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المملكة العربية السعودية
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قسم الرياضيات

OR: $E(X\wedge t) = E(X) - E(X-t)_+$.

$$E(X-t)_+ = \int_t^\infty S_X(x)\,dx = \int_t^\infty \frac{\theta^2}{(\theta+x)^2}dx = -\frac{\theta^2}{\theta+x}\Big|_t^\infty = \frac{\theta^2}{\theta+t} \Rightarrow E(X) = E(X-t)_+\Big|_{t=0} = \theta.$$

$$\Rightarrow E(X\wedge t) = \theta - \frac{\theta^2}{t+\theta}$$

MID2 Actu466 Solution

1) (2+1+1=4 marks) Suppose a random variable X has a Pareto distribution with parameters $\alpha = 2$ and θ:

$$F_X(x) = 1 - \frac{\theta^2}{(x+\theta)^2} \quad \text{for } x \geq 0.$$

a) Show that $E(X \wedge t) = \theta\left(1 - \frac{\theta}{t+\theta}\right)$ for any positive real number t.

b) Deduce $E(X)$.

c) Find the distribution of cX for a positive constant c.

a) $f(x) = \dfrac{2\theta^2}{(x+\theta)^3}$; $x \geq 0$.

$$E(X\wedge t) = \int_0^\infty (x\wedge t) f(x)\,dx = \int_0^t x f(x)\,dx + \int_t^\infty t f(x)\,dx$$

$$\int_0^t x f(x)\,dx = \int_0^t 2\theta^2 \frac{x}{(x+\theta)^3}dx = \int_\theta^{t+\theta} 2\theta^2 \frac{y-\theta}{y^3}dy \qquad y = x+\theta.$$

$$= 2\theta^2 \int_\theta^{t+\theta}\left(\frac{1}{y^2} - \frac{\theta}{y^3}\right)dy = 2\theta^2\left(-\frac{1}{y} + \frac{\theta}{2y^2}\right)\Big|_\theta^{t+\theta}$$

$$= 2\theta^2\left(-\frac{1}{t+\theta} + \frac{\theta}{2(t+\theta)^2} + \frac{1}{\theta} - \frac{\theta}{2\theta^2}\right)$$

$$= 2\theta^2\left(\frac{-2(t+\theta)+\theta}{2(t+\theta)^2} + \frac{1}{2\theta}\right) = 2\theta^2\left(\frac{-2t-\theta}{2(t+\theta)^2} + \frac{1}{2\theta}\right).$$

$$\Rightarrow E(X\wedge t) = 2\theta^2\left(\frac{-2t-\theta}{2(t+\theta)^2} + \frac{1}{2\theta}\right) + t\frac{\theta^2}{(t+\theta)^2}.$$

$$= \frac{-2t\theta^2 - \theta^3}{(t+\theta)^2} + \theta + \frac{t\theta^2}{(t+\theta)^2}.$$

$$= \frac{-t\theta^2 - \theta^3}{(t+\theta)^2} + \theta = \theta - \frac{\theta^2(t+\theta)}{(t+\theta)^2}$$

$$= \theta - \frac{\theta^2}{t+\theta} = \theta\left(1 - \frac{\theta}{t+\theta}\right).$$

b) $E(X) = \lim_{t\to\infty} E(X\wedge t) = \theta$.

c) $F_{cX}(y) = F_X(y/c) = 1 - \dfrac{\theta^2}{\left(\frac{y}{c}+\theta\right)^2} = 1 - \dfrac{(c\theta)^2}{(y+c\theta)^2}$.

$cX \sim \text{Pareto}(\alpha = 2, c\theta)$.

2) (2+2+2+2+2=10 marks) An insurance policy is subject to an ordinary deductible of d. The cdf of the loss amount X is given in **Exercise 1**.

a) Compute the cdf and pdf for Y^L.

b) Compute the cdf and pdf for Y^P.

d) Compute the mean of Y^L and Y^P.

e) Compute the loss elimination ratio.

f) Deduce the loss elimination ratio after a uniform inflation of 100r %.

a)
$$F_X(x) = 1-\left(\frac{\theta}{x+\theta}\right)^2 ; \quad f_X(x) = \frac{2\theta^2}{(x+\theta)^3}.$$

$$F_{Y^L}(y) = F_X(y+d) = 1-\left(\frac{\theta}{y+d+\theta}\right)^2 ; \quad y \geq 0.$$

$$f_{Y^L}(y) = \begin{cases} 0 & y<0 \\ 1-\left(\frac{\theta}{d+\theta}\right)^2 & y=0 \\ \frac{2\theta^2}{(y+d+\theta)^3} & y>0. \end{cases}$$

b)
$$F_{Y^P}(y) = \frac{\left(1-\left(\frac{\theta}{y+d+\theta}\right)^2\right)-\left(1-\left(\frac{\theta}{d+\theta}\right)^2\right)}{\left(\frac{\theta}{d+\theta}\right)^2} ; \quad y \geq 0$$

$$= 1-\left(\frac{d+\theta}{y+d+\theta}\right)^2, \quad Y^P \sim \text{Pareto}(\alpha=2, d+\theta).$$

$$f_{Y^P}(y) = \begin{cases} 0 & y<0 \\ \frac{2\theta^2}{(y+d+\theta)^3} \Big/ \frac{\theta^2}{(d+\theta)^2} = \frac{2(\theta+d)^2}{(y+\theta+d)^3}. & y \geq 0 \end{cases}$$

d)
$$E(Y^L) = E(X - X\wedge d) = \theta - \theta\left(1-\frac{\theta}{d+\theta}\right) = \frac{\theta^2}{d+\theta}.$$

$$E(Y^P) = \theta+d \quad (\text{since } Y^P \sim \text{Pareto}(2, \theta+d).$$

$$\text{OR} \quad E(Y^P) = \frac{E\,Y^L}{S_X(d)} = \frac{\theta^2}{d+\theta} \Big/ \frac{\theta^2}{(\theta+d)^2} = \theta+d.$$

e)
$$LER = \frac{E(X)-E(Y^L)}{E(X)} = \frac{E(X\wedge d)}{E(X)} = \frac{\theta\left(1-\frac{\theta}{d+\theta}\right)}{\theta}$$

$$= 1-\frac{\theta}{d+\theta}.$$

f) we replace X by $(1+r)X$ and since $(1+r)X \sim \text{Pareto}(2, (1+r)\theta)$. Then

$$LER = 1-\frac{(1+r)\theta}{d+(1+r)\theta}.$$

3) (3 marks) Consider two insurance contracts. One has a policy limit of u. The second has a coinsurance α. Losses in both contracts follow the same distribution in **Exercise 1** with parameter θ.

Find the relationship between u, α and θ so that the expected loss per cost is the same for the two contracts.

policy limit of $u \longrightarrow E(Y_1^L) = E(X \wedge u) = \theta\left(1 - \frac{\theta}{u+\theta}\right)$.

Coinsurance $\alpha \longrightarrow E(Y_2^L) = E(\alpha X) = \alpha\theta$.

$$EY_1^L = EY_2^L \Rightarrow \theta\left(1 - \frac{\theta}{u+\theta}\right) = \alpha\theta$$

$$\Rightarrow \boxed{1 - \frac{\theta}{u+\theta} = \alpha}$$

4) (1+2=3 marks) Individual losses have a Pareto distribution with parameter θ (as in **Exercise 1**). The number of losses when there is no deductible has a negative binomial distribution with parameters r and p.

a) Determine the expected number of cost-per payments when a deductible d is applied.

b) Determine the expected total cost-per payment.

a) $N^P = \sum_{i=1}^{N} Z_i$, $Z = 1_{(X \geq d)}$, $E(Z) = S_X(d) = \left(\frac{\theta}{\theta+d}\right)^2$

$$E(N^P) = E(N)E(Z) = \frac{r(1-p)}{p}\left(\frac{\theta}{\theta+d}\right)^2.$$

b) $S = \sum_{i=1}^{N^P} Y_i^P \rightarrow E(S) = E(N^P)E(Y^P)$

$$= \frac{r(1-p)}{p}\left(\frac{\theta}{\theta+d}\right)^2(\theta+d)$$

$$= r(1-p)/p \; \frac{\theta^2}{\theta+d}.$$

5) (1+2+2=5 marks) The number of losses follows a geometric distribution with parameter p.

a) Solve the equation $\frac{1-p}{p} = t$ where t is known.

Using the data $X_1, X_2, \ldots X_n$ of the number of losses for the last n years, compute the estimate for the parameter p, by applying:

b) Method of moments,

c) Maximum likelihood method.

a) $\frac{1-p}{p} = t \Rightarrow \frac{1}{p} - 1 = t \Rightarrow \frac{1}{p} = 1+t \Rightarrow p = \frac{1}{1+t}$.

b) $E(X) = \bar{X} \Rightarrow \frac{1-p}{p} = \bar{X} \Rightarrow \boxed{\hat{p} = \frac{1}{1+\bar{X}}}$

c) $L(p) = p(1-p)^{x_1} \cdots p(1-p)^{x_n} = p^n (1-p)^{n\bar{X}}$

$$\ell(p) = \ln(L(p)) = n\ln(p) + n\bar{x}\ln(1-p).$$

$$\frac{\partial}{\partial p}\ell(p) = 0 = \frac{n}{p} - \frac{n\bar{x}}{1-p} = 0$$

$$\Rightarrow \frac{n}{p} = +\frac{n\bar{X}}{1-p}$$

$$\Rightarrow \frac{1-p}{p} = \bar{X} \Rightarrow \boxed{\hat{p} = \frac{1}{1+\bar{X}}}$$