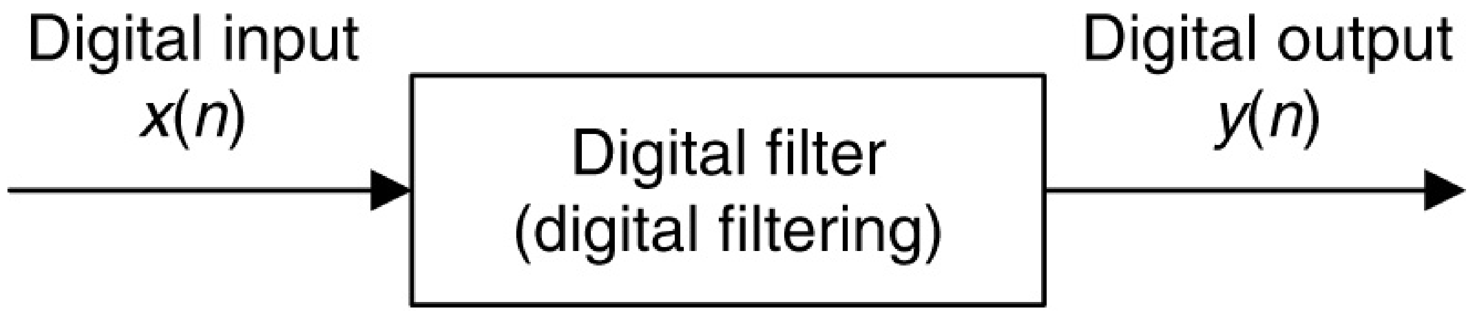
Digital Filters

Analog Filters	Digital Filters
Cheap	Costly
Fast	Slow
Larger dynamic range	
Low performance	**Very high performance**

Digital Filtering: Realization



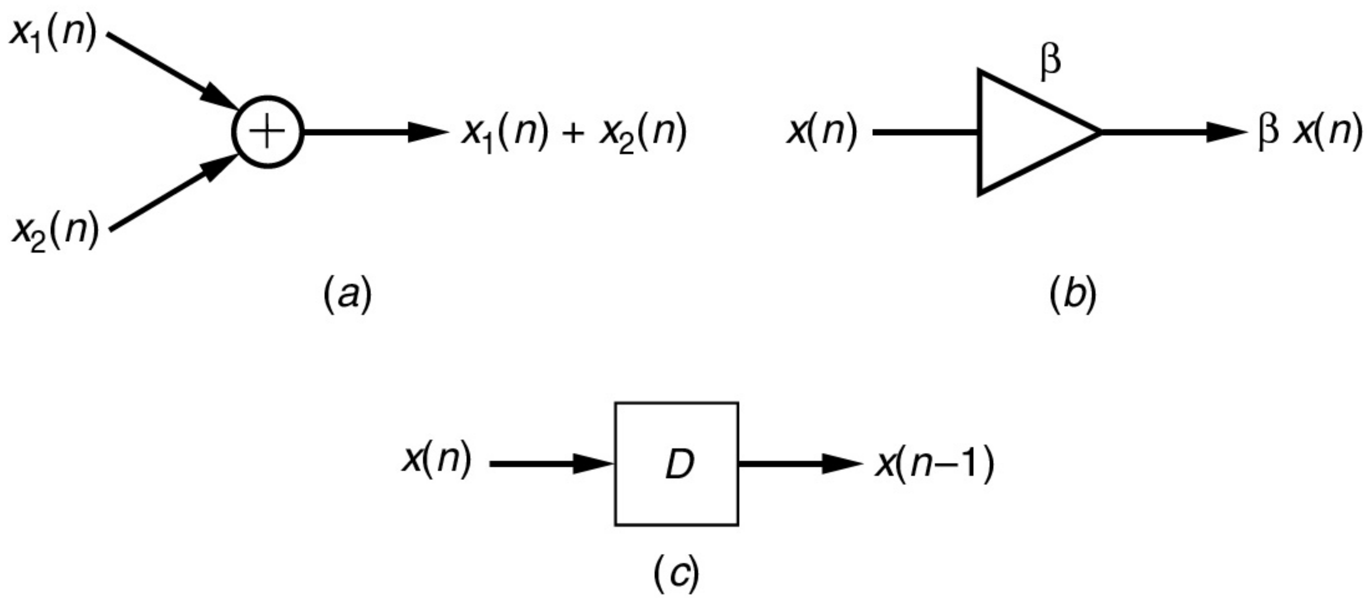
Digital Filtering: $$y(n) = \sum_{i=0}^{M} b_i x(n-i) - \sum_{j=1}^{N} a_j y(n-j).$$

Matlab Implementation:
3-tap (2^{nd} order) IIR filter

```
>> B = [0 1];A = [1 0 -0.5];
>> x = [1 0.5 0.25 0.125];
>> y = filter(B, A, x)
   y =
   0   1.0000   0.5000   0.7500
```

Adder, Multiplier & Delay

Three components of Filters



(*a*) Adder, (*b*) multiplier, (*c*) delay.

Digital Filters: Response (Impulse, Step, Frequency)

Input signal ⊗ impulse res. = output sig.

Filter Kernel

Convolution = weighted sum of input samples.

Finite Impulse Response (FIR) filters

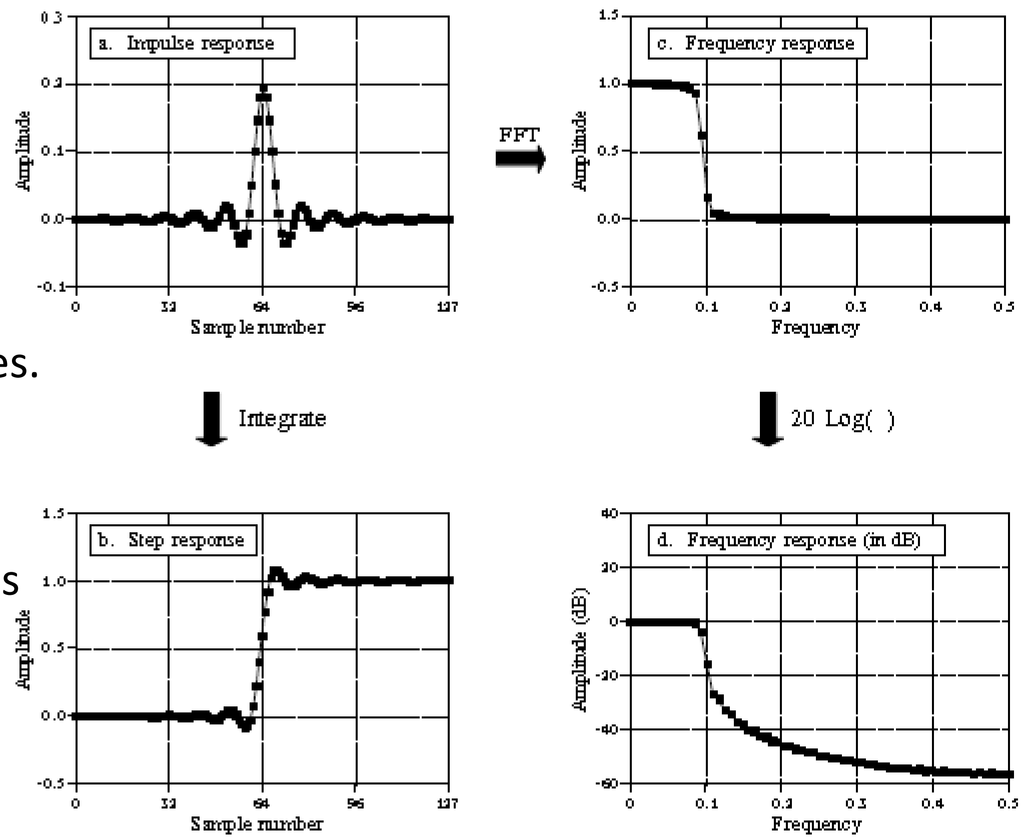
Recursion = input sample + previous outputs

Impulse response of recursive filter

Exponentially decaying sinusoids

→ Infinitely long

Infinite Impulse Response (IIR) filters

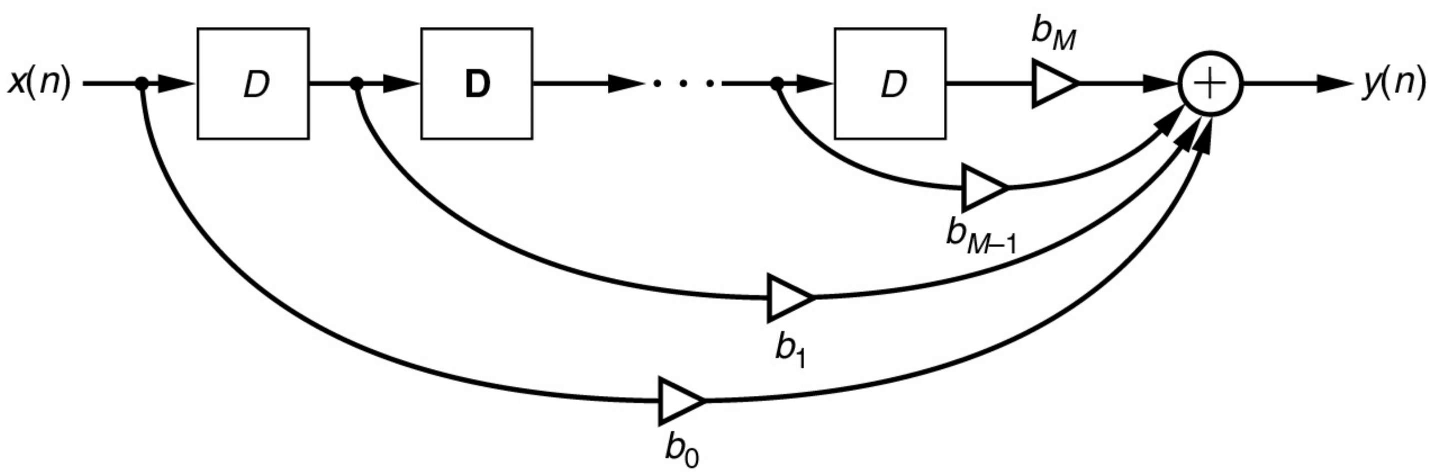


FIR (Finite Impulse Response) Filter

$$y(n) = b_0 x(n) + b_1 x(n-1) + \ldots.. + b_M x(n-M)$$

$$y(n) = \sum_{j=0}^{M} b_j x(n-j)$$

Convolution



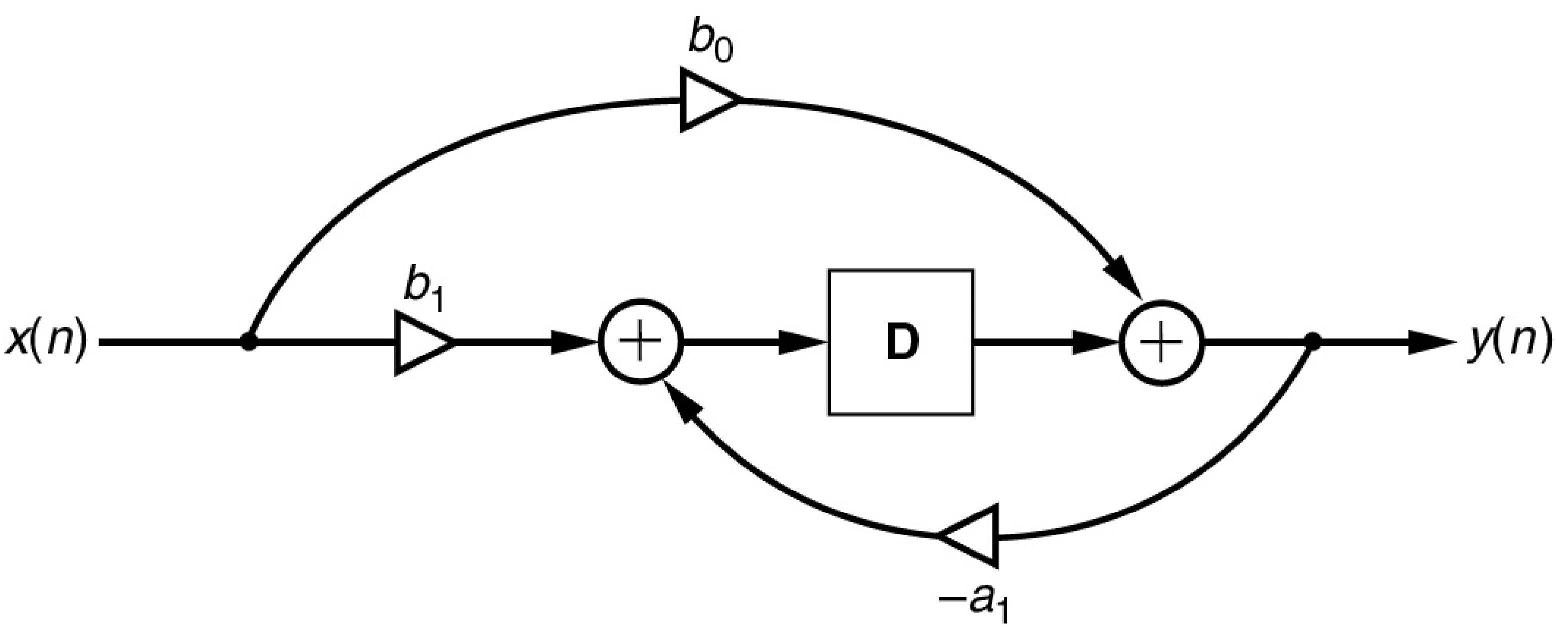
IIR (Infinite Impulse Response) Filter

First-order IIR filter.

$$y(n) + a_1 y(n-1) = b_0 x(n) + b_1 x(n-1)$$

$$\Rightarrow y(n) = b_0 x(n) + b_1 x(n-1) - a_1 y(n-1)$$

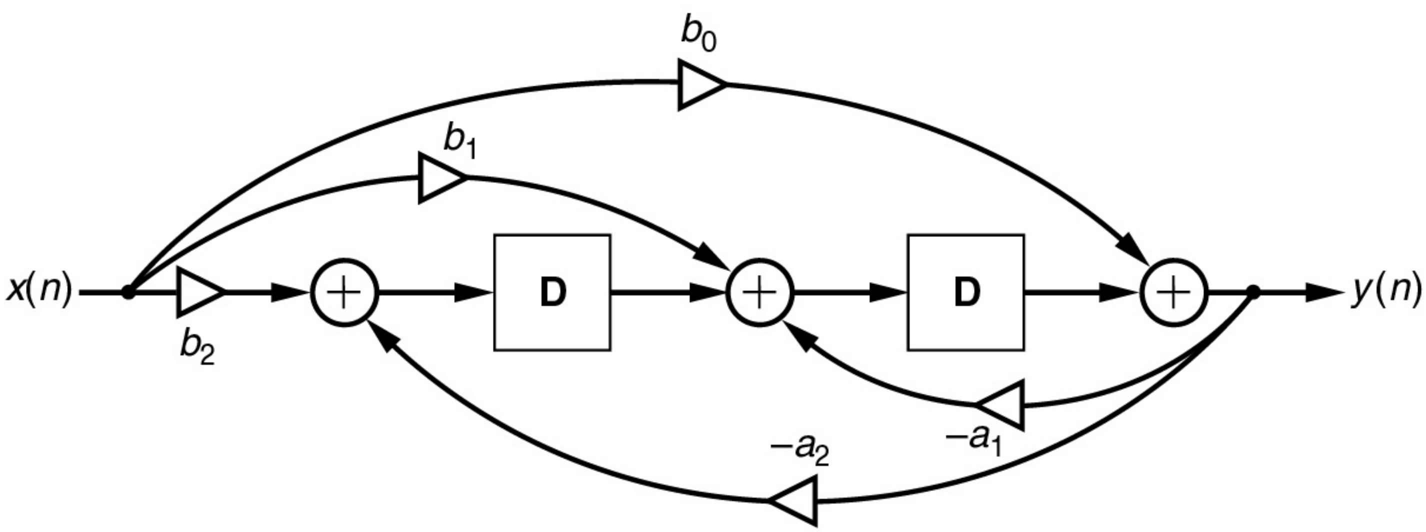
$$\Rightarrow y(n) = b_0 x(n) + \mathsf{D}\{b_1 x(n) - a_1 y(n)\}$$



IIR (Infinite Impulse Response) Filter

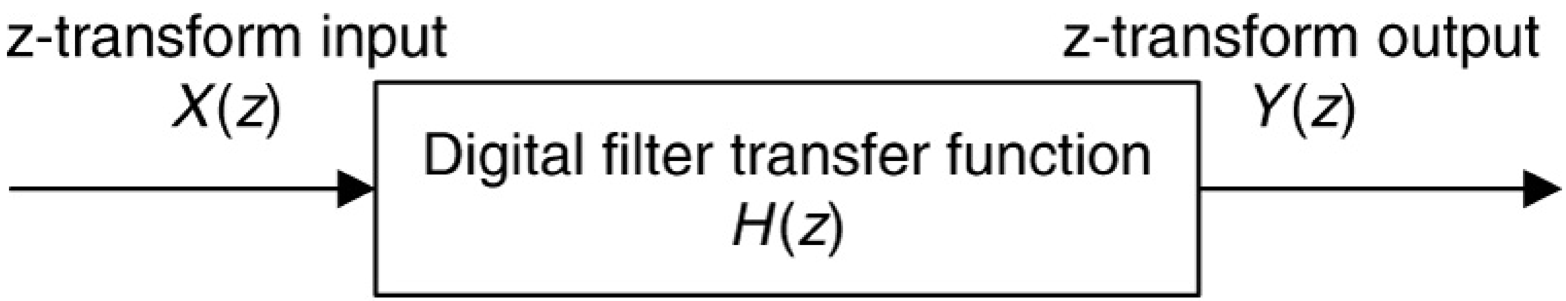
Second-order IIR filter.

$$y(n)+a_1y(n-1)+a_2y(n-2)=b_0x(n)+b_1x(n-1)+b_2x(n-2)$$



(b)

Transfer Function



Differential Equation:

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) - a_1 y(n-1) - \cdots - a_N y(n-N).$$

z- Transform:

$$Y(z) = b_0 X(z) + b_1 X(z) z^{-1} + \cdots + b_M X(z) z^{-M} - a_1 Y(z) z^{-1} - \cdots - a_N Y(z) z^{-N}$$

Transfer Function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Example: Transfer Function

Given: $$y(n) = x(n) - x(n-2) - 1.3y(n-1) - 0.36y(n-2)$$

z- Transform: $$Y(z) = X(z) - X(z)z^{-2} - 1.3Y(z)z^{-1} - 0.36Y(z)z^{-2}.$$

Rearrange: $$Y(z)(1 + 1.3z^{-1} + 0.36z^{-2}) = (1 - z^{-2})X(z)$$

Transfer Function: $$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$$

Given: $$H(z) = \frac{z^2 - 1}{z^2 + 1.3z + 0.36}$$

Rearrange: $$H(z) = \frac{(z^2 - 1)/z^2}{(z^2 + 1.3z + 0.36)/z^2} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}.$$

Differential Equation: $$y(n) = x(n) - x(n-2) - 1.3y(n-1) - 0.36y(n-2)$$

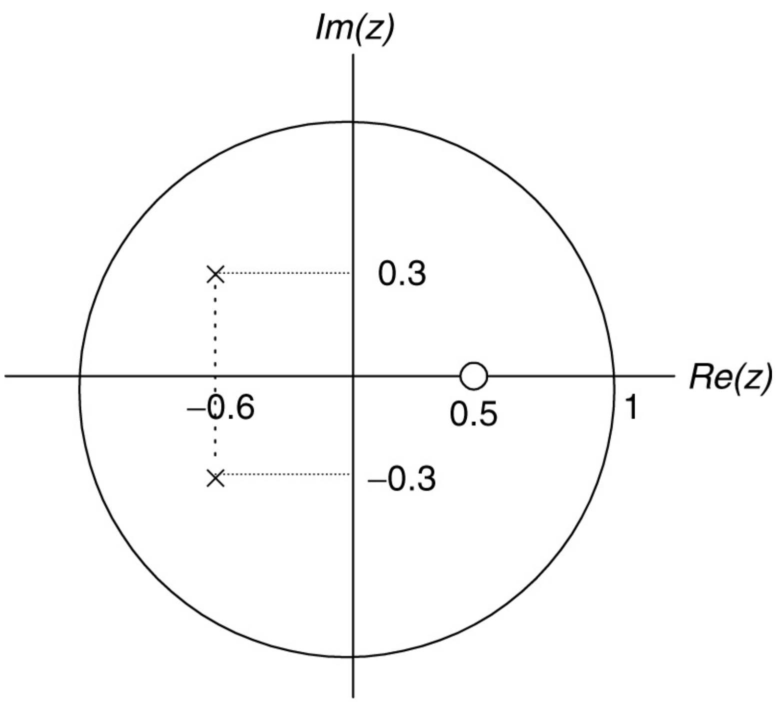
Pole – Zero from Transfer Function

$$H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}}$$

$$H(z) = \frac{(z^{-1} - 0.5z^{-2})z^2}{(1 + 1.2z^{-1} + 0.45z^{-2})z^2} = \frac{z - 0.5}{z^2 + 1.2z + 0.45} \qquad \frac{(z - 0.5)}{(z + 0.6 - j0.3)(z + 0.6 + j0.3)}$$

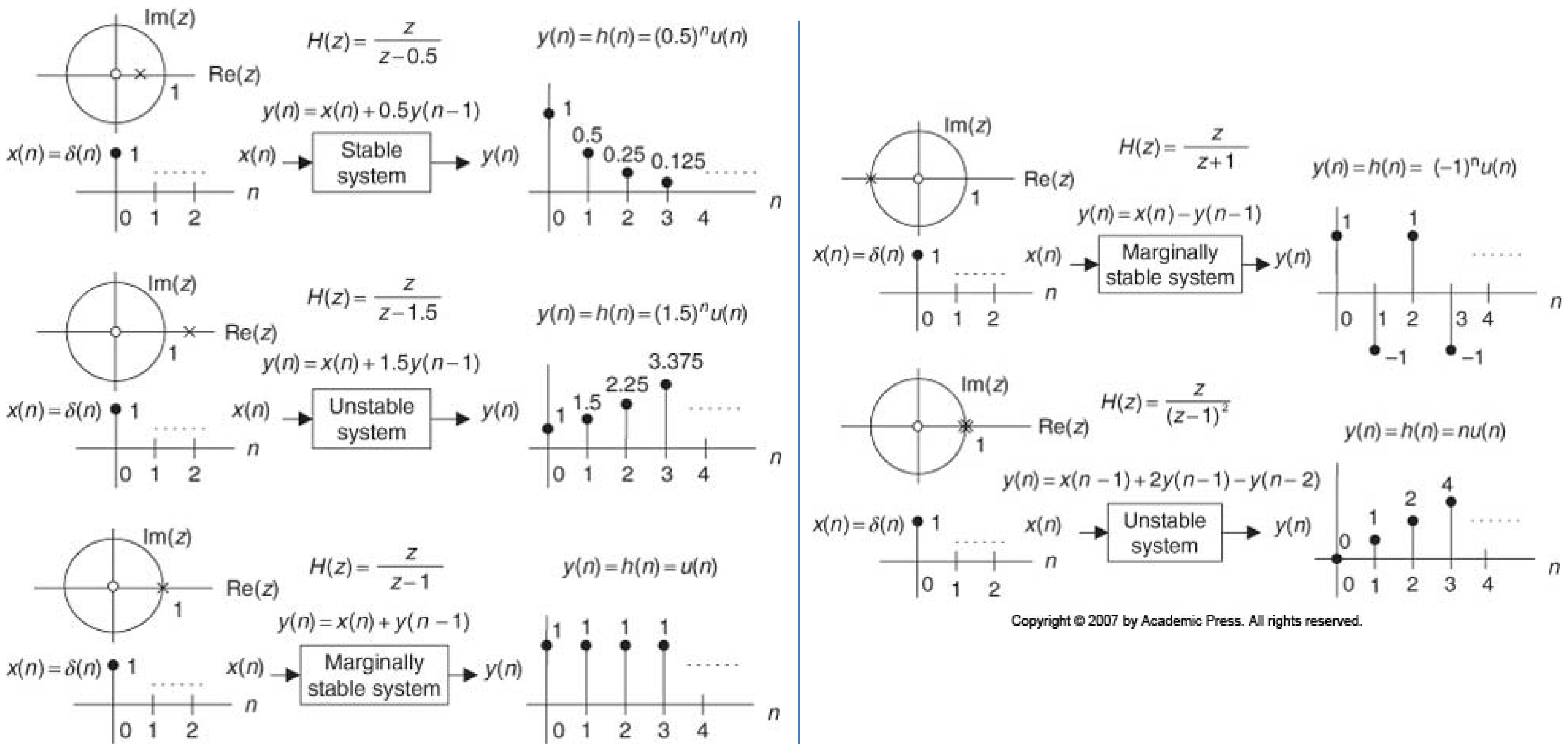
The system is stable.

The zeros do not affect system stability.



System Stability

Depends on poles' location



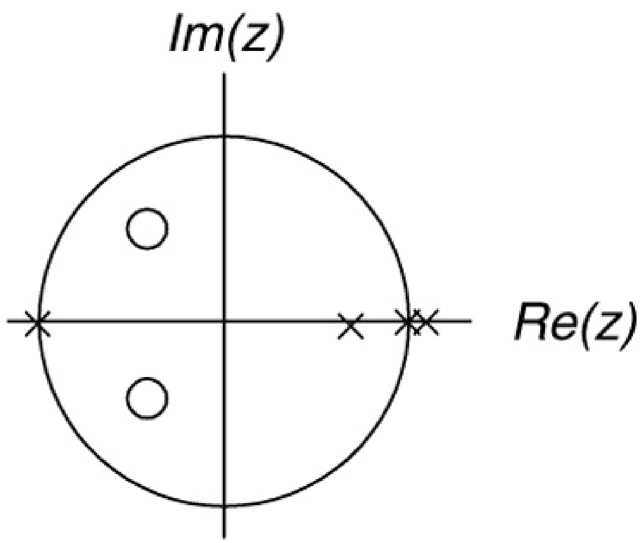
Example: System Stability

$$H(z) = \frac{z^2 + z + 0.5}{(z-1)^2(z+1)(z-0.6)}$$

Zeros are $z = -0.5 \pm j0.5$.

Poles: $z = 1, \ |z| = 1$; $z = 1, \ |z| = 1$; $z = -1, \ |z| = 1$; $z = 0.6, |z| = 0.6 < 1$.

Since the outermost pole is multiple order (2nd order) at z = 1 and is on the unit circle, the system is unstable.



Digital Filter: Frequency Response

$$H(z)|_{z=e^{j\omega T}} = H(e^{j\omega T}) = |H(e^{j\omega T})| \angle H(e^{j\omega T})$$

Magnitude frequency response

Phase response

Putting $\Omega = \omega T$

$$H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}} = |H(e^{j\Omega})| \angle H(e^{j\Omega})$$

Example: Given $y(n) = 0.5x(n) + 0.5x(n-1)$ Sampling rate = 8k Hz

Transfer function: $H(z) = \frac{Y(z)}{X(z)} = 0.5 + 0.5z^{-1}$.

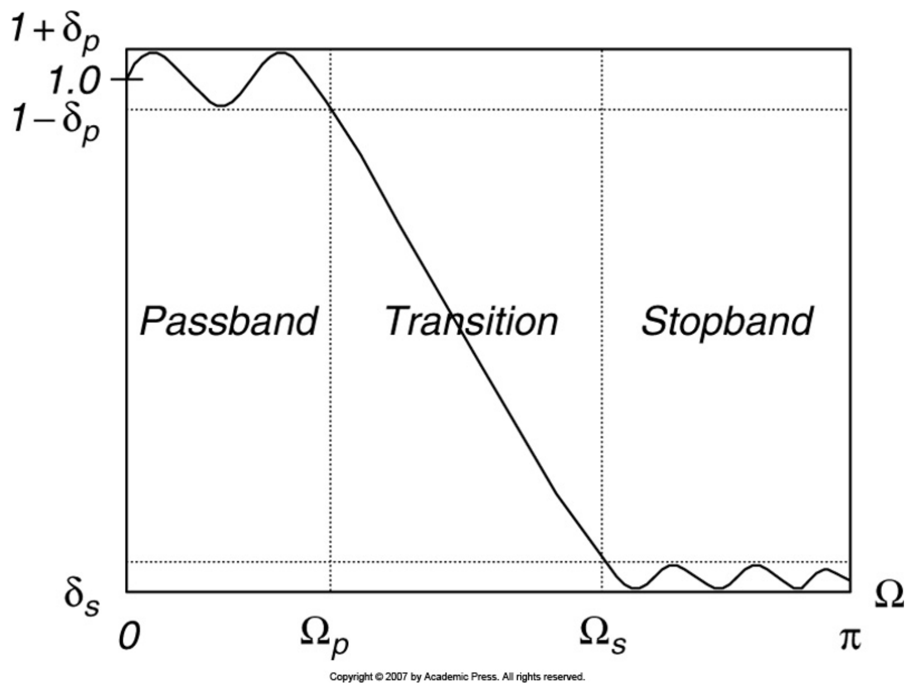
Frequency response: $H(e^{j\Omega}) = 0.5 + 0.5e^{-j\Omega}$

$= 0.5 + 0.5\cos(\Omega) - j0.5\sin(\Omega)$

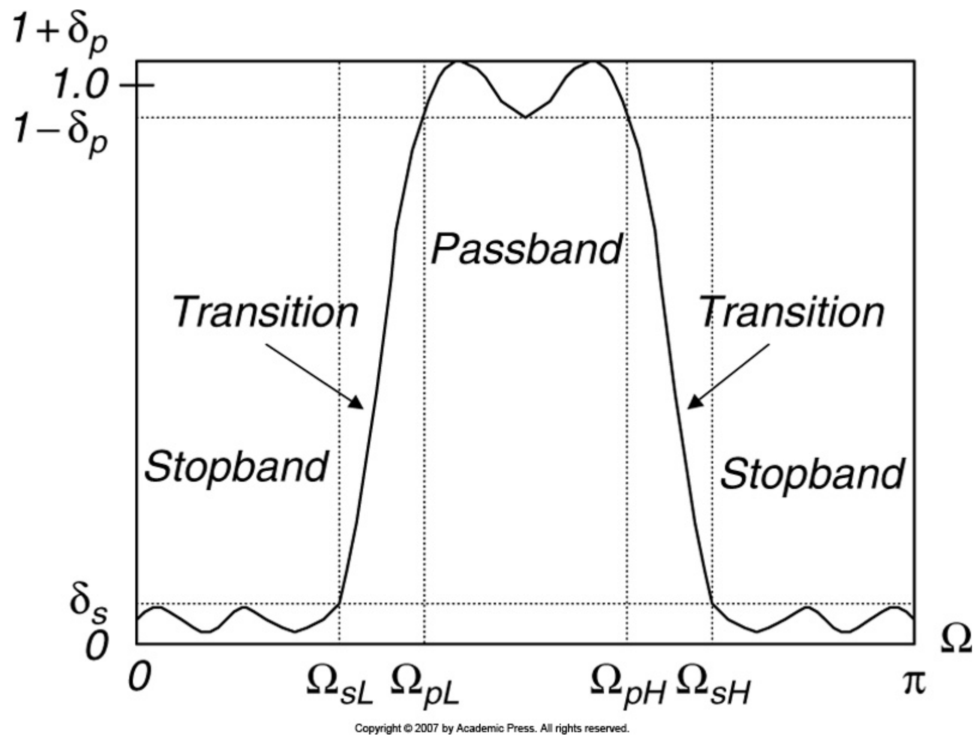
Complete Plot!

$|H(e^{j\Omega})| = \sqrt{(0.5 + 0.5\cos(\Omega))^2 + (0.5\sin(\Omega))^2}$ and $\angle H(e^{j\Omega}) = \tan^{-1}\left(\frac{-0.5\sin(\Omega)}{0.5 + 0.5\cos(\Omega)}\right)$

Digital Filter: Frequency Response – contd.



Low Pass Filter (LPF)

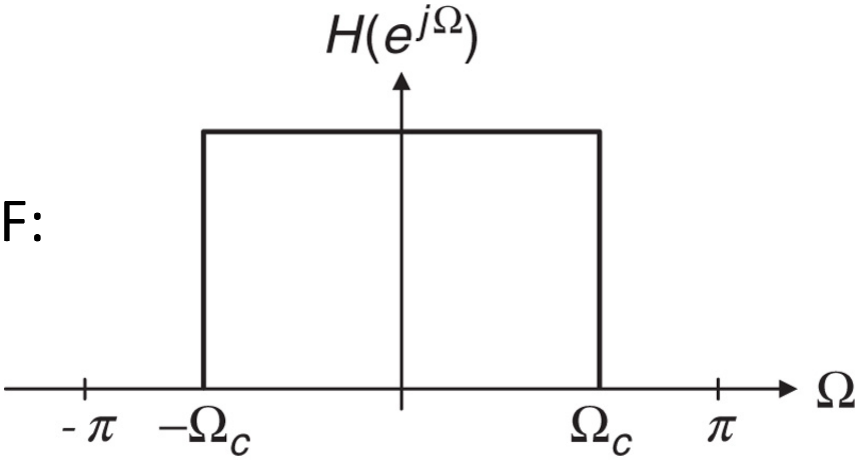


Band Pass Filter (BPF)

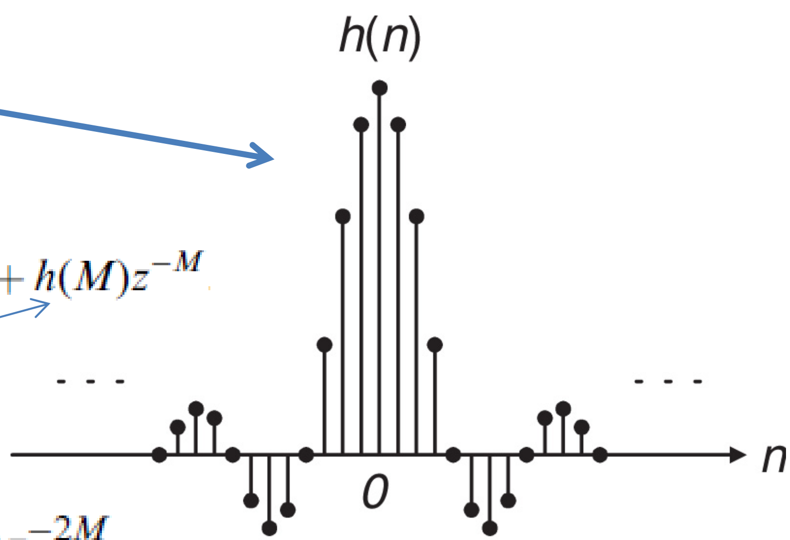
Matlab: Frequency Response [h, w] = freqz(B, A, N)

Impulse Response of FIR Filters

Frequency response of ideal LPF:



Impulse response of ideal LPF:



After truncating 2M+1 major components:

$$H(z) = h(M)z^{M} + \cdots + h(1)z^{1} + h(0) + h(1)z^{-1} + \cdots + h(M)z^{-M}$$

symmetric

Making causal,

$$H(z) = b_0 + b_1 z^{-1} + \cdots + b_{2M}(2M)z^{-2M}$$

Where, $b_n = h(n - M)$ for $n = 0, 1, \ldots, 2M.$

Ideal Low Pass Filter

Impulse Response:

$$h(n) = \begin{cases} \frac{\Omega_c}{\pi} & n = 0 \\ \frac{\sin(\Omega_c n)}{n\pi} \text{ for } n \neq 0 & -M \leq n \leq M \end{cases}$$

Example:

3-tap FIR LPF with cutoff freq. = 800 Hz and sampling rate = 8k Hz.

$$\Omega_c = 2\pi f_c T_s = 2\pi \times 800/8000 = 0.2\pi \text{ radians}$$

$$2M + 1 = 3$$

$$h(0) = \frac{\Omega_c}{\pi} \quad \text{for } n = 0$$

$$h(n) = \frac{\sin(\Omega_c n)}{n\pi} = \frac{\sin(0.2\pi n)}{n\pi}, \quad \text{for } n \neq 1$$

$$h(0) = \frac{0.2\pi}{\pi} = 0.2$$

$$h(1) = \frac{\sin[0.2\pi \times 1]}{1 \times \pi} = 0.1871$$

Using symmetry: $h(-1) = h(1) = 0.1871$

Ideal Low Pass Filter – contd.

Delaying h(n) by M = 1 sample,

$$b_0 = h(0-1) = h(-1) = 0.1871$$
$$b_1 = h(1-1) = h(0) = 0.2$$
$$b_2 = h(2-1) = h(1) = 0.1871$$

Filter coefficients

Transfer function $H(z) = 0.1871 + 0.2z^{-1} + 0.1871z^{-2}$

Differential Eq: $y(n) = 0.1871x(n) + 0.2x(n-1) + 0.1871x(n-2)$

Frequency response
$$H(e^{j\Omega}) = 0.1871 + 0.2e^{-j\Omega} + 0.1871e^{-j2\Omega}$$
$$= e^{-j\Omega}(0.1871e^{j\Omega} + 0.2 + 0.1871e^{-j\Omega})$$
$$= e^{-j\Omega}(0.2 + 0.3742\cos(\Omega))$$

$e^{jx} + e^{-jx} = 2\cos(x)$

Magnitude: $|H(e^{j\Omega})| = |0.2 + 0.3472\cos\Omega|$

Complete Plot!

Phase: and $\angle H(e^{j\Omega}) = \begin{cases} -\Omega & \text{if } 0.2 + 0.3472\cos\Omega > 0 \\ -\Omega + \pi & \text{if } 0.2 + 0.3472\cos\Omega < 0 \end{cases}$

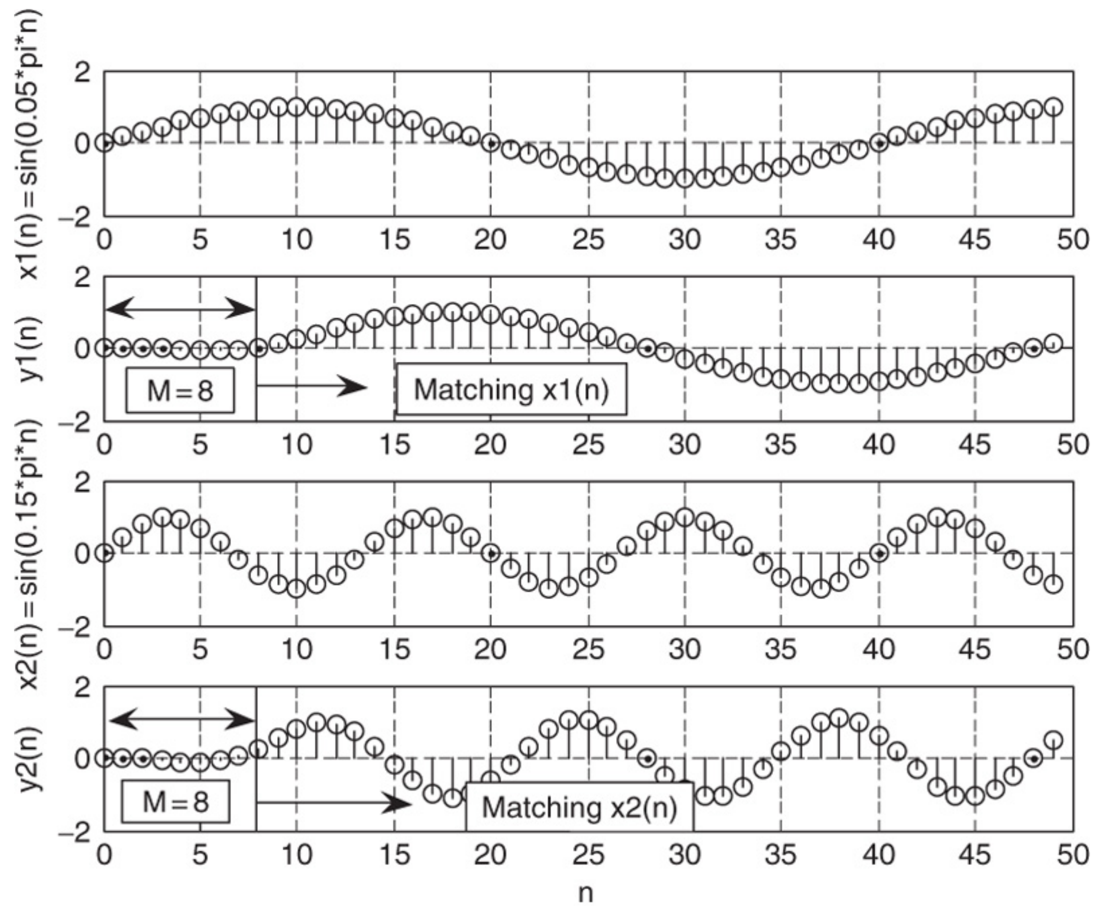
Linear Phase

If filter has linear phase property, the output will simply be a delayed version of input.

Let, 17-tap FIR filter with linear phase property.

8 samples delay

8 samples delay



Nonlinear Phase

Input: $x(n) = x_1(n) + x_2(n) = \sin(0.05\pi n)u(n) - \frac{1}{3}\sin(0.15\pi n)u(n)$

Linear phase filter output: $y_1(n) = \sin[0.05\pi(n-8)] - \frac{1}{3}\sin[0.15\pi(n-8)]$

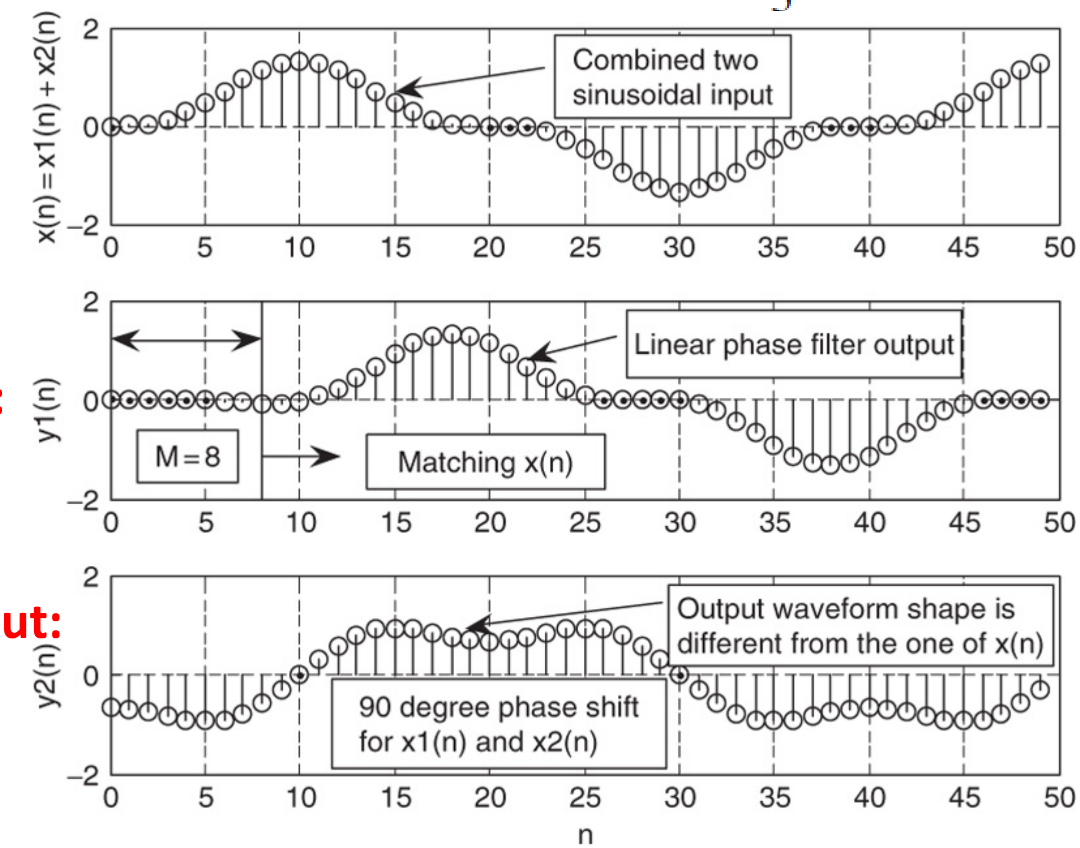
90 d phase delay filter output: $y_2(n) = \sin(0.05\pi n - \pi/2) - \frac{1}{3}\sin(0.15\pi n - \pi/2)$

Input:

Linear phase filter output:

90 d phase delay filter output:

Distorted!



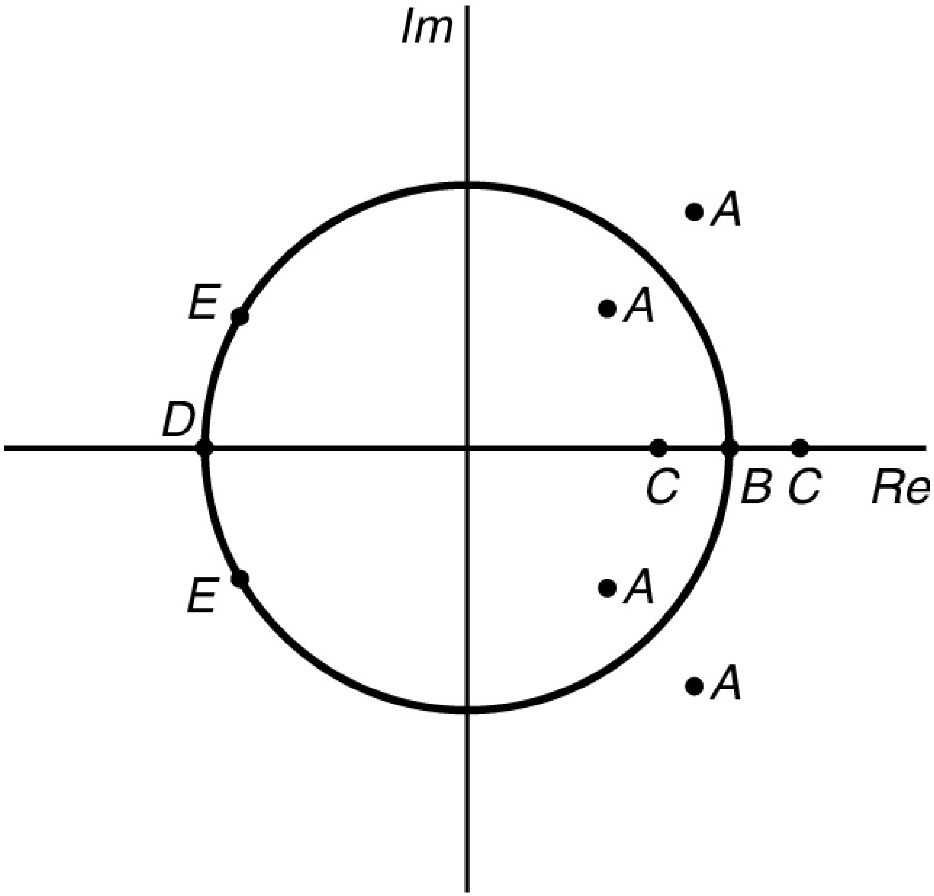
Linear Phase: Zero Placement

• A single zero can be either at z = 1 or z = -1. (B or D)

• Real zeros not on the unit circle always occur in pairs with r and r^{-1}. (C)

• If the zero is complex, its conjugate is also zero. (E) [on the unit circle]

• Complex zeros not on the unit circle always occur in quadruples with r and r^{-1}. (A)



Example: FIR Filtering With Window Method

Problem: Design a 5-tap FIR band reject filter with a lower cutoff frequency of 2,000 Hz, an upper cutoff frequency of 2,400 Hz, and a sampling rate of 8,000 Hz using the Hamming window method.

Solution:

$$\Omega_L = 2\pi f_L T = 2\pi \times 2000/8000 = 0.5\pi \text{ radians}$$

$$\Omega_H = 2\pi f_H T = 2\pi \times 2400/8000 = 0.6\pi \text{ radians}$$

$$2M + 1 = 5$$

$M = 2$

$$h(n) = \begin{cases} \frac{\pi - \Omega_H + \Omega_L}{\pi} & n = 0 \\ -\frac{\sin(\Omega_H n)}{n\pi} + \frac{\sin(\Omega_L n)}{n\pi} \quad n \neq 0 & -2 \leq n \leq 2. \end{cases}$$

$$h(0) = \frac{\pi - \Omega_H + \Omega_L}{\pi} = \frac{\pi - 0.6\pi + 0.5\pi}{\pi} = 0.9$$

$$h(1) = \frac{\sin[0.5\pi \times 1]}{1 \times \pi} - \frac{\sin[0.6\pi \times 1]}{1 \times \pi} = 0.01558$$

$$h(2) = \frac{\sin[0.5\pi \times 2]}{2 \times \pi} - \frac{\sin[0.6\pi \times 2]}{2 \times \pi} = 0.09355$$

$\Rightarrow$

$$h(-1) = h(1) = 0.01558$$

$$h(-2) = h(2) = 0.09355$$

Symmetry

Example: Window Method – contd.

Hamming window function

$$w_{ham}(0) = 0.54 + 0.46\cos\left(\frac{0 \times \pi}{2}\right) = 1.0$$

$$w_{ham}(1) = 0.54 + 0.46\cos\left(\frac{1 \times \pi}{2}\right) = 0.54$$

$$w_{ham}(2) = 0.54 + 0.46\cos\left(\frac{2 \times \pi}{2}\right) = 0.08$$

⇨

$$w_{ham}(-1) = w_{ham}(1) = 0.54$$
$$w_{ham}(-2) = w_{ham}(2) = 0.08.$$

Symmetry

Windowed impulse response

$$h_w(0) = h(0)w_{ham}(0) = 0.9 \times 1 = 0.9$$
$$h_w(1) = h(1)w_{ham}(1) = 0.01558 \times 0.54 = 0.00841$$
$$h_w(2) = h(2)w_{ham}(2) = 0.09355 \times 0.08 = 0.00748$$
$$h_w(-1) = h(-1)w_{ham}(-1) = 0.00841$$
$$h_w(-2) = h(-2)w_{ham}(-2) = 0.00748$$

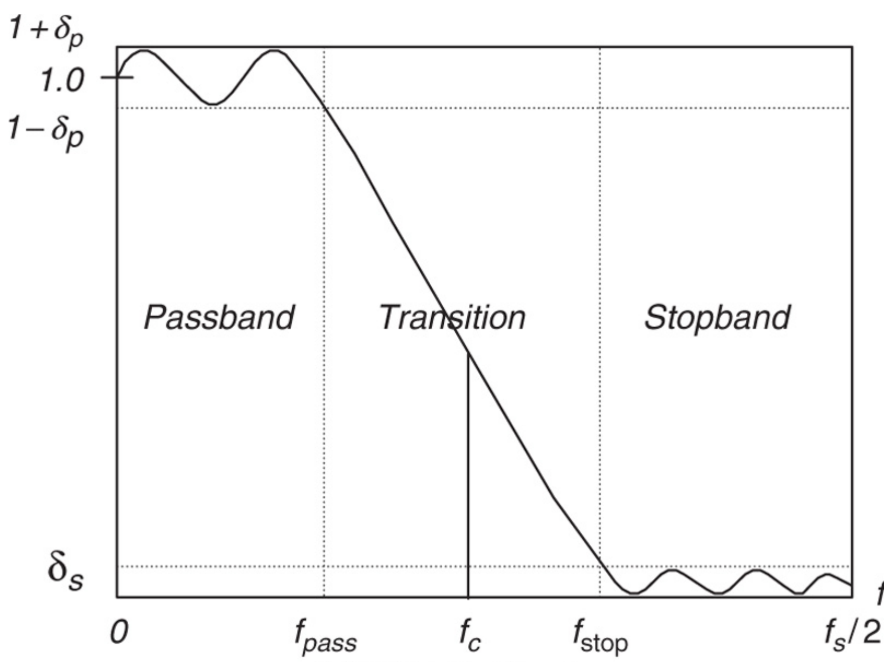
By delaying $h_w(n)$ by M = 2 samples,

$$b_0 = b_4 = 0.00748,\ b_1 = b_3 = 0.00841,\ \text{and}\ b_2 = 0.9$$

$$H(z) = 0.00748 + 0.00841z^{-1} + 0.9z^{-2} + 0.00841z^{-3} + 0.00748z^{-4}$$

FIR Filter Length Estimation

Window Type	Window Function $w(n)$, $-M \le n \le M$	Window Length, N	Passband Ripple (dB)	Stopband Attenuation (dB)
Rectangular	1	$N = 0.9/\Delta f$	0.7416	21
Hanning	$0.5 + 0.5\cos\left(\frac{\pi n}{M}\right)$	$N = 3.1/\Delta f$	0.0546	44
Hamming	$0.54 + 0.46\cos\left(\frac{\pi n}{M}\right)$	$N = 3.3/\Delta f$	0.0194	53
Blackman	$0.42 + 0.5\cos\left(\frac{n\pi}{M}\right) + 0.08\cos\left(\frac{2n\pi}{M}\right)$	$N = 5.5/\Delta f$	0.0017	74



$$\Delta f = |f_{stop} - f_{pass}|/f_s$$

$$f_c = (f_{pass} + f_{stop})/2$$

$$\delta_p\ dB = 20 \cdot \log_{10}(1 + \delta_p)$$

$$\delta_s\ dB = -20\log_{10}(\delta_s)$$

Example: FIR Filter Length Estimation

Problem:

Design a BPF with

Lower stopband = 0–500 Hz
Passband = 1,600–2,300 Hz
Upper stopband = 3,500–4,000 Hz
Stopband attenuation = 50 dB
Passband ripple = 0.05 dB
Sampling rate = 8,000 Hz

Use Hamming window

Solution:

$$\Delta f_1 = |1600 - 500|/8000 = 0.1375$$

$$\Delta f_2 = |3500 - 2300|/8000 = 0.15$$

⇒

$$N_1 = 3.3/0.1375 = 24$$

$$N_2 = 3.3/0.15 = 22$$

Choose nearest higher odd N = 25

Cutoff frequencies:

$$f_1 = (1600 + 500)/2 = 1050\,\text{Hz}$$

$$f_2 = (3500 + 2300)/2 = 2900\,\text{Hz}.$$

Normalized ⇒

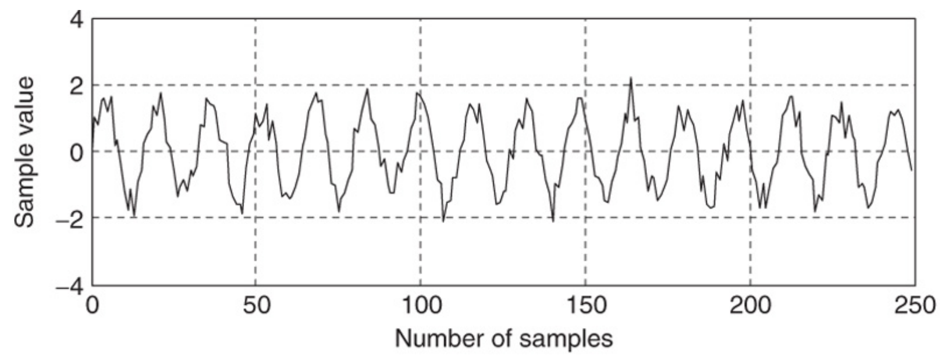
$$\Omega_L = \frac{1050 \times 2\pi}{8000} = 0.2625\pi \text{ radians}$$

$$\Omega_H = \frac{2900 \times 2\pi}{8000} = 0.725\pi \text{ radians}$$

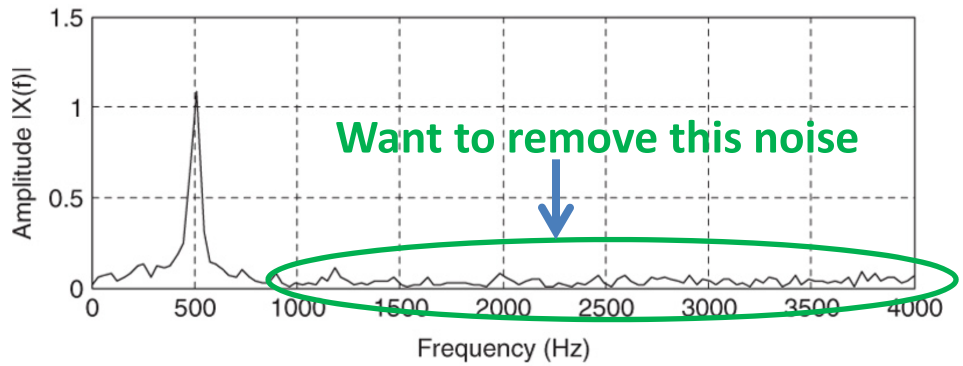
Now design the filter with hint from slide 14.

Application: Noise Reduction

Input waveform: sinusoid + broadband noise



Spectrum:



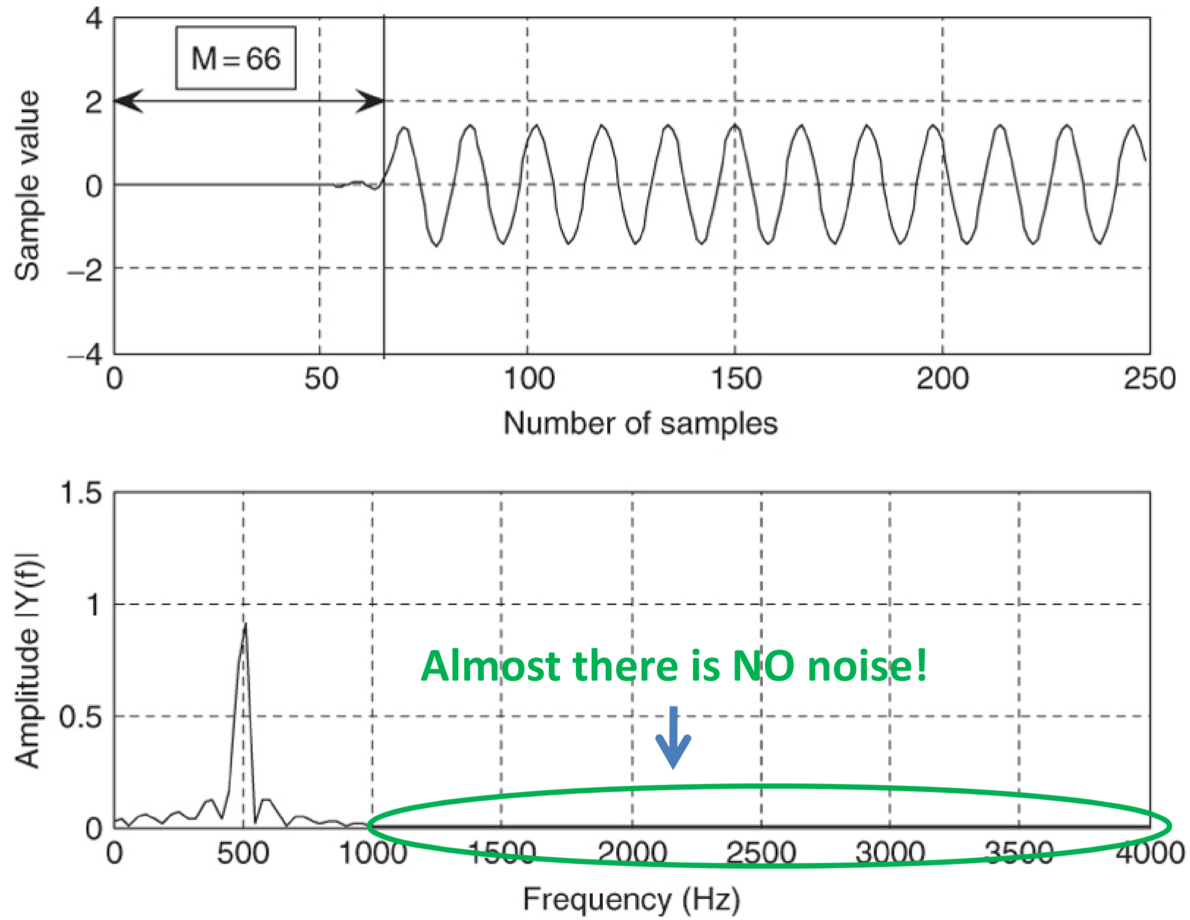
Specification: LPF

Pass band frequency [0 – 800 Hz]
Stop band frequency [1000 – 4000 Hz]

Pass band ripple < 0.02 dB
Stop band attenuation = 50 dB

Application: Noise Reduction –contd.

133- tap FIR filter, so a delay of 66



However, noise reduction in real world scenario is not so easy!

Frequency Sampling Design Method

Simple to design

Filter length = 2M + 1

$$H_k \text{ at } \Omega_k = \frac{2\pi k}{(2M+1)} \quad \text{for } k = 0, 1, \ldots, M$$

Magnitude response in the range [0 ~ π]

Calculate FIR filter coefficients:

$$h(n) = \frac{1}{2M+1}\left\{H_0 + 2\sum_{k=1}^{M} H_k \cos\left(\frac{2\pi k(n-M)}{2M+1}\right)\right\}$$

for $n = 0, 1, \ldots, M.$

Use the symmetry:

$$h(n) = h(2M - n) \text{ for } n = M+1, \ldots, 2M.$$

Example: Frequency Sampling Design Method

Problem: Design a linear phase lowpass FIR filter with 7 taps and a cutoff frequency of $\Omega_c = 0.3\pi$ radian using the frequency sampling method.

Solution:

$N = 2M + 1 = 7 \rightarrow M = 3 \Rightarrow \Omega_k = \frac{2\pi}{7}k$ radians, $k = 0, 1, 2, 3.$

for $\Omega_0 = 0$ radians, $H_0 = 1.0$

for $\Omega_1 = \frac{2}{7}\pi$ radians, $H_1 = 1.0$

for $\Omega_2 = \frac{4}{7}\pi$ radians, $H_2 = 0.0$

for $\Omega_3 = \frac{6}{7}\pi$ radians, $H_3 = 0.0.$

$$\Rightarrow h(n) = \frac{1}{7}\left\{1 + 2\sum_{k=1}^{3} H_k \cos\left[2\pi k(n-3)/7\right]\right\}, \; n = 0, 1, \ldots, 3.$$

$$= \frac{1}{7}\{1 + 2\cos\left[2\pi(n-3)/7\right]\}$$

$$h(0) = \frac{1}{7}\{1 + 2\cos(-6\pi/7)\} = -0.11456$$

$$h(1) = \frac{1}{7}\{1 + 2\cos(-4\pi/7)\} = 0.07928$$

$$h(2) = \frac{1}{7}\{1 + 2\cos(-2\pi/7)\} = 0.32100$$

$$h(3) = \frac{1}{7}\{1 + 2\cos(-0 \times \pi/7)\} = 0.42857.$$

By symmetry:

$$h(4) = h(2) = 0.32100$$

$$h(5) = h(1) = 0.07928$$

$$h(6) = h(0) = -0.11456.$$

Coefficient Quantization Effect

Filter coefficients are usually truncated or rounded off for the application.

Transfer function with infinite precision:

$$H(z) = \sum_{n=0}^{K} b_n z^{-n} = b_0 + b_1 z^{-1} + \ldots + b_K z^{-K}$$

Transfer function with quantized precision:

$$H^q(z) = \sum_{n=0}^{K} b_n^q z^{-n} = b_0^q + b_1^q z^{-1} + \ldots + b_K^q z^{-K}$$

Error of the magnitude frequency response:

$$\left| H(e^{j\Omega}) - H^q(e^{j\Omega}) \right| = \sum_{n=0}^{K} \left| (b_n - b_n^q) e^{-jn\Omega} \right|$$

$$< \sum_{n=0}^{K} \left| b_n - b_n^q \right| < (K+1) \cdot 2^{-B-1}$$

K = tap

Example

25 – tap FIR filter; 7 bits used for fraction

Error is bounded by < 25 / 256 = 0.0977

Let infinite precision coeff. = 0.00759455135346

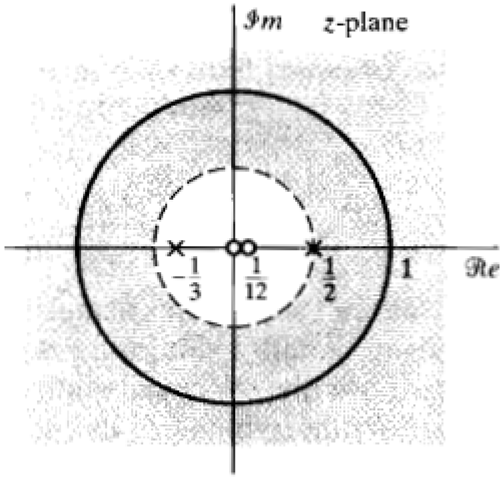
$$0.00759455135346 \times 2^7 = 0.9721 = 1 (\text{rounded up to the integer})$$

Quantized coeff. = 1 / 2^7 = 0.0078125

Complementary Example - I

Consider a signal that is the sum of two real exponentials:

$$x[n] = \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n].$$



The z-transform is then

$$X(z) = \sum_{n=-\infty}^{\infty} \left\{ \left(\tfrac{1}{2}\right)^n u[n] + \left(-\tfrac{1}{3}\right)^n u[n] \right\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n u[n] z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\tfrac{1}{3}\right)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\tfrac{1}{3} z^{-1}\right)^n$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2\left(1 - \frac{1}{12} z^{-1}\right)}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{3} z^{-1}\right)}$$

$$= \frac{2z\left(z - \frac{1}{12}\right)}{\left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)}.$$

z transform of $a^n u(n) = \dfrac{z}{z-a}$

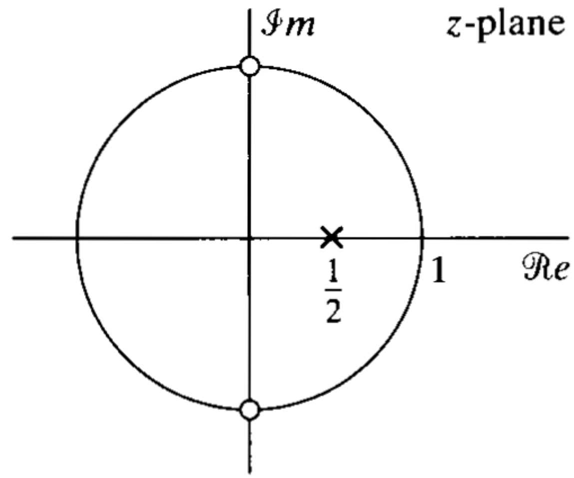
$$= \frac{1}{1 - a z^{-1}}$$

Complementary Example - II

Given:

$$y[n] = \left(\tfrac{1}{2}\right)^n x[n]$$

Given:

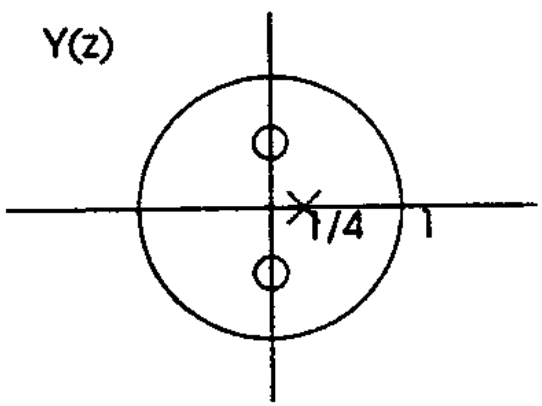


$$X(z) = \frac{z^2+1}{z-\frac{1}{2}}$$

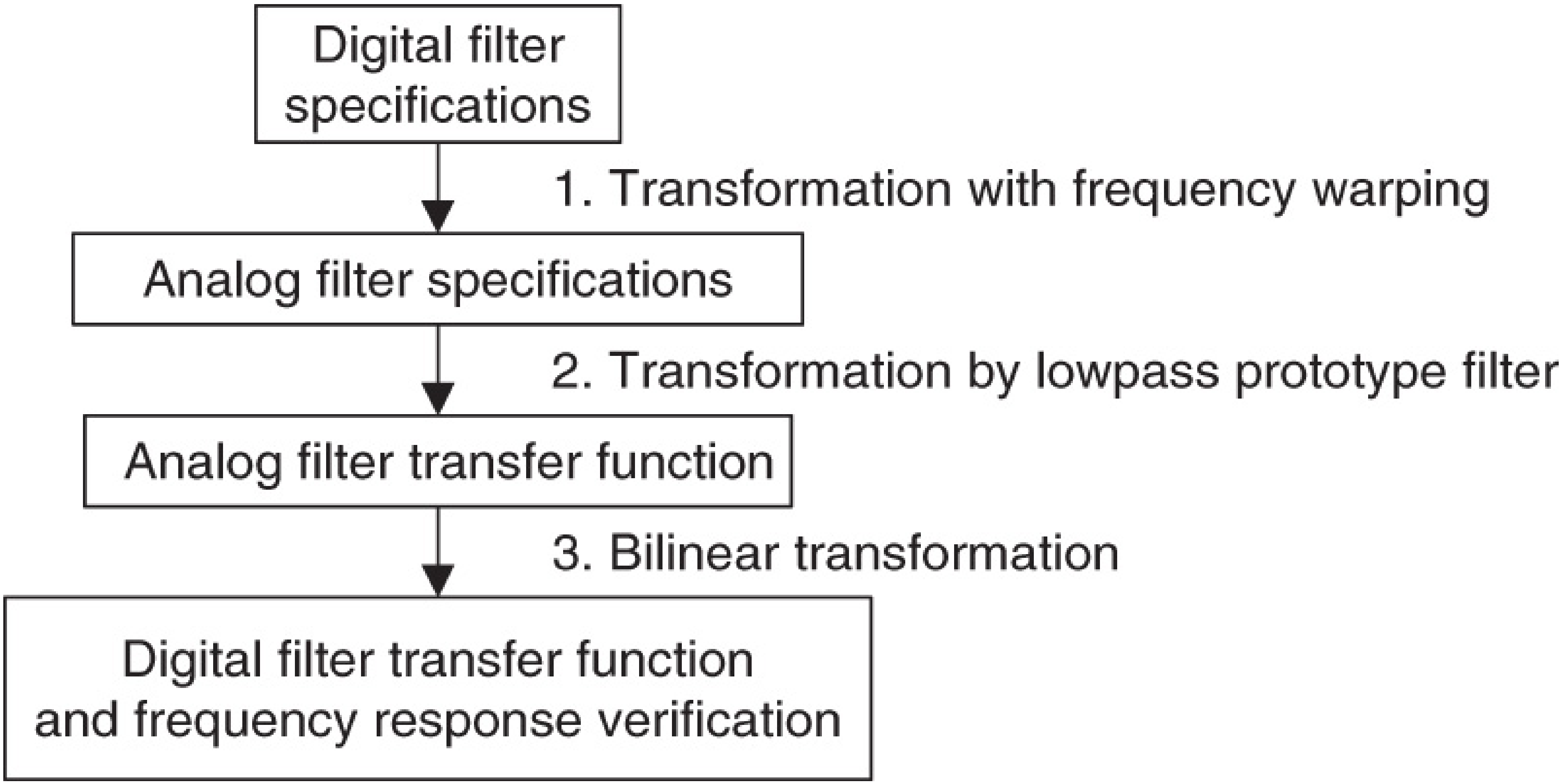
$$y[n] = \left(\frac{1}{2}\right)^n x[n] \Rightarrow Y(z) = X(2z) = \frac{4z^2+1}{2z-\frac{1}{2}}$$

zeros $\pm\frac{1}{2}j$

poles $\frac{1}{4}, \infty$



IIR Filter Design: Bilinear Transformation Method



Bilinear Transformation Method

For LPF and HPF: $\omega_a = \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right)$

For BPF and BRF: $\omega_{al} = \frac{2}{T}\tan\left(\frac{\omega_l T}{2}\right),\ \omega_{ah} = \frac{2}{T}\tan\left(\frac{\omega_h T}{2}\right)$

$$\omega_0 = \sqrt{\omega_{al}\omega_{ah}},\ W = \omega_{ah} - \omega_{al}$$

Frequency Warping

From LPF to LPF: $H(s) = H_P(s)\big|_{s=\frac{s}{\omega_a}}$

From LPF to HPF: $H(s) = H_P(s)\big|_{s=\frac{\omega_a}{s}}$

From LPF to BPF: $H(s) = H_P(s)\big|_{s=\frac{s^2+\omega_0^2}{sW}}$

From LPF to BRF: $H(s) = H_P(s)\big|_{s=\frac{sW}{s^2+\omega_0^2}}$

Prototype Transformation

Obtained Transfer Function: $H(z) = H(s)\big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$

Example 1: Bilinear Transformation Method

Problem: Design a first-order digital highpass Chebyshev filter with a cutoff frequency of 3 kHz and 1 dB ripple on passband using a sampling frequency of 8,000 Hz.

Solution:

$$\omega_d = 2\pi f = 2\pi(3000) = 6000\pi \text{ rad/sec, and } T = 1/f_s = 1/8000 \text{ sec.}$$

$$\omega_a = \frac{2}{T}\tan\left(\frac{\omega_d T}{2}\right) = 16000 \times \tan\left(\frac{6000\pi/8000}{2}\right) = 3.8627 \times 10^4 \text{ rad/sec.}$$

First-order LP Chebyshev filter prototype: $H_P(s) = \dfrac{1.9652}{s + 1.9625}$

Applying transformation LPF to HPF: $H(s) = H_P(s)\big|_{\frac{\omega_a}{s}} = \dfrac{1.9652}{\frac{\omega_a}{s} + 1.9652} = \dfrac{1.9652s}{1.9652s + 3.8627 \times 10^4}$

$$\Rightarrow H(s) = \frac{s}{s + 1.9656 \times 10^4}$$

Applying BLT: $H(z) = \dfrac{s}{s + 1.9656 \times 10^4}\Bigg|_{s=16000(z-1)/(z+1)}$

$$H(z) = \frac{0.4487 - 0.4487z^{-1}}{1 + 0.1025z^{-1}}$$

Example 2: Bilinear Transformation Method

Problem: Design a second-order digital bandpass Butterworth filter with the following specifications:

- an upper cutoff frequency of 2.6 kHz and
- a lower cutoff frequency of 2.4 kHz,
- a sampling frequency of 8,000 Hz.

Solution:

$$\omega_h = 2\pi f_h = 2\pi(2600) = 5200\pi \text{ rad/sec}$$

$$\omega_l = 2\pi f_l = 2\pi(2400) = 4800\pi \text{ rad/sec, and } T = 1/f_s = 1/8000 \text{ sec.}$$

$$\omega_{ah} = \frac{2}{T}\tan\left(\frac{\omega_h T}{2}\right) = 16000 \times \tan\left(\frac{5200\pi/8000}{2}\right) = 2.6110 \times 10^4 \text{ rad/sec}$$

$$\omega_{al} = 16000 \times \tan\left(\frac{\omega_l T}{2}\right) = 16000 \times \tan(0.3\pi) = 2.2022 \times 10^4 \text{ rad/sec}$$

$$W = \omega_{ah} - \omega_{al} = 26110 - 22022 = 4088 \text{ rad/sec}$$

$$\omega_0^2 = \omega_{ah} \times \omega_{al} = 5.7499 \times 10^8$$

A first-order LPF prototype will produce second-order BPF prototype.

Example 2: Bilinear Transformation Method

Contd.

1st order LPF prototype: $$H_P(s) = \frac{1}{s+1}$$

Applying transformation LPF to BPF: $$H(s) = H_P(s)\Big|_{\frac{s^2+\omega_0^2}{sW}} = \frac{Ws}{s^2 + Ws + \omega_0^2} = \frac{4088s}{s^2 + 4088s + 5.7499 \times 10^8}$$

Applying BLT: $$H(z) = \frac{4088s}{s^2 + 4088s + 5.7499 \times 10^8}\Bigg|_{s=16000(z-1)/(z+1)}$$

$$\Downarrow$$

$$\boxed{H(z) = \frac{0.0730 - 0.0730z^{-2}}{1 + 0.7117z^{-1} + 0.8541z^{-2}}}$$

Pole Zero Placement Method

Second-Order BPF Design

r: controls bandwidth

θ: controls central frequency

Location of poles & zeros:

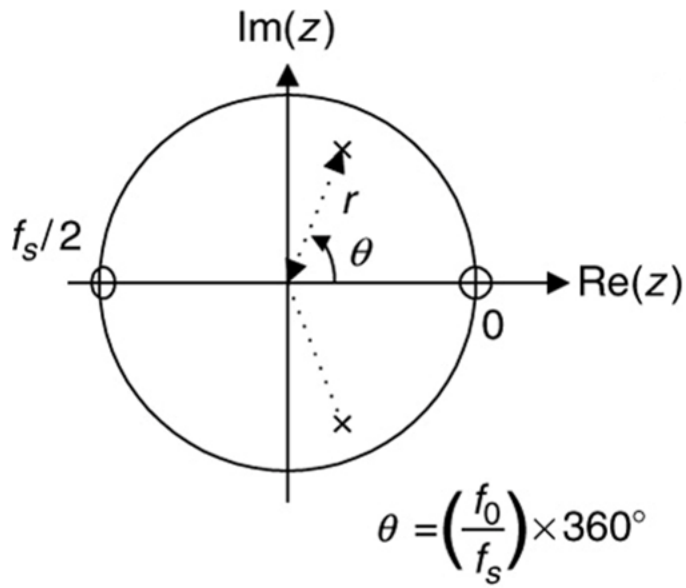
controls magnitude

Location of pole:

determines stability

Number of zero:

determines phase linearity



$$r \approx 1 - (BW_{3dB}/f_s) \times \pi \qquad \theta = \left(\frac{f_0}{f_s}\right) \times 360^0$$

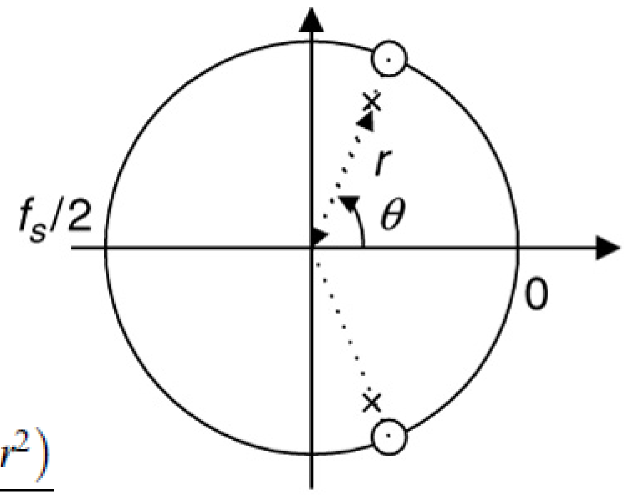
$$K = \frac{(1-r)\sqrt{1 - 2r\cos 2\theta + r^2}}{2|\sin\theta|}$$

$$H(z) = \frac{K(z-1)(z+1)}{(z - re^{j\theta})(z - re^{-j\theta})} = \frac{K(z^2-1)}{(z^2 - 2rz\cos\theta + r^2)}$$

Pole Zero Placement Method

Second-Order BRF Design

$$r \approx 1 - (BW_{3dB}/f_s) \times \pi \qquad \theta = \left(\frac{f_0}{f_s}\right) \times 360^0$$



$$H(z) = \frac{K(z - e^{j\theta})(z + e^{-j\theta})}{(z - re^{j\theta})(z - re^{-j\theta})} = \frac{K(z^2 - 2z\cos\theta + 1)}{(z^2 - 2rz\cos\theta + r^2)}$$

$$K = \frac{(1 - 2r\cos\theta + r^2)}{(2 - 2\cos\theta)}$$

Example

Sampling rate = 8,000 Hz

3 dB bandwidth: $BW = 100$ Hz

Narrow passband centered at $f_0 = 1{,}500$ Hz.

$$K = \frac{(1 - 2 \times 0.9607\cos 67.5^0 + 0.9607^2)}{(2 - 2\cos 67.5^0)} = 0.9620$$

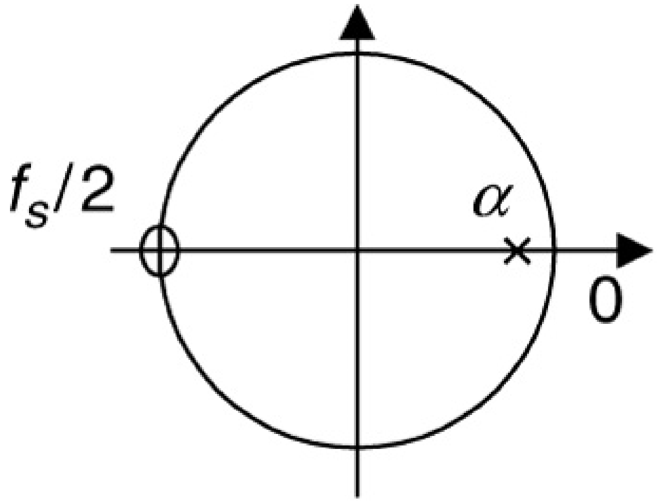
$$r \approx 1 - (100/8000) \times \pi = 0.9607$$

$$\theta = \left(\frac{1500}{8000}\right) \times 360^0 = 67.5^0$$

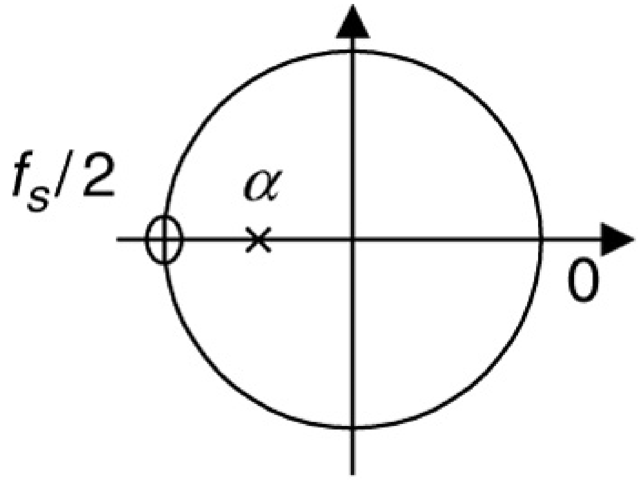
$$H(z) = \frac{0.9620(z^2 - 2z\cos 67.5^0 + 1)}{(z^2 - 2 \times 0.9607z\cos 67.5^0 + 0.9607^2)} = \frac{0.9620 - 0.7363z^{-1} + 0.9620z^{-2}}{1 - 0.7353z^{-1} + 0.9229}.$$

Pole Zero Placement Method

First-Order LPF Design



When $f_c < f_s/4$, $\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi$.



When $f_c > f_s/4$, $\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi)$.

$$H(z) = \frac{K(z+1)}{(z-\alpha)}$$

$$K = \frac{(1-\alpha)}{2}$$

Example

Sampling rate $= 8{,}000$ Hz

3 dB cutoff frequency: $f_c = 100$ Hz

100 Hz < $f_s/4 = 2{,}000$ Hz

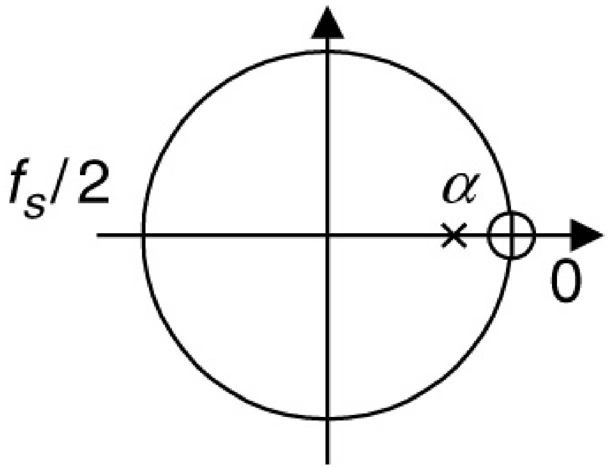
$$\alpha \approx 1 - 2 \times (100/8000) \times \pi = 0.9215$$

$$K = \frac{(1-0.9215)}{2} = 0.03925$$

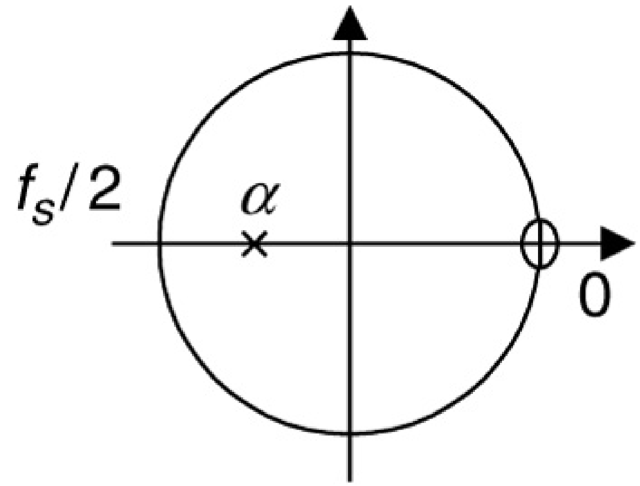
$$H(z) = \frac{0.03925(z+1)}{(z-0.9215)} = \frac{0.03925 + 0.03925z^{-1}}{1 - 0.9215z^{-1}}.$$

Pole Zero Placement Method

First-Order **HPF** Design



When $f_c < f_s/4$, $\alpha \approx 1 - 2 \times (f_c/f_s) \times \pi$



When $f_c > f_s/4$, $\alpha \approx -(1 - \pi + 2 \times (f_c/f_s) \times \pi)$

$$H(z) = \frac{K(z-1)}{(z-\alpha)} \qquad K = \frac{(1+\alpha)}{2}$$

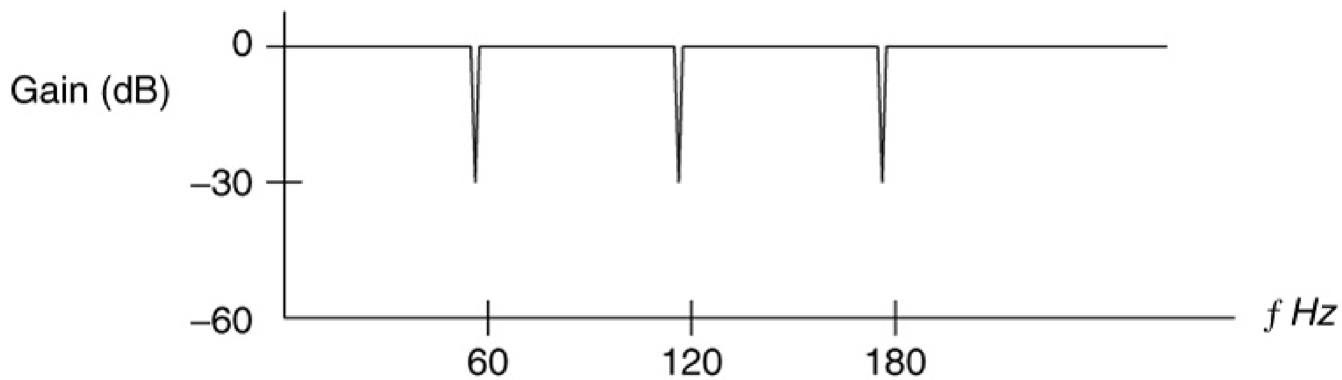
Practice examples.

Application: 60 – Hz Hum Eliminator

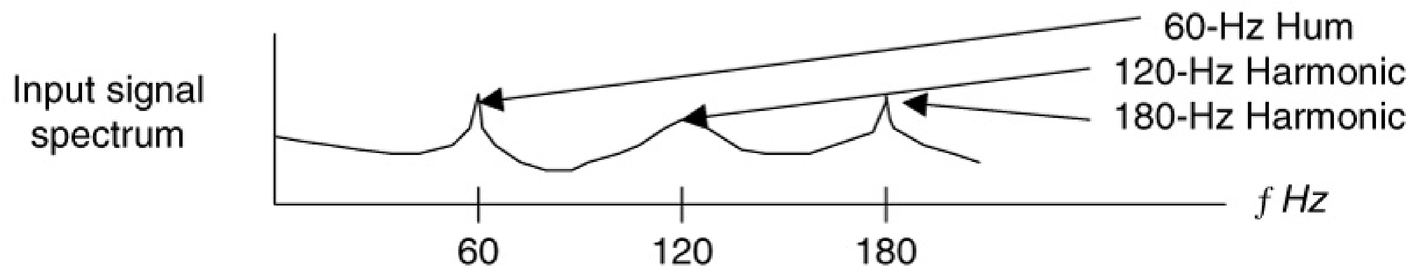
Hum noise: created by poor power supply or electromagnetic interference and characterized by a frequency of 60 Hz and its harmonics.

Hum eliminator

Frequency response of Hum eliminator

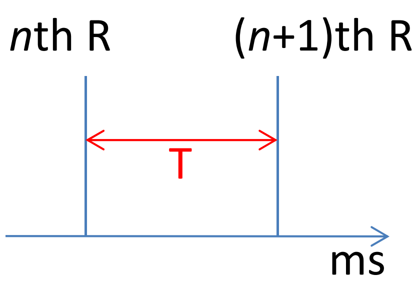


Corrupted by hum & harmonics

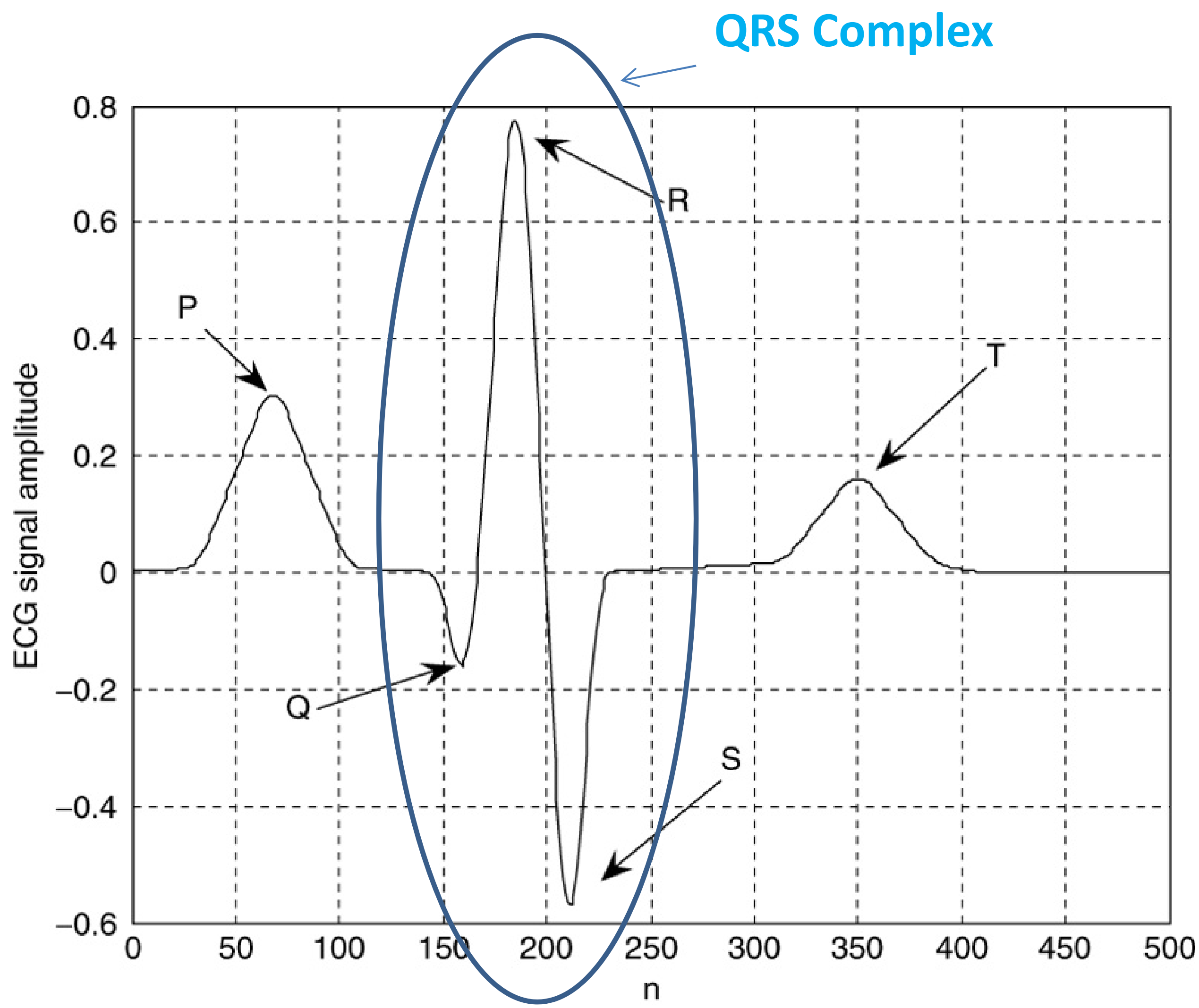


ECG Pulse

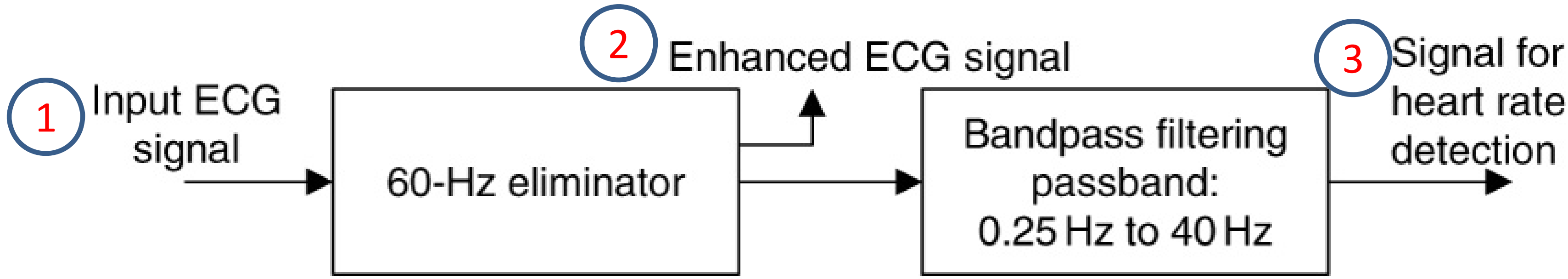
ECG + Hum → makes difficult to analyze.



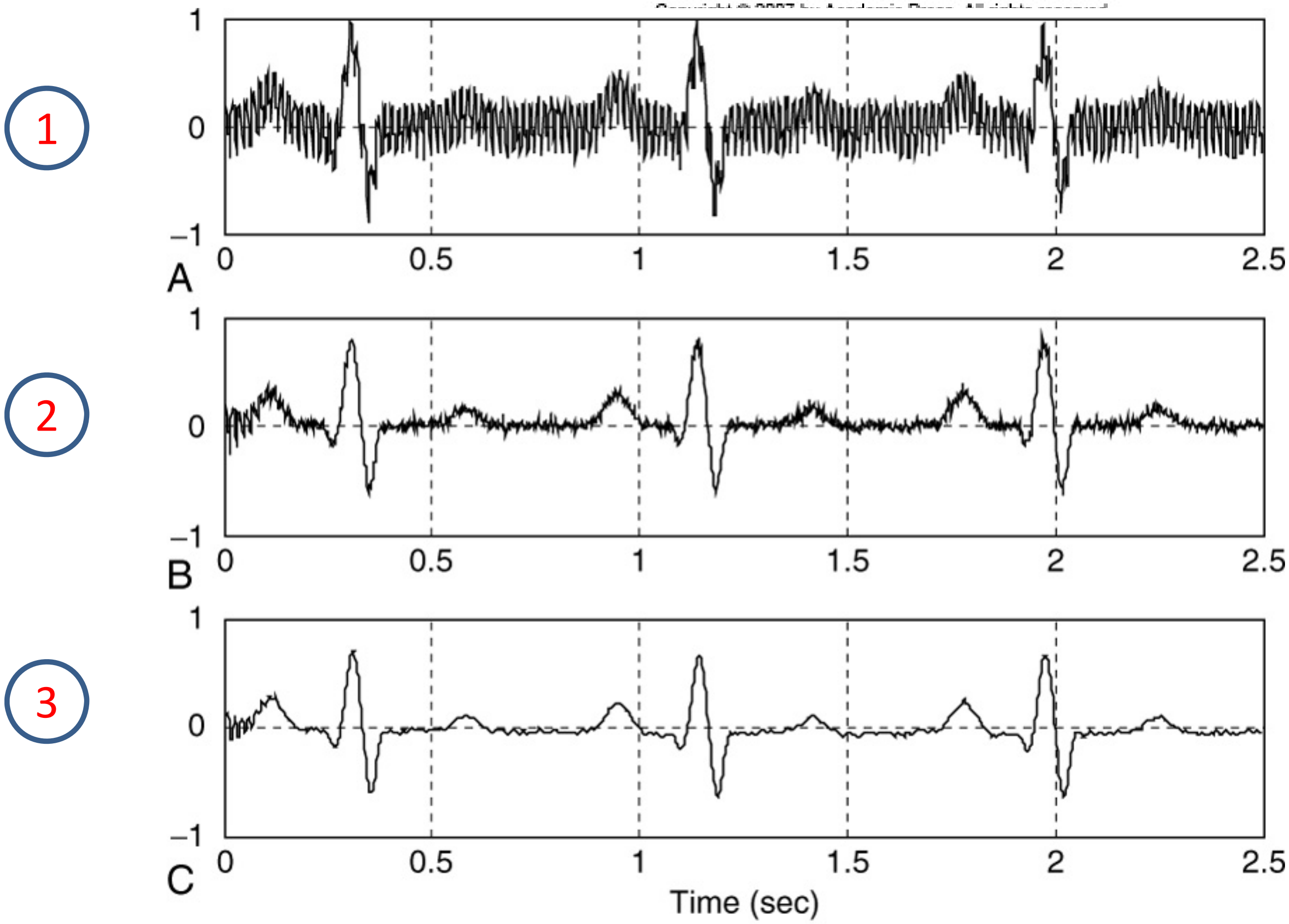
Heart beat /min = 60000 / T



Heart Beat Detection Using ECG Pulse



To filter muscle noise ≈ 40 Hz



Practice example