if and only if all poles of the closed-loop transfer function T(z) are inside the unity circle.

- The digital system is marginally stable if poles of multiplicity one of the closed-loop transfer function *T*(*z*) are on the unity circle and other are inside the unity circle.



Example

Study the stability of the closed-loop system in the figure.
Where $G_1(s) = \frac{1}{s+2}$ and T=0.5s



SOLUTION

$$G(z) = (1 - z^{-1})\, z\left(\frac{G(s)}{s}\right) = \frac{0.316}{z - 0.368}$$

$$T(z) = \frac{G(z)}{1 + G(z)} = \frac{0.316}{z - 0.05}$$

since the pole is inside the unity circle then the system is stable.

MATLAB Code:

```
>> T=1;Num=1;Den=[1 2];
>> T=0.5;Num=1;Den=[1 2];
>> sysc=tf(Num,Den);
>> sysd=c2d(sysc,T,'zoh')

Transfer function:
  0.3161
----------
z - 0.3679
 Sampling time (seconds): 0.5
```

```
>> sysclD=feedback(sysd,1)

Transfer function:
  0.3161
-----------
z - 0.05182

Sampling time (seconds): 0.5
```

- We can check the stability with regards to the sampling period T:

$$G(s) = \frac{(1-e^{-Ts})}{s(s+2)} = 0.5(1-e^{-Ts})\left(\frac{1}{s} - \frac{1}{s+2}\right) \xrightarrow{\text{Taking the Z-Transform}} G(z) = \frac{0.5\,(z-1)}{z}\left[\frac{z}{z-1} - \frac{z}{z-e^{-2T}}\right] = 0.5\frac{1-e^{-2T}}{z-e^{-2T}}$$

$$T(z) = \frac{G(z)}{1+G(z)} = \frac{0.5\frac{1-e^{-2T}}{z-e^{-2T}}}{1+0.5\frac{1-e^{-2T}}{z-e^{-2T}}} = \frac{0.5(1-e^{-2T})}{z-(1.5e^{-2T}-0.5)}$$

The pole is $(1.5e^{-2T}-0.5)$ The system is stable for all T >0.

- Let $G(s) = \frac{10}{s+1} \rightarrow G(z) = 10\,\frac{(1-e^{-T})}{z-e^{-T}}$ and $T(z) = \frac{10\,(1-e^{-T})}{z-(11e^{-T}-10)}$

The system is stable for: 0 < T< 0.2.

$$for\; T = 0 \quad \rightarrow 11e^{-0} - 10 = 1 \qquad and \qquad for\; T = 0.2 \quad \rightarrow 11e^{-0.2} - 10 = -1$$

The pole is $(11e^{-T}-10)$, monotonically decreases from +1 to -1 for 0 < *T* < 0.2. For 0.2 < *T* <∞, $(11e^{-T}-10)$ monotonically decreases from - 1 to -10. Thus, the pole of *T(z)* will be inside the unit circle, and the system will be stable if 0 < *T* < 0.2. In terms of frequency, where f= 1/T, the system will be stable as long as the sampling frequency is 1/0.2 = 5 hertz or greater.

6.1 Stability via S-plane (Routh-Hurwitz criterion)

- We can study the stability of digital system using the Routh-Hurwitz criterion. Indeed, we have just to find the transformation from z-Domain to s-domain.

- We have: $Z = e^{sT}$, we cannot use the transformation: $G(s) = G(Z)|_{Z=e^{sT}}$ with a nonlinear operator.

- The most used transformation is the Bilinear Transformation, where: $Z = \frac{s+1}{s-1}$ This transformation verifies the mapping from

s-domain to z-domain saw previously.

Example: Let the characteristic equation of a system be: $D(z) = z^3 - z^2 - 0.2\,z + 0.1 = 0$

In s-domain for $Z = \frac{s+1}{s-1}$, this is equivalent to : $s^3 - 19\,s^2 - 45\,s - 17 = 0.$

TABLE 13.3 Routh table for Example 13.8

s^3	1	−45
s^2	−19	−17
s^1	−45.89	0
s^0	−17	0

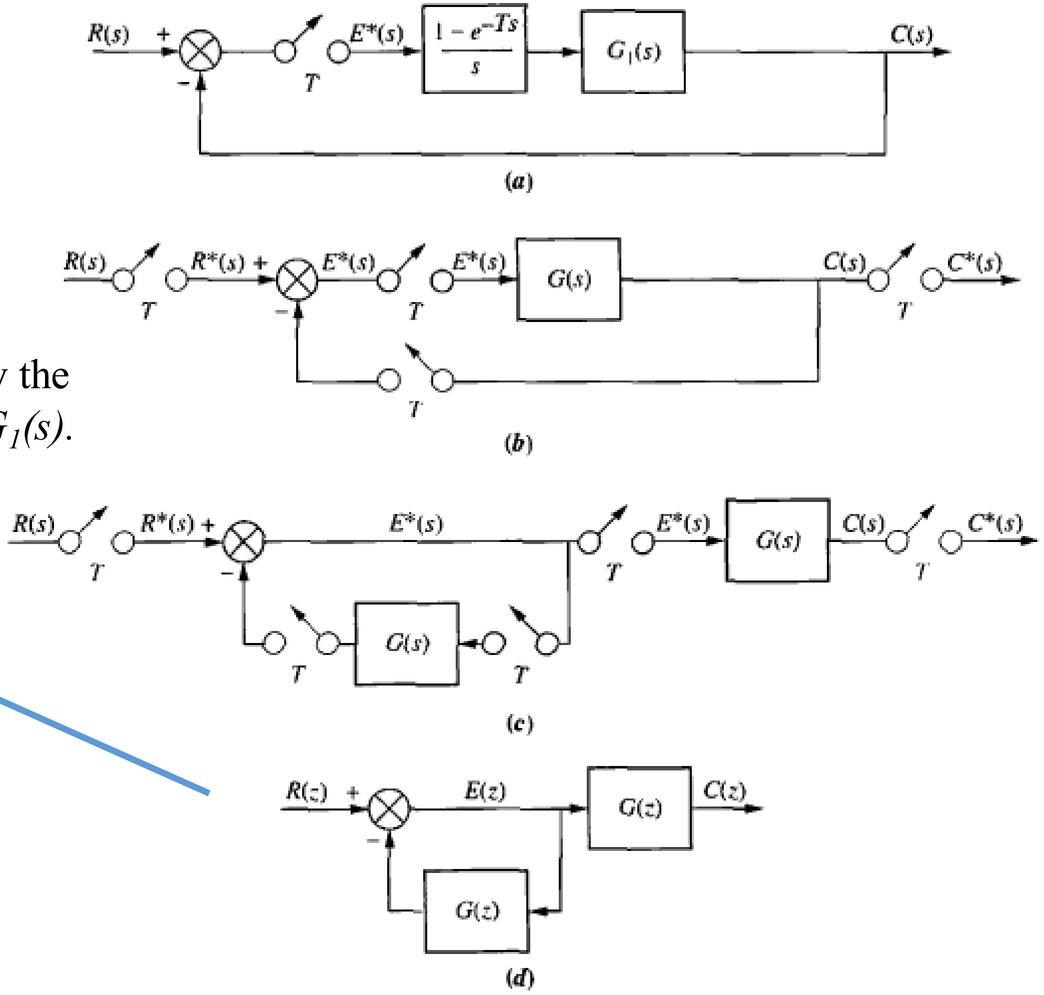
Thus system is unstable and has 1 pole outside the unity circle.
No pole on the unity circle and two poles inside the unity circle.

7 Steady-State Errors

- We examine the effect of sampling upon the steady-state error for digital systems. the placement of the sampler changes the open-loop transfer function.

- we assume the typical placement of the sampler after the error and in the position of the cascade controller, and we derive our conclusions accordingly about the steady-state error of digital systems.

- Consider the digital system in Figure, where the digital computer is represented by the sampler and zero-order hold. The transfer function of the plant is represented by $G_1(s)$.



- we have: $E(z) = R(z) - C(z)$,

  or: $$E(z) = \frac{R(z)}{1 + G(z)}$$

$$E(z) = R(z) - G(z)E(z)$$
$$E(z) + G(z)E(z) = R(z)$$
$$E(z)[1 + G(z)] = R(z)$$

- Using the final value theorem (for discrete signals see slide 13):

$$e_{ss}^{*} = e^{*}(\infty) = \lim_{z\to 1}(1 - z^{-1})\,E(z) = \lim_{z\to 1}(1 - z^{-1})\,\frac{R(z)}{1 + G(z)}$$

7.1 Unit Step Input: $\boldsymbol{R(z) = \frac{z}{z-1}}$

Thus, the steady state error is: $$e_{ss} = \lim_{z\to1}\left(\frac{z-1}{z}\right)\frac{\left(\frac{z}{z-1}\right)}{1+G(z)} = \frac{1}{1+\lim_{z\to1} G(z)}$$

Defining the static error constant K_p as: $$K_p = \lim_{z\to1} G(z) \qquad e_{ss} = \frac{1}{1+K_p}$$

7.2 Unit Ramp Input: $\boldsymbol{R(z) = \frac{T\,z}{(z-1)^2}}$

$$\Rightarrow \lim_{z\to1}\left(\frac{z-1}{z}\right)\frac{\left(\frac{Tz}{(z-1)^2}\right)}{1+G(z)} \Rightarrow \lim_{z\to1}\left(\frac{T}{z-1}\right)\frac{1}{1+G(z)} \Rightarrow \lim_{z\to1}\frac{1}{\frac{1}{T}(z-1)+\frac{1}{T}(z-1)G(z)}$$

0

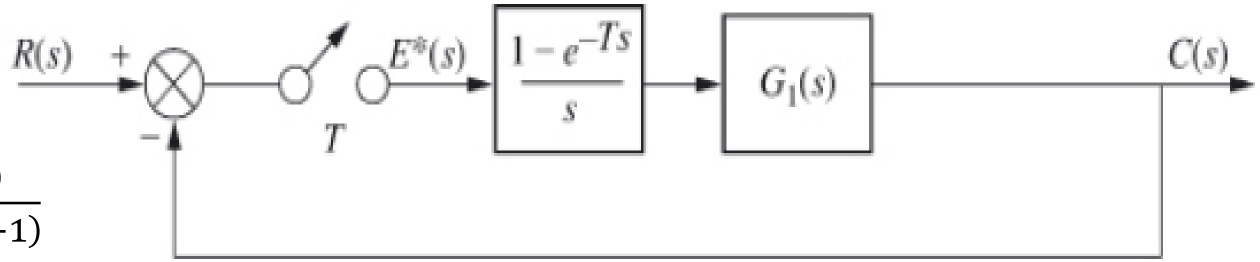
The steady state error for this case is: $$e_{ss} = e(\infty) = \frac{1}{K_v} \qquad \text{Where} \qquad K_v = \frac{1}{T}\lim_{z\to1}(z-1)G(z)$$

7.3 Unit Parabolic Input: $\boldsymbol{R(z) = \frac{T^2\,z(z+1)}{2\,(z-1)^3}}$

The steady state error for this case is: $$e_{ss} = e(\infty) = \frac{1}{K_a} \qquad \text{Where} \qquad K_a = \frac{1}{T^2}\lim_{z\to1}(z-1)^2G(z)$$

- The equations developed above for $e^*(\infty)$, K_p, K_v, and K_a are similar to the equations developed for analog systems.
- Multiple pole placement at the origin of the S-plane reduced steady-state errors to zero in the analog case.
- Multiple pole placement at $z = 1$ reduces the steady-state error to zero for digital systems. $s = 0$ maps into $z = 1$ under $z = e^{Ts}$

Example



For step, ramp, and parabolic inputs, find the steady-state error for the feedback control system shown in Figure if: $G_1(s) = \frac{10}{s\,(s+1)}$

SOLUTION

First find *G(s),* the product of the z.o.h. and the plant. $G(s) = \dfrac{10(1-e^{-Ts})}{s^2(s+1)} = 10(1-e^{-Ts})\left[\dfrac{1}{s^2} - \dfrac{1}{s} + \dfrac{1}{s+1}\right]$

The z-transform is then:

$$G(z) = 10(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right] = 10\left[\frac{T}{z-1} - 1 + \frac{z-1}{z-e^{-T}}\right]$$

Thus:

1. For a step input: $K_p = \lim_{z\to 1} G(z) = \infty \rightarrow e_{ss} = e^*(\infty) = \frac{1}{1+K_p} = 0$

2. For a ramp input: $K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = 10 \rightarrow e_{ss} = e^*(\infty) = \frac{1}{K_v} = 0.1$

3. For a parabolic input: $K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G_1(z) = 0 \rightarrow e_{ss} = e^*(\infty) = \frac{1}{K_a} = \infty$

8 Transient Response on the Z-Plane

- On the s-plane: vertical lines were lines of constant settling time, horizontal lines were lines of constant peak time, and radial lines were lines of constant percent overshoot.

$$T_r = \frac{1.8}{\omega}, \qquad T_s = \frac{4}{\sigma}, \qquad T_p = \frac{\pi}{\omega}, \qquad \%OS = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

- The transformation used to z-domain is: $z = e^{sT}$, for $s = \sigma + j\omega$ we obtain $z = e^{\sigma T}e^{j\omega T} = re^{j\omega T}$ **constant settling time** are concentric circles of radius r. for σ is positive, the circle has a larger radius than the unit circle. if σ is negative, the circle has a smaller radius than the unit circle.

- lines of **constant peak time** are radial lines at an angle of θ_1. If σ is negative, that section of the radial line lies inside the unit circle. If σ is positive, that section of the radial line lies outside the unit circle.
  for $s = \sigma + j\omega$ we obtain $z = e^{\sigma T}e^{j\omega T} = e^{\sigma T}e^{j\theta_1}, \omega T = \theta_1 = \pi\frac{T}{T_p}$

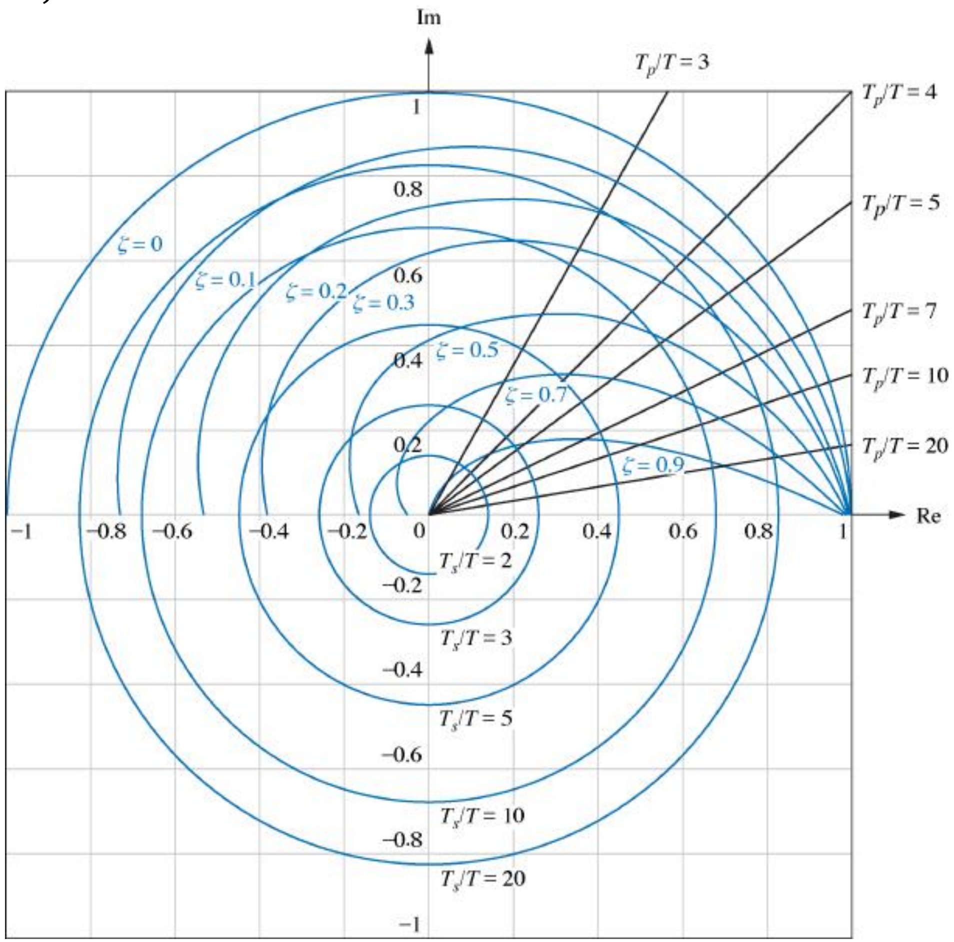


Fig (a) Constant damping ratio, normalized (to the sampling interval) settling time, and normalized peak time plots on the z-plane

- For **constant percent overshoot** we obtain curves on the z-plane. radial lines in the S-plane are represented by

$$\frac{\sigma}{\omega} = -\tan(sin^{-1}\xi) = -\frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}} = \frac{\xi}{\sqrt{1-\xi^2}}$$

Radius and phase depend on ξ

$$s = \sigma + j\omega = -\omega_n\xi + j\omega_n\sqrt{1-\xi^2} \xrightarrow{\text{Transforming to Z-plane}} z = e^{sT} = e^{-\zeta\omega_n T}e^{j\omega_n\sqrt{1-\xi^2}T}$$

- Thus, given a desired damping ratio, ξ, curves can then be used as constant percent overshoot curves on the z-plane through a range of ωT (see previous slide).

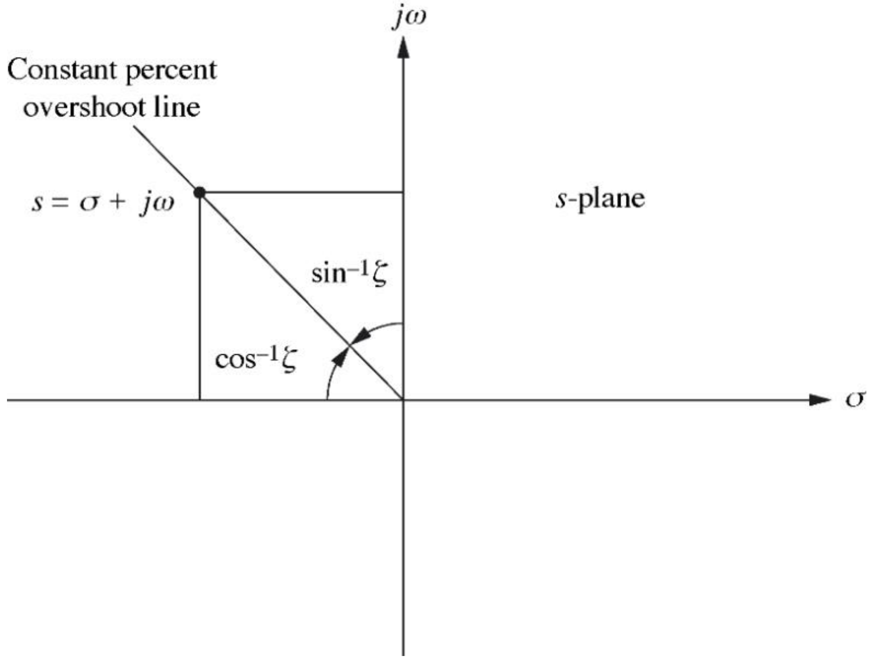
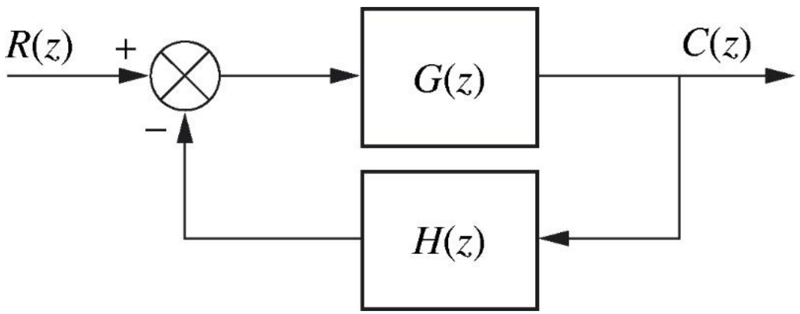


Fig. The s-plane sketch of constant percent overshoot line

9 Design Gain (P-Controller) Design via Root Locus

- The objective is to plot root locus and determine the gain required for stability as well as the gain required to meet a transient response requirement.
- the open-loop and closed-loop transfer functions for the generic digital system shown in Figure (a) are identical to the continuous system except for a change in variables from *s* to z, we can use the same rules for plotting a root locus.
- the region of stability on the z-plane is within the unit circle and not the left half-plane. Thus, in order to determine stability, we must search for the intersection of the root locus with the unit circle rather than the imaginary axis.
- we derived the curves of constant settling time, peak time, and damping ratio. In order to design a digital system for transient response, we find the intersection of the root locus with the appropriate curves as they appear on the *z*-plane.



Fig(a) Generic digital feedback

Example (Stability Design via Root Locus)

Sketch the root locus for the system shown in Figure (a). Also, determine the range of gain, *K*, for stability from the root locus plot.

SOLUTION

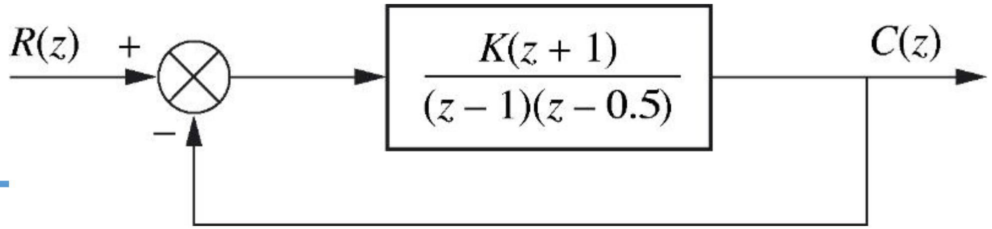
- Treat the system as if *z* were *s*, and sketch the root locus. The result is shown in Figure (b) using Matlab.

- search along the unit circle for 180°. Identification of the gain, K, at this point yields the range of gain for stability.

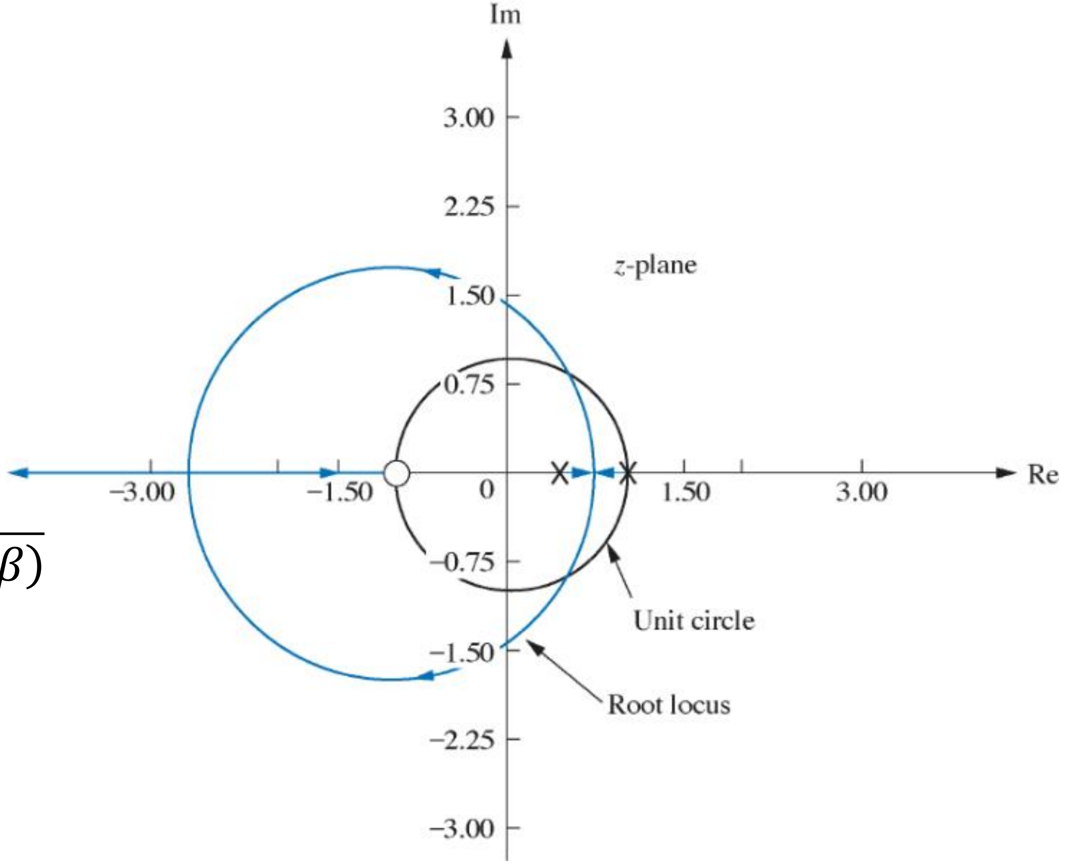
- We find that the intersection of the root locus with the unit circle is $1\angle 60^\circ$. The gain at this point is 0.5. Hence, the range of gain for stability is $0 < K < 0.5$.

- if the open-loop transfer function is given by: $K\,G(z) = K\dfrac{z+\gamma}{(z+\alpha)(z+\beta)}$

The root locus is a circle with center $z_0 = (-\gamma, 0)$ with the radius $r = \sqrt{(\gamma-\alpha)(\gamma-\beta)}$



Fig(a) Digital feedback control



Fig(b) Root locus for the system of fig(a)

Example (Transient Response Design via Gain Adjustment)

For the system of the previous example, find the value of gain, *K,* to yield a damping ratio of 0.7.

SOLUTION

- Figure (a) shows the constant damping ratio curves superimposed over the root locus for the system as determined from the last example.

- Draw a radial line from the origin to the intersection of the root locus with the 0.7 damping ratio curve (a 16.62° line).

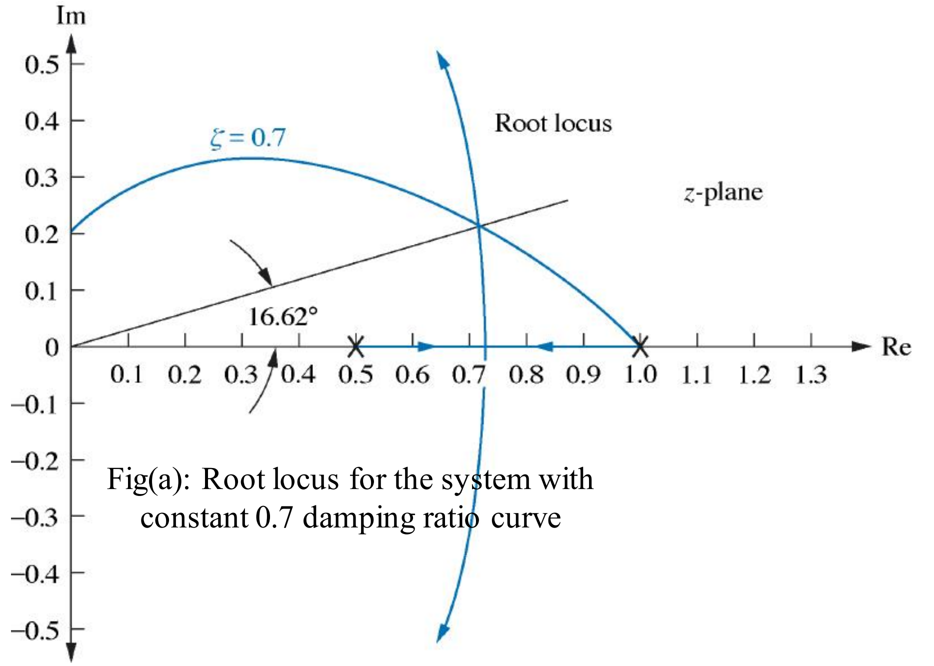
- We obtain the gain by searching along a 16.62° line for 180°, the intersection with the root locus. K = 0.0627 at 0.719 +j 0.215, the point where the 0.7 damping ratio curve intersects the root locus.

- We can now check our design by finding the unit sampled step response of the system Using our design, K = 0.0627.

Input: $R(z) = z/(z - 1)$, sampled step

the sampled output $C(z) = \dfrac{R(z)G(z)}{1+G(z)} = \dfrac{0.0627\,z^2 + 0.0627\,z}{z^3 - 2.4373\,z^2 + 2\,z - 0.5627}$

- Since the overshoot is approximately 5%, the requirement of a 0.7 damping ratio has been met.



Fig(a): Root locus for the system with constant 0.7 damping ratio curve

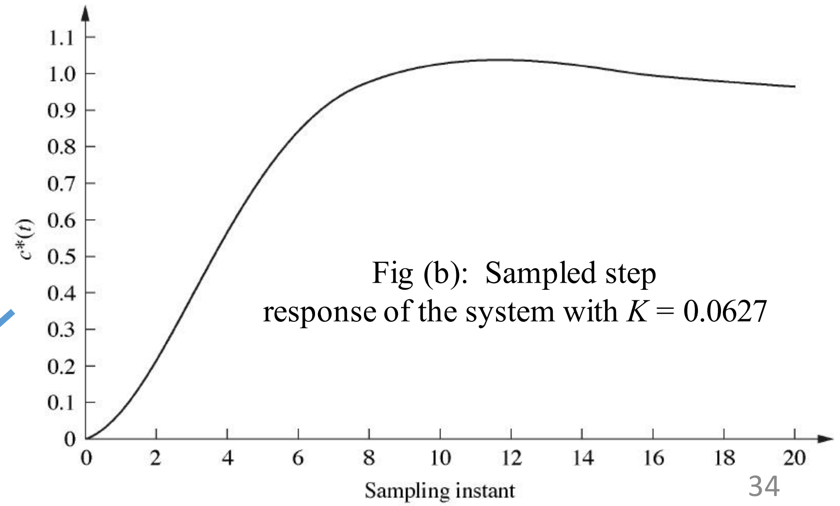


Fig (b): Sampled step response of the system with $K = 0.0627$

Note: Valid only at integer values of sampling instant

10 Cascade Compensation via the s-Plane

- Rather than designing directly in the z-domain, we can design on the s-plane, using S-plane analysis, and then convert the continuous compensator to a digital compensator using the bilinear.

- A bilinear transformation that yields a digital transfer function whose output response at the sampling instants is approximately the same as the equivalent analog transfer function is called the ***Tustin transformation***.

- ***Tustin transformation*** is used to transform the continuous compensator, $G_c(s)$, to the digital compensator, $G_c(z)$, by:

Tustin transformation

$$\boxed{s = \frac{2(z-1)}{T(z+1)}}$$

Inverse Tustin transformation

$$\boxed{z = \frac{-\left(s+\frac{2}{T}\right)}{\left(s-\frac{2}{T}\right)} = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}}$$

- As the sampling interval, *T*, gets smaller (higher sampling rate), the designed digital compensator's output yields a closer match to the analog compensator.

Example (Digital Cascade Compensator Design)

For the digital control system of Figure(a), where the plant $G_p(s)$ is given, design a digital lead compensator, $G_c(z)$, as shown in Figure (b), so that the system will operate with 20% overshoot and a settling time of 1.1 seconds. Create your design in the s-domain and transform the compensator to the z-domain (Sampling period T=0.01 second).

$$G_p(s) = \frac{1}{s(s+6)(s+10)}$$

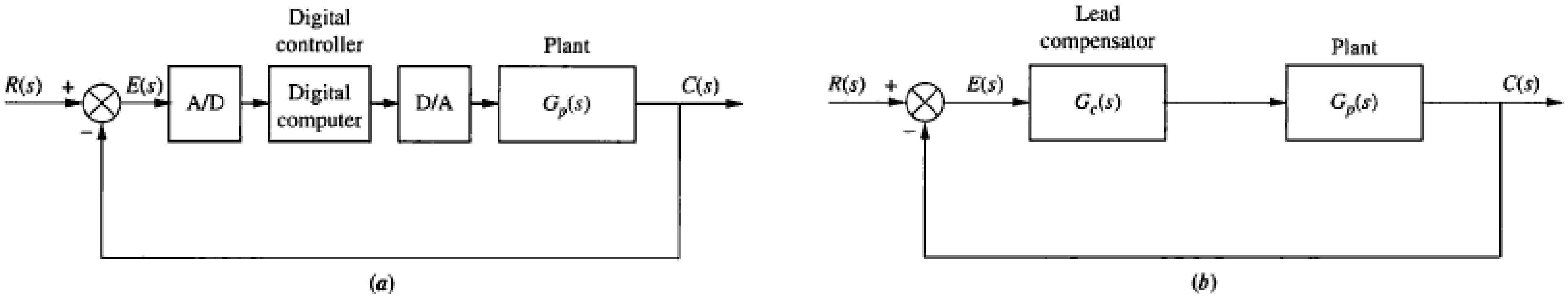


FIGURE a. Digital control system showing the digital computer performing compensation; b. continuous system used for design;

SOLUTION

Using Figure(b), design a lead compensator using the techniques described previously. The design was created as part of an Example, where we found that the lead compensator was (see previous chapters)

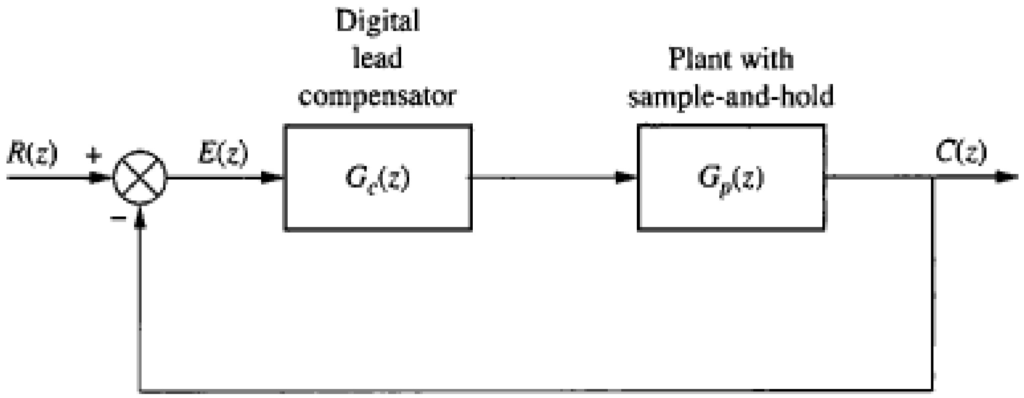
$$G_c(s) = \frac{1977(s+6)}{(s+29.1)}$$

- Using ***Tustin transformation*** :
- We have the analog compensator transfer function $$G_c(s) = \frac{1977(s+6)}{(s+29.1)}$$

  Using ***Tustin transformation*** with T = 0.01 $$s = \frac{2(z-1)}{T(z+1)}$$ Yields to the digital compensator TF $$G_c(z) = \frac{1778z - 1674}{z - 0.746}$$

- The z-transform of the plant and zero-order hold, with T = 0.01 second, is $$G_p(z) = \frac{(1.602 \times 10^{-7}z^2) + (6.156 \times 10^{-7}z) + (1.478 \times 10^{-7})}{z^3 - 2.847z^2 + 2.699z - 0.8521}$$

- The transformed digital system



- The time response in Figure (T = 0.01 s) shows that the compensated closed-loop system meets the transient response requirements. The figure also shows the response for a compensator designed with sampling times at the extremes of Astrom and Wittenmark's guideline.

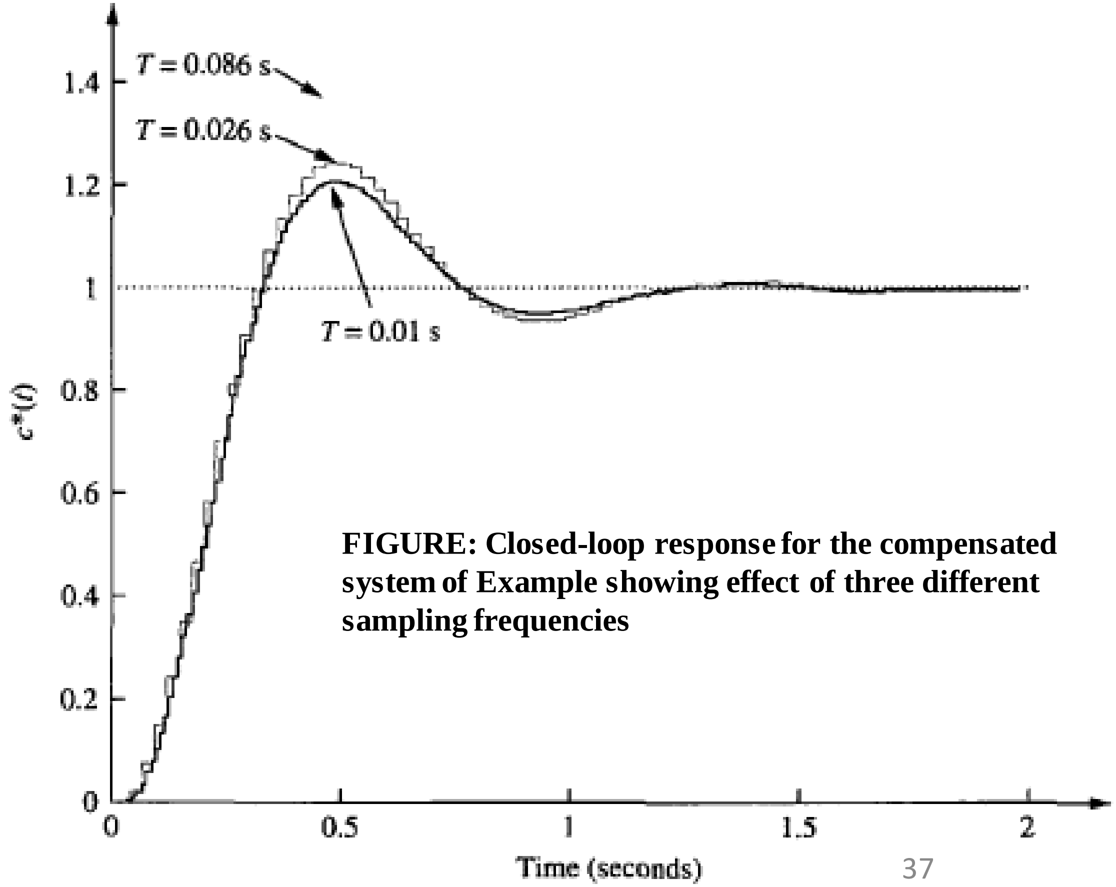


FIGURE: Closed-loop response for the compensated system of Example showing effect of three different sampling frequencies