CHAPTER 5

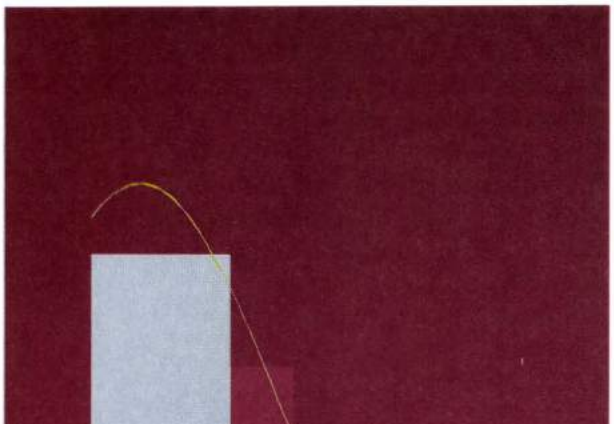
INTEGRALS

INTRODUCTION

The two most important tools in calculus are the derivative, considered in previous chapters, and the *definite integral*, defined in Section 5.4. The derivative was motivated by the problems of finding the slope of a tangent line and defining velocity. The definite integral arises naturally when we consider the problem of finding the area of a region in the xy-plane. However, this is merely one application. As we shall see in later chapters, the uses for definite integrals are as abundant and varied as those for derivatives.

The principal result in this chapter is the *fundamental theorem of calculus*, proved in Section 5.6. This outstanding theorem enables us to find exact values of definite integrals by using an *antiderivative* or *indefinite integral*. Each of these concepts is defined in Section 5.1; the procedure may be regarded as a reverse procedure to finding the derivative of a function. Thus, in addition to providing an important evaluation process, the fundamental theorem shows that there is a relationship between derivatives and integrals—a key result in calculus.

The chapter closes with a discussion of methods of *numerical integration*, used for approximating definite integrals that cannot be evaluated by means of the fundamental theorem. These methods are readily programmable for use with calculators and computers and are employed in a wide variety of applied fields.



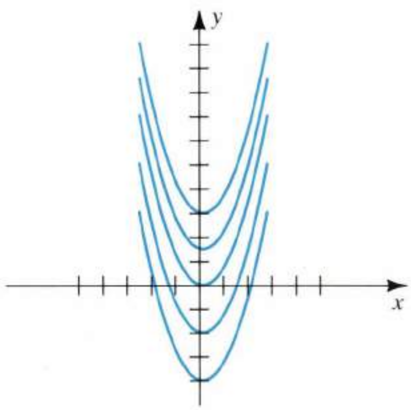
5.1 ANTIDERIVATIVES AND INDEFINITE INTEGRALS

In our previous work we solved problems of the following type: *Given a function f, find the derivative f'.* We shall now consider a related problem: *Given a function f, find a function F such that $F' = f$.* In the next definition we give F a special name.

Definition **(5.1)**

> A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for every x in I.

FIGURE 5.1



We shall also call $F(x)$ an antiderivative of $f(x)$. The process of finding F, or $F(x)$, is called **antidifferentiation**.

To illustrate, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, because

$$F'(x) = D_x\,(x^2) = 2x = f(x).$$

There are many other antiderivatives of $2x$, such as $x^2 + 2$, $x^2 - \frac{5}{3}$, and $x^2 + \sqrt{3}$. In general, if C is *any* constant, then $x^2 + C$ is an antiderivative of $2x$, because

$$D_x\,(x^2 + C) = 2x + 0 = 2x.$$

Thus, there is a *family of antiderivatives* of $2x$ of the form $F(x) = x^2 + C$, where C is any constant. Graphs of several members of this family are sketched in Figure 5.1.

The next illustration contains other examples of antiderivatives, where C is a constant.

ILLUSTRATION

$f(x)$	ANTIDERIVATIVES OF $f(x)$
x^2	$\frac{1}{3}x^3,\ \ \frac{1}{3}x^3 + 8,\ \ \frac{1}{3}x^3 + C$
$8x^3$	$2x^4,\ \ 2x^4 - \sqrt[3]{7},\ \ 2x^4 + C$
$\cos x$	$\sin x,\ \ \sin x + \frac{4}{9},\ \ \sin x + C$

As in the preceding illustration, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any constant C. The next theorem states that *every* antiderivative is of this form.

Theorem **(5.2)**

> Let F be an antiderivative of f on an interval I. If G is any antiderivative of f on I, then
>
> $$G(x) = F(x) + C$$
>
> for some constant C and every x in I.

PROOF If F and G are antiderivatives of f, let H be the function defined by

$$H(x) = G(x) - F(x)$$

for every x in I. We will show that H is a constant function on I; that is, $G(x) - F(x) = C$ for some C, or, equivalently, $G(x) = F(x) + C$.

Let a and b be any numbers in I such that $a < b$. To show that H is constant on I, it suffices to prove that $H(a) = H(b)$. Since F and G are antiderivatives of f,

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$$

for every x in I. Since $H(x)$ is differentiable, H is continuous, by Theorem (3.11). Applying the mean value theorem (4.12) to H on the interval $[a, b]$, there exists a number c in (a, b) such that

$$H'(c) = \frac{H(b) - H(a)}{b - a}.$$

Since c is in I, $H'(c) = 0$, and thus

$$H(b) - H(a) = 0, \quad \text{or} \quad H(a) = H(b),$$

which is what we wished to prove. ▬

We refer to the constant C in Theorem (5.2) as an **arbitrary constant**. If $F(x)$ is an antiderivative of $f(x)$, then *all* antiderivatives of $f(x)$ can be obtained from $F(x) + C$ by letting C range through the set of real numbers. We shall employ the following notation for a family of antiderivatives of this type.

Definition **(5.3)**

The notation

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$ and C is an arbitrary constant, denotes the family of all antiderivatives of $f(x)$ on an interval I.

The symbol $\int$ used in Definition (5.3) is an **integral sign**. We call $\int f(x)\,dx$ the **indefinite integral** of $f(x)$. The expression $f(x)$ is the **integrand**, and C is the **constant of integration**. The process of finding $F(x) + C$, when given $\int f(x)\,dx$, is referred to as **indefinite integration**, **evaluating the integral**, or **integrating $f(x)$**. The adjective *indefinite* is used because $\int f(x)\,dx$ represents a *family* of antiderivatives, not any *specific* function. Later in the chapter, when we discuss definite integrals, we shall give reasons for using the integral sign and the differential dx that appears to the right of the integrand $f(x)$. At present we shall not interpret $f(x)\,dx$ as the product of $f(x)$ and the differential dx. We shall regard dx merely as a symbol that specifies the independent variable x, which we refer to as the **variable of integration**. If we use a different variable of integration, such as t, we write

$$\int f(t)\,dt = F(t) + C,$$

where $F'(t) = f(t)$.

ILLUSTRATION

- $\int x^4\, dx = \frac{1}{5}x^5 + C$ because $D_x\left(\frac{1}{5}x^5\right) = x^4$.
- $\int t^{-3}\, dt = -\frac{1}{2}t^{-2} + C$ because $D_t\left(-\frac{1}{2}t^{-2}\right) = t^{-3}$.
- $\int \cos u\, du = \sin u + C$ because $D_u \sin u = \cos u$.

Note that, in general,

$$\int [D_x f(x)]\, dx = f(x) + C$$

because $f'(x) = D_x f(x)$. This allows us to use any derivative formula to obtain a corresponding formula for an indefinite integral, as illustrated in the next table. As shown in Formula (1), it is customary to abbreviate $\int 1\, dx$ by $\int dx$.

Find [illegible] whose [illegible] is

Brief table of indefinite integrals **(5.4)**

DERIVATIVE $D_x[f(x)]$	**INDEFINITE INTEGRAL** $\int D_x[f(x)]\, dx = f(x) + C$
$D_x(x) = 1$	**(1)** $\int 1\, dx = \int dx = x + C$
$D_x\left(\dfrac{x^{r+1}}{r+1}\right) = x^r \ (r \neq -1)$	**(2)** $\int x^r\, dx = \dfrac{x^{r+1}}{r+1} + C \ (r \neq -1)$
$D_x(\sin x) = \cos x$	**(3)** $\int \cos x\, dx = \sin x + C$
$D_x(-\cos x) = \sin x$	**(4)** $\int \sin x\, dx = -\cos x + C$
$D_x(\tan x) = \sec^2 x$	**(5)** $\int \sec^2 x\, dx = \tan x + C$
$D_x(-\cot x) = \csc^2 x$	**(6)** $\int \csc^2 x\, dx = -\cot x + C$
$D_x(\sec x) = \sec x \tan x$	**(7)** $\int \sec x \tan x\, dx = \sec x + C$
$D_x(-\csc x) = \csc x \cot x$	**(8)** $\int \csc x \cot x\, dx = -\csc x + C$

Formula (2) is called the *power rule for indefinite integration*. As in the following illustration, it is often necessary to rewrite an integrand before applying the power rule or one of the trigonometric formulas.

ILLUSTRATION

- $\int x^3 \cdot x^5\, dx = \int x^8\, dx = \dfrac{x^{8+1}}{8+1} = \dfrac{1}{9}x^9 + C$
- $\int \dfrac{1}{x^3}\, dx = \int x^{-3}\, dx = \dfrac{x^{-3+1}}{-3+1} = -\dfrac{1}{2x^2} + C$
- $\int \sqrt[3]{x^2}\, dx = \int x^{2/3}\, dx = \dfrac{x^{2/3+1}}{\frac{2}{3}+1} = \dfrac{3}{5}x^{5/3} + C$
- $\int \dfrac{\tan x}{\sec x}\, dx = \int \cos x \dfrac{\sin x}{\cos x}\, dx = \int \sin x\, dx = -\cos x + C$

It is a good idea to check indefinite integrations (such as those in the preceding illustration) by differentiating the final expression to see if either the integrand or an equivalent form of the integrand is obtained.

The next theorem indicates that differentiation and indefinite integration are inverse processes, because each, in a sense, undoes the other. In statement (i) it is assumed that f is differentiable, and in (ii) that f has an antiderivative on some interval.

Theorem **(5.5)**

(i) $\int [D_x f(x)]\, dx = f(x) + C$

(ii) $D_x\left[\int f(x)\, dx\right] = f(x)$

PROOF We have already proved (i). To prove (ii), let F be an antiderivative of f and write

$$D_x\left[\int f(x)\, dx\right] = D_x\,[F(x) + C] = F'(x) + 0 = f(x).$$

EXAMPLE 1 Verify Theorem (5.5) for the special case $f(x) = x^2$.

SOLUTION

(i) If we first differentiate x^2 and then integrate,

$$\int D_x\,(x^2)\, dx = \int 2x\, dx = x^2 + C.$$

(ii) If we first integrate x^2 and then differentiate,

$$D_x \int x^2\, dx = D_x\left(\frac{x^3}{3} + C\right) = x^2.$$

The next theorem is useful for evaluating many types of indefinite integrals. In the statements we assume that $f(x)$ and $g(x)$ have antiderivatives on an interval I.

Theorem **(5.6)**

(i) $\int cf(x)\, dx = c\int f(x)\, dx$ for any constant c

(ii) $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

(iii) $\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx$

PROOF We shall prove (ii). The proofs of (i) and (iii) are similar. If F and G are antiderivatives of f and g, respectively,

$$D_x\,[F(x) + G(x)] = F'(x) + G'(x) = f(x) + g(x).$$

Hence, by Definition (5.3),

$$\int [f(x) + g(x)]\, dx = F(x) + G(x) + C,$$

where C is an arbitrary constant. Similarly,

$$\int f(x)\,dx + \int g(x)\,dx = F(x) + C_1 + G(x) + C_2$$

for arbitrary constants C_1 and C_2. These give us the same family of antiderivatives, since for any special case we can choose values of the constants such that $C = C_1 + C_2$. This proves (ii).

EXAMPLE 2 Evaluate $\int (5x^3 + 2\cos x)\,dx$.

SOLUTION We first use (ii) and (i) of Theorem (5.6) and then formulas from (5.4):

$$\begin{aligned}\int (5x^3 + 2\cos x)\,dx &= \int 5x^3\,dx + \int 2\cos x\,dx \\ &= 5\int x^3\,dx + 2\int \cos x\,dx \\ &= 5\left(\frac{x^4}{4} + C_1\right) + 2(\sin x + C_2) \\ &= \tfrac{5}{4}x^4 + 5C_1 + 2\sin x + 2C_2 \\ &= \tfrac{5}{4}x^4 + 2\sin x + C\end{aligned}$$

where $C = 5C_1 + 2C_2$.

In Example 2 we added the two constants $5C_1$ and $2C_2$ to obtain one arbitrary constant C. We can always manipulate arbitrary constants in this way, so it is not necessary to introduce a constant for each indefinite integration as we did in Example 2. Instead, if an integrand is a sum, *we integrate each term of the sum without introducing constants and then add one arbitrary constant C after the last integration.* We also often bypass the step $\int cf(x)\,dx = c\int f(x)\,dx$, as in the next example.

EXAMPLE 3 Evaluate $\int \left(8t^3 - 6\sqrt{t} + \dfrac{1}{t^3}\right) dt$.

SOLUTION First we find an antiderivative for each of the three terms in the integrand and then add an arbitrary constant C. We rewrite $\sqrt{t}$ as $t^{1/2}$ and $1/t^3$ as t^{-3} and then use the power rule for integration:

$$\begin{aligned}\int \left(8t^3 - 6\sqrt{t} + \frac{1}{t^3}\right) dt &= \int (8t^3 - 6t^{1/2} + t^{-3})\,dt \\ &= 8\cdot\frac{t^4}{4} - 6\cdot\frac{t^{3/2}}{\frac{3}{2}} + \frac{t^{-2}}{-2} + C \\ &= 2t^4 - 4t^{3/2} - \frac{1}{t^2} + C\end{aligned}$$

EXAMPLE 4 Evaluate $\int \dfrac{(x^2 - 1)^2}{x^2}\,dx$.

SOLUTION First we change the form of the integrand, because the degree of the numerator is greater than or equal to the degree of the denominator. We then find an antiderivative for each term, adding an arbitrary constant C after the last integration:

$$\begin{aligned}\int \frac{(x^2-1)^2}{x^2}\,dx &= \int \frac{x^4-2x^2+1}{x^2}\,dx \\ &= \int (x^2-2+x^{-2})\,dx \\ &= \frac{x^3}{3} - 2x + \frac{x^{-1}}{-1} + C \\ &= \frac{1}{3}x^3 - 2x - \frac{1}{x} + C\end{aligned}$$

EXAMPLE 5 Evaluate $\displaystyle\int \frac{1}{\cos u \cot u}\,du$.

SOLUTION We use trigonometric identities to change the integrand and then apply formula (7) from Table (5.4):

$$\begin{aligned}\int \frac{1}{\cos u \cot u}\,du &= \int \sec u \tan u\,du \\ &= \sec u + C\end{aligned}$$

An applied problem may be stated in terms of a **differential equation**—that is, an equation that involves derivatives or differentials of an unknown function. A function f is a **solution** of a differential equation if it satisfies the equation—that is, if substitution of f for the unknown function produces a true statement. To **solve** a differential equation means to find all solutions. Sometimes, in addition to the differential equation, we may know certain values of f or f', called **initial conditions**.

Indefinite integrals are useful for solving certain differential equations, because if we are given a derivative $f'(x)$ we can integrate and use Theorem (5.5)(i) to obtain an equation involving the unknown function f:

$$\int f'(x)\,dx = f(x) + C$$

If we are also given an initial condition for f, it may be possible to find $f(x)$ explicitly, as in the next example.

EXAMPLE 6 Solve the differential equation

$$f'(x) = 6x^2 + x - 5$$

subject to the initial condition $f(0) = 2$.

SOLUTION We proceed as follows:

$$\begin{aligned}f'(x) &= 6x^2 + x - 5 \\ \int f'(x)\,dx &= \int (6x^2 + x - 5)\,dx \\ f(x) &= 2x^3 + \tfrac{1}{2}x^2 - 5x + C\end{aligned}$$

for some number C. (It is unnecessary to add a constant of integration to *each* side of the equation.) Letting $x = 0$ and using the given initial condition $f(0) = 2$ gives us

$$f(0) = 0 + 0 - 0 + C, \quad \text{or} \quad 2 = C.$$

Hence the solution f of the differential equation with the initial condition $f(0) = 2$ is

$$f(x) = 2x^3 + \tfrac{1}{2}x^2 - 5x + 2.$$

The given equation can also be stated in terms of the differentials of $y = f(x)$ by writing

$$\frac{dy}{dx} = 6x^2 + x - 5, \quad \text{or} \quad dy = (6x^2 + x - 5)\,dx.$$

In this case we integrate as follows:

$$\int dy = \int (6x^2 + x - 5)\,dx$$
$$y = 2x^3 + \tfrac{1}{2}x^2 - 5x + C$$

The constant C may be found by letting $x = 0$ and using $y = f(0) = 2$.

If we are given a *second* derivative $f''(x)$, then we must employ two successive indefinite integrals to find $f(x)$. First we use Theorem (5.5)(i) as follows:

$$\int f''(x)\,dx = \int [D_x f'(x)]\,dx = f'(x) + C$$

After finding $f'(x)$, we proceed as in Example 6.

EXAMPLE 7 Solve the differential equation

$$f''(x) = 5 \cos x + 2 \sin x$$

subject to the initial conditions $f(0) = 3$ and $f'(0) = 4$.

SOLUTION We proceed as follows:

$$f''(x) = 5 \cos x + 2 \sin x$$
$$\int f''(x)\,dx = \int (5 \cos x + 2 \sin x)\,dx$$
$$f'(x) = 5 \sin x - 2 \cos x + C$$

Letting $x = 0$ and using the initial condition $f'(0) = 4$ gives us

$$f'(0) = 5 \sin 0 - 2 \cos 0 + C$$
$$4 = 0 - 2 \cdot 1 + C, \quad \text{or} \quad C = 6.$$

Hence

$$f'(x) = 5 \sin x - 2 \cos x + 6.$$

We integrate a second time:

$$\int f'(x)\,dx = \int (5 \sin x - 2 \cos x + 6)\,dx$$

$$f(x) = -5 \cos x - 2 \sin x + 6x + D$$

Letting $x = 0$ and using the initial condition $f(0) = 3$, we find that

$$f(0) = -5 \cos 0 - 2 \sin 0 + 6 \cdot 0 + D$$

$$3 = -5 - 0 + 0 + D, \quad \text{or} \quad D = 8.$$

Therefore, the solution of the differential equation with the given initial conditions is

$$f(x) = -5 \cos x - 2 \sin x + 6x + 8.$$

Suppose a point P is moving on a coordinate line with an acceleration $a(t)$ at time t, and the corresponding velocity is $v(t)$. By Definition (4.21), $a(t) = v'(t)$ and hence

$$\int a(t)\,dt = \int v'(t)\,dt = v(t) + C$$

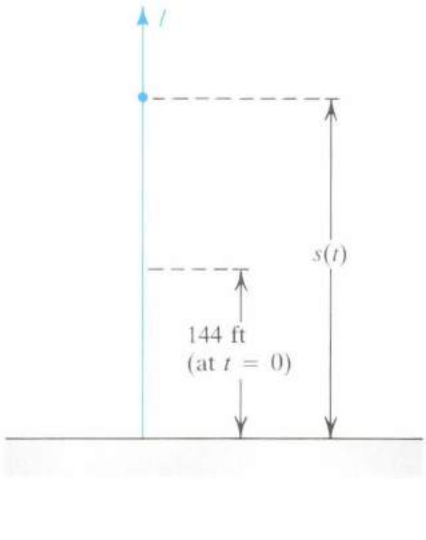
for some constant C.

Similarly, if we know $v(t)$, then since $v(t) = s'(t)$, where s is the position function of P, we can find a formula that involves $s(t)$ by indefinite integration:

$$\int v(t)\,dt = \int s'(t)\,dt = s(t) + D$$

for some constant D. In the next example we shall use this technique to find the position function for an object that is moving under the influence of gravity. Understanding the problem requires knowledge of a fact from physics: An object on or near the surface of the earth is acted upon by a force—*gravity*—that produces a constant acceleration, denoted by g. The approximation to g that is employed for most problems is 32 ft/sec^2, or 980 cm/sec^2.

FIGURE 5.2



EXAMPLE 8 A stone is thrown vertically upward from a position 144 feet above the ground with an initial velocity of 96 ft/sec. Disregarding air resistance, find

(a) the stone's distance above the ground after t seconds

(b) the length of time that the stone rises

(c) when and with what velocity the stone strikes the ground

SOLUTION The motion of the stone may be represented by a point moving on a vertical coordinate line l with origin at ground level and positive direction upward (see Figure 5.2).

(a) The stone's distance above the ground at time t is $s(t)$, and the initial conditions are $s(0) = 144$ and $v(0) = 96$. Since the velocity is decreasing,

$v'(t) < 0$; that is, the acceleration is negative. Hence, by the remarks preceding this example;

$$a(t) = v'(t) = -32$$
$$\int v'(t)\,dt = \int -32\,dt$$
$$v(t) = -32t + C$$

for some number C. Substituting 0 for t and using the fact that $v(0) = 96$ gives us $96 = 0 + C = C$ and, consequently,

$$v(t) = -32t + 96.$$

Since $s'(t) = v(t)$, we obtain

$$s'(t) = -32t + 96$$
$$\int s'(t)\,dt = \int (-32t + 96)\,dt$$
$$s(t) = -16t^2 + 96t + D$$

for some number D. Letting $t = 0$ and using the fact that $s(0) = 144$ leads to $144 = 0 + 0 + D$, or $D = 144$. It follows that the distance from the ground to the stone at time t is given by

$$s(t) = -16t^2 + 96t + 144.$$

(b) The stone will rise until $v(t) = 0$—that is, until

$$-32t + 96 = 0, \quad \text{or} \quad t = 3.$$

(c) The stone will strike the ground when $s(t) = 0$—that is, when

$$-16t^2 + 96t + 144 = 0.$$

An equivalent equation is $t^2 - 6t - 9 = 0$. Applying the quadratic formula, we obtain $t = 3 \pm 3\sqrt{2}$. The solution $3 - 3\sqrt{2}$ is extraneous, since t is nonnegative. Hence the stone strikes the ground after $3 + 3\sqrt{2}$ sec. The velocity at that time is

$$\begin{aligned} v(3 + 3\sqrt{2}) &= -32(3 + 3\sqrt{2}) + 96 \\ &= -96\sqrt{2} \approx -135.8 \text{ ft/sec}. \end{aligned}$$

In economic applications, if a marginal function is known (see page 226), then we can use indefinite integration to find the function, as illustrated in the next example.

EXAMPLE 9 A manufacturer finds that the marginal cost (in dollars) associated with the production of x units of a photocopier component is given by $30 - 0.02x$. If the cost of producing one unit is \$35, find the cost function and the cost of producing 100 units.

SOLUTION If C is the cost function, then the marginal cost is the rate of change of C with respect to x—that is,

$$C'(x) = 30 - 0.02x.$$

Hence

$$\int C'(x)\,dx = \int (30 - 0.02x)\,dx$$

and
$$C(x) = 30x - 0.01x^2 + K$$

for some K. Letting $x = 1$ and using $C(1) = 35$, we obtain

$$35 = 30 - 0.01 + K, \quad \text{or} \quad K = 5.01.$$

Consequently,

$$C(x) = 30x - 0.01x^2 + 5.01.$$

In particular, the cost of producing 100 units is

$$C(100) = 3000 - 100 + 5.01 = \$2905.01.$$

EXERCISES 5.1

Exer. 1–42: Evaluate.

1 $\int (4x + 3)\,dx$

2 $\int (4x^2 - 8x + 1)\,dx$

3 $\int (9t^2 - 4t + 3)\,dt$

4 $\int (2t^3 - t^2 + 3t - 7)\,dt$

5 $\int \left(\frac{1}{z^3} - \frac{3}{z^2}\right) dz$

6 $\int \left(\frac{4}{z^7} - \frac{7}{z^4} + z\right) dz$

7 $\int \left(3\sqrt{u} + \frac{1}{\sqrt{u}}\right) du$

8 $\int (\sqrt{u^3} - \frac{1}{2}u^{-2} + 5)\,du$

9 $\int (2v^{5/4} + 6v^{1/4} + 3v^{-4})\,dv$

10 $\int (3v^5 - v^{5/3})\,dv$

11 $\int (3x - 1)^2\,dx$

12 $\int \left(x - \frac{1}{x}\right)^2 dx$

13 $\int x(2x + 3)\,dx$

14 $\int (2x - 5)(3x + 1)\,dx$

15 $\int \frac{8x - 5}{\sqrt[3]{x}}\,dx$

16 $\int \frac{2x^2 - x + 3}{\sqrt{x}}\,dx$

17 $\int \frac{x^3 - 1}{x - 1}\,dx, \quad x \neq 1$

18 $\int \frac{x^3 + 3x^2 - 9x - 2}{x - 2}\,dx, \quad x \neq 2$

19 $\int \frac{(t^2 + 3)^2}{t^6}\,dt$

20 $\int \frac{(\sqrt{t} + 2)^2}{t^3}\,dt$

21 $\int \frac{3}{4}\cos u\,du$

22 $\int -\frac{1}{5}\sin u\,du$

23 $\int \frac{7}{\csc x}\,dx$

24 $\int \frac{1}{4\sec x}\,dx$

25 $\int (\sqrt{t} + \cos t)\,dt$

26 $\int (\sqrt[3]{t^2} - \sin t)\,dt$

27 $\int \frac{\sec t}{\cos t}\,dt$

28 $\int \frac{1}{\sin^2 t}\,dt$

29 $\int (\csc v \cot v \sec v)\,dv$

30 $\int (4 + 4\tan^2 v)\,dv$

31 $\int \frac{\sec w \sin w}{\cos w}\,dw$

32 $\int \frac{\csc w \cos w}{\sin w}\,dw$

33 $\int \frac{(1 + \cot^2 z)\cot z}{\csc z}\,dz$

34 $\int \frac{\tan z}{\cos z}\,dz$

35 $\int D_x \sqrt{x^2 + 4}\,dx$

36 $\int D_x \sqrt[3]{x^3 - 8}\,dx$

37 $\int \frac{d}{dx}(\sin \sqrt[3]{x})\,dx$

38 $\int \frac{d}{dx}(\sqrt{\tan x})\,dx$

39 $D_x \int (x^3\sqrt{x - 4})\,dx$

40 $D_x \int (x^4 \sqrt[3]{x^2 + 9})\,dx$

41 $\frac{d}{dx}\int \cot x^3\,dx$

42 $\frac{d}{dx}\int \cos\sqrt{x^2 + 1}\,dx$

Exer. 43–48: Evaluate the integral if a and b are constants.

43 $\int a^2\,dx$

44 $\int ab\,dx$

45 $\int (at + b)\,dt$

46 $\int \left(\frac{a}{b^2}t\right) dt$

47 $\int (a + b)\,du$

48 $\int (b - a^2)\,du$

Exer. 49–56: Solve the differential equation subject to the given conditions.

49 $f'(x) = 12x^2 - 6x + 1; \quad f(1) = 5$

50 $f'(x) = 9x^2 + x - 8; \quad f(-1) = 1$

51 $\frac{dy}{dx} = 4x^{1/2}; \quad y = 21 \text{ if } x = 4$

52 $\dfrac{dy}{dx} = 5x^{-1/3}$; $y = 70$ if $x = 27$

53 $f''(x) = 4x - 1$; $f'(2) = -2$; $f(1) = 3$

54 $f''(x) = 6x - 4$; $f'(2) = 5$; $f(2) = 4$

55 $\dfrac{d^2y}{dx^2} = 3 \sin x - 4 \cos x$; $y = 7$ and $y' = 2$ if $x = 0$

56 $\dfrac{d^2y}{dx^2} = 2 \cos x - 5 \sin x$; $y = 2 + 6\pi$ and $y' = 3$ if $x = \pi$

Exer. 57–58: If a point is moving on a coordinate line with the given acceleration $a(t)$ and initial conditions, find $s(t)$.

57 $a(t) = 2 - 6t$; $v(0) = -5$; $s(0) = 4$

58 $a(t) = 3t^2$; $v(0) = 20$; $s(0) = 5$

59 A projectile is fired vertically upward from ground level with a velocity of 1600 ft/sec. Disregarding air resistance, find

(a) its distance $s(t)$ above ground at time t

(b) its maximum height

60 An object is dropped from a height of 1000 feet. Disregarding air resistance, find

(a) the distance it falls in t seconds

(b) its velocity at the end of 3 seconds

(c) when it strikes the ground

61 A stone is thrown directly downward from a height of 96 feet with an initial velocity of 16 ft/sec. Find

(a) its distance above the ground after t seconds

(b) when it strikes the ground

(c) the velocity at which it strikes the ground

62 A gravitational constant for objects near the surface of the moon is 5.3 ft/sec^2.

(a) If an astronaut on the moon throws a stone directly upward with an initial velocity of 60 ft/sec, find its maximum altitude.

(b) If, after returning to Earth, the astronaut throws the same stone directly upward with the same initial velocity, find the maximum altitude.

63 If a projectile is fired vertically upward from a height of s_0 feet above the ground with a velocity of v_0 ft/sec, prove that if air resistance is disregarded, its distance $s(t)$ above the ground after t seconds is given by $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$, where g is a gravitational constant.

64 A ball rolls down an inclined plane with an acceleration of 2 ft/sec^2.

(a) If the ball is given no initial velocity, how far will it roll in t seconds?

(b) What initial velocity must be given for the ball to roll 100 feet in 5 seconds?

65 If an automobile starts from rest, what constant acceleration will enable it to travel 500 feet in 10 seconds?

66 If a car is traveling at a speed of 60 mi/hr, what constant (negative) acceleration will enable it to stop in 9 seconds?

67 A small country has natural gas reserves of 100 billion ft^3. If $A(t)$ denotes the total amount of natural gas consumed after t years, then dA/dt is the *rate of consumption*. If the rate of consumption is predicted to be $5 + 0.01t$ billion ft^3/year, in approximately how many years will the country's natural gas reserves be depleted?

68 Refer to Exercise 67. Based on U.S. Department of Energy statistics, the rate of consumption of gasoline in the United States (in billions of gallons per year) is approximated by $dA/dt = 2.74 - 0.11t - 0.01t^2$, with $t = 0$ corresponding to the year 1980. Estimate the number of gallons of gasoline consumed in the United States between 1980 and 1984.

69 A sportswear manufacturer determines that the marginal cost in dollars of producing x warm-up suits is given by $20 - 0.015x$. If the cost of producing one suit is \$25, find the cost function and the cost of producing 50 suits.

70 If the marginal cost function of a product is given by $2/x^{1/3}$ and if the cost of producing 8 units is \$20, find the cost function and the cost of producing 64 units.

5.2 CHANGE OF VARIABLES IN INDEFINITE INTEGRALS

The formulas for indefinite integrals in Table (5.4) are limited in scope, because we cannot use them directly to evaluate integrals such as

$$\int \sqrt{5x + 7}\, dx \quad \text{or} \quad \int \cos 4x\, dx.$$

In this section we shall develop a simple but powerful method for changing the variable of integration so that these integrals (and many others) can be evaluated by using the formulas in Table (5.4).

To justify this method, we shall apply formula (i) of Theorem (5.5) to a *composite* function. We intend to consider several functions f, g, and F, so it will simplify our work if we state the formula in terms of a function h as follows:

$$\int [D_x\, h(x)]\, dx = h(x) + C$$

Suppose that F is an antiderivative of a function f and that g is a differentiable function such that $g(x)$ is in the domain of F for every x in some interval. If we let h denote the composite function $F \circ g$, then $h(x) = F(g(x))$ and hence

$$\int [D_x\, F(g(x))]\, dx = F(g(x)) + C.$$

Applying the chain rule (3.33) to the integrand $D_x\, F(g(x))$ and using the fact that $F' = f$, we obtain

$$D_x\, F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Substitution in the preceding indefinite integral gives us

$$(*) \qquad \int f(g(x))g'(x)\, dx = F(g(x)) + C.$$

We can employ the following device to help remember this formula:

$$\text{Let} \quad u = g(x) \quad \text{and} \quad du = g'(x)\, dx.$$

Note that once we have introduced the variable $u = g(x)$, the differential du of u is determined by using (ii) of Definition (3.28). If we *formally substitute* into the last integration formula, we obtain

$$\int f(u)\, du = F(u) + C.$$

This has the same *form* as the integral in Definition (5.3); however, u represents a *function*, not an independent variable x, as before. This indicates that $g'(x)\, dx$ in $(*)$ may be regarded as the product of $g'(x)$ and dx. Since the variable x has been replaced by a new variable u, finding indefinite integrals in this way is referred to as a **change of variable**, or as the **method of substitution**. We may summarize our discussion as follows, where we assume that f and g have the properties described previously.

Method of substitution **(5.7)**

If F is an antiderivative of f, then

$$\int f(g(x))g'(x)\, dx = F(g(x)) + C.$$

If $u = g(x)$ and $du = g'(x)\, dx$, then

$$\int f(u)\, du = F(u) + C.$$

After making the substitution $u = g(x)$ as indicated in (5.7), it may be necessary to insert a constant factor k into the integrand in order to arrive at the proper form $\int f(u)\, du$. We must then also multiply by $1/k$ to maintain equality, as illustrated in the following examples.

EXAMPLE 1 Evaluate $\int \sqrt{5x+7}\,dx$.

SOLUTION We let $u = 5x + 7$ and calculate du:

$$u = 5x + 7, \quad du = 5\,dx$$

Since du contains the factor 5, the integral is not in the proper form $\int f(u)\,du$ required by (5.7). However, we can *introduce* the factor 5 into the integrand, provided we also multiply by $\frac{1}{5}$. Doing this and using (i) of Theorem (5.6) gives us

$$\begin{aligned}\int \sqrt{5x+7}\,dx &= \int \sqrt{5x+7}\left(\tfrac{1}{5}\right)5\,dx \\ &= \tfrac{1}{5}\int \sqrt{5x+7}\;5\,dx.\end{aligned}$$

We now substitute and use the power rule for integration:

$$\begin{aligned}\int \sqrt{5x+7}\,dx &= \tfrac{1}{5}\int \sqrt{u}\,du \\ &= \tfrac{1}{5}\int u^{1/2}\,du \\ &= \tfrac{1}{5}\,\frac{u^{3/2}}{\frac{3}{2}} + C \\ &= \tfrac{2}{15}u^{3/2} + C \\ &= \tfrac{2}{15}(5x+7)^{3/2} + C\end{aligned}$$

In the future, after inserting a factor k into an integrand, as in Example 1, we shall simply multiply the integral by $1/k$, skipping the intermediate steps of first writing $(1/k)k$ and then bringing $1/k$ *outside*—that is, to the left of—the integral sign.

EXAMPLE 2 Evaluate $\int \cos 4x\,dx$.

SOLUTION We make the substitution

$$u = 4x, \quad du = 4\,dx.$$

Since du contains the factor 4, we adjust the integrand by multiplying by 4 and compensate by multiplying the integral by $\frac{1}{4}$ before substituting:

$$\begin{aligned}\int \cos 4x\,dx &= \tfrac{1}{4}\int (\cos 4x)4\,dx \\ &= \tfrac{1}{4}\int \cos u\,du \\ &= \tfrac{1}{4}\sin u + C \\ &= \tfrac{1}{4}\sin 4x + C\end{aligned}$$

It is not always easy to decide what substitution $u = g(x)$ is needed to transform an indefinite integral into a form that can be readily evaluated.

It may be necessary to try several different possibilities before finding a suitable substitution. In most cases *no* substitution will simplify the integrand properly. The following guidelines may be helpful.

Guidelines for changing variables in indefinite integrals **(5.8)**

1. Decide on a reasonable substitution $u = g(x)$.
2. Calculate $du = g'(x)\,dx$.
3. Using 1 and 2, try to transform the integral into a form that involves only the variable u. If necessary, introduce a *constant* factor k into the integrand and compensate by multiplying the integral by $1/k$. If any part of the resulting integrand contains the variable x, use a different substitution in 1.
4. Evaluate the integral obtained in 3, obtaining an antiderivative involving u.
5. Replace u in the antiderivative obtained in guideline 4 by $g(x)$. The final result should contain only the variable x.

The following examples illustrate the use of the guidelines.

EXAMPLE 3 Evaluate $\int (2x^3 + 1)^7 x^2\,dx$.

SOLUTION If an integrand involves an expression raised to a power, such as $(2x^3 + 1)^7$, we often substitute u for the expression. Thus, we let

$$u = 2x^3 + 1, \quad du = 6x^2\,dx.$$

Comparing $du = 6x^2\,dx$ with $x^2\,dx$ in the integral suggests that we introduce the factor 6 into the integrand. Doing this and compensating by multiplying the integral by $\frac{1}{6}$, we obtain the following:

$$\begin{aligned}\int (2x^3 + 1)^7 x^2\,dx &= \tfrac{1}{6}\int (2x^3 + 1)^7 6x^2\,dx \\ &= \tfrac{1}{6}\int u^7\,du \\ &= \frac{1}{6}\left(\frac{u^8}{8}\right) + C \\ &= \tfrac{1}{48}(2x^3 + 1)^8 + C\end{aligned}$$

A substitution in an indefinite integral can sometimes be made in several different ways. To illustrate, another method for evaluating the integral in Example 3 is to consider

$$u = 2x^3 + 1, \quad du = 6x^2\,dx, \quad \tfrac{1}{6}\,du = x^2\,dx.$$

We then substitute $\frac{1}{6}\,du$ for $x^2\,dx$,

$$\int (2x^3 + 1)^7 x^2\,dx = \int u^7 \tfrac{1}{6}\,du = \tfrac{1}{6}\int u^7\,du,$$

and integrate as before.

EXAMPLE 4 Evaluate $\int x\sqrt[3]{7-6x^2}\,dx$.

SOLUTION Note that the integrand contains the term $x\,dx$. If the factor x were missing or if x were raised to a higher power, the problem would be more complicated. For integrands that involve a radical, we often substitute for the expression under the radical sign. Thus, we let

$$u = 7 - 6x^2, \quad du = -12x\,dx.$$

Next we introduce the factor -12 into the integrand, compensate by multiplying the integral by $-\frac{1}{12}$, and proceed as follows:

$$\begin{aligned}\int x\sqrt[3]{7-6x^2}\,dx &= -\tfrac{1}{12}\int \sqrt[3]{7-6x^2}(-12)x\,dx\\ &= -\tfrac{1}{12}\int \sqrt[3]{u}\,du = -\tfrac{1}{12}\int u^{1/3}\,du\\ &= -\frac{1}{12}\left(\frac{u^{4/3}}{4/3}\right) + C = -\frac{1}{16}u^{4/3} + C\\ &= -\tfrac{1}{16}(7-6x^2)^{4/3} + C\end{aligned}$$

We could also have written

$$u = 7 - 6x^2 \quad du = -12x\,dx, \quad -\tfrac{1}{12}\,du = x\,dx$$

and substituted directly for $x\,dx$. Thus,

$$\int \sqrt[3]{7-6x^2}\,x\,dx = \int \sqrt[3]{u}\left(-\tfrac{1}{12}\right)du = -\tfrac{1}{12}\int \sqrt[3]{u}\,du.$$

The remainder of the solution would proceed exactly as before.

EXAMPLE 5 Evaluate $\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx$.

SOLUTION Let

$$u = x^3 - 3x + 1, \quad du = (3x^2-3)\,dx = 3(x^2-1)\,dx$$

and proceed as follows:

$$\begin{aligned}\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx &= \frac{1}{3}\int \frac{3(x^2-1)}{(x^3-3x+1)^6}\,dx\\ &= \frac{1}{3}\int \frac{1}{u^6}\,du = \frac{1}{3}\int u^{-6}\,du\\ &= \frac{1}{3}\left(\frac{u^{-5}}{-5}\right) + C = -\frac{1}{15}\left(\frac{1}{u^5}\right) + C\\ &= -\frac{1}{15}\frac{1}{(x^3-3x+1)^5} + C\end{aligned}$$

EXAMPLE 6 Evaluate $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.

SOLUTION We wish to use the formula $\int \cos u\, du = \sin u + C$, so let us make the substitution

$$u = \sqrt{x} = x^{1/2}, \quad du = \frac{1}{2}x^{-1/2}\, dx = \frac{1}{2\sqrt{x}}\, dx.$$

If we introduce the factor $\frac{1}{2}$ into the integrand and compensate by multiplying the integral by 2, we obtain

$$\begin{aligned}\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx &= 2\int \cos\sqrt{x}\left(\frac{1}{2}\cdot\frac{1}{\sqrt{x}}\right) dx\\ &= 2\int \cos u\, du = 2\sin u + C\\ &= 2\sin\sqrt{x} + C.\end{aligned}$$

EXAMPLE 7 Evaluate $\int \cos^3 5x \sin 5x\, dx$.

SOLUTION The form of the integrand suggests that we use the power rule $\int u^3\, du = \frac{1}{4}u^4 + C$. Thus, we let

$$u = \cos 5x, \quad du = -5 \sin 5x\, dx.$$

The form of du indicates that we should introduce the factor -5 into the integrand, multiply the integral by $-\frac{1}{5}$, and then integrate as follows:

$$\begin{aligned}\int \cos^3 5x \sin 5x\, dx &= -\tfrac{1}{5}\int \cos^3 5x(-5\sin 5x)\, dx\\ &= -\tfrac{1}{5}\int u^3\, du\\ &= -\frac{1}{5}\left(\frac{u^4}{4}\right) + C\\ &= -\tfrac{1}{20}\cos^4 5x + C\end{aligned}$$

EXERCISES 5.2

Exer. 1–8: Evaluate the integral using the given substitution, and express the answer in terms of x.

1 $\int x(2x^2+3)^{10}\, dx;\quad u = 2x^2 + 3$

2 $\int \frac{x}{(x^2+5)^3}\, dx;\quad u = x^2 + 5$

3 $\int x^2\sqrt[3]{3x^3+7}\, dx;\quad u = 3x^3 + 7$

4 $\int \frac{5x}{\sqrt{x^2-3}}\, dx;\quad u = x^2 - 3$

5 $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\, dx;\quad u = 1 + \sqrt{x}$

6 $\int \frac{1}{(5x-4)^{10}}\, dx;\quad u = 5x - 4$

7 $\int \sqrt{x}\cos\sqrt{x^3}\, dx;\quad u = x^{3/2}$

8 $\int \tan x \sec^2 x\, dx;\quad u = \tan x$

Exer. 9–48: Evaluate the integral.

9 $\int \sqrt{3x-2}\, dx$

10 $\int \sqrt[4]{2x+5}\, dx$

11 $\int \sqrt[3]{8t+5}\, dt$

12 $\int \frac{1}{\sqrt{4-5t}}\, dt$

13 $\int (3z+1)^4\, dz$

14 $\int (2z^2-3)^5 z\, dz$

15 $\int v^2\sqrt{v^3-1}\, dv$

16 $\int v\sqrt{9-v^2}\, dv$

17 $\int \frac{x}{\sqrt[3]{1-2x^2}}\, dx$

18 $\int (3-x^4)^3 x^3\, dx$

19 $\int (s^2 + 1)^2 \, ds$

20 $\int (3 - s^3)^2 \, s \, ds$

21 $\int \frac{(\sqrt{x} + 3)^4}{\sqrt{x}} \, dx$

22 $\int \left(1 + \frac{1}{x}\right)^{-3} \left(\frac{1}{x^2}\right) dx$

23 $\int \frac{t - 2}{(t^2 - 4t + 3)^3} \, dt$

24 $\int \frac{t^2 + t}{(4 - 3t^2 - 2t^3)^4} \, dt$

25 $\int 3 \sin 4x \, dx$

26 $\int 4 \cos \frac{1}{2}x \, dx$

27 $\int \cos (4x - 3) \, dx$

28 $\int \sin (1 + 6x) \, dx$

29 $\int v \sin (v^2) \, dv$

30 $\int \frac{\cos \sqrt[3]{v}}{\sqrt[3]{v^2}} \, dv$

31 $\int \cos 3x \sqrt[3]{\sin 3x} \, dx$

32 $\int \frac{\sin 2x}{\sqrt{1 - \cos 2x}} \, dx$

33 $\int (\sin x + \cos x)^2 \, dx$ (*Hint:* $\sin 2\theta = 2 \sin \theta \cos \theta$)

34 $\int \frac{\sin 4x}{\cos 2x} \, dx$ (*Hint:* $\sin 2\theta = 2 \sin \theta \cos \theta$)

35 $\int \sin x (1 + \cos x)^2 \, dx$

36 $\int \sin^3 x \cos x \, dx$

37 $\int \frac{\sin x}{\cos^4 x} \, dx$

38 $\int \sin 2x \sec^5 2x \, dx$

39 $\int \frac{\cos t}{(1 - \sin t)^2} \, dt$

40 $\int (2 + 5 \cos t)^3 \sin t \, dt$

41 $\int \sec^2 (3x - 4) \, dx$

42 $\int \frac{\csc 2x}{\sin 2x} \, dx$

43 $\int \sec^2 3x \tan 3x \, dx$

44 $\int \frac{1}{\tan 4x \sin 4x} \, dx$

45 $\int \frac{1}{\sin^2 5x} \, dx$

46 $\int \frac{x}{\cos^2 (x^2)} \, dx$

47 $\int x \cot (x^2) \csc (x^2) \, dx$

48 $\int \sec \left(\frac{x}{3}\right) \tan \left(\frac{x}{3}\right) dx$

Exer. 49–52: Solve the differential equation subject to the given conditions.

49 $f'(x) = \sqrt[3]{3x + 2}$; $f(2) = 9$

50 $\frac{dy}{dx} = x\sqrt{x^2 + 5}$; $y = 12$ if $x = 2$

51 $f''(x) = 16 \cos 2x - 3 \sin x$; $f(0) = -2$; $f'(0) = 4$

52 $f''(x) = 4 \sin 2x + 16 \cos 4x$; $f(0) = 6$; $f'(0) = 1$

Exer. 53–56: Evaluate the integral by (a) the method of substitution and (b) expanding the integrand. In what way do the constants of integration differ?

53 $\int (x + 4)^2 \, dx$

54 $\int (x^2 + 4)^2 x \, dx$

55 $\int \frac{(\sqrt{x} + 3)^2}{\sqrt{x}} \, dx$

56 $\int \left(1 + \frac{1}{x}\right)^2 \frac{1}{x^2} \, dx$

57 A charged particle is moving on a coordinate line in a magnetic field such that its velocity (in cm/sec) at time t is given by $v(t) = \frac{1}{2} \sin (3t - \frac{1}{4}\pi)$. Show that the motion is simple harmonic (see page 223).

58 The acceleration of a particle that is moving on a coordinate line is given by $a(t) = k \cos (\omega t + \phi)$ for constants k, ω, and ϕ and time t (in seconds). Show that the motion is simple harmonic (see page 223).

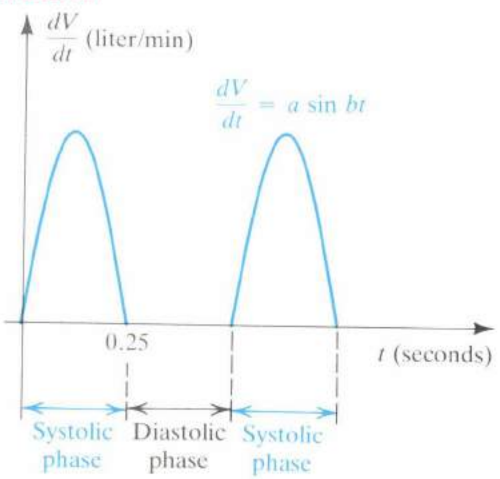
59 A reservoir supplies water to a community. In summer the demand A for water (in ft^3/day) changes according to the formula $dA/dt = 4000 + 2000 \sin (\frac{1}{90}\pi t)$ for time t (in days), with $t = 0$ corresponding to the beginning of summer. Estimate the total water consumption during 90 days of summer.

60 The pumping action of the heart consists of the systolic phase, in which blood rushes from the left ventricle into the aorta, and the diastolic phase, during which the heart muscle relaxes. The graph shown in the figure is sometimes used to model one complete cycle of the process. For a particular individual, the systolic phase lasts $\frac{1}{4}$ second and has a maximum flow rate dV/dt of 8 L/min, where V is the volume of blood in the heart at time t.

(a) Show that $dV/dt = 8 \sin (240\pi t)$ L/min.

(b) Estimate the total amount of blood pumped into the aorta during a systolic phase.

EXERCISE 60



61 The rhythmic process of breathing consists of alternating periods of inhaling and exhaling. For an adult, one complete cycle normally takes place every 5 seconds. If V denotes the volume of air in the lungs at time t, then dV/dt is the flow rate.

(a) If the maximum flow rate is 0.6 L/sec, find a formula $dV/dt = a \sin bt$ that fits the given information.

(b) Use part (a) to estimate the amount of air inhaled during one cycle.

62 Many animal populations fluctuate over 10-year cycles. Suppose that the rate of growth of a rabbit population is given by $dN/dt = 1000 \cos (\frac{1}{5}\pi t)$ rabbits/yr, where N denotes the number in the population at time t (in years) and $t = 0$ corresponds to the beginning of a cycle. If the population after 5 years is estimated to be 3000 rabbits, find a formula for N at time t and estimate the maximum population.

63 Show, by evaluating in three different ways, that

$$\begin{aligned}\int \sin x \cos x \, dx &= \tfrac{1}{2} \sin^2 x + C \\ &= -\tfrac{1}{2} \cos^2 x + D \\ &= -\tfrac{1}{4} \cos 2x + E.\end{aligned}$$

How can all three answers be correct?

5.3 SUMMATION NOTATION AND AREA

In this section we shall lay the foundation for the definition of the *definite integral*. At the outset, it is virtually impossible to see any connection between definite integrals and indefinite integrals. In Section 5.6, however, we show that there is a very close relationship: *Indefinite integrals can be used to evaluate definite integrals.*

In our development of the definite integral we shall employ sums of many numbers. To express such sums compactly, it is convenient to use **summation notation**. Given a collection of numbers $\{a_1, a_2, \ldots, a_n\}$, the symbol $\sum_{k=1}^{n} a_k$ represents their sum as follows.

Summation notation **(5.9)**

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

The Greek capital letter Σ (sigma) indicates a sum, and a_k represents the kth term of the sum. The letter k is the **index of summation**, or the **summation variable**, and assumes successive integer values. The integers 1 and n indicate the extreme values of the summation variable.

EXAMPLE 1 Evaluate $\sum_{k=1}^{4} k^2(k-3)$.

SOLUTION Comparing the sum with (5.9), we see that $a_k = k^2(k-3)$ and $n = 4$. To find the sum, we substitute 1, 2, 3, and 4 for k and add the resulting terms. Thus,

$$\begin{aligned}\sum_{k=1}^{4} k^2(k-3) &= 1^2(1-3) + 2^2(2-3) + 3^2(3-3) + 4^2(4-3) \\ &= (-2) + (-4) + 0 + 16 = 10.\end{aligned}$$

Letters other than k can be used for the summation variable. To illustrate,

$$\sum_{k=1}^{4} k^2(k-3) = \sum_{i=1}^{4} i^2(i-3) = \sum_{j=1}^{4} j^2(j-3) = 10.$$

If $a_k = c$ for every k, then

$$\sum_{k=1}^{2} a_k = a_1 + a_2 = c + c = 2c = \sum_{k=1}^{2} c$$
$$\sum_{k=1}^{3} a_k = a_1 + a_2 + a_3 = c + c + c = 3c = \sum_{k=1}^{3} c.$$

In general, the following result is true for every positive integer n.

Theorem **(5.10)**

$$\boxed{\sum_{k=1}^{n} c = nc}$$

The domain of the summation variable does not have to begin at 1. For example,

$$\sum_{k=4}^{8} a_k = a_4 + a_5 + a_6 + a_7 + a_8.$$

EXAMPLE 2 Evaluate $\sum_{k=0}^{3} \dfrac{2^k}{(k+1)}$.

SOLUTION

$$\sum_{k=0}^{3} \frac{2^k}{(k+1)} = \frac{2^0}{(0+1)} + \frac{2^1}{(1+1)} + \frac{2^2}{(2+1)} + \frac{2^3}{(3+1)}$$
$$= 1 + 1 + \tfrac{4}{3} + 2 = \tfrac{16}{3}$$

Theorem **(5.11)**

If n is any positive integer and $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are sets of real numbers, then

(i) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(ii) $\sum_{k=1}^{n} ca_k = c\left(\sum_{k=1}^{n} a_k\right)$ for every real number c

(iii) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

PROOF To prove (i), we begin with

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n).$$

Rearranging terms on the right we obtain

$$\sum_{k=1}^{n} (a_k + b_k) = (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n)$$
$$= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

For (ii),

$$\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + ca_3 + \cdots + ca_n$$
$$= c(a_1 + a_2 + a_3 + \cdots + a_n) = c\left(\sum_{k=1}^{n} a_k\right).$$

To prove (iii), write $a_k - b_k = a_k + (-1)b_k$ and use (i) and (ii).

The formulas in the following theorem will be useful later in this section. They may be proved by mathematical induction (see Appendix I).

Theorem (5.12)

(i) $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

(ii) $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE 3 Evaluate $\displaystyle\sum_{k=1}^{100} k$ and $\displaystyle\sum_{k=1}^{20} k^2$.

SOLUTION Using (i) and (ii) of Theorem (5.12), we obtain

$$\sum_{k=1}^{100} k = 1 + 2 + \cdots + 100 = \frac{100(101)}{2} = 5050$$

and

$$\sum_{k=1}^{20} k^2 = 1^2 + 2^2 + \cdots + 20^2 = \frac{20(21)(41)}{6} = 2870.$$

EXAMPLE 4 Express $\displaystyle\sum_{k=1}^{n} (k^2 - 4k + 3)$ in terms of n.

SOLUTION We use Theorems (5.11), (5.12), and (5.10):

$$\sum_{k=1}^{n} (k^2 - 4k + 3) = \sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3$$
$$= \frac{n(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} + 3n$$
$$= \tfrac{1}{3}n^3 - \tfrac{3}{2}n^2 + \tfrac{7}{6}n$$

The definition of the definite integral (to be given in Section 5.4) is closely related to the areas of certain regions in a coordinate plane. We can easily calculate the area if the region is bounded by lines. For example, the area of a rectangle is the product of its length and width. The area

of a triangle is one-half the product of an altitude and the corresponding base. The area of any polygon can be found by subdividing it into triangles.

In order to find areas of regions whose boundaries involve graphs of functions, however, we utilize a limiting process and then use methods of calculus. In particular, let us consider a region R in a coordinate plane, bounded by the vertical lines $x = a$ and $x = b$, by the x-axis, and by the graph of a function f that is continuous and nonnegative on the closed interval $[a, b]$. A region of this type is illustrated in Figure 5.3. Since $f(x) \geq 0$ for every x in $[a, b]$, no part of the graph lies below the x-axis. For convenience, we shall refer to R as **the region under the graph of f from a to b**. We wish to define the area A of R.

FIGURE 5.3 Region under the graph of f

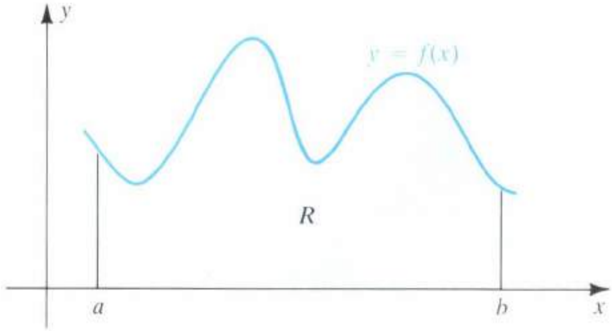
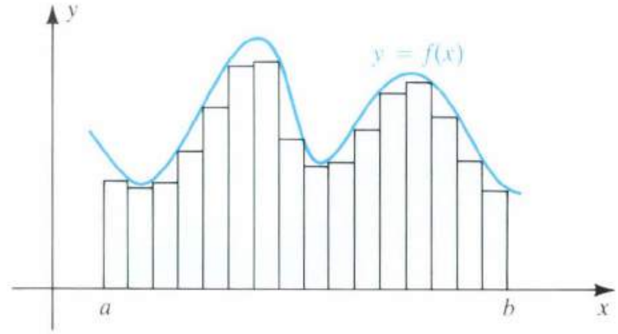


FIGURE 5.4 An inscribed rectangular polygon



To arrive at a satisfactory definition of A, we shall consider many rectangles of equal width such that each rectangle lies completely under the graph of f and intersects the graph in at least one point, as illustrated in Figure 5.4. The boundary of the region formed by the totality of these rectangles is called an **inscribed rectangular polygon**. We shall use the following notation:

$$A_{\text{IP}} = \text{area of an inscribed rectangular polygon}$$

If the width of the rectangles in Figure 5.4 is small, then it appears that

$$A_{\text{IP}} \approx A.$$

This suggests that we let the width of the rectangles approach zero and define A as a limiting value of the areas A_{IP} of the corresponding inscribed rectangular polygons. The notation discussed next will allow us to carry out this procedure rigorously.

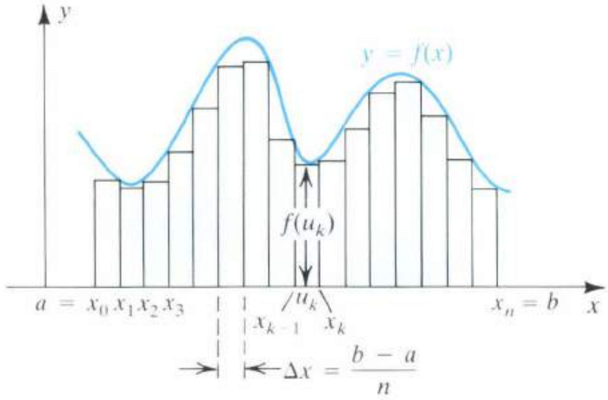
If n is any positive integer, divide the interval $[a, b]$ into n subintervals, all having the same length $\Delta x = (b - a)/n$. We can do this by choosing numbers $x_0, x_1, x_2, \ldots, x_n$, with $a = x_0$, $b = x_n$, and

$$x_k - x_{k-1} = \frac{b - a}{n} = \Delta x$$

for $k = 1, 2, \ldots, n$, as indicated in Figure 5.5. Note that

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\,\Delta x, \quad x_3 = a + 3\,\Delta x, \quad \ldots$$
$$x_k = a + k\,\Delta x, \quad \ldots, \quad x_n = a + n\,\Delta x = b.$$

FIGURE 5.5



The function f is continuous on each subinterval $[x_{k-1}, x_k]$, and hence, by the extreme value theorem (4.3), f takes on a minimum value at some number u_k in $[x_{k-1}, x_k]$. For each k, let us construct a rectangle of width $\Delta x = x_k - x_{k-1}$ and height equal to the minimum distance $f(u_k)$ from the x-axis to the graph of f (see Figure 5.5). The area of the kth rectangle is $f(u_k)\,\Delta x$. The area A_{IP} of the resulting inscribed rectangular polygon is the sum of the areas of the n rectangles; that is,

$$A_{\text{IP}} = f(u_1)\,\Delta x + f(u_2)\,\Delta x + \cdots + f(u_n)\,\Delta x.$$

Using summation notation, we may write

$$A_{\text{IP}} = \sum_{k=1}^{n} f(u_k)\,\Delta x,$$

where $f(u_k)$ is the minimum value of f on $[x_{k-1}, x_k]$.

If n is very large, or, equivalently, if Δx is very small, then the sum A_{IP} of the rectangular areas should approximate the area of the region R. Intuitively we know that if there exists a number A such that $\sum_{k=1}^{n} f(u_k)\,\Delta x$ gets closer to A as Δx gets closer to 0 (but $\Delta x \neq 0$), we can call A the **area** of R and write

$$A = \lim_{\Delta x \to 0} A_{\text{IP}} = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(u_k)\,\Delta x.$$

The meaning of this *limit of sums* is not the same as that of the limit of a function, introduced in Chapter 2. To eliminate the word *closer* and arrive at a satisfactory definition of A, let us take a slightly different point of view. If A denotes the area of the region R, then the difference

$$A - \sum_{k=1}^{n} f(u_k)\,\Delta x$$

is the area of the portion in Figure 5.5 that lies *under* the graph of f and *over* the inscribed rectangular polygon. This number may be regarded as the error in using the area of the inscribed rectangular polygon to approximate A. We should be able to make this error as small as desired by choosing the width Δx of the rectangles sufficiently small. This is the motivation for the following definition of the area A of R. The notation is the same as that used in the preceding discussion.

Definition (5.13)

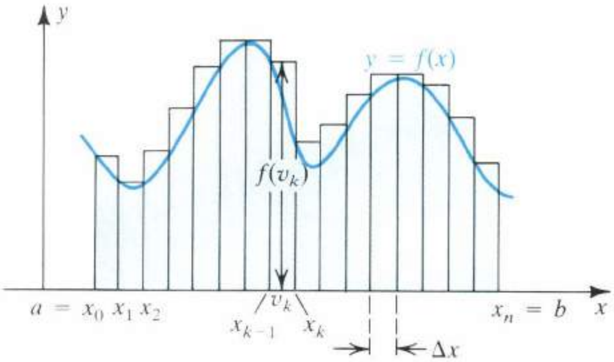
> Let f be continuous and nonnegative on $[a, b]$. Let A be a real number, and let $f(u_k)$ be the minimum value of f on $[x_{k-1}, x_k]$. The notation
>
> $$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(u_k)\,\Delta x$$
>
> means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < \Delta x < \delta$, then
>
> $$A - \sum_{k=1}^{n} f(u_k)\,\Delta x < \epsilon.$$

If A is the indicated limit and we let $\epsilon = 10^{-9}$, then Definition (5.13) states that by using rectangles of sufficiently small width Δx, we can make the difference between A and the area of the inscribed polygon less than one-billionth of a square unit. Similarly, if $\epsilon = 10^{-12}$, we can make this difference less than one-trillionth of a square unit. In general, the difference can be made less than *any* preassigned ϵ.

If f is continuous on $[a, b]$, it is shown in more advanced texts that a number A satisfying Definition (5.13) actually exists. We shall call A **the area under the graph of f from a to b**.

The area A may also be obtained by means of **circumscribed rectangular polygons** of the type illustrated in Figure 5.6. In this case we select the number v_k in each interval $[x_{k-1}, x_k]$ such that $f(v_k)$ is the *maximum* value of f on $[x_{k-1}, x_k]$.

FIGURE 5.6 A circumscribed rectangular polygon



Let

$$A_{CP} = \text{area of a circumscribed rectangular polygon.}$$

Using summation notation, we have

$$A_{CP} = \sum_{k=1}^{n} f(v_k)\,\Delta x,$$

where $f(v_k)$ is the maximum value of f on $[x_{k-1}, x_k]$. Note that

$$\sum_{k=1}^{n} f(u_k)\,\Delta x \le A \le \sum_{k=1}^{n} f(v_k)\,\Delta x.$$

The limit of A_{CP} as $\Delta x \to 0$ is defined as in (5.13). The only change is that we use

$$\sum_{k=1}^{n} f(v_k)\,\Delta x - A < \epsilon,$$

since we want this difference to be nonnegative. It can be proved that the same number A is obtained using either inscribed or circumscribed rectangles.

FIGURE 5.7

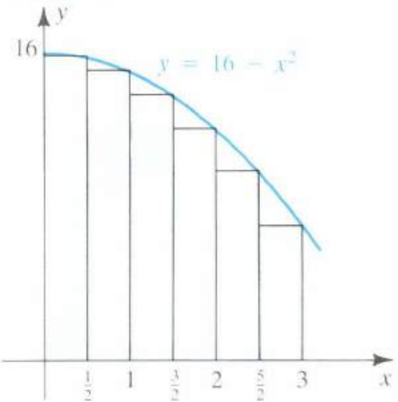
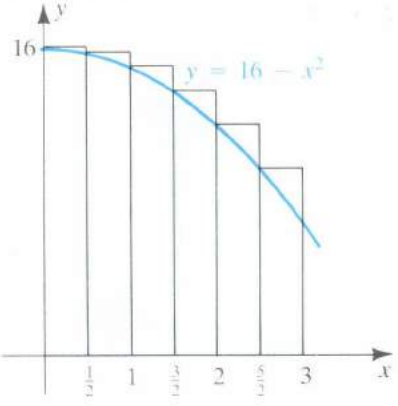


FIGURE 5.8



EXAMPLE 5 Let $f(x) = 16 - x^2$, and let R be the region under the graph of f from 0 to 3. Approximate the area A of R using

(a) an inscribed rectangular polygon with $\Delta x = \frac{1}{2}$

(b) a circumscribed rectangular polygon with $\Delta x = \frac{1}{2}$

SOLUTION

(a) The graph of f and the inscribed rectangular polygon with $\Delta x = \frac{1}{2}$ are sketched in Figure 5.7 (with different scales on the x- and y-axes). Note that f is decreasing on $[0, 3]$, and hence the minimum value $f(u_k)$ on the kth subinterval occurs at the right-hand endpoint of the subinterval. Since there are six rectangles to consider, the formula for A_{IP} is

$$\begin{aligned} A_{IP} &= \sum_{k=1}^{6} f(u_k)\,\Delta x \\ &= f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} + f(3)\cdot\tfrac{1}{2} \\ &= \tfrac{63}{4}\cdot\tfrac{1}{2} + 15\cdot\tfrac{1}{2} + \tfrac{55}{4}\cdot\tfrac{1}{2} + 12\cdot\tfrac{1}{2} + \tfrac{39}{4}\cdot\tfrac{1}{2} + 7\cdot\tfrac{1}{2} \\ &= \tfrac{293}{8} = 36.625. \end{aligned}$$

(b) The graph of f and the circumscribed rectangular polygon are sketched in Figure 5.8. Since f is decreasing on $[0, 3]$, the maximum value $f(v_k)$ occurs at the left-hand endpoint of the kth subinterval. Hence

$$\begin{aligned} A_{CP} &= \sum_{k=1}^{6} f(v_k)\,\Delta x \\ &= f(0)\cdot\tfrac{1}{2} + f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} \\ &= 16\cdot\tfrac{1}{2} + \tfrac{63}{4}\cdot\tfrac{1}{2} + 15\cdot\tfrac{1}{2} + \tfrac{55}{4}\cdot\tfrac{1}{2} + 12\cdot\tfrac{1}{2} + \tfrac{39}{4}\cdot\tfrac{1}{2} \\ &= \tfrac{329}{8} = 41.125. \end{aligned}$$

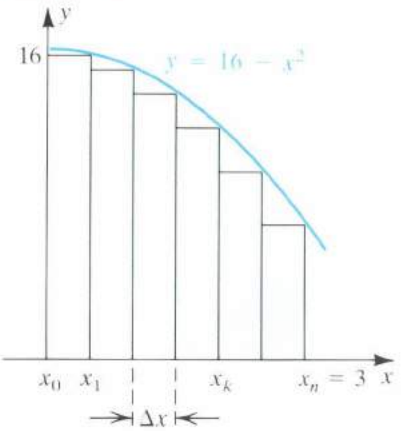
It follows that $36.625 < A < 41.125$. In the next example we prove that $A = 39$.

EXAMPLE 6 If $f(x) = 16 - x^2$, find the area of the region under the graph of f from 0 to 3.

SOLUTION The region was considered in Example 5 and is resketched in Figure 5.9, on the following page. If the interval $[0, 3]$ is divided into n equal subintervals, then the length Δx of each subinterval is $3/n$. Employing the notation used in Figure 5.5, with $a = 0$ and $b = 3$, we have

$$x_0 = 0, \quad x_1 = \Delta x, \quad x_2 = 2\,\Delta x, \quad \ldots, \quad x_k = k\,\Delta x, \quad \ldots, \quad x_n = n\,\Delta x = 3.$$

FIGURE 5.9



Since $\Delta x = 3/n$,

$$x_k = k\,\Delta x = k\,\frac{3}{n} = \frac{3k}{n}.$$

Since f is decreasing on $[0, 3]$, the number u_k in $[x_{k-1}, x_k]$ at which f takes on its minimum value is always the right-hand endpoint x_k of the subinterval; that is, $u_k = x_k = 3k/n$. Thus,

$$f(u_k) = f\left(\frac{3k}{n}\right) = 16 - \left(\frac{3k}{n}\right)^2 = 16 - \frac{9k^2}{n^2},$$

and the summation in Definition (5.13) is

$$\begin{aligned}\sum_{k=1}^{n} f(u_k)\,\Delta x &= \sum_{k=1}^{n}\left[\left(16 - \frac{9k^2}{n^2}\right)\cdot\frac{3}{n}\right]\\ &= \frac{3}{n}\sum_{k=1}^{n}\left(16 - \frac{9k^2}{n^2}\right),\end{aligned}$$

where the last equality follows from (ii) of Theorem (5.11). (Note that $3/n$ does not contain the summation variable k.) We next use Theorems (5.11), (5.10), and (5.12) to obtain

$$\begin{aligned}\sum_{k=1}^{n} f(u_k)\,\Delta x &= \frac{3}{n}\left(\sum_{k=1}^{n} 16 - \frac{9}{n^2}\sum_{k=1}^{n} k^2\right)\\ &= \frac{3}{n}\left[n\cdot 16 - \frac{9}{n^2}\,\frac{n(n+1)(2n+1)}{6}\right]\\ &= 48 - \frac{9}{2}\,\frac{(n+1)(2n+1)}{n^2}.\end{aligned}$$

To find the area of the region, we let Δx approach 0. Since $\Delta x = 3/n$, we can accomplish this by letting n increase without bound. Although our discussion of limits involving infinity in Section 2.4 was concerned with a real variable x, a similar discussion can be given if the variable is an integer n. Assuming that this is true and that we can replace $\Delta x \to 0$ by $n \to \infty$, we obtain

No n^2 in numerator.
n dominates ok

$$\begin{aligned}\lim_{\Delta x \to 0}\sum_{k=1}^{n} f(u_k)\,\Delta x &= \lim_{n\to\infty}\left[48 - \frac{9}{2}\,\frac{(n+1)(2n+1)}{n^2}\right]\\ &= 48 - \tfrac{9}{2}\cdot 2 = 39.\end{aligned}$$

Thus, the area of the region is 39 square units.

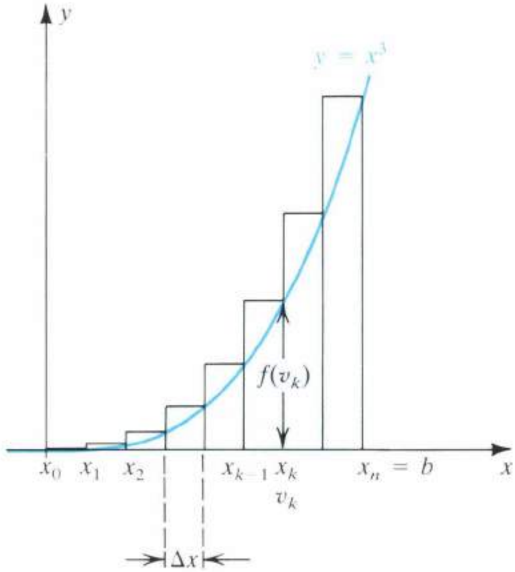
The area in the preceding example may also be found by using circumscribed rectangular polygons. In this case we select, in each subinterval $[x_{k-1}, x_k]$, the number $v_k = (k-1)(3/n)$ at which f takes on its maximum value.

The next example illustrates the use of circumscribed rectangles in finding an area.

EXAMPLE 7 If $f(x) = x^3$, find the area under the graph of f from 0 to b for any $b > 0$.

SOLUTION Subdividing the interval $[0, b]$ into n equal parts (see Figure 5.10), we obtain a circumscribed rectangular polygon such that $\Delta x = b/n$ and $x_k = k\,\Delta x$.

FIGURE 5.10



Since f is an increasing function, the maximum value $f(v_k)$ in the interval $[x_{k-1}, x_k]$ occurs at the right-hand endpoint; that is,

$$v_k = x_k = k\,\Delta x = k\,\frac{b}{n} = \frac{bk}{n}.$$

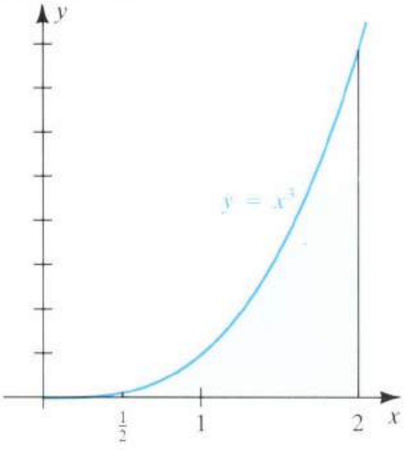
The sum of the areas of the circumscribed rectangles is

$$\begin{aligned}\sum_{k=1}^{n} f(v_k)\,\Delta x &= \sum_{k=1}^{n}\left[\left(\frac{bk}{n}\right)^3 \cdot \frac{b}{n}\right] = \sum_{k=1}^{n} \frac{b^4}{n^4}\,k^3\\ &= \frac{b^4}{n^4}\sum_{k=1}^{n} k^3 = \frac{b^4}{n^4}\left[\frac{n(n+1)}{2}\right]^2\\ &= \frac{b^4}{4}\cdot\frac{n^2(n+1)^2}{n^4},\end{aligned}$$

where we have used Theorem (5.12)(iii). If we let Δx approach 0, then n increases without bound and the expression involving n approaches 1. It follows that the area under the graph is

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(v_k)\,\Delta x = \frac{b^4}{4}.$$

FIGURE 5.11



EXAMPLE 8 If $f(x) = x^3$, find the area A of the region under the graph of f from $\frac{1}{2}$ to 2.

SOLUTION The region is sketched in Figure 5.11.

If we let A_1 = area under the graph of f from 0 to $\frac{1}{2}$

and $\qquad A_2$ = area under the graph of f from 0 to 2,

the area A can be found by subtracting A_1 from A_2:

$$A = A_2 - A_1$$

In Example 7 we found that the area under the graph of $y = x^3$ from 0 to b is $\frac{1}{4}b^4$. Hence, using $b = \frac{1}{2}$ for A_1 and $b = 2$ for A_2 yields

$$A = \frac{2^4}{4} - \frac{(\frac{1}{2})^4}{4} = 4 - \frac{1}{64} \approx 3.98.$$

EXERCISES 5.3

Exer. 1–8: Evaluate the sum. Replace #'s.

1 $\sum_{j=1}^{4} (j^2 + 1)$

2 $\sum_{j=1}^{4} (2^j + 1)$

3 $\sum_{k=0}^{5} k(k-1)$

4 $\sum_{k=0}^{4} (k-2)(k-3)$

5 $\sum_{n=1}^{10} [1 + (-1)^n]$

6 $\sum_{n=1}^{4} (-1)^n\left(\frac{1}{n}\right)$

7 $\sum_{i=1}^{50} 10$

8 $\sum_{k=1}^{1000} 2$

Exer. 9–12: Express the sum in terms of n (see Example 4).

9 $\sum_{k=1}^{n} (k^2 + 3k + 5)$

10 $\sum_{k=1}^{n} (3k^2 - 2k + 1)$

11 $\sum_{k=1}^{n} (k^3 + 2k^2 - k + 4)$

12 $\sum_{k=1}^{n} (3k^3 + k)$

Exer. 13–18: Express in summation notation.

13 $1 + 5 + 9 + 13 + 17$

14 $2 + 5 + 8 + 11 + 14$

15 $\frac{1}{2} + \frac{2}{5} + \frac{3}{8} + \frac{4}{11}$

16 $\frac{1}{4} + \frac{2}{9} + \frac{3}{14} + \frac{4}{19}$

17 $1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \cdots + (-1)^n\frac{x^{2n}}{2n}$

18 $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}$

Exer. 19–22: Let A be the area under the graph of the given function f from a to b. Approximate A by dividing $[a, b]$ into subintervals of equal length Δx and using (a) A_{IP} and (b) A_{CP}.

19 $f(x) = 3 - x$; $\quad a = -2$, $b = 2$; $\quad \Delta x = 1$

20 $f(x) = x + 2$; $\quad a = -1$, $b = 4$; $\quad \Delta x = 1$

21 $f(x) = x^2 + 1$; $\quad a = 1$, $b = 3$; $\quad \Delta x = \frac{1}{2}$

22 $f(x) = 4 - x^2$; $\quad a = 0$, $b = 2$; $\quad \Delta x = \frac{1}{2}$

c 23 $f(x) = \sqrt{\sin x}$; $\quad a = 0$, $b = 1.5$; $\quad \Delta x = 0.15$

c 24 $f(x) = \frac{1}{\sqrt{x^3 + 1}}$; $\quad a = 0$, $b = 3$; $\quad \Delta x = 0.3$

Exer. 25–30: Refer to Examples 6 and 7. Find the area under the graph of the given function f from 0 to b using (a) inscribed rectangles and (b) circumscribed rectangles.

25 $f(x) = 2x + 3$; $\quad b = 4$

26 $f(x) = 8 - 3x$; $\quad b = 2$

27 $f(x) = 9 - x^2$; $\quad b = 3$

28 $f(x) = x^2$; $\quad b = 5$

29 $f(x) = x^3 + 1$; $\quad b = 2$

30 $f(x) = 4x + x^3$; $\quad b = 2$

Exer. 31–32: Refer to Example 7. Find the area under the graph of f corresponding to the interval (a) $[1, 3]$ and (b) $[a, b]$.

31 $f(x) = x^3$

32 $f(x) = x^3 + 2$

5.4 THE DEFINITE INTEGRAL

In Section 5.3 we defined the area under the graph of a function f from a to b as a limit of the form

$$\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(w_k)\,\Delta x.$$

In our discussion we restricted f and Δx as follows:

1. The function f is continuous on the closed interval $[a, b]$.

2. $f(x)$ is nonnegative for every x in $[a, b]$.

3. All the subintervals $[x_{k-1}, x_k]$ have the same length Δx.

4. The number w_k is chosen such that $f(w_k)$ is always the minimum (or maximum) value of f on $[x_{k-1}, x_k]$.

There are many applications involving this type of limit in which one or more of these conditions is not satisfied. Thus, it is desirable to allow the following changes in 1–4:

1′. The function f may be discontinuous at some numbers in $[a, b]$.

2′. $f(x)$ may be negative for some x in $[a, b]$.

3′. The lengths of the subintervals $[x_{k-1}, x_k]$ may be different.

4′. The number w_k may be *any* number in $[x_{k-1}, x_k]$.

Note that if 2′ is true, part of the graph lies under the x-axis, and therefore the limit is no longer the area under the graph of f.

Let us introduce some new terminology and notation. A **partition** P of a closed interval $[a, b]$ is any decomposition of $[a, b]$ into subintervals of the form

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

for a positive integer n and numbers x_k such that

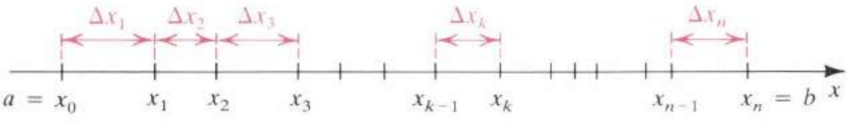
$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

The length of the kth subinterval $[x_{k-1}, x_k]$ will be denoted by Δx_k; that is,

$$\Delta x_k = x_k - x_{k-1}.$$

A typical partition of $[a, b]$ is illustrated in Figure 5.12. The largest of the numbers $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ is the **norm** of the partition P and is denoted by $\|P\|$.

FIGURE 5.12 A partition of $[a, b]$



EXAMPLE 1 The numbers $\{1, 1.7, 2.2, 3.3, 4.1, 4.5, 5, 6\}$ determine a partition P of the interval $[1, 6]$. Find the lengths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ of the subintervals in P and the norm of the partition.

SOLUTION The lengths Δx_k of the subintervals are found by subtracting successive numbers in P. Thus,

$$\Delta x_1 = 0.7, \quad \Delta x_2 = 0.5, \quad \Delta x_3 = 1.1, \quad \Delta x_4 = 0.8, \quad \Delta x_5 = 0.4,$$
$$\Delta x_6 = 0.5, \quad \Delta x_7 = 1.0.$$

The norm of P is the largest of these numbers. Hence

$$\|P\| = \Delta x_3 = 1.1.$$

The following concept, named after the mathematician G. F. B. Riemann (1826–1866), is fundamental to the definition of the definite integral.

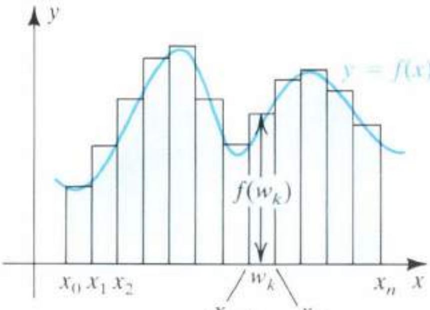
Definition (5.14)

Let f be defined on a closed interval $[a, b]$, and let P be a partition of $[a, b]$. A **Riemann sum** of f (or $f(x)$) for P is any expression R_P of the form

$$R_P = \sum_{k=1}^{n} f(w_k)\,\Delta x_k,$$

where w_k is in $[x_{k-1}, x_k]$ and $k = 1, 2, \ldots, n$.

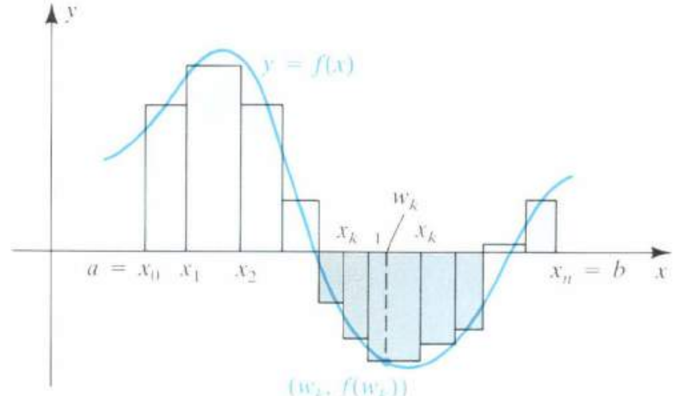
FIGURE 5.13



In Definition (5.14), $f(w_k)$ is not necessarily a maximum or minimum value of f on $[x_{k-1}, x_k]$. If we construct a rectangle of length $|f(w_k)|$ and width Δx_k, as illustrated in Figure 5.13, the rectangle may be neither inscribed nor circumscribed. Moreover, since $f(x)$ may be negative, certain terms of the Riemann sum R_P may be negative. Consequently, R_P does not always represent a sum of areas of rectangles.

We may interpret the Riemann sum R_P in (5.14) geometrically, as follows. For each subinterval $[x_{k-1}, x_k]$, construct a horizontal line segment through the point $(w_k, f(w_k))$, thereby obtaining a collection of rectangles. If $f(w_k)$ is positive, the rectangle lies above the x-axis, as illustrated by the lighter rectangles in Figure 5.14, and the product $f(w_k)\,\Delta x_k$ is the area of this rectangle. If $f(w_k)$ is negative, then the rectangle lies below the x-axis, as illustrated by the darker rectangles in Figure 5.14. In this case the product $f(w_k)\,\Delta x_k$ is the *negative* of the area of a rectangle. It follows that R_P is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis.

FIGURE 5.14



EXAMPLE 2 Let $f(x) = 8 - \frac{1}{2}x^2$, and let P be the partition of $[0, 6]$ into the five subintervals determined by

$$x_0 = 0, \quad x_1 = 1.5, \quad x_2 = 2.5, \quad x_3 = 4.5, \quad x_4 = 5, \quad x_5 = 6.$$

Find the norm of the partition and the Riemann sum R_P if

$$w_1 = 1, \quad w_2 = 2, \quad w_3 = 3.5, \quad w_4 = 5, \quad w_5 = 5.5.$$

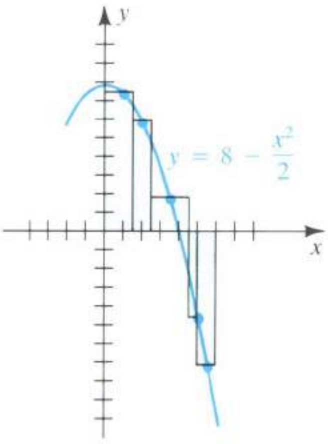


FIGURE 5.15

SOLUTION The graph of f is sketched in Figure 5.15, where we have also shown the points that correspond to w_k and the rectangles of lengths $|f(w_k)|$ for $k = 1, 2, 3, 4$, and 5. Thus,

$$\Delta x_1 = 1.5, \quad \Delta x_2 = 1, \quad \Delta x_3 = 2, \quad \Delta x_4 = 0.5, \quad \Delta x_5 = 1.$$

The norm $\|P\|$ of the partition is Δx_3, or 2.

Using Definition (5.14) with $n = 5$, we have

$$\begin{aligned} R_P &= \sum_{k=1}^{5} f(w_k)\,\Delta x_k \\ &= f(w_1)\,\Delta x_1 + f(w_2)\,\Delta x_2 + f(w_3)\,\Delta x_3 + f(w_4)\,\Delta x_4 + f(w_5)\,\Delta x_5 \\ &= f(1)(1.5) + f(2)(1) + f(3.5)(2) + f(5)(0.5) + f(5.5)(1) \\ &= (7.5)(1.5) + (6)(1) + (1.875)(2) + (-4.5)(0.5) + (-7.125)(1) \\ &= 11.625. \end{aligned}$$

We shall not always specify the number n of subintervals in a partition P of $[a, b]$. A Riemann sum (5.14) will then be written

$$R_P = \sum_k f(w_k)\,\Delta x_k,$$

and we will assume that terms of the form $f(w_k)\,\Delta x_k$ are to be summed over all subintervals $[x_{k-1}, x_k]$ of the partition P.

Using the same approach as in Definition (5.13), we next define

$$\lim_{\|P\| \to 0} \sum_k f(w_k)\,\Delta x_k = L$$

for a real number L.

Definition **(5.15)**

Let f be defined on a closed interval $[a, b]$, and let L be a real number. The statement

$$\lim_{\|P\| \to 0} \sum_k f(w_k)\,\Delta x_k = L$$

means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if P is a partition of $[a, b]$ with $\|P\| < \delta$, then

$$\left| \sum_k f(w_k)\,\Delta x_k - L \right| < \epsilon$$

for any choice of numbers w_k in the subintervals $[x_{k-1}, x_k]$ of P. The number L is a **limit of (Riemann) sums**.

For every $\delta > 0$ there are infinitely many partitions P of $[a, b]$ with $\|P\| < \delta$. Moreover, for each such partition P there are infinitely many ways of choosing the number w_k in $[x_{k-1}, x_k]$. Consequently, an infinite number of different Riemann sums may be associated with *each* partition P. However, if the limit L exists, then for any $\epsilon > 0$, every Riemann sum is within ϵ units of L, provided a small enough norm is chosen. Although Definition (5.15) differs from the definition of the limit of a function, we may use a proof similar to that given for the uniqueness theorem in Appendix II to show that if the limit L exists, then it is unique.

We next define the definite integral as a limit of a sum, where w_k and Δx_k have the same meanings as in Definition (5.15).

Definition **(5.16)**

Let f be defined on a closed interval $[a, b]$. The **definite integral of f from a to b**, denoted by $\int_a^b f(x)\,dx$, is

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_k f(w_k)\,\Delta x_k,$$

provided the limit exists.

If the limit in Definition (5.16) exists, then f is **integrable** on $[a, b]$, and we say that the definite integral $\int_a^b f(x)\,dx$ **exists**. The process of finding the limit is called **evaluating the integral**. Note that the value of a definite integral is a *real number*, not a family of antiderivatives, as was the case for indefinite integrals.

The integral sign in Definition (5.16), which may be thought of as an elongated letter S (the first letter of the word *sum*), is used to indicate the connection between definite integrals and Riemann sums. The numbers a and b are the **limits of integration**, a being the **lower limit** and b the **upper limit**. In this context *limit* refers to the smallest or largest number in the interval $[a, b]$ and is not related to definitions of limits given earlier in the text. The expression $f(x)$, which appears to the right of the integral sign, is the *integrand*, as it is with indefinite integrals. The differential symbol dx that follows $f(x)$ may be associated with the increment Δx_k of a Riemann sum of f. This association will be useful in later applications.

EXAMPLE 3 Express the following limit of sums as a definite integral on the interval $[3, 8]$:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} (5w_k^3 + \sqrt{w_k} - 4 \sin w_k)\,\Delta x_k,$$

where w_k and Δx_k are as in Definition (5.15).

SOLUTION The given limit of sums has the form stated in Definition (5.16), with

$$f(x) = 5x^3 + \sqrt{x} - 4 \sin x.$$

Hence the limit can be expressed as the definite integral

$$\int_3^8 (5x^3 + \sqrt{x} - 4 \sin x)\,dx.$$

Letters other than x may be used in the notation for the definite integral. If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_a^b f(s)\,ds = \int_a^b f(t)\,dt$$

and so on. For this reason the letter x in Definition (5.16) is called a **dummy variable**.

Whenever an interval $[a, b]$ is employed, we assume that $a < b$. Consequently, Definition (5.16) does not take into account the cases in which the lower limit of integration is greater than or equal to the upper limit. The definition may be extended to include the case where the lower limit is greater than the upper limit, as follows.

Definition **(5.17)**

If $c > d$, then $\int_c^d f(x)\,dx = -\int_d^c f(x)\,dx$.

Definition (5.17) may be phrased as follows: *Interchanging the limits of integration changes the sign of the integral.*

The case in which the lower and upper limits of integration are equal is covered by the next definition.

Definition **(5.18)**

If $f(a)$ exists, then $\int_a^a f(x)\,dx = 0$.

If f is integrable, then the limit in Definition (5.16) exists for every choice of w_k in $[x_{k-1}, x_k]$. This allows us to specialize w_k if we wish to do so. For example, we could always choose w_k as the smallest number x_{k-1} in the subinterval or as the largest number x_k or as the midpoint of the subinterval or as the number that always produces the minimum or maximum value in $[x_{k-1}, x_k]$. Moreover, since the limit is independent of the partition P of $[a, b]$ (provided that $\|P\|$ is sufficiently small), we may specialize the partitions to the case in which all the subintervals $[x_{k-1}, x_k]$ have the same length Δx. A partition of this type is a **regular partition**.

If a regular partition of $[a, b]$ contains n subintervals, then $\Delta x = (b - a)/n$. In this case the symbol $\|P\| \to 0$ is equivalent to $\Delta x \to 0$ or $n \to \infty$, and Definition (5.16) takes on the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\,\Delta x.$$

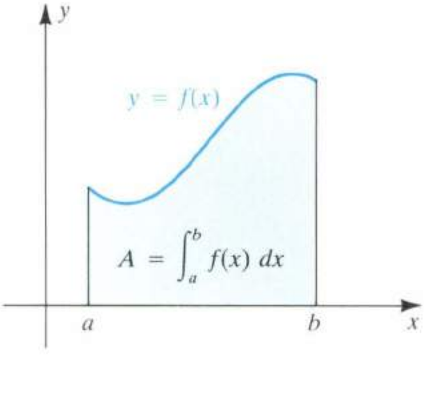
The following theorem is a first application of these special Riemann sums. Many other applications will be discussed in Chapter 6.

Theorem **(5.19)**

If f is integrable and $f(x) \geq 0$ for every x in $[a, b]$, then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)\,dx.$$

FIGURE 5.16



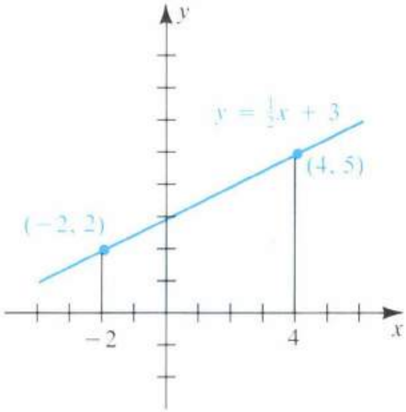
PROOF From the preceding section we know that the area A is a limit of sums $\sum_k f(u_k)\,\Delta x$, where $f(u_k)$ is the minimum value of f on $[x_{k-1}, x_k]$. Since these are Riemann sums, the conclusion follows from Definition (5.16). ■

Theorem (5.19) is illustrated in Figure 5.16. It is important to keep in mind that area is merely our first application of the definite integral. *There are many instances where $\int_a^b f(x)\,dx$ does not represent the area of a region.*

In fact, if $f(x) < 0$ for some x in $[a, b]$, then the definite integral may be negative or zero.

If f is continuous and $f(x) \geq 0$ on $[a, b]$, then Theorem (5.19) can be used to evaluate $\int_a^b f(x)\,dx$, provided we can find the area of the region under the graph of f from a to b. This will be true if the graph is a line or part of a circle, as in the following examples. (We consider more complicated definite integrals later in this chapter.) When evaluating a definite integral using this empirical technique, remember that the area of the region and the value of the integral are *numerically equal*; that is, if the area is A square units, the value of the integral is the real number A.

FIGURE 5.17

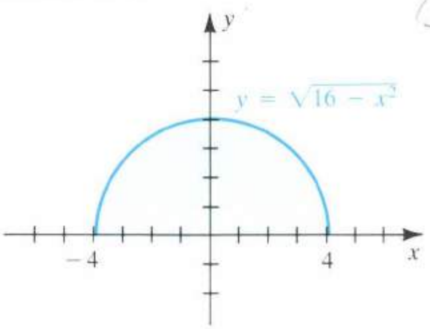


EXAMPLE 4 Evaluate $\int_{-2}^{4} (\frac{1}{2}x + 3)\,dx$.

SOLUTION If $f(x) = \frac{1}{2}x + 3$, then the graph of f is the line sketched in Figure 5.17. By Theorem (5.19), the value of the integral is numerically equal to the area of the region under this line from $x = -2$ to $x = 4$. The region is a trapezoid with bases parallel to the y-axis of length 2 and 5 and altitude on the x-axis of length 6. Using the formula for the area of a trapezoid, we obtain

$$\int_{-2}^{4} (\tfrac{1}{2}x + 3)\,dx = \tfrac{1}{2}(2 + 5)6 = 21.$$

FIGURE 5.18



EXAMPLE 5 Evaluate $\int_{-4}^{4} \sqrt{16 - x^2}\,dx$.

SOLUTION If $f(x) = \sqrt{16 - x^2}$, then the graph of f is the semicircle shown in Figure 5.18. By Theorem (5.19), the value of the integral is numerically equal to the area of the region under this semicircle from $x = -4$ to $x = 4$. Hence

$$\int_{-4}^{4} \sqrt{16 - x^2}\,dx = \tfrac{1}{2} \cdot \pi(4)^2 = 8\pi.$$

EXAMPLE 6 Evaluate

(a) $\int_{4}^{-4} \sqrt{16 - x^2}\,dx$ **(b)** $\int_{4}^{4} \sqrt{16 - x^2}\,dx$

SOLUTION

(a) Using Definition (5.17) and Example 5, we have

$$\int_{4}^{-4} \sqrt{16 - x^2}\,dx = -\int_{-4}^{4} \sqrt{16 - x^2}\,dx = -8\pi.$$

(b) By Definition (5.18),

$$\int_{4}^{4} \sqrt{16 - x^2}\,dx = 0.$$

The next theorem states that functions that are continuous on closed intervals are integrable. This fact will play a crucial role in the proof of the fundamental theorem of calculus in Section 5.6.

Theorem **(5.20)**

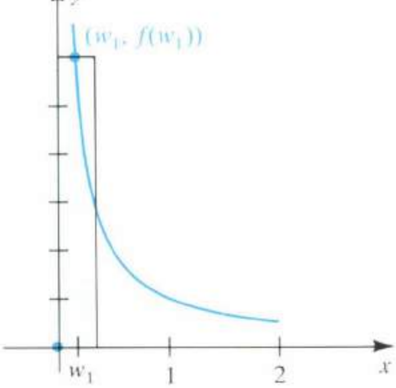
If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

A proof of Theorem (5.20) may be found in texts on advanced calculus.

Definite integrals of discontinuous functions may or may not exist, depending on the types of discontinuities. In particular, *functions that have infinite discontinuities on a closed interval are not integrable on that interval.* To illustrate, consider the piecewise-defined function f with domain $[0, 2]$ such that

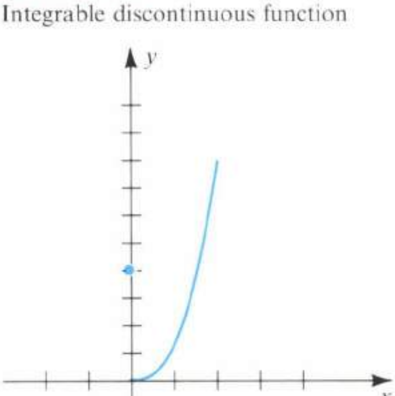
$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{1}{x} & \text{if } 0 < x \le 2 \end{cases}.$$

FIGURE 5.19
Nonintegrable discontinuous function



The graph of f is sketched in Figure 5.19. Note that $\lim_{x \to 0^+} f(x) = \infty$. If M is any (large) positive number, then in the first subinterval $[x_0, x_1]$ of any partition P of $[a, b]$, we can find a number w_1 such that $f(w_1) > M/\Delta x_1$, or, equivalently, $f(w_1)\,\Delta x_1 > M$. It follows that there are Riemann sums $\sum_k f(w_k)\,\Delta x_k$ that are arbitrarily large, and hence the limit in Definition (5.16) cannot exist. Thus, f is not integrable. A similar argument can be given for *any* function that has an infinite discontinuity in $[a, b]$. Consequently, *if a function f is integrable on $[a, b]$, then it is bounded on $[a, b]$; that is, there is a real number M such that $|f(x)| \le M$ for every x in $[a, b]$.*

FIGURE 5.20
Integrable discontinuous function



As an illustration of a discontinuous function that *is* integrable, consider the piecewise-defined function f with domain $[0, 2]$ such that

$$f(x) = \begin{cases} 4 & \text{if } x = 0 \\ x^3 & \text{if } 0 < x \le 2 \end{cases}$$

The graph of f is sketched in Figure 5.20. Note that f has a jump discontinuity at $x = 0$. From Example 7 of Section 5.3, the area under the graph of $y = x^3$ from 0 to 2 is $2^4/4 = 4$. Thus, by Theorem (5.19), $\int_0^2 x^3\,dx = 4$. We can also show that $\int_0^2 f(x)\,dx = 4$. Hence f is integrable.

We have shown that a function that is discontinuous on a closed interval may or may not be integrable. However, by Theorem (5.20), functions that are *continuous* on a closed interval are *always* integrable.

EXERCISES 5.4

Exer. 1–4: The given numbers determine a partition P of an interval. (a) Find the length of each subinterval of P. (b) Find the norm $\|P\|$ of the partition.

1 $\{0, 1.1, 2.6, 3.7, 4.1, 5\}$

2 $\{2, 3, 3.7, 4, 5.2, 6\}$

3 $\{-3, -2.7, -1, 0.4, 0.9, 1\}$

4 $\{1, 1.6, 2, 3.5, 4\}$

Exer. 5–6: Let P be the partition of $[1, 5]$ determined by $\{1, 3, 4, 5\}$. Find a Riemann sum R_P for the given function f by choosing, in each subinterval of P, (a) the right-hand endpoint, (b) the left-hand endpoint, and (c) the midpoint.

5 $f(x) = 2x + 3$

6 $f(x) = 3 - 4x$

7 If $f(x) = 8 - \frac{1}{2}x^2$ and P is the regular partition of $[0, 6]$ into six subintervals, find a Riemann sum R_P of f by choosing the midpoint of each subinterval.

8 If $f(x) = 8 - \frac{1}{2}x^2$ and P is the partition of $[0, 6]$ determined by $\{0, 1.5, 3, 4.5, 6\}$, find a Riemann sum R_P of f by choosing the numbers 1, 2, 4, and 5 in the subintervals of P.

9 If $f(x) = x^3$ and P is the partition of $[-2, 4]$ determined by $\{-2, 0, 1, 3, 4\}$, find a Riemann sum R_P of f by choosing the numbers -1, 1, 2, and 4 in the subintervals of P.

10 If $f(x) = \sqrt{x}$ and P is the partition of $[1, 16]$ determined by $\{1, 3, 5, 7, 9, 16\}$, find a Riemann sum R_P of f by choosing the numbers 1, 4, 5, 9, and 9 in the subintervals of P.

c 11 If $f(x) = x^2\sqrt{\cos x}$ and P is the regular partition of $[0, 1]$ into ten subintervals, find a Riemann sum R_P of f by choosing the midpoint of each subinterval of P.

c 12 If $f(x) = \sin(\cos x)$ and P is the partition of $[-1, 1]$ determined by $\{-1, -0.65, -0.23, 0.51, 0.85, 1\}$, find a Riemann sum R_P of f by choosing the numbers -0.75, -0.5, 0, 0.6, 0.9 in the subintervals of P.

Exer. 13–16: Use Definition (5.16) to express each limit as a definite integral on the given interval $[a, b]$.

13 $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (3w_k^2 - 2w_k + 5)\,\Delta x_k;\quad [-1, 2]$

14 $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \pi(w_k^2 - 4)\,\Delta x_k;\quad [2, 3]$

15 $\lim_{\|P\| \to 0} \sum_{k=1}^{n} 2\pi w_k(1 + w_k^3)\,\Delta x_k;\quad [0, 4]$

16 $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\sqrt[3]{w_k} + 4w_k)\,\Delta x_k;\quad [-4, -3]$

Exer. 17–22: Given $\int_1^4 \sqrt{x}\,dx = \frac{14}{3}$, evaluate the integral.

17 $\int_4^1 \sqrt{x}\,dx$

18 $\int_1^4 \sqrt{s}\,ds$

19 $\int_1^4 \sqrt{t}\,dt$

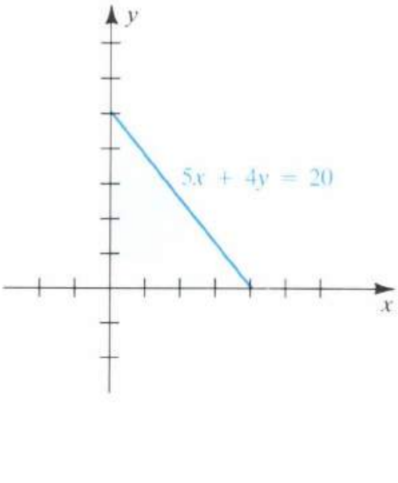
20 $\int_1^4 \sqrt{x}\,dx + \int_4^1 \sqrt{x}\,dx$

21 $\int_4^4 \sqrt{x}\,dx + \int_4^1 \sqrt{x}\,dx$

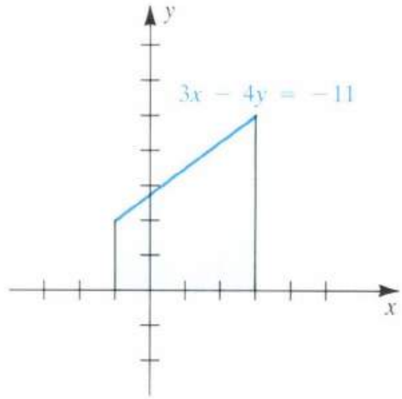
22 $\int_4^4 \sqrt{x}\,dx$

Exer. 23–26: Express the area of the region in the figure as a definite integral.

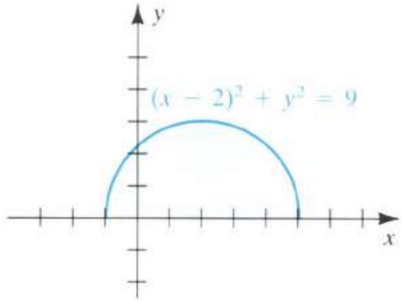
23



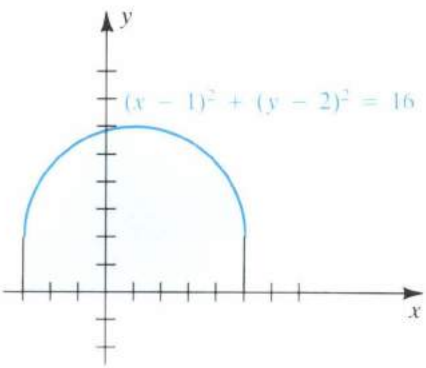
24



25



26



Exer. 27–36: Evaluate the definite integral by regarding it as the area under the graph of a function.

27 $\int_{-1}^{5} 6\,dx$

28 $\int_{-2}^{3} 4\,dx$

29 $\int_{-3}^{2} (2x + 6)\,dx$

30 $\int_{-1}^{2} (7 - 3x)\,dx$

31 $\int_{0}^{3} |x - 1|\,dx$

32 $\int_{-1}^{4} |x|\,dx$

33 $\int_0^3 \sqrt{9 - x^2}\,dx$

34 $\int_0^a \sqrt{a^2 - x^2}\,dx,\quad a > 0$

35 $\int_{-2}^{2} (3 + \sqrt{4 - x^2})\,dx$

36 $\int_{-2}^{2} (3 - \sqrt{4 - x^2})\,dx$

5.5 PROPERTIES OF THE DEFINITE INTEGRAL

This section contains some fundamental properties of the definite integral. Most of the proofs are difficult and have been placed in Appendix II.

Theorem (5.21) If c is a real number, then

$$\int_a^b c\,dx = c(b-a).$$

PROOF Let f be the constant function defined by $f(x) = c$ for every x in $[a, b]$. If P is a partition of $[a, b]$, then for every Riemann sum of f,

$$\sum_k f(w_k)\,\Delta x_k = \sum_k c\,\Delta x_k = c\sum_k \Delta x_k = c(b-a).$$

(The last equality is true because the sum $\sum_k \Delta x_k$ is the length of the interval $[a, b]$.) Consequently,

$$\left|\sum_k f(w_k)\,\Delta x_k - c(b-a)\right| = |c(b-a) - c(b-a)| = 0,$$

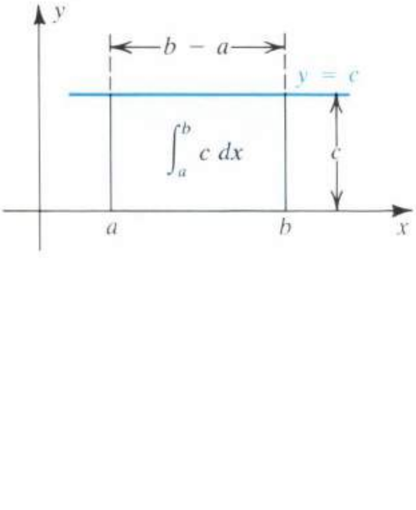
which is less than any positive number ϵ *regardless* of the size of $\|P\|$. Thus, by Definition (5.15), with $L = c(b-a)$,

$$\lim_{\|P\|\to 0} \sum_k f(w_k)\,\Delta x_k = \lim_{\|P\|\to 0} \sum_k c\,\Delta x_k = c(b-a).$$

By Definition (5.16), this means that

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a).$$

FIGURE 5.21



Note that if $c > 0$, then Theorem (5.21) agrees with Theorem (5.19): As illustrated in Figure 5.21, the graph of f is the horizontal line $y = c$, and the region under the graph from a to b is a rectangle with sides of lengths c and $b - a$. Hence the area $\int_a^b f(x)\,dx$ of the rectangle is $c(b-a)$.

EXAMPLE 1 Evaluate $\int_{-2}^{3} 7\,dx$.

SOLUTION Using Theorem (5.21) yields

$$\int_{-2}^{3} 7\,dx = 7[3-(-2)] = 7(5) = 35.$$

If $c = 1$ in Theorem (5.21), we shall abbreviate the integrand as follows:

$$\int_a^b dx = b - a$$

If a function f is integrable on $[a, b]$ and c is a real number, then, by Theorem (5.11)(ii), a Riemann sum of the function cf may be written

$$\sum_k cf(w_k)\,\Delta x_k = c\sum_k f(w_k)\,\Delta x_k.$$

We can prove that the limit of the sums on the left of the last equation is equal to c times the limit of the sums on the right. This gives us the next theorem. A proof may be found in Appendix II.

Theorem **(5.22)**

If f is integrable on $[a, b]$ and c is any real number, then cf is integrable on $[a, b]$ and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Theorem (5.22) is sometimes stated as follows: *A constant factor in the integrand may be taken outside the integral sign.* It is *not* permissible to take expressions involving variables outside the integral sign in this manner.

If two functions f and g are defined on $[a, b]$, then, by Theorem (5.11)(i), a Riemann sum of $f + g$ may be written

$$\sum_k [f(w_k) + g(w_k)]\,\Delta x_k = \sum_k f(w_k)\,\Delta x_k + \sum_k g(w_k)\,\Delta x_k.$$

We can show that if f and g are integrable, then the limit of the sums on the left may be found by adding the limits of the two sums on the right. This fact is stated in integral form in (i) of the next theorem. A proof of (i) may be found in Appendix II. The analogous result for differences is stated in (ii) of the theorem.

Theorem **(5.23)**

If f and g are integrable on $[a, b]$, then $f + g$ and $f - g$ are integrable on $[a, b]$ and

(i) $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(ii) $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Theorem (5.23)(i) may be extended to any finite number of functions. Thus, if $f_1, f_2, \ldots, f_n$ are integrable on $[a, b]$, then so is their sum and

$$\begin{aligned}\int_a^b [f_1(x) + f_2(x) + \cdots + f_n(x)]\,dx \\ = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)\,dx.\end{aligned}$$

EXAMPLE 2 It will follow from the results in Section 5.6 that

$$\int_0^2 x^3\,dx = 4 \quad \text{and} \quad \int_0^2 x\,dx = 2.$$

Use these facts to evaluate $\int_0^2 (5x^3 - 3x + 6)\,dx$.

SOLUTION We may proceed as follows:

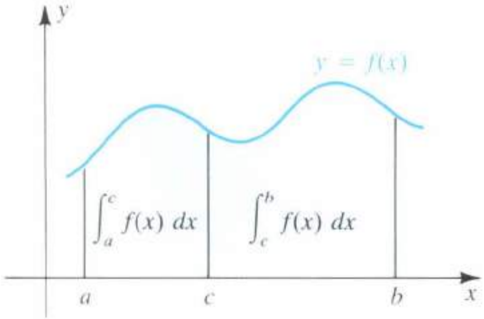
$$\begin{aligned}\int_0^2 (5x^3 - 3x + 6)\,dx &= \int_0^2 5x^3\,dx - \int_0^2 3x\,dx + \int_0^2 6\,dx \\ &= 5\int_0^2 x^3\,dx - 3\int_0^2 x\,dx + 6(2 - 0) \\ &= 5(4) - 3(2) + 12 = 26\end{aligned}$$

If f is continuous on $[a, b]$ and $f(x) \geq 0$ for every x in $[a, b]$, then, by Theorem (5.19), the integral $\int_a^b f(x)\,dx$ is the area under the graph of f from a to b. Similarly, if $a < c < b$, then the integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are the areas under the graph of f from a to c and from c to b, respectively, as illustrated in Figure 5.22. Since the area from a to b is the sum of the two smaller areas, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The next theorem shows that the preceding equality is true under a more general hypothesis. The proof is given in Appendix II.

FIGURE 5.22



Theorem **(5.24)**

> If $a < c < b$ and if f is integrable on both $[a, c]$ and $[c, b]$, then f is integrable on $[a, b]$ and
>
> $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The following result is a generalization of Theorem (5.24) to the case where c is not necessarily between a and b.

Theorem **(5.25)**

> If f is integrable on a closed interval and if a, b, and c are any three numbers in the interval, then
>
> $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

PROOF If a, b, and c are all different, then there are six possible ways of arranging these three numbers. The theorem should be verified for each of these cases and also for the cases in which two or all three of the numbers are equal. We shall verify one case. Suppose $c < a < b$. By Theorem (5.24),

$$\int_c^b f(x)\,dx = \int_c^a f(x)\,dx + \int_a^b f(x)\,dx,$$

which, in turn, may be written

$$\int_a^b f(x)\,dx = -\int_c^a f(x)\,dx + \int_c^b f(x)\,dx.$$

The conclusion of the theorem now follows from the fact that interchanging the limits of integration changes the sign of the integral (see Definition (5.17)). ■

EXAMPLE 3 Express as one integral:

$$\int_2^7 f(x)\,dx - \int_5^7 f(x)\,dx$$

SOLUTION First we interchange the limits of the second integral using Definition (5.17), and then we use Theorem (5.25) with $a = 2$, $b = 5$, and $c = 7$:

$$\begin{aligned}\int_2^7 f(x)\,dx - \int_5^7 f(x)\,dx &= \int_2^7 f(x)\,dx + \int_7^5 f(x)\,dx \\ &= \int_2^5 f(x)\,dx\end{aligned}$$

If f and g are continuous on $[a, b]$ and $f(x) \geq g(x) \geq 0$ for every x in $[a, b]$, then the area under the graph of f from a to b is greater than or equal to the area under the graph of g from a to b. The corollary to the next theorem is a generalization of this fact to arbitrary integrable functions. The proof of the theorem is given in Appendix II.

Theorem **(5.26)**

If f is integrable on $[a, b]$ and $f(x) \geq 0$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \geq 0.$$

Corollary **(5.27)**

If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for every x in $[a, b]$, then

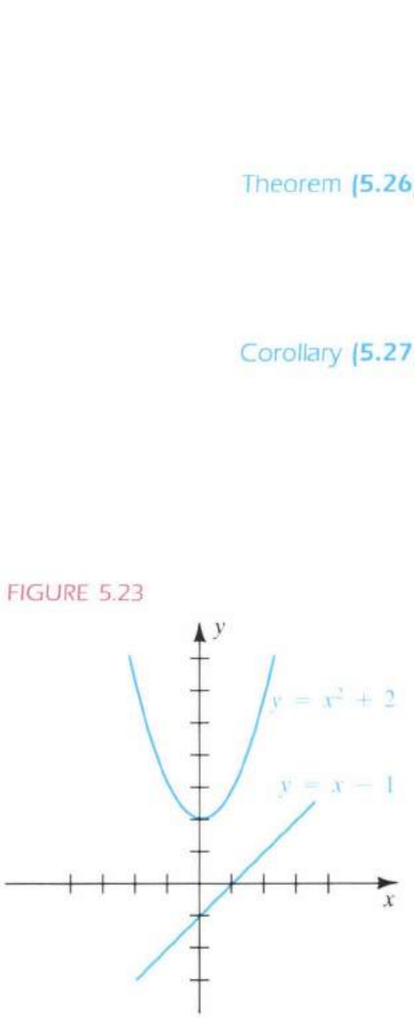
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

PROOF By Theorem (5.23), $f - g$ is integrable. Moreover, since $f(x) \geq g(x)$, $f(x) - g(x) \geq 0$ for every x in $[a, b]$. Hence, by Theorem (5.26),

$$\int_a^b [f(x) - g(x)]\,dx \geq 0.$$

Applying Theorem (5.23)(ii) leads to the desired conclusion.

FIGURE 5.23



EXAMPLE 4 Show that $\int_{-1}^2 (x^2 + 2)\,dx \geq \int_{-1}^2 (x - 1)\,dx$.

SOLUTION The graphs of $y = x^2 + 2$ and $y = x - 1$ are sketched in Figure 5.23. Since

$$x^2 + 2 \geq x - 1$$

for every x in $[-1, 2]$, the conclusion follows from Corollary (5.27).

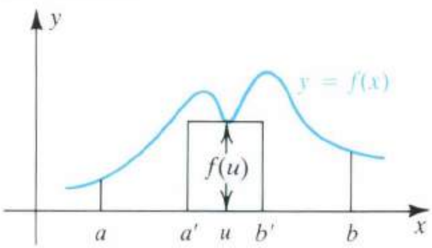
Suppose, in Theorem (5.26), that f is continuous and that, in addition to the condition $f(x) \geq 0$, we have $f(c) > 0$ for some c in $[a, b]$. In this case $\lim_{x \to c} f(x) > 0$, and, by Theorem (2.6), there is a subinterval $[a', b']$

of $[a, b]$ throughout which $f(x)$ is positive. If $f(u)$ is the minimum value of f on $[a', b']$ (see Figure 5.24), then the area under the graph of f from a to b is at least as large as the area $f(u)(b' - a')$ of the pictured rectangle. Consequently, $\int_a^b f(x)\,dx > 0$. It now follows, as in the proof of (5.27), that if f and g are continuous on $[a, b]$, if $f(x) \geq g(x)$ throughout $[a, b]$, and if $f(x) > g(x)$ for some x in $[a, b]$, then $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$. This fact will be used in the proof of the next theorem.

Mean value theorem for definite integrals (5.28)

> If f is continuous on a closed interval $[a, b]$, then there is a number z in the open interval (a, b) such that
>
> $$\int_a^b f(x)\,dx = f(z)(b - a).$$

FIGURE 5.24



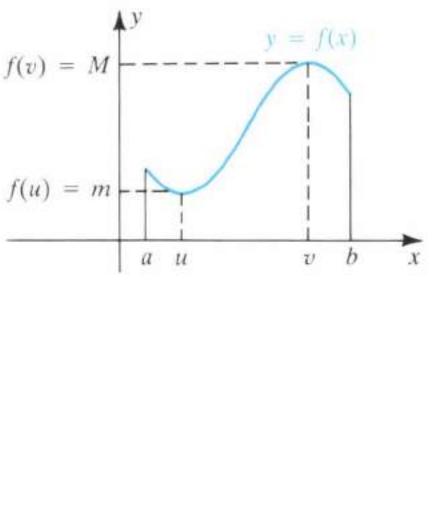
PROOF If f is a constant function, then $f(x) = c$ for some number c, and by Theorem (5.21),

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b - a) = f(z)(b - a)$$

for every number z in (a, b).

Next assume that f is not a constant function and suppose that m and M are the minimum and maximum values of f, respectively, on $[a, b]$. Let $f(u) = m$ and $f(v) = M$ for some u and v in $[a, b]$. This is illustrated in Figure 5.25 for the case in which $f(x)$ is positive throughout $[a, b]$. Since f is not a constant function, $m < f(x) < M$ for some x in $[a, b]$. Hence, by the remark immediately preceding this theorem,

FIGURE 5.25



$$\int_a^b m\,dx < \int_a^b f(x)\,dx < \int_a^b M\,dx.$$

Applying Theorem (5.21) yields

$$m(b - a) < \int_a^b f(x)\,dx < M(b - a).$$

Dividing by $b - a$ and recalling that $m = f(u)$ and $M = f(v)$ gives us

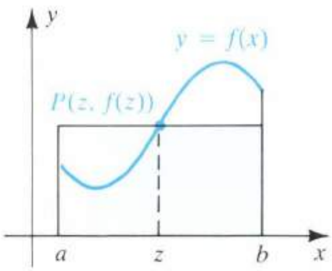
$$f(u) < \frac{1}{b - a}\int_a^b f(x)\,dx < f(v).$$

Since $[1/(b - a)]\int_a^b f(x)\,dx$ is a number between $f(u)$ and $f(v)$, it follows from the intermediate value theorem (2.26) that there is a number z, with $u < z < v$, such that

$$f(z) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Multiplying both sides by $b - a$ gives us the conclusion of the theorem.

FIGURE 5.26



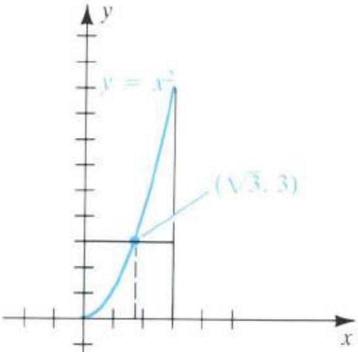
The number z of Theorem (5.28) is not necessarily unique; however, the theorem guarantees that *at least* one number z will produce the desired result.

The mean value theorem has an interesting geometric interpretation if $f(x) \geq 0$ on $[a, b]$. In this case $\int_a^b f(x)\,dx$ is the area under the graph of f from a to b. If, as in Figure 5.26, a horizontal line is drawn through

the point $P(z, f(z))$, then the area of the rectangular region bounded by this line, the x-axis, and the lines $x = a$ and $x = b$ is $f(z)(b - a)$, which, according to Theorem (5.28), is the same as the area under the graph of f from a to b.

EXAMPLE 5 It will follow from the results of Section 5.6 that $\int_0^3 x^2\,dx = 9$. Find a number z that satisfies the conclusion of the mean value theorem (5.28) for this definite integral.

FIGURE 5.27



SOLUTION The graph of $f(x) = x^2$ for $0 \le x \le 3$ is sketched in Figure 5.27. By the mean value theorem, there is a number z between 0 and 3 such that

$$\int_0^3 x^2\,dx = f(z)(3 - 0) = z^2(3).$$

This implies that

$$9 = 3z^2, \quad \text{or} \quad z^2 = 3.$$

The solutions of the last equation are $z = \pm\sqrt{3}$; however, $-\sqrt{3}$ is not in $[0, 3]$. The number $z = \sqrt{3}$ satisfies the conclusion of the theorem.

If we consider the horizontal line through $P(\sqrt{3}, 3)$, then the area of the rectangle bounded by this line, the x-axis, and the lines $x = 0$ and $x = 3$ is equal to the area under the graph of f from $x = 0$ to $x = 3$ (see Figure 5.27).

In statistics the term **arithmetic mean** is used for the **average** of a set of numbers. Thus, the arithmetic mean of two numbers a and b is $(a + b)/2$, the arithmetic mean of three numbers a, b, and c is $(a + b + c)/3$, and so on. To see the relationship between arithmetic means and the word *mean* used in mean value theorem, let us rewrite the conclusion of (5.28) as

$$f(z) = \frac{1}{b - a}\int_a^b f(x)\,dx$$

and express the definite integral as a limit of sums. If we specialize Definition (5.16) by using a regular partition P with n subintervals, then

$$f(z) = \frac{1}{b - a}\lim_{n\to\infty}\sum_{k=1}^{n} f(w_k)\,\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left[f(w_k)\frac{\Delta x}{b - a}\right]$$

for any number w_k in the kth subinterval of P and $\Delta x = (b - a)/n$. Since $\Delta x/(b - a) = 1/n$, we obtain

$$f(z) = \lim_{n\to\infty}\sum_{k=1}^{n}\left[f(w_k)\frac{\Delta x}{b - a}\right] = \lim_{n\to\infty}\sum_{k=1}^{n}\left[f(w_k)\frac{1}{n}\right],$$

or

$$f(z) = \lim_{n\to\infty}\left[\frac{f(w_1) + f(w_2) + \cdots + f(w_n)}{n}\right].$$

This shows that we may regard the number $f(z)$ in the mean value theorem (5.28) as a limit of the arithmetic means (averages) of the function values $f(w_1), f(w_2), \ldots, f(w_n)$ as n increases without bound. This is the motivation for the next definition.

Definition (5.29)

Let f be continuous on $[a, b]$ The **average value** f_{av} of f on $[a, b]$ is

$$f_{av} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Note that, by the mean value theorem for definite integrals, if f is continuous on $[a, b]$, then

$$f_{av} = f(z) \quad \text{for some } z \text{ in } [a, b].$$

EXAMPLE 6 Given $\int_0^3 x^2\,dx = 9$, find the average value of f on $[0, 3]$.

SOLUTION By Definition (5.29), with $a = 0$, $b = 3$, and $f(x) = x^2$,

$$f_{av} = \frac{1}{3-0} \int_0^3 x^2\,dx = \frac{1}{3} \cdot 9 = 3.$$

In the interval $[0, 3]$, the function values $f(x) = x^2$ range from $f(0) = 0$ to $f(3) = 9$. Note that the function f *takes on* its average value 3 at the number $z = \sqrt{3}$.

EXERCISES 5.5

Exer. 1–6: Evaluate the integral.

1 $\int_{-2}^4 5\,dx$ 2 $\int_1^{10} \sqrt{2}\,dx$ 3 $\int_6^2 3\,dx$

4 $\int_4^{-3} dx$ 5 $\int_{-1}^1 dx$ 6 $\int_2^2 100\,dx$

Exer. 7–10: It will follow from the results in Section 5.6 that

$$\int_1^4 x^2\,dx = 21 \quad \text{and} \quad \int_1^4 x\,dx = \tfrac{15}{2}.$$

Use these facts to evaluate the integral.

7 $\int_1^4 (3x^2 + 5)\,dx$ 8 $\int_1^4 (6x - 1)\,dx$

9 $\int_1^4 (2 - 9x - 4x^2)\,dx$ 10 $\int_1^4 (3x + 2)^2\,dx$

Exer. 11–16: Verify the inequality without evaluating the integrals.

11 $\int_1^2 (3x^2 + 4)\,dx \ge \int_1^2 (2x^2 + 5)\,dx$

12 $\int_1^4 (2x + 2)\,dx \le \int_1^4 (3x + 1)\,dx$

13 $\int_2^4 (x^2 - 6x + 8)\,dx \le 0$

14 $\int_2^4 (5x^2 - x + 1)\,dx \ge 0$

15 $\int_0^{2\pi} (1 + \sin x)\,dx \ge 0$

16 $\int_{-\pi/3}^{\pi/3} (\sec x - 2)\,dx \le 0$

Exer. 17–22: Express as one integral.

17 $\int_5^1 f(x)\,dx + \int_{-3}^5 f(x)\,dx$

18 $\int_4^1 f(x)\,dx + \int_6^4 f(x)\,dx$

19 $\int_c^d f(x)\,dx + \int_e^c f(x)\,dx$

20 $\int_{-2}^6 f(x)\,dx - \int_{-2}^2 f(x)\,dx$

21 $\int_c^{c+h} f(x)\,dx - \int_c^h f(x)\,dx$

22 $\int_c^m f(x)\,dx - \int_d^m f(x)\,dx$

Exer. 23–30: The given integral $\int_a^b f(x)\,dx$ may be verified using the results in Section 5.6. (a) Find a number z that satisfies the conclusion of the mean value theorem (5.28). (b) Find the average value of f on $[a, b]$.

23 $\int_0^3 3x^2\,dx = 27$ 24 $\int_{-4}^{-1} \frac{3}{x^2}\,dx = \frac{9}{4}$

25 $\int_{-2}^1 (x^2 + 1)\,dx = 6$

26 $\int_{-1}^3 (3x^2 - 2x + 3)\,dx = 32$

27 $\int_{-1}^8 3\sqrt{x+1}\,dx = 54$ 28 $\int_{-2}^{-1} \frac{8}{x^3}\,dx = -3$

29 $\int_1^2 (4x^3 - 1)\,dx = 14$ 30 $\int_1^4 (2 + 3\sqrt{x})\,dx = 20$

c **Exer. 31–32: The given integral may be verified using results in Section 5.6. Use Newton's method to approximate, to three decimal places, a number z that satisfies the conclusion of the mean value theorem (5.28).**

31 $\int_{-2}^{3} (8x^3 + 3x - 1)\, dx = 132.5$

32 $\int_{\pi/6}^{\pi/4} (1 - \cos 4x)\, dx = \dfrac{\pi}{12} + \dfrac{\sqrt{3}}{8}$

33 Let f and g be integrable on $[a, b]$. If c and d are any real numbers, prove that

$$\int_a^b [cf(x) + dg(x)]\, dx = c \int_a^b f(x)\, dx + d \int_a^b g(x)\, dx.$$

34 If f is continuous on $[a, b]$, prove that

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx.$$

(*Hint:* $-|f(x)| \le f(x) \le |f(x)|$.)

5.6 THE FUNDAMENTAL THEOREM OF CALCULUS

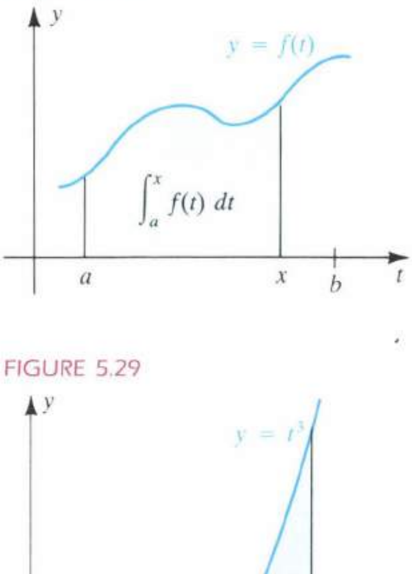
This section contains one of the most important theorems in calculus. In addition to being useful in evaluating definite integrals, the theorem also exhibits the relationship between derivatives and definite integrals. This theorem, aptly called *the fundamental theorem of calculus*, was discovered independently by Sir Isaac Newton (1642–1727) in England and by Gottfried Wilhelm Leibniz (1646–1716) in Germany. It is primarily because of this discovery that both men are credited with the invention of calculus.

To avoid confusion in the following discussion, we shall use t as the independent variable and denote the definite integral of f from a to b by $\int_a^b f(t)\, dt$. If f is continuous on $[a, b]$ and $a \le x \le b$, then f is continuous on $[a, x]$; therefore, by Theorem (5.20), f is integrable on $[a, x]$. Consequently, the formula

$$G(x) = \int_a^x f(t)\, dt$$

determines a function G with domain $[a, b]$, since for each x in $[a, b]$ there corresponds a unique number $G(x)$.

FIGURE 5.28

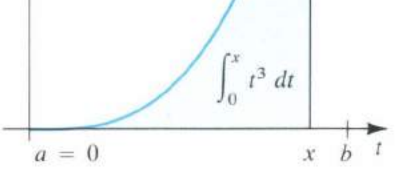


To obtain a geometric interpretation of $G(x)$, suppose that $f(t) \ge 0$ for every t in $[a, b]$. In this case we see from Theorem (5.19) that $G(x)$ is the area of the region under the graph of f from a to x (see Figure 5.28).

As a specific illustration, consider $f(t) = t^3$ with $a = 0$ and $b > 0$ (see Figure 5.29). In Example 7 of Section 5.3 we proved that the area under the graph of f from 0 to b is $\frac{1}{4}b^4$. Hence the area from 0 to x is

$$G(x) = \int_0^x t^3\, dt = \tfrac{1}{4}x^4.$$

FIGURE 5.29



This gives us an explicit form for the function G if $f(t) = t^3$. Note that in this illustration,

$$G'(x) = D_x\left(\tfrac{1}{4}x^4\right) = x^3 = f(x).$$

Thus, by Definition (5.1), G is an antiderivative of f. This result is not an accident. Part I of the next theorem brings out the remarkable fact that if f is *any* continuous function and $G(x) = \int_a^x f(t)\, dt$, then G is an antiderivative of f. Part II of the theorem shows how *any* antiderivative may be used to find the value of $\int_a^b f(x)\, dx$.

Fundamental theorem of calculus **(5.30)**

Suppose f is continuous on a closed interval $[a, b]$.

Part I If the function G is defined by

$$G(x) = \int_a^x f(t)\,dt$$

for every x in $[a, b]$, then G is an antiderivative of f on $[a, b]$.

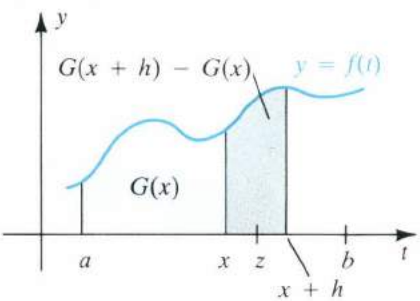
Part II If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

PROOF To establish Part I, we must show that if x is in $[a, b]$, then $G'(x) = f(x)$; that is,

$$\lim_{h \to 0} \frac{G(x+h) - G(x)}{h} = f(x).$$

FIGURE 5.30



Before giving a formal proof, let us consider some geometric aspects of this limit. If $f(x) \geq 0$ throughout $[a, b]$, then $G(x)$ is the area under the graph of f from a to x, as illustrated in Figure 5.30. If $h > 0$, then the difference $G(x+h) - G(x)$ is the area under the graph of f from x to $x + h$, and the number h is the length of the interval $[x, x+h]$. We shall show that

$$\frac{G(x+h) - G(x)}{h} = f(z)$$

for some number z between x and $x + h$. Apparently, if $h \to 0$, then $z \to x$ and $f(z) \to f(x)$, which is what we wish to prove.

Let us now give a rigorous proof that $G'(x) = f(x)$. If x and $x + h$ are in $[a, b]$, then using the definition of G together with Definition (5.17) and Theorem (5.24) yields

$$\begin{aligned} G(x+h) - G(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt. \end{aligned}$$

Consequently, if $h \neq 0$,

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\,dt.$$

If $h > 0$, then, by the mean value theorem (5.28), there is a number z in the open interval $(x, x+h)$ such that

$$\int_x^{x+h} f(t)\,dt = f(z)h$$

and, therefore,
$$\frac{G(x+h) - G(x)}{h} = f(z).$$

Since $x < z < x + h$, it follows from the continuity of f that

$$\lim_{h \to 0^+} f(z) = \lim_{z \to x^+} f(z) = f(x)$$

and hence $$\lim_{h\to 0^+} \frac{G(x+h) - G(x)}{h} = \lim_{h\to 0^+} f(z) = f(x).$$

If $h < 0$, then we may prove in similar fashion that

$$\lim_{h\to 0^-} \frac{G(x+h) - G(x)}{h} = f(x).$$

The two preceding one-sided limits imply that

$$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = f(x).$$

This completes the proof of Part I.

To prove Part II, let F be any antiderivative of f and let G be the special antiderivative defined in Part I. From Theorem (5.2) we know that there is a constant C such that

$$G(x) = F(x) + C$$

for every x in $[a, b]$. Hence, from the definition of G,

$$\int_a^x f(t)\, dt = F(x) + C$$

for every x in $[a, b]$. If we let $x = a$ and use the fact that $\int_a^a f(t)\, dt = 0$, we obtain $0 = F(a) + C$, or $C = -F(a)$. Consequently,

$$\int_a^x f(t)\, dt = F(x) - F(a).$$

This is an identity for every x in $[a, b]$, so we may substitute b for x, obtaining

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Replacing the variable t by x gives us the conclusion of Part II. ■

We often denote the difference $F(b) - F(a)$ either by $F(x)]_a^b$ or by $[F(x)]_a^b$. Part II of the fundamental theorem may then be expressed as follows.

Corollary (5.31)

If f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\, dx = F(x)\Big]_a^b = F(b) - F(a).$$

The formula in Corollary (5.31) is also valid if $a \geq b$. If $a > b$, then, by Definition (5.17),

$$\begin{aligned}\int_a^b f(x)\, dx &= -\int_b^a f(x)\, dx \\ &= -[F(a) - F(b)] \\ &= F(b) - F(a).\end{aligned}$$

If $a = b$, then, by Definition (5.18),

$$\int_a^a f(x)\, dx = 0 = F(a) - F(a).$$

Corollary (5.31) allows us to evaluate a definite integral very easily if we can find an antiderivative of the integrand. For example, since an anti-

derivative of x^3 is $\frac{1}{4}x^4$, we have

$$\int_0^b x^3\,dx = \frac{1}{4}x^4\Big]_0^b = \frac{1}{4}b^4 - \frac{1}{4}(0)^4 = \frac{1}{4}b^4.$$

Those who doubt the importance of the fundamental theorem should compare this simple computation with the limit of sums calculation discussed in Example 7 of Section 5.3.

EXAMPLE 1 Evaluate $\int_{-2}^{3} (6x^2 - 5)\,dx$.

SOLUTION An antiderivative of $6x^2 - 5$ is $F(x) = 2x^3 - 5x$. Applying Corollary (5.31), we get

$$\begin{aligned}\int_{-2}^{3} (6x^2 - 5)\,dx &= \Big[2x^3 - 5x\Big]_{-2}^{3} \\ &= [2(3)^3 - 5(3)] - [2(-2)^3 - 5(-2)] \\ &= [54 - 15] - [-16 + 10] = 45.\end{aligned}$$

Note that if $F(x) + C$ is used in place of $F(x)$ in Corollary (5.31), the same result is obtained, since

$$\begin{aligned}\Big[F(x) + C\Big]_a^b &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) = \Big[F(x)\Big]_a^b.\end{aligned}$$

In particular, since

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$, we obtain the following theorem.

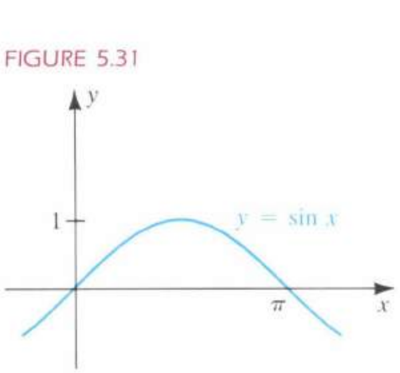
Theorem **(5.32)**

$$\boxed{\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b}$$

Theorem (5.32) states that *a definite integral can be evaluated by evaluating the corresponding indefinite integral.* As with previous cases, when using Theorem (5.32) it is unnecessary to include the constant of integration C for the indefinite integral.

EXAMPLE 2 Find the area A of the region between the graph of $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$.

FIGURE 5.31



SOLUTION The region is sketched in Figure 5.31. Applying Theorems (5.19) and (5.32) gives us the following:

$$\begin{aligned}A = \int_0^\pi \sin x\,dx &= \left[\int \sin x\,dx\right]_0^\pi \\ &= \Big[-\cos x\Big]_0^\pi \\ &= -\cos\pi - (-\cos 0) \\ &= -(-1) + 1 = 2\end{aligned}$$

By Theorem (5.32), we can use any formula for indefinite integration to obtain a formula for definite integrals. To illustrate, using Table (5.4), we obtain

$$\int_a^b x^r\,dx = \left[\frac{x^{r+1}}{r+1}\right]_a^b \quad \text{if } r \neq -1$$

$$\int_a^b \sin x\,dx = \Big[-\cos x\Big]_a^b$$

$$\int_a^b \sec^2 x\,dx = \Big[\tan x\Big]_a^b.$$

EXAMPLE 3 Evaluate $\int_{-1}^{2} (x^3+1)^2\,dx$.

SOLUTION We first square the integrand and then apply the power rule to each term as follows:

$$\begin{aligned}\int_{-1}^{2} (x^3+1)^2\,dx &= \int_{-1}^{2} (x^6+2x^3+1)\,dx \\ &= \left[\frac{x^7}{7} + 2\cdot\frac{x^4}{4} + x\right]_{-1}^{2} \\ &= \left[\frac{2^7}{7} + 2\cdot\frac{2^4}{4} + 2\right] - \left[\frac{(-1)^7}{7} + 2\frac{(-1)^4}{4} + (-1)\right] \\ &= \tfrac{405}{14}\end{aligned}$$

EXAMPLE 4 Evaluate $\int_1^4 \left(5x - 2\sqrt{x} + \dfrac{32}{x^3}\right) dx$.

SOLUTION We begin by changing the form of the integrand so that the power rule may be applied to each term. Thus,

$$\begin{aligned}\int_1^4 (5x - 2x^{1/2} + 32x^{-3})\,dx &= \left[5\left(\frac{x^2}{2}\right) - 2\left(\frac{x^{3/2}}{3/2}\right) + 32\left(\frac{x^{-2}}{-2}\right)\right]_1^4 \\ &= \left[\frac{5}{2}x^2 - \frac{4}{3}x^{3/2} - \frac{16}{x^2}\right]_1^4 \\ &= \left[\frac{5}{2}(4)^2 - \frac{4}{3}(4)^{3/2} - \frac{16}{4^2}\right] - \left[\frac{5}{2} - \frac{4}{3} - 16\right] \\ &= \tfrac{259}{6}.\end{aligned}$$

The method of substitution developed for indefinite integrals may also be used to evaluate a definite integral. We could use (5.7) to find an indefinite integral (that is, an antiderivative) and then apply the fundamental theorem of calculus. Another method, which is sometimes shorter, is to change the limits of integration. Using (5.7) together with the fundamental theorem gives us the following formula, with $F' = f$:

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\Big]_a^b$$

The number on the right may be written

$$F(g(b)) - F(g(a)) = F(u)\Big]_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u)\,du.$$

This gives us the following result, provided f and g' are integrable.

Theorem **(5.33)**

If $u = g(x)$, then $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$

Theorem (5.33) states that after making the substitution $u = g(x)$ and $du = g'(x)\,dx$, we may use the values of g that correspond to $x = a$ and $x = b$, respectively, as the limits of the integral involving u. It is then unnecessary to return to the variable x after integrating. This technique is illustrated in the next example.

EXAMPLE 5 Evaluate $\displaystyle\int_2^{10} \frac{3}{\sqrt{5x-1}}\,dx.$

SOLUTION Let us begin by writing the integral as

$$3\int_2^{10} \frac{1}{\sqrt{5x-1}}\,dx.$$

The expression $\sqrt{5x-1}$ in the integrand suggests the following substitution:

$$u = 5x - 1, \quad du = 5\,dx$$

The form of du indicates that we should introduce the factor 5 into the integrand and then compensate by multiplying the integral by $\frac{1}{5}$, as follows:

$$3\int_2^{10} \frac{1}{\sqrt{5x-1}}\,dx = \frac{3}{5}\int_2^{10} \frac{1}{\sqrt{5x-1}}\,5\,dx$$

We next calculate the values of $u = 5x - 1$ that correspond to the limits of integration $x = 2$ and $x = 10$:

(i) If $x = 2$, then $u = 5(2) - 1 = 9$.

(ii) If $x = 10$, then $u = 5(10) - 1 = 49$.

Substituting in the integrand and changing the limits of integration as in Theorem (5.33) gives us

$$\begin{aligned}3\int_2^{10} \frac{1}{\sqrt{5x-1}}\,dx &= \frac{3}{5}\int_2^{10} \frac{1}{\sqrt{5x-1}}\,5\,dx \\ &= \frac{3}{5}\int_9^{49} \frac{1}{\sqrt{u}}\,du = \frac{3}{5}\int_9^{49} u^{-1/2}\,du \\ &= \left[\left(\frac{3}{5}\right)\frac{u^{1/2}}{1/2}\right]_9^{49} = \frac{6}{5}[49^{1/2} - 9^{1/2}] = \frac{24}{5}.\end{aligned}$$

EXAMPLE 6 Evaluate $\int_0^{\pi/4} (1 + \sin 2x)^3 \cos 2x \, dx$.

SOLUTION The integrand suggests the power rule $\int_a^b u^3 \, du = \left[\frac{1}{4}u^4\right]_a^b$. Thus, we let

$$u = 1 + \sin 2x, \quad du = 2 \cos 2x \, dx.$$

The form of du indicates that we should introduce the factor 2 into the integrand and multiply the integral by $\frac{1}{2}$, as follows:

$$\int_0^{\pi/4} (1 + \sin 2x)^3 \cos 2x \, dx = \tfrac{1}{2} \int_0^{\pi/4} (1 + \sin 2x)^3 \, 2 \cos 2x \, dx$$

We next calculate the values of $u = 1 + \sin 2x$ that correspond to the limits of integration $x = 0$ and $x = \pi/4$:

(i) If $x = 0$, then $u = 1 + \sin 0 = 1 + 0 = 1$.

(ii) If $x = \dfrac{\pi}{4}$, then $u = 1 + \sin \dfrac{\pi}{2} = 1 + 1 = 2$.

Substituting in the integrand and changing the limits of integration gives us

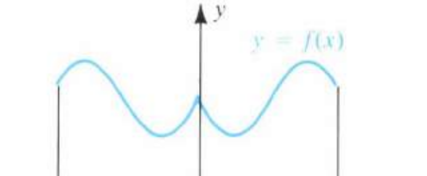
$$\begin{aligned}\int_0^{\pi/4} (1 + \sin 2x)^3 \cos 2x \, dx &= \tfrac{1}{2} \int_1^2 u^3 \, du \\ &= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 \\ &= \tfrac{1}{8}[16 - 1] = \tfrac{15}{8}.\end{aligned}$$

The following theorem illustrates a useful technique for evaluating certain definite integrals.

Theorem **(5.34)**

> Let f be continuous on $[-a, a]$.
>
> **(i)** If f is an even function,
>
> $$\int_{-a}^{a} f(x) \, dx = 2 \int_0^a f(x) \, dx.$$
>
> **(ii)** If f is an odd function,
>
> $$\int_{-a}^{a} f(x) \, dx = 0.$$

FIGURE 5.32



PROOF We shall prove (i). If f is an even function, then the graph of f is symmetric with respect to the y-axis. As a special case, if $f(x) \geq 0$ for every x in $[0, a]$, we have a situation similar to that in Figure 5.32, and hence the area under the graph of f from $x = -a$ to $x = a$ is twice that from $x = 0$ to $x = a$. This gives us the formula in (i).

To show that the formula is true if $f(x) < 0$ for some x, we may proceed as follows. Using, successively, Theorem (5.24), Definition (5.17),

and Theorem (5.22), we have

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{0}^{-a} f(x)(-dx) + \int_{0}^{a} f(x)\,dx.\end{aligned}$$

Since f is even, $f(-x) = f(x)$, and the last equality may be written

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{-a} f(-x)(-dx) + \int_{0}^{a} f(x)\,dx.$$

If, in the first integral on the right, we substitute $u = -x$, $du = -dx$ and observe that $u = a$ when $x = -a$, we obtain

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx.$$

The last two integrals on the right are equal, since the variables are *dummy variables*, and, therefore,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

EXAMPLE 7 Evaluate

(a) $\int_{-1}^{1} (x^4 + 3x^2 + 1)\,dx$ **(b)** $\int_{-1}^{1} (x^5 + 3x^3 + x)\,dx$

(c) $\int_{-5}^{5} (2x^3 + 3x^2 + 7x)\,dx$

SOLUTION

(a) Since the integrand determines an even function, we may apply Theorem (5.34)(i):

$$\begin{aligned}\int_{-1}^{1} (x^4 + 3x^2 + 1)\,dx &= 2\int_{0}^{1} (x^4 + 3x^2 + 1)\,dx \\ &= 2\left[\frac{x^5}{5} + x^3 + x\right]_0^1 = \frac{22}{5}\end{aligned}$$

(b) The integrand is odd, so we apply Theorem (5.34)(ii):

$$\int_{-1}^{1} (x^5 + 3x^3 + x)\,dx = 0$$

(c) The function given by $2x^3 + 7x$ is odd but the function given by $3x^2$ is even, so we apply Theorem (5.34)(ii) *and* (i):

$$\begin{aligned}\int_{-5}^{5} (2x^3 + 3x^2 + 7x)\,dx &= \int_{-5}^{5} (2x^3 + 7x)\,dx + \int_{-5}^{5} 3x^2\,dx \\ &= 0 + 2\int_{0}^{5} 3x^2\,dx \\ &= 2\Big[x^3\Big]_0^5 = 250\end{aligned}$$

The technique of defining a function by means of a definite integral, as in Part I of the fundamental theorem of calculus (5.30), will have a

very important application in Chapter 7, when we consider logarithmic functions. Recall, from (5.30), that if f is continuous on $[a, b]$ and $G(x) = \int_a^x f(t)\,dt$ for $a \le x \le b$, then G is an antiderivative of f; that is, $D_x\,G(x) = f(x)$. This may be stated in integral form as follows:

$$D_x \int_a^x f(t)\,dt = f(x)$$

The preceding formula is generalized in the next theorem.

Theorem **(5.35)**

Let f be continuous on $[a, b]$. If $a \le c \le b$, then for every x in $[a, b]$,

$$D_x \int_c^x f(t)\,dt = f(x).$$

PROOF If F is an antiderivative of f, then

$$\begin{aligned} D_x \int_c^x f(t)\,dt &= D_x\,[F(x) - F(c)] \\ &= D_x\,F(x) - D_x\,F(c) \\ &= f(x) - 0 = f(x). \end{aligned}$$

EXAMPLE 8 If $G(x) = \int_1^x \frac{1}{t}\,dt$ and $x > 0$, find $G'(x)$.

SOLUTION We shall apply Theorem (5.35) with $c = 1$ and $f(x) = 1/x$. If we choose a and b such that $0 < a \le 1 \le b$, then f is continuous on $[a, b]$. Hence, by Theorem (5.35), for every x in $[a, b]$,

$$G'(x) = D_x \int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

In (5.29) we defined the *average value* f_{av} of a function f on $[a, b]$ as follows:

$$f_{\text{av}} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

The next example indicates why this terminology is appropriate in applications.

EXAMPLE 9 Suppose a point P moving on a coordinate line has a continuous velocity function v. Show that the average value of v on $[a, b]$ equals the average velocity during the time interval $[a, b]$.

SOLUTION By Definition (5.29) with $f = v$,

$$v_{\text{av}} = \frac{1}{b-a} \int_a^b v(t)\,dt.$$

If s is the position function of P, then $s'(t) = v(t)$; that is, $s(t)$ is an antiderivative of $v(t)$. Hence, by the fundamental theorem of calculus,

$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(t)\Big]_a^b = s(b) - s(a).$$

Substituting in the formula for v_{av} gives us

$$v_{\text{av}} = \frac{s(b) - s(a)}{b - a},$$

which is the average velocity of P on $[a, b]$ (see (3.2)).

Results similar to that in Example 9 occur in discussions of average acceleration, average marginal cost, average marginal revenue, and many other applications of the derivative (see Exercises 45–50).

EXERCISES 5.6

Exer. 1–36: Evaluate the integral.

1 $\int_1^4 (x^2 - 4x - 3)\,dx$

2 $\int_{-2}^3 (5 + x - 6x^2)\,dx$

3 $\int_{-2}^3 (8z^3 + 3z - 1)\,dz$

4 $\int_0^2 (z^4 - 2z^3)\,dz$

5 $\int_7^{12} dx$

6 $\int_{-6}^{-1} 8\,dx$

7 $\int_1^2 \frac{5}{x^6}\,dx$

8 $\int_1^4 \sqrt{16x^5}\,dx$

9 $\int_4^9 \frac{t - 3}{\sqrt{t}}\,dt$

10 $\int_{-1}^{-2} \frac{2t - 7}{t^3}\,dt$

11 $\int_{-8}^8 (\sqrt[3]{s^2} + 2)\,ds$

12 $\int_1^0 s^2(\sqrt[3]{s} - \sqrt{s})\,ds$

13 $\int_{-1}^0 (2x + 3)^2\,dx$

14 $\int_1^2 (4x^{-5} - 5x^4)\,dx$

15 $\int_3^2 \frac{x^2 - 1}{x - 1}\,dx$

16 $\int_0^{-1} \frac{x^3 + 8}{x + 2}\,dx$

17 $\int_1^1 (4x^2 - 5)^{100}\,dx$

18 $\int_5^5 \sqrt[3]{x^2 + \sqrt{x^5 + 1}}\,dx$

19 $\int_1^3 \frac{2x^3 - 4x^2 + 5}{x^2}\,dx$

20 $\int_{-2}^{-1} \left(x - \frac{1}{x}\right)^2 dx$

21 $\int_{-3}^6 |x - 4|\,dx$

22 $\int_{-1}^5 |2x - 3|\,dx$

23 $\int_1^4 \sqrt{5 - x}\,dx$

24 $\int_1^5 \sqrt[3]{2x - 1}\,dx$

25 $\int_{-1}^1 (v^2 - 1)^3\,v\,dv$

26 $\int_{-2}^0 \frac{v^2}{(v^3 - 2)^2}\,dv$

27 $\int_0^1 \frac{1}{(3 - 2x)^2}\,dx$

28 $\int_0^4 \frac{x}{\sqrt{x^2 + 9}}\,dx$

29 $\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x} + 1)^3}\,dx$

30 $\int_0^1 (3 - x^4)^3\,x^3\,dx$

31 $\int_{\pi/2}^{\pi} \cos\left(\frac{1}{3}x\right)dx$

32 $\int_0^{\pi/2} 3\sin\left(\frac{1}{2}x\right)dx$

33 $\int_{\pi/4}^{\pi/3} (4\sin 2\theta + 6\cos 3\theta)\,d\theta$

34 $\int_{\pi/6}^{\pi/4} (1 - \cos 4\theta)\,d\theta$

35 $\int_{-\pi/6}^{\pi/6} (x + \sin 5x)\,dx$

36 $\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx$

Exer. 37–40: (a) Find a number z that satisfies the conclusion of the mean value theorem (5.28) for the given integral $\int_a^b f(x)\,dx$. (b) Find the average value of f on $[a, b]$.

37 $\int_0^4 \frac{x}{\sqrt{x^2 + 9}}\,dx$

38 $\int_{-2}^0 \sqrt[3]{x + 1}\,dx$

39 $\int_0^5 \sqrt{x + 4}\,dx$

40 $\int_{-3}^2 \sqrt{6 - x}\,dx$

Exer. 41–44: Find the derivative without integrating.

41 $D_x \int_0^3 \sqrt{x^2 + 16}\,dx$

42 $D_x \int_0^1 x\sqrt{x^2 + 4}\,dx$

43 $D_x \int_0^x \frac{1}{t + 1}\,dt$

44 $D_x \int_0^x \frac{1}{\sqrt{1 - t^2}}\,dt, \quad |x| < 1$

45 A point P is moving on a coordinate line with a continuous acceleration function a. If v is the velocity function, then the *average acceleration* on a time interval $[t_1, t_2]$ is

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1}.$$

Show that the average acceleration is equal to the average value of a on $[t_1, t_2]$.

46 If a function f has a continuous derivative on $[a, b]$, show that the average rate of change of $f(x)$ with respect to x on $[a, b]$ (see Definition (3.4)) is equal to the average value of f' on $[a, b]$.

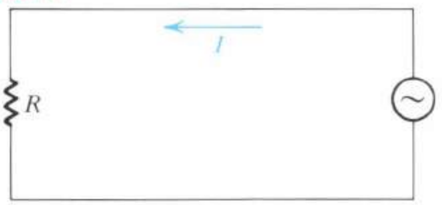
47 The vertical distribution of velocity of the water in a river may be approximated by $v = c(d - y)^{1/6}$, where v is the velocity (in m/sec) at a depth of y meters below the water surface, d is the depth of the river, and c is a positive constant.

(a) Find a formula for the average velocity v_{av} in terms of d and c.

(b) If v_0 is the velocity at the surface, show that $v_{av} = \frac{6}{7}v_0$.

48 In the electrical circuit shown in the figure, the alternating current I is given by $I = I_M \sin \omega t$, where t is the time and I_M is the maximum current. The rate P at which heat is being produced in the resistor of R ohms is given by $P = I^2R$. Compute the *average rate* of production of heat over one complete cycle (from $t = 0$ to $t = 2\pi/\omega$). (*Hint:* Use the half-angle formula for the sine.)

EXERCISE 48



49 If a ball is dropped from a height of s_0 feet above the ground and air resistance is negligible, then the distance it falls in t seconds is $16t^2$ feet. Use Definition (5.29) to show that the average velocity for the ball's journey to the ground is $4\sqrt{s_0}$ ft/sec.

50 A meteorologist determines that the temperature T (in °F) on a cold winter day is given by

$$T = \tfrac{1}{20}t(t - 12)(t - 24),$$

where t is time (in hours) and $t = 0$ corresponds to midnight. Find the average temperature between 6 A.M. and 12 noon.

51 If g is differentiable and f is continuous for every x, prove that

$$D_x \int_a^{g(x)} f(t)\, dt = f(g(x))g'(x).$$

52 Extend the formula in Exercise 51 to

$$D_x \int_{k(x)}^{g(x)} f(t)\, dt = f(g(x))g'(x) - f(k(x))k'(x).$$

Exer. 53–56: Use Exercises 51 and 52 to find the derivative.

53 $D_x \int_2^{x^4} \dfrac{t}{\sqrt{t^3 + 2}}\, dt$

54 $D_x \int_0^{x^2} \sqrt[3]{t^4 + 1}\, dt$

55 $D_x \int_{3x}^{x^3} (t^3 + 1)^{10}\, dt$

56 $D_x \int_{1/x}^{\sqrt{x}} \sqrt{t^4 + t^2 + 4}\, dt$

5.7 NUMERICAL INTEGRATION

To evaluate a definite integral $\int_a^b f(x)\, dx$ by means of the fundamental theorem of calculus, we need an antiderivative of f. If an antiderivative cannot be found, then we may use numerical methods to approximate the integral to any desired degree of accuracy. For example, if the norm of a partition of $[a, b]$ is small, then, by Definition (5.16), the definite integral can be approximated by any Riemann sum of f. In particular, if we use a regular partition with $\Delta x = (b - a)/n$, then

$$\int_a^b f(x)\, dx \approx \sum_{k=1}^{n} f(w_k)\, \Delta x,$$

where w_k is any number in the kth subinterval $[x_{k-1}, x_k]$ of the partition. Of course, the accuracy of the approximation depends on the nature of f and the magnitude of Δx. It may be necessary to make Δx very small to obtain a desired degree of accuracy. This, in turn, means that n is large, and hence the preceding sum contains many terms. Figure 5.5 illustrates the case in which $f(w_k)$ is the minimum value of f on $[x_{k-1}, x_k]$. In this case the error involved in the approximation is numerically the same as the area of the region that lies under the graph of f and over the inscribed rectangles.

If we let $w_k = x_{k-1}$ (that is, if f is evaluated at the left-hand endpoint of each subinterval $[x_{k-1}, x_k]$), then

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(x_{k-1})\,\Delta x.$$

If we let $w_k = x_k$ (that is, if f is evaluated at the right-hand endpoint of $[x_{k-1}, x_k]$), then

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

Another and usually more accurate approximation can be obtained by using the average of the last two approximations—that is,

$$\tfrac{1}{2}\left[\sum_{k=1}^{n} f(x_{k-1})\,\Delta x + \sum_{k=1}^{n} f(x_k)\,\Delta x\right].$$

With the exception of $f(x_0)$ and $f(x_n)$, each function value $f(x_k)$ appears twice, and hence we may write the last expression as

$$\frac{\Delta x}{2}\left[f(x_0) + \sum_{k=1}^{n-1} 2f(x_k) + f(x_n)\right].$$

Since $\Delta x = (b-a)/n$, this gives us the following rule.

Trapezoidal rule **(5.36)**

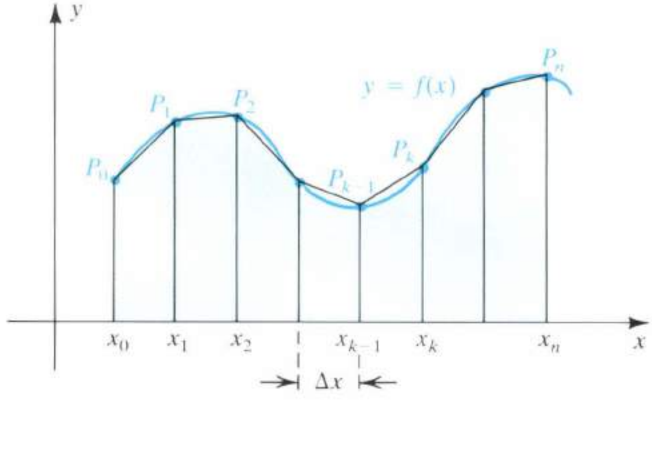
Let f be continuous on $[a, b]$. If a regular partition of $[a, b]$ is determined by $a = x_0, x_1, \ldots, x_n = b$, then

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

The term *trapezoidal* comes from the case in which $f(x)$ is nonnegative on $[a, b]$. As illustrated in Figure 5.33, if P_k is the point with x-coordinate x_k on the graph of $y = f(x)$, then for each $k = 1, 2, \ldots, n$, the points on the x-axis with x-coordinates x_{k-1} and x_k, together with P_{k-1} and P_k, are vertices of a trapezoid having area

$$\frac{\Delta x}{2}\left[f(x_{k-1}) + f(x_k)\right].$$

FIGURE 5.33



The sum of the areas of these trapezoids is the same as the sum in Rule (5.36). Hence, in geometric terms, the trapezoidal rule gives us an approximation to the area under the graph of f from a to b by means of trapezoids instead of the rectangles associated with Riemann sums.

The next result provides information about the maximum error that can occur if the trapezoidal rule is used to approximate a definite integral. The proof is omitted.

Error estimate for the trapezoidal rule **(5.37)**

If f'' is continuous and if M is a positive real number such that $|f''(x)| \le M$ for every x in $[a, b]$, then the error involved in using the trapezoidal rule (5.36) is not greater than

$$\frac{M(b-a)^3}{12n^2}.$$

EXAMPLE 1 Use the trapezoidal rule with $n = 10$ to approximate $\int_1^2 \frac{1}{x}\,dx$. Estimate the maximum error in the approximation.

SOLUTION It is convenient to arrange our work as follows. Each $f(x_k)$ was obtained with a calculator and is accurate to nine decimal places. The column labeled m contains the coefficient of $f(x_k)$ in the trapezoidal rule (5.36). Thus, $m = 1$ for $f(x_0)$ or $f(x_n)$, and $m = 2$ for the remaining $f(x_k)$.

k	x_k	$f(x_k)$	m	$mf(x_k)$
0	1.0	1.000000000	1	1.000000000
1	1.1	0.909090909	2	1.818181818
2	1.2	0.833333333	2	1.666666666
3	1.3	0.769230769	2	1.538461538
4	1.4	0.714285714	2	1.428571428
5	1.5	0.666666667	2	1.333333334
6	1.6	0.625000000	2	1.250000000
7	1.7	0.588235294	2	1.176470588
8	1.8	0.555555556	2	1.111111112
9	1.9	0.526315789	2	1.052631578
10	2.0	0.500000000	1	0.500000000

The sum of the numbers in the last column is 13.875428062.

Since

$$\frac{b-a}{2n} = \frac{2-1}{2(10)} = \frac{1}{20},$$

it follows from (5.36) that

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{20}(13.875428062) \approx 0.693771403.$$

The error in the approximation may be estimated by means of (5.37). Since $f(x) = 1/x$, we have $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$. The maximum value of $f''(x)$ on the interval $[1, 2]$ occurs at $x = 1$ because f'' is a decreasing function. Hence

$$|f''(x)| \leq \frac{2}{(1)^3} = 2.$$

Applying (5.37) with $M = 2$, we see that the maximum error is not greater than

$$\frac{2(2-1)^3}{12(10)^2} = \frac{1}{600} < 0.002.$$

In Chapter 7 we shall see that the integral in Example 1 equals the natural logarithm of 2, denoted by ln 2, which to nine decimal places is 0.693147181. To obtain this approximation by means of the trapezoidal rule, it is necessary to use a very large value of n.

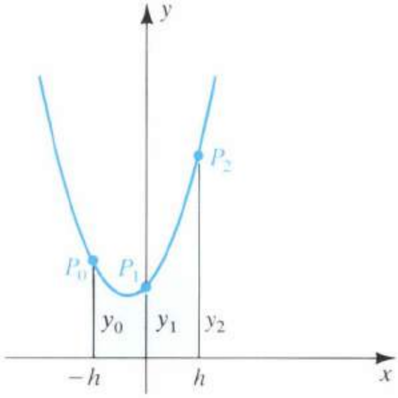
The following rule is often more accurate than the trapezoidal rule.

Simpson's rule **(5.38)**

Let f be continuous on $[a, b]$, and let n be an even integer. If a regular partition is determined by $a = x_0, x_1, \ldots, x_n = b$, then

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

FIGURE 5.34



The idea behind the proof of Simpson's rule is that instead of using trapezoids to approximate the graph of f, we use portions of graphs of equations of the form $y = cx^2 + dx + e$ for constants c, d, and e; that is, we use portions of parabolas or lines. If $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, and $P_2(x_2, y_2)$ are points on the parabola such that $x_0 < x_1 < x_2$, then substituting the coordinates of P_0, P_1, and P_2, respectively, into the equation $y = cx^2 + dx + e$ gives us three equations that may be solved for c, d, and e. As a special case, suppose h, y_0, y_1, and y_2 are positive, and consider the points $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$, as illustrated in Figure 5.34.

The area A under the graph of the equation from $-h$ to h is

$$A = \int_{-h}^{h} (cx^2 + dx + e)\,dx = \left[\frac{cx^3}{3} + \frac{dx^2}{2} + ex\right]_{-h}^{h} = \frac{h}{3}(2ch^2 + 6e).$$

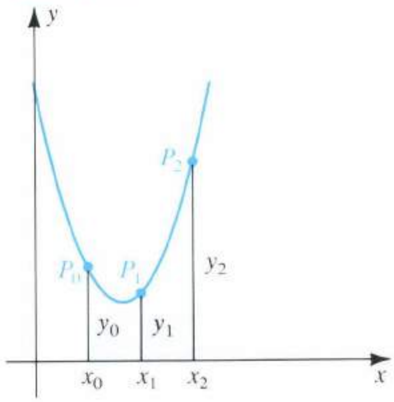
Since the coordinates of $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$ are solutions of $y = cx^2 + dx + e$, we have, by substitution,

$$\begin{aligned} y_0 &= ch^2 - dh + e \\ y_1 &= e \\ y_2 &= ch^2 + dh + e. \end{aligned}$$

Thus,

$$y_0 + 4y_1 + y_2 = 2ch^2 + 6e$$

FIGURE 5.35

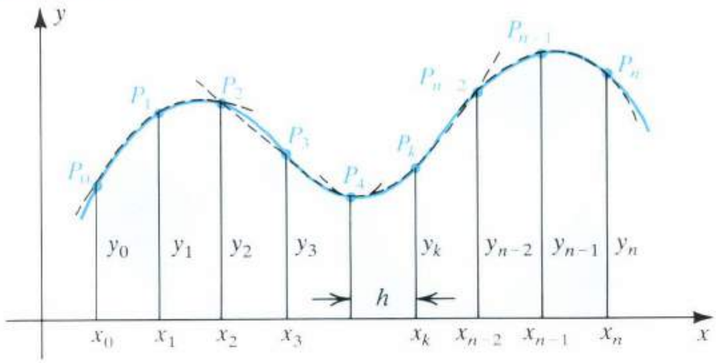


and
$$A = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

If the points P_0, P_1, and P_2 are translated horizontally, as illustrated in Figure 5.35, then the area under the graph remains the same. Consequently, the preceding formula for A is true for *any* points P_0, P_1, and P_2, provided $x_1 - x_0 = x_2 - x_1$.

If $f(x) \geq 0$ on $[a, b]$, then Simpson's rule is obtained by regarding the definite integral as the area under the graph of f from a to b. Thus, suppose n is an even integer and $h = (b - a)/n$. We divide $[a, b]$ into n subintervals, each of length h, by choosing numbers $a = x_0, x_1, \ldots, x_n = b$. Let $P_k(x_k, y_k)$ be the point on the graph of f with x-coordinate x_k, as illustrated in Figure 5.36.

FIGURE 5.36



If the arc through P_0, P_1, and P_2 is approximated by the graph of an equation $y = cx^2 + dx + e$, then, as we have seen, the area under the graph of f from x_0 to x_2 is approximated by

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

If we use the arc through P_2, P_3, and P_4, then the area under the graph of f from x_2 to x_4 is approximately

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

We continue in this manner until we reach the last triple of points P_{n-2}, P_{n-1}, P_n and the corresponding approximation to the area under the graph:

$$\frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Summing these approximations give us

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n),$$

which is the same as the sum in (5.38). If f is negative for some x in $[a, b]$, then negatives of areas may be used to establish Simpson's rule.

The following result is analogous to (5.37).

Error estimate for Simpson's rule **(5.39)**

> If $f^{(4)}$ is continuous and if M is a positive real number such that $|f^{(4)}(x)| \leq M$ for every x in $[a, b]$, then the error involved in using Simpson's rule (5.38) is not greater than
>
> $$\frac{M(b-a)^5}{180n^4}.$$

EXAMPLE 2 Use Simpson's rule with $n = 10$ to approximate $\int_1^2 \frac{1}{x}\, dx$. Estimate the maximum error in the approximation.

SOLUTION This is the same integral considered in Example 1. We arrange our work as follows. The column labeled m contains the coefficient of $f(x_k)$ in Simpson's rule (5.38).

k	x_k	$f(x_k)$	m	$mf(x_k)$
0	1.0	1.000000000	1	1.000000000
1	1.1	0.909090909	4	3.636363636
2	1.2	0.833333333	2	1.666666666
3	1.3	0.769230769	4	3.076923076
4	1.4	0.714285714	2	1.428571428
5	1.5	0.666666667	4	2.666666668
6	1.6	0.625000000	2	1.250000000
7	1.7	0.588235294	4	2.352941176
8	1.8	0.555555556	2	1.111111112
9	1.9	0.526315789	4	2.105263156
10	2.0	0.500000000	1	0.500000000

The sum of the numbers in the last column is 20.794506918. Since

$$\frac{b-a}{3n} = \frac{2-1}{30},$$

it follows from (5.38) that

$$\int_1^2 \frac{1}{x}\, dx \approx \left(\frac{1}{30}\right)(20.794506918) \approx 0.693150231.$$

We shall use (5.39) to estimate the error in the approximation. If $f(x) = 1/x$, we can verify that $f^{(4)}(x) = 24/x^5$. The maximum value of $f^{(4)}(x)$ on the interval $[1, 2]$ occurs at $x = 1$, and hence

$$|f^{(4)}(x)| \leq \frac{24}{(1)^5} = 24.$$

Applying (5.39) with $M = 24$, we see that the maximum error in the approximation is not greater than

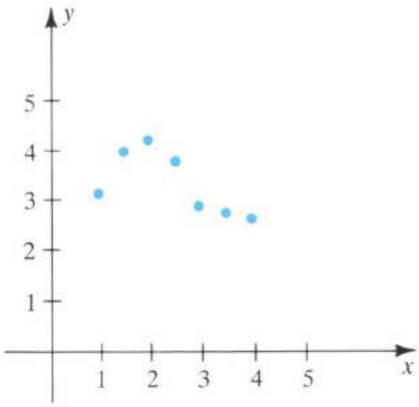
$$\frac{24(2-1)^5}{180(10)^4} = \frac{2}{150{,}000} < 0.00002.$$

Note that this estimated error is much less than that obtained using the trapezoidal rule in Example 1.

An important aspect of numerical integration is that it can be used to approximate the definite integral of a function that is described by means of a table or graph. To illustrate, suppose it is found experimentally that two physical variables x and y are related as shown in the following table.

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	3.1	4.0	4.2	3.8	2.9	2.8	2.7

FIGURE 5.37



The points (x, y) are plotted in Figure 5.37. If we regard y as a function of x, say $y = f(x)$ with f continuous, then the definite integral $\int_1^4 f(x)\,dx$ may represent a physical quantity. In the present illustration, *the integral may be approximated without knowing an explicit form for $f(x)$*. In particular, if the trapezoidal rule (5.36) is used, with $n = 6$ and $(b-a)/(2n) = (4-1)/12 = 0.25$, then

$$\int_1^4 f(x)\,dx \approx 0.25[3.1 + 2(4.0) + 2(4.2) + 2(3.8) + 2(2.9) + 2(2.8) + 2.7],$$

or

$$\int_1^4 f(x)\,dx \approx 10.3.$$

The number of subdivisions is even, so we could also approximate the integral by means of Simpson's rule.

EXERCISES 5.7

Exer. 1–12: Approximate the definite integral for the stated value of n by using (a) the trapezoidal rule and (b) Simpson's rule. (Approximate each $f(x_k)$ to four decimal places, and round off answers to two decimal places, whenever appropriate.)

1 $\int_1^3 (x^2+1)\,dx; \quad n = 4$

2 $\int_1^5 x^3\,dx; \quad n = 4$

3 $\int_1^{1.6} (2x-1)\,dx; \quad n = 6$

4 $\int_2^{3.2} (\frac{1}{2}x+1)\,dx; \quad n = 6$

c 5 $\int_1^4 \frac{1}{x}\,dx; \quad n = 6$

c 6 $\int_0^3 \frac{1}{1+x}\,dx; \quad n = 8$

c 7 $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx; \quad n = 4$

c 8 $\int_2^3 \sqrt{1+x^3}\,dx; \quad n = 4$

c 9 $\int_0^2 \frac{1}{4+x^2}\,dx; \quad n = 6$

c 10 $\int_0^{0.6} \frac{1}{\sqrt{4-x^2}}\,dx; \quad n = 6$

c 11 $\int_0^\pi \sqrt{\sin x}\,dx; \quad n = 6$

c 12 $\int_0^\pi \sin\sqrt{x}\,dx; \quad n = 4$

Exer. 13–16: Estimate the maximum error in approximating the definite integral for the stated value of n when using (a) the trapezoidal rule and (b) Simpson's rule.

13 $\int_{-2}^{3} \left(\frac{1}{360}x^6 + \frac{1}{60}x^5\right) dx;\quad n = 4$

14 $\int_{0}^{3} \left(-\frac{1}{12}x^4 + \frac{2}{3}x^3\right) dx;\quad n = 4$

15 $\int_{1}^{5} \frac{1}{x^2}\, dx;\quad n = 8$

16 $\int_{1}^{4} \frac{1}{35}x^{7/2}\, dx;\quad n = 6$

Exer. 17–20: Find the least integer n such that the error in approximating the definite integral is less than E when using (a) the trapezoidal rule and (b) Simpson's rule.

17 $\int_{1}^{8} 81x^{8/3}\, dx;\quad E = 0.001$

18 $\int_{1}^{2} \frac{1}{120x^2}\, dx;\quad E = 0.001$

19 $\int_{1/2}^{1} \frac{1}{x}\, dx;\quad E = 0.0001$

20 $\int_{0}^{3} \frac{1}{x+1}\, dx;\quad E = 0.001$

Exer. 21–24: Suppose the table of values for x and y was obtained empirically. Assuming that $y = f(x)$ and f is continuous, approximate $\int_2^4 f(x)\, dx$ by means of (a) the trapezoidal rule and (b) Simpson's rule.

21

x	2.0	2.5	3.0	3.5	4.0
y	3	2	4	3	5

22

x	2.0	3.0	4.0
y	5	4	3

c 23

x	y
2.00	4.12
2.25	3.76
2.50	3.21
2.75	3.58
3.00	3.94
3.25	4.15
3.50	4.69
3.75	5.44
4.00	7.52

c 24

x	y
2.0	12.1
2.2	11.4
2.4	9.7
2.6	8.4
2.8	6.3
3.0	6.2
3.2	5.8
3.4	5.4
3.6	5.1
3.8	5.9
4.0	5.6

c 25 Use the trapezoidal rule with $(b - a)/n = 0.1$ to show that

$$\int_{1}^{2.7} \frac{1}{x}\, dx < 1 < \int_{1}^{2.8} \frac{1}{x}\, dx.$$

c 26 It will follow from our work in Chapter 8 that

$$\int_{0}^{1} \frac{1}{x^2 + 1}\, dx = \frac{\pi}{4}.$$

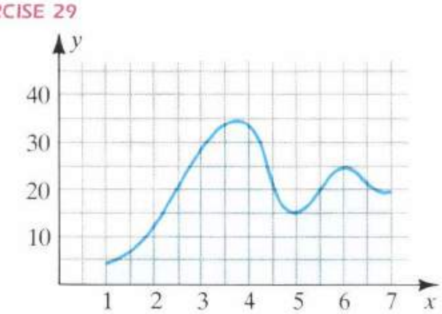
Use this fact and Simpson's rule with $n = 8$ to approximate π to four decimal places.

27 If $f(x)$ is a polynomial of degree less than 4, prove that Simpson's rule gives the exact value of $\int_a^b f(x)\, dx$.

28 Suppose that f is continuous and that both f and f'' are nonnegative throughout $[a, b]$. Prove that $\int_a^b f(x)\, dx$ is less than the approximation given by the trapezoidal rule.

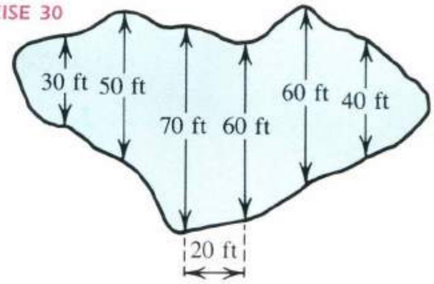
29 The graph in the figure was recorded by an instrument used to measure a physical quantity. Estimate y-coordinates of points on the graph, and approximate the area of the shaded region by using (with $n = 6$) (a) the trapezoidal rule and (b) Simpson's rule.

EXERCISE 29



30 A man-made lake has the shape illustrated in the figure, with adjacent measurements 20 feet apart. Use the trapezoidal rule to estimate the surface area of the lake.

EXERCISE 30



c 31 An important aspect of water management is the production of reliable data on *streamflow*, the number of cubic meters of water passing through a cross section of a stream or river per second. A first step in this computation is the determination of the average velocity $\bar{v}_x$ at a distance x meters from the river bank. If k is the

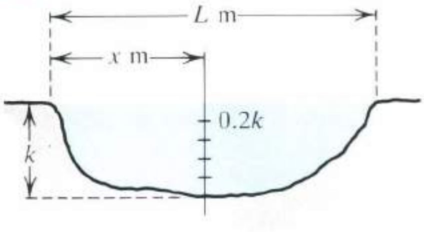
depth of the stream at a point x meters from the bank and $v(y)$ is the velocity (in m/sec) at a depth of y meters (see figure), then

$$\bar{v}_x = \frac{1}{k}\int_0^k v(y)\,dy$$

(see Definition (5.29)). With the *six-point method*, velocity readings are taken at the surface; at depths $0.2k$, $0.4k$, $0.6k$, and $0.8k$; and near the river bottom. The trapezoidal rule is then used to estimate $\bar{v}_x$. Given the data in the following table, estimate $\bar{v}_x$.

y (m)	0	$0.2k$	$0.4k$	$0.6k$	$0.8k$	k
$v(y)$ (m/sec)	0.28	0.23	0.19	0.17	0.13	0.02

EXERCISE 31



c 32 Refer to Exercise 31. The streamflow F (in m^3/sec) can be approximated using the formula

$$F = \int_0^L \bar{v}_x\, h(x)\,dx,$$

where $h(x)$ is the depth of the stream at a distance x meters from the bank and L is the length of the cross section. Given the data in the following table, use Simpson's rule to estimate F.

x (m)	0	3	6	9	12
$h(x)$ (m)	0	0.51	0.73	1.61	2.11
$\bar{v}_x$ (m/sec)	0	0.09	0.18	0.21	0.36

x (m)	15	18	21	24
$h(x)$ (m)	2.02	1.53	0.64	0
$\bar{v}_x$ (m/sec)	0.32	0.19	0.11	0

c **Exer. 33–34: Use Simpson's rule with $n = 8$ to approximate the average value of f on the given interval.**

33 $f(x) = \dfrac{1}{x^4+1}$; $[0, 4]$

34 $f(x) = \sqrt{\cos x}$; $[-1, 1]$

c **Exer. 35–36: If f is determined by the given differential equation and initial condition $f(0)$, approximate $f(1)$ using the trapezoidal rule with $n = 10$.**

35 $f'(x) = \dfrac{\sqrt{x}}{x^2+1}$; $f(0) = 1$

36 $f'(x) = \sqrt{\tan x}$; $f(0) = 2$

5.8 REVIEW EXERCISES

Exer. 1–42: Evaluate.

1 $\displaystyle\int \frac{8x^2-4x+5}{x^4}\,dx$

2 $\displaystyle\int (3x^5+2x^3-x)\,dx$

3 $\displaystyle\int 100\,dx$

4 $\displaystyle\int x^{3/5}(2x-\sqrt{x})\,dx$

5 $\displaystyle\int (2x+1)^7\,dx$

6 $\displaystyle\int \sqrt[3]{5x+1}\,dx$

$\displaystyle\int (1-2x^2)^3\,x\,dx$

8 $\displaystyle\int \frac{(1+\sqrt{x})^2}{\sqrt[3]{x}}\,dx$

9 $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

10 $\displaystyle\int (x^2+4)^2\,dx$

11 $\displaystyle\int (3-2x-5x^3)\,dx$

12 $\displaystyle\int (x+x^{-1})^2\,dx$

13 $\displaystyle\int (4x+1)(4x^2+2x-7)^2\,dx$

14 $\displaystyle\int \frac{\sqrt[4]{1-(1/x)}}{x^2}\,dx$

15 $\displaystyle\int (2x^{-3}-3x^2)\,dx$

16 $\displaystyle\int (x^{3/2}+x^{-3/2})\,dx$

17 $\displaystyle\int_0^1 \sqrt[3]{8x^7}\,dx$

18 $\displaystyle\int_1^2 \frac{x^2-x-6}{x+2}\,dx$

19 $\displaystyle\int_0^1 \frac{x^2}{(1+x^3)^2}\,dx$

20 $\displaystyle\int_1^9 \sqrt{2x+7}\,dx$

21 $\displaystyle\int_1^2 \frac{x+1}{\sqrt{x^2+2x}}\,dx$

22 $\displaystyle\int_1^2 \frac{x^2+2}{x^2}\,dx$

23 $\displaystyle\int_0^2 x^2\sqrt{x^3+1}\,dx$

24 $\displaystyle\int_1^1 3x^2\sqrt{x^3+x}\,dx$

25 $\displaystyle\int_0^1 (2x-3)(5x+1)\,dx$

26 $\displaystyle\int_{-1}^1 (x^2+1)^2\,dx$

27 $\displaystyle\int_0^4 \sqrt{3x}(\sqrt{x}+\sqrt{3})\,dx$

28 $\displaystyle\int_{-1}^1 (x+1)(x+2)(x+3)\,dx$

29 $\displaystyle\int \sin(3-5x)\,dx$

30 $\displaystyle\int x^2\cos(2x^3)\,dx$

31 $\displaystyle\int \cos 3x \sin^4 3x\,dx$

32 $\displaystyle\int \frac{\sin(1/x)}{x^2}\,dx$

33 $\displaystyle\int \frac{\cos 3x}{\sin^3 3x}\,dx$

34 $\int (3 \cos 2\pi t - 5 \sin 4\pi t)\, dt$

35 $\int_0^{\pi/2} \cos x \sqrt{3 + 5 \sin x}\, dx$

36 $\int_{-\pi/4}^{0} (\sin x + \cos x)^2\, dx$

37 $\int_0^{\pi/4} \sin 2x \cos^2 2x\, dx$

38 $\int_{\pi/6}^{\pi/4} (\sec x + \tan x)(1 - \sin x)\, dx$

39 $\int D_x \sqrt[5]{x^4 + 2x^2 + 1}\, dx$

40 $\int_0^{\pi/2} D_x (x \sin^3 x)\, dx$

41 $D_x \int_0^1 (x^3 + x^2 - 7)^5\, dx$

42 $D_x \int_0^x (t^2 + 1)^{10}\, dt$

Exer. 43–44: Solve the differential equation subject to the given conditions.

43 $\dfrac{d^2y}{dx^2} = 6x - 4;\quad y = 4$ and $y' = 5$ if $x = 2$

44 $f''(x) = x^{1/3} - 5;\quad f'(1) = 2;\quad f(1) = -8$

Exer. 45–46: Let $f(x) = 9 - x^2$ for $-2 \le x \le 3$, and let P be the regular partition of $[-2, 3]$ into five subintervals.

45 Find the Riemann sum R_P if f is evaluated at the midpoint of each subinterval of P.

46 Find (a) A_{IP} and (b) A_{CP}.

Exer. 47–48: Verify the inequality without evaluating the integrals.

47 $\int_0^1 x^2\, dx \ge \int_0^1 x^3\, dx$

48 $\int_1^2 x^2\, dx \le \int_1^2 x^3\, dx$

Exer. 49–50: Express as one integral.

49 $\int_c^e f(x)\, dx + \int_a^b f(x)\, dx - \int_c^b f(x)\, dx - \int_a^d f(x)\, dx$

50 $\int_a^d f(x)\, dx - \int_t^b f(x)\, dx - \int_g^g f(x)\, dx + \int_m^b f(x)\, dx + \int_t^a f(x)\, dx$

51 A stone is thrown directly downward from a height of 900 feet with an initial velocity of 30 ft/sec.

(a) Determine the stone's distance above ground after t seconds.

(b) Find its velocity after 5 seconds.

(c) Determine when it strikes the ground.

52 Given $\int_1^4 (x^2 + 2x - 5)\, dx$, find

(a) a number z that satisfies the conclusion of the mean value theorem for integrals (5.28)

(b) the average value of $x^2 + 2x - 5$ on $[1, 4]$

c 53 Evaluate $\int_0^{10} \sqrt{1 + x^4}\, dx$ by using

(a) the trapezoidal rule, with $n = 5$

(b) Simpson's rule, with $n = 8$

(Use approximations to four decimal places for $f(x_k)$, and round off answers to two decimal places.)

c 54 To monitor the thermal pollution of a river, a biologist takes hourly temperature readings (in °F) from 9 A.M. to 5 P.M. The results are shown in the following table.

Time of day	9	10	11	12	1
Temperature	75.3	77.0	83.1	84.8	86.5

Time of day	2	3	4	5
Temperature	86.4	81.1	78.6	75.1

Use Simpson's rule and Definition (5.29) to estimate the average water temperature between 9 A.M. and 5 P.M.

CHAPTER

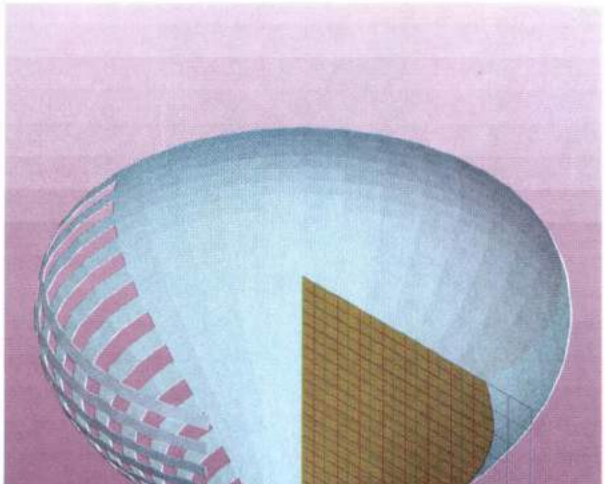
6

APPLICATIONS OF THE DEFINITE INTEGRAL

INTRODUCTION

In this chapter we discuss some of the uses for the definite integral. We begin by reconsidering the application that motivated the definition of this mathematical concept—determining the area of a region in the xy-plane. Then, in turn, we use definite integrals to find volumes, lengths of graphs, surface areas of solids, work done by a variable force, and moments and the center of mass (the balance point) of a flat plate. The reason definite integrals are applicable is that each of these quantities is expressible as a limit of sums. Moreover, because of the multitude of other quantities that can be similarly expressed, there is no end to the types of applications. In the last section we further illustrate the versatility of the definite integral by considering a variety of uses that include the following: finding the force exerted by oil on one end of a storage tank, measuring cardiac output and blood flow in arteries, estimating the future wealth of a corporation, calculating the thickness of the ozone layer, determining the amount of radon gas in a home, and finding the number of calories burned during a workout on an exercise bicycle.

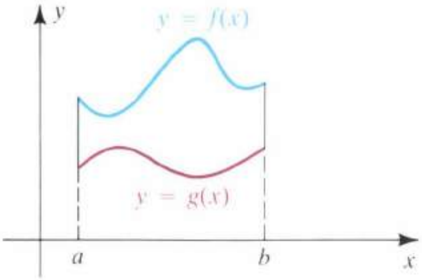
As you proceed through this chapter and whenever you encounter definite integrals in applied courses, keep the following nine words in mind: *limit of sums, limit of sums, limit of sums.*



6.1 AREA

If a function f is continuous and $f(x) \geq 0$ on $[a, b]$, then, by Theorem (5.19), the area of the region under the graph of f from a to b is given by the definite integral $\int_a^b f(x)\,dx$. In this section we shall consider the region that lies *between* the graphs of two functions.

FIGURE 6.1



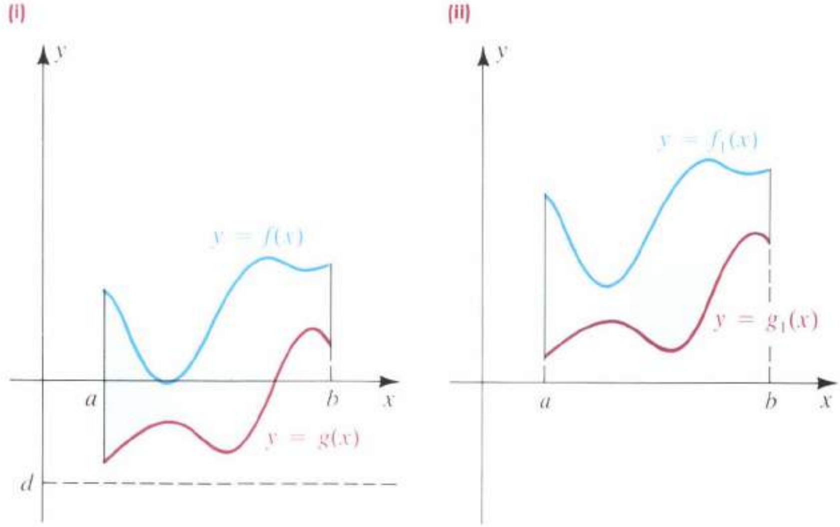
If f and g are continuous and $f(x) \geq g(x) \geq 0$ for every x in $[a, b]$, then the area A of the region R bounded by the graphs of f, g, $x = a$, and $x = b$ (see Figure 6.1) can be found by subtracting the area of the region under the graph of g (the **lower boundary** of R) from the area of the region under the graph of f (the **upper boundary** of R), as follows:

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

This formula for A is also true if f or g is negative for some x in $[a, b]$. To verify this fact, choose a *negative* number d that is less than the minimum value of g on $[a, b]$, as illustrated in Figure 6.2(i). Next, consider the functions f_1 and g_1, defined as follows:

$$f_1(x) = f(x) - d = f(x) + |d|$$
$$g_1(x) = g(x) - d = g(x) + |d|$$

FIGURE 6.2



The graphs of f_1 and g_1 can be obtained by vertically shifting the graphs of f and g a distance $|d|$. If A is the area of the region in Figure 6.2(ii), then

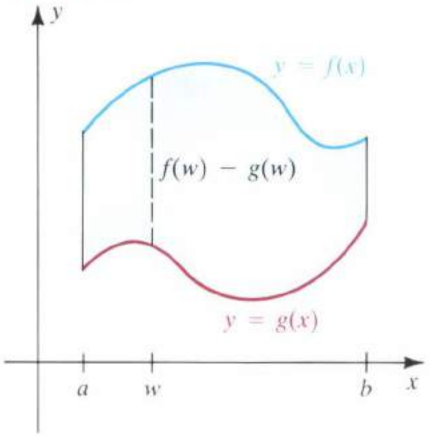
$$\begin{aligned} A &= \int_a^b [f_1(x) - g_1(x)]\,dx \\ &= \int_a^b \{[f(x) - d] - [g(x) - d]\}\,dx \\ &= \int_a^b [f(x) - g(x)]\,dx. \end{aligned}$$

We may summarize our discussion as follows.

Theorem (6.1)

> If f and g are continuous and $f(x) \geq g(x)$ for every x in $[a, b]$, then the area A of the region bounded by the graphs of f, g, $x = a$, and $x = b$ is
> $$A = \int_a^b [f(x) - g(x)]\, dx.$$

FIGURE 6.3

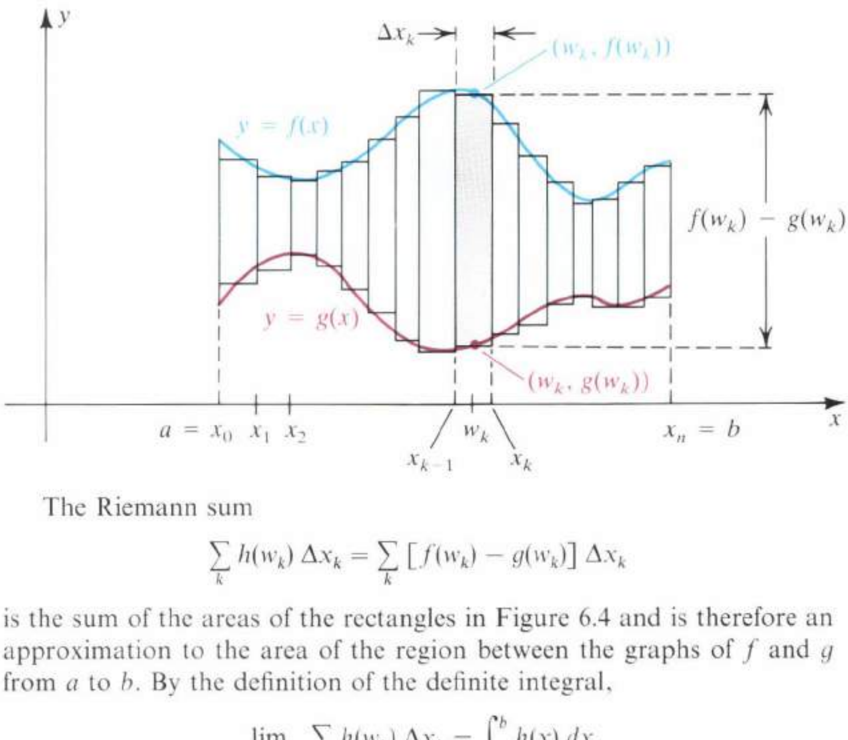


We may interpret the formula for A in Theorem (6.1) as a limit of sums. If we let $h(x) = f(x) - g(x)$ and if w is in $[a, b]$, then $h(w)$ is the vertical distance between the graphs of f and g for $x = w$ (see Figure 6.3). As in our discussion of Riemann sums in Chapter 5, let P denote a partition of $[a, b]$ determined by $a = x_0, x_1, \ldots, x_n = b$. For each k, let $\Delta x_k = x_k - x_{k-1}$, and let w_k be any number in the kth subinterval $[x_{k-1}, x_k]$ of P. By the definition of h,

$$h(w_k)\, \Delta x_k = [f(w_k) - g(w_k)]\, \Delta x_k,$$

which is the area of the rectangle of length $f(w_k) - g(w_k)$ and width Δx_k shown in Figure 6.4.

FIGURE 6.4



The Riemann sum

$$\sum_k h(w_k)\, \Delta x_k = \sum_k [f(w_k) - g(w_k)]\, \Delta x_k$$

is the sum of the areas of the rectangles in Figure 6.4 and is therefore an approximation to the area of the region between the graphs of f and g from a to b. By the definition of the definite integral,

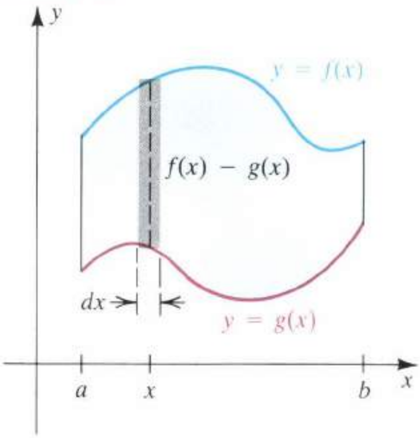
$$\lim_{\|P\| \to 0} \sum_k h(w_k)\, \Delta x_k = \int_a^b h(x)\, dx.$$

Since $h(x) = f(x) - g(x)$, we obtain the following corollary of Theorem (6.1).

Corollary (6.2)

> $$A = \lim_{\|P\| \to 0} \sum_k [f(w_k) - g(w_k)]\, \Delta x_k = \int_a^b [f(x) - g(x)]\, dx$$

FIGURE 6.5



We may employ the following intuitive method for remembering this limit of sums formula (see Figure 6.5):

1. Use dx for the width Δx_k of a typical vertical rectangle.
2. Use $f(x) - g(x)$ for the length $f(w_k) - g(w_k)$ of the rectangle.
3. Regard the symbol $\int_a^b$ as an operator that takes a limit of sums of the rectangular areas $[f(x) - g(x)]\,dx$.

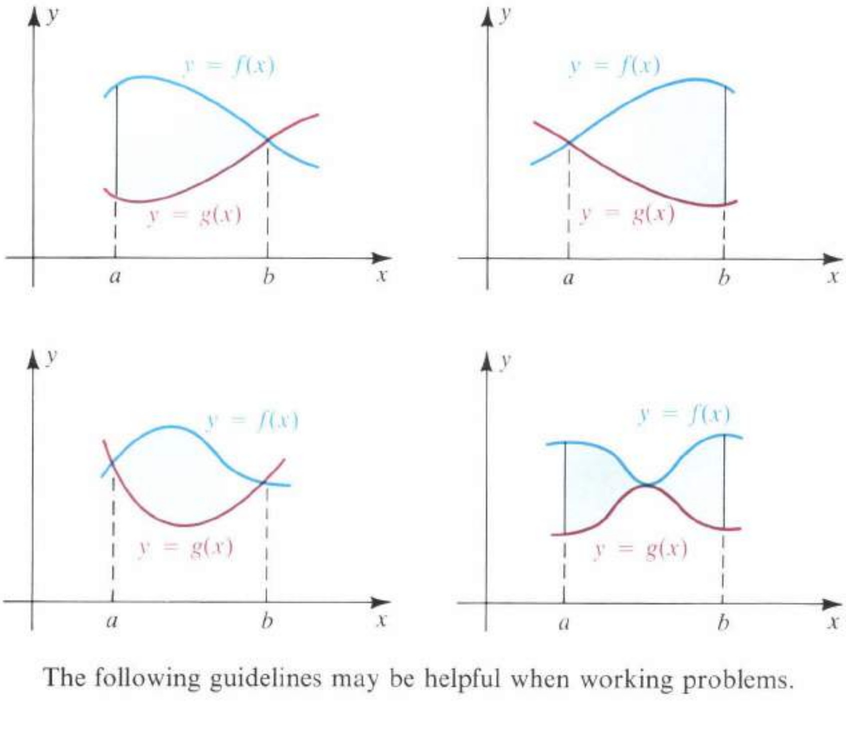
This method allows us to interpret the area formula in Theorem (6.1) as follows:

$$A = \underbrace{\int_a^b}_{\text{limit of sums}} \underbrace{[f(x) - g(x)]}_{\text{length of a rectangle}} \underbrace{dx}_{\text{width of a rectangle}}$$

When using this technique, we visualize summing areas of vertical rectangles by moving through the region from left to right. Later in this section we consider different types of regions, finding areas by using *horizontal* rectangles and integrating with respect to y.

Let us call a region an $\boldsymbol{R_x}$ **region** (for integration with respect to x) if it lies between the graphs of two equations $y = f(x)$ and $y = g(x)$, with f and g continuous, and $f(x) \geq g(x)$ for every x in $[a, b]$, where a and b are the smallest and largest x-coordinates, respectively, of the points (x, y) in the region. The regions in Figures 6.1 through 6.5 are R_x regions. Several others are sketched in Figure 6.6. Note that the graphs of $y = f(x)$ and $y = g(x)$ may intersect one or more times; however, $f(x) \geq g(x)$ throughout the interval.

FIGURE 6.6 R_x regions



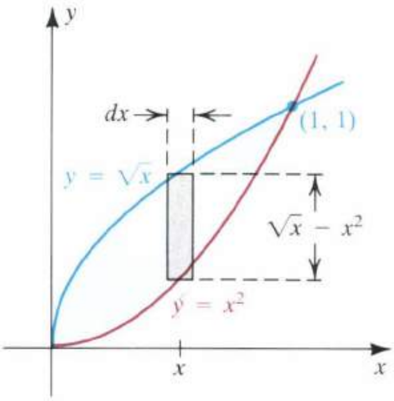
The following guidelines may be helpful when working problems.

Guidelines for finding the area of an R_x region **(6.3)**

1. Sketch the region, labeling the upper boundary $y = f(x)$ and the lower boundary $y = g(x)$. Find the smallest value $x = a$ and the largest value $x = b$ for points (x, y) in the region.
2. Sketch a typical vertical rectangle and label its width dx.
3. Express the area of the rectangle in guideline 2 as $[f(x) - g(x)]\,dx$.
4. Apply the limit of sums operator $\int_a^b$ to the expression in guideline 3 and evaluate the integral.

EXAMPLE 1 Find the area of the region bounded by the graphs of the equations $y = x^2$ and $y = \sqrt{x}$.

FIGURE 6.7



SOLUTION Following guidelines 1–3, we sketch and label the region and show a typical vertical rectangle (see Figure 6.7). The points (0, 0) and (1, 1) at which the graphs intersect can be found by solving the equations $y = x^2$ and $y = \sqrt{x}$ simultaneously. Referring to the figure, we obtain the following facts:

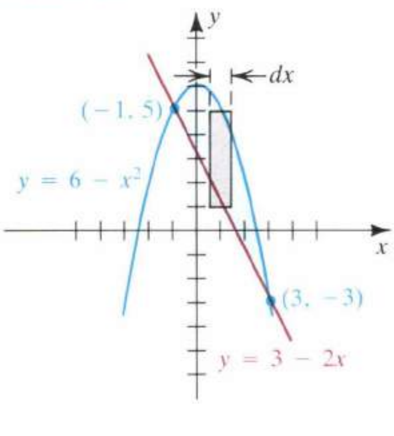
upper boundary:	$y = \sqrt{x}$
lower boundary:	$y = x^2$
width of rectangle:	dx
length of rectangle:	$\sqrt{x} - x^2$
area of rectangle:	$(\sqrt{x} - x^2)\,dx$

Next, we follow guideline 4 with $a = 0$ and $b = 1$, remembering that applying $\int_0^1$ to the expression $(\sqrt{x} - x^2)\,dx$ represents taking a limit of sums of areas of vertical rectangles. This gives us

$$A = \int_0^1 (\sqrt{x} - x^2)\,dx = \int_0^1 (x^{1/2} - x^2)\,dx$$
$$= \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

EXAMPLE 2 Find the area of the region bounded by the graphs of $y + x^2 = 6$ and $y + 2x - 3 = 0$.

FIGURE 6.8



SOLUTION The region and a typical rectangle are sketched in Figure 6.8. The points of intersection $(-1, 5)$ and $(3, -3)$ of the two graphs may be found by solving the two given equations simultaneously. To apply guideline 1 we must label the upper and lower boundaries $y = f(x)$ and $y = g(x)$, respectively, and hence we solve each of the given equations for y in terms of x, as shown in Figure 6.8. This gives us the following:

upper boundary:	$y = 6 - x^2$
lower boundary:	$y = 3 - 2x$
width of rectangle:	dx
length of rectangle:	$(6 - x^2) - (3 - 2x)$
area of rectangle:	$[(6 - x^2) - (3 - 2x)]\,dx$

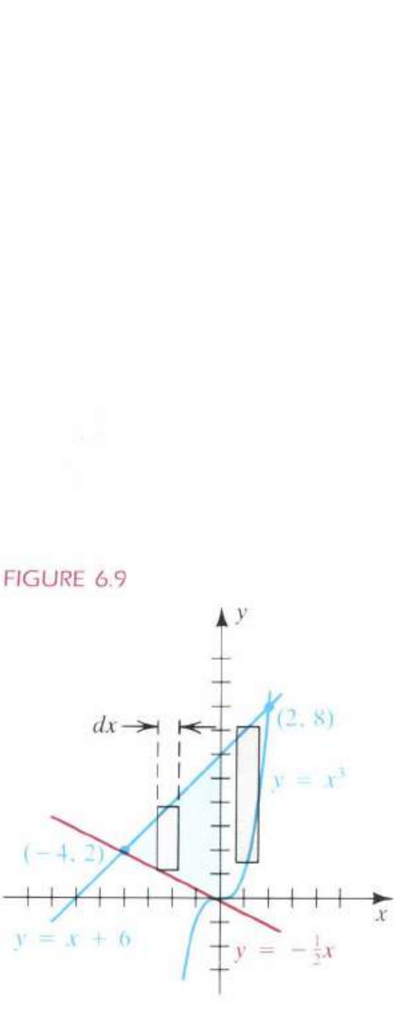
We next use guideline 4, with $a = -1$ and $b = 3$, regarding $\int_{-1}^{3}$ as an operator that takes a limit of sums of areas of rectangles. Thus,

$$\begin{aligned} A &= \int_{-1}^{3} [(6 - x^2) - (3 - 2x)]\,dx \\ &= \int_{-1}^{3} (3 - x^2 + 2x)\,dx \\ &= \left[3x - \frac{x^3}{3} + x^2\right]_{-1}^{3} \\ &= [9 - \tfrac{27}{3} + 9] - [-3 - (-\tfrac{1}{3}) + 1] = \tfrac{32}{3}. \end{aligned}$$

The following example illustrates that it is sometimes necessary to subdivide a region into several R_x regions and then use more than one definite integral to find the area.

EXAMPLE 3 Find the area of the region R bounded by the graphs of $y - x = 6$, $y - x^3 = 0$, and $2y + x = 0$.

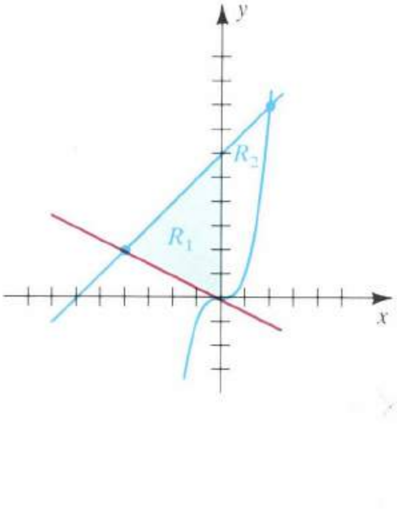
FIGURE 6.9



SOLUTION The graphs and region are sketched in Figure 6.9. Each equation has been solved for y in terms of x, and the boundaries have been labeled as in guideline 1. Typical vertical rectangles are shown extending from the lower boundary to the upper boundary of R. Since the lower boundary consists of portions of two different graphs, the area cannot be found by using only one definite integral. However, if R is divided into two R_x regions, R_1 and R_2, as shown in Figure 6.10, then we can determine the area of each and add them together. Let us arrange our work as follows.

	REGION R_1	REGION R_2
upper boundary:	$y = x + 6$	$y = x + 6$
lower boundary:	$y = -\frac{1}{2}x$	$y = x^3$
width of rectangle:	dx	dx
length of rectangle:	$(x + 6) - (-\frac{1}{2}x)$	$(x + 6) - x^3$
area of rectangle:	$[(x + 6) - (-\frac{1}{2}x)]\,dx$	$[(x + 6) - x^3]\,dx$

FIGURE 6.10



Applying guideline 4, we find the areas A_1 and A_2 of R_1 and R_2:

$$\begin{aligned} A_1 &= \int_{-4}^{0} [(x + 6) - (-\tfrac{1}{2}x)]\,dx \\ &= \int_{-4}^{0} \left(\frac{3}{2}x + 6\right) dx = \left[\frac{3}{2}\left(\frac{x^2}{2}\right) + 6x\right]_{-4}^{0} \\ &= 0 - (12 - 24) = 12 \end{aligned}$$

$$\begin{aligned} A_2 &= \int_{0}^{2} [(x + 6) - x^3]\,dx \\ &= \left[\frac{x^2}{2} + 6x - \frac{x^4}{4}\right]_{0}^{2} \\ &= (2 + 12 - 4) - 0 = 10 \end{aligned}$$

The area A of the entire region R is

$$A = A_1 + A_2 = 12 + 10 = 22.$$

FIGURE 6.11

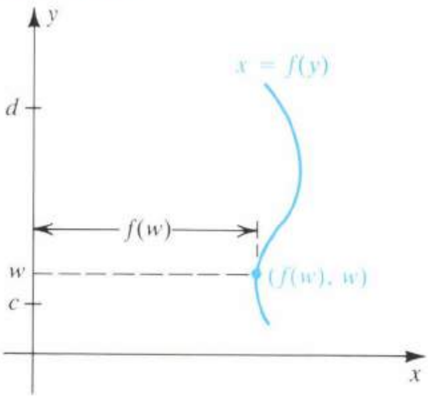
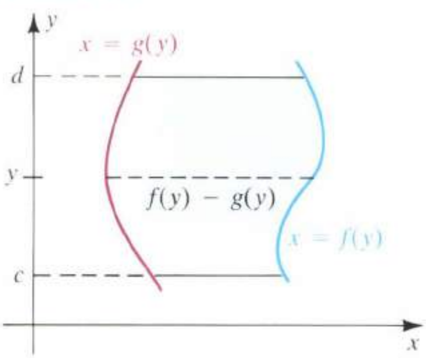


FIGURE 6.12



We have now evaluated many integrals similar to those in Example 3. For this reason we sometimes merely *set up* an integral; that is, we express it in the proper form but do not find its numerical value.

If we consider an equation of the form $x = f(y)$, where f is continuous for $c \le y \le d$, then we *reverse the roles of x and y in the previous discussion, treating y as the independent variable and x as the dependent variable*. A typical graph of $x = f(y)$ is sketched in Figure 6.11. Note that if a value w is assigned to y, then $f(w)$ *is an x-coordinate* of the corresponding point on the graph.

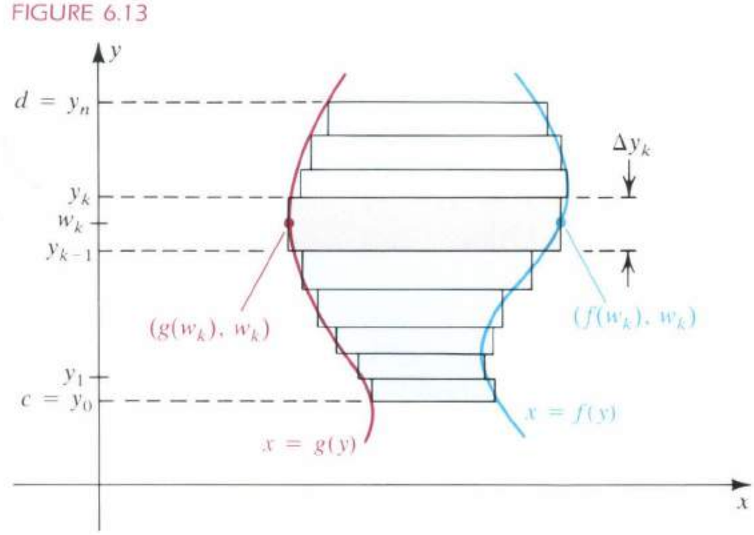
An $\boldsymbol{R_y}$ **region** is a region that lies between the graphs of two equations of the form $x = f(y)$ and $x = g(y)$, with f and g continuous, and $f(y) \ge g(y)$ for every y in $[c, d]$, where c and d are the smallest and largest y-coordinates, respectively, of points in the region. One such region is illustrated in Figure 6.12. We call the graph of f the **right boundary** of the region and the graph of g the **left boundary**. For any y, the number $f(y) - g(y)$ is the horizontal distance between these boundaries, as shown in Figure 6.12.

We can use limits of sums to find the area A of an R_y region. We begin by selecting points on the y-axis with y-coordinates $c = y_0, y_1, \ldots, y_n = d$, obtaining a partition of the interval $[c, d]$ into subintervals of width $\Delta y_k = y_k - y_{k-1}$. For each k we choose a number w_k in $[y_{k-1}, y_k]$ and consider horizontal rectangles that have areas $[f(w_k) - g(w_k)]\,\Delta y_k$, as illustrated in Figure 6.13. This leads to

$$A = \lim_{\|P\| \to 0} \sum_k [f(w_k) - g(w_k)]\,\Delta y_k = \int_c^d [f(y) - g(y)]\,dy.$$

The last equality follows from the definition of the definite integral.

FIGURE 6.13



Using notation similar to that for R_x regions, we represent the width Δy_k of a horizontal rectangle by dy and the length $f(w_k) - g(w_k)$ of the rectangle by $f(y) - g(y)$ in the following guidelines.

Guidelines for finding the area of an R_y region **(6.4)**

1. Sketch the region, labeling the right boundary $x = f(y)$ and the left boundary $x = g(y)$. Find the smallest value $y = c$ and the largest value $y = d$ for points (x, y) in the region.
2. Sketch a typical horizontal rectangle and label its width dy.
3. Express the area of the rectangle in guideline 2 as $[f(y) - g(y)]\,dy$.
4. Apply the limit of sums operator $\int_c^d$ to the expression in guideline 3 and evaluate the integral.

In guideline 4, we visualize summing areas of horizontal rectangles by moving from the lowest point of the region to the highest point.

EXAMPLE 4 Find the area of the region bounded by the graphs of the equations $2y^2 = x + 4$ and $y^2 = x$.

FIGURE 6.14

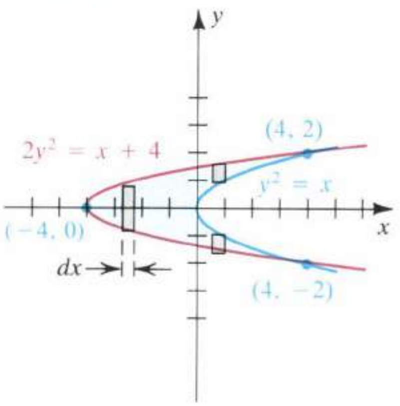
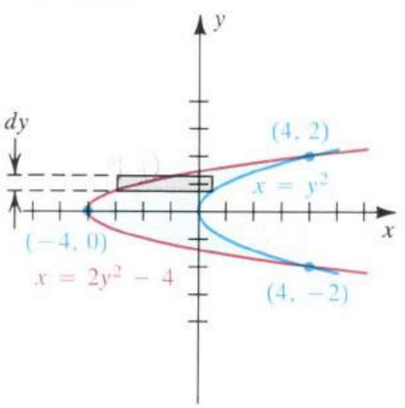


FIGURE 6.15



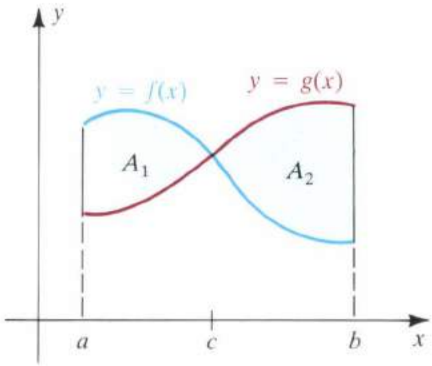
SOLUTION The region is sketched in Figures 6.14 and 6.15. Figure 6.14 illustrates the use of vertical rectangles (integration with respect to x), and Figure 6.15 illustrates the use of horizontal rectangles (integration with respect to y). Referring to Figure 6.14, we see that several integrations with respect to x are required to find the area. However, for Figure 6.15, we need only one integration with respect to y. Thus we apply Guidelines (6.4), solving each equation for x in terms of y. Referring to Figure 6.15, we obtain the following:

$$\begin{aligned} \text{right boundary:} &\quad x = y^2 \\ \text{left boundary:} &\quad x = 2y^2 - 4 \\ \text{width of rectangle:} &\quad dy \\ \text{length of rectangle:} &\quad y^2 - (2y^2 - 4) \\ \text{area of rectangle:} &\quad [y^2 - (2y^2 - 4)]\,dy \end{aligned}$$

We could now use guideline 4 with $c = -2$ and $d = 2$, finding A by applying the operator $\int_{-2}^{2}$ to $[y^2 - (2y^2 - 4)]\,dy$. Another method is to use the symmetry of the region with respect to the x-axis and find A by doubling the area of the part that lies above the x-axis. Thus,

$$\begin{aligned} A &= \int_{-2}^{2} [y^2 - (2y^2 - 4)]\,dy \\ &= 2\int_0^2 (4 - y^2)\,dy \\ &= 2\left[4y - \frac{y^3}{3}\right]_0^2 \\ &= 2\left(8 - \tfrac{8}{3}\right) = \tfrac{32}{3} \end{aligned}$$

FIGURE 6.16

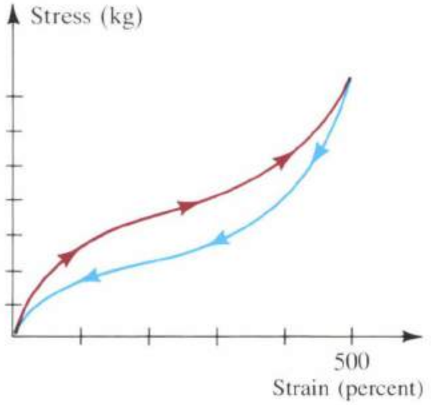


Throughout this section we have assumed that the graphs of the functions (or equations) do not cross one another in the interval under discussion. If the graphs of f and g cross at one point $P(c, d)$, with $a < c < b$, and we wish to find the area bounded by the graphs from $x = a$ to $x = b$, then the methods developed in this section may still be used; however, *two* integrations are required, one corresponding to the interval $[a, c]$ and the other to $[c, b]$. This is illustrated in Figure 6.16, with $f(x) \geq g(x)$ on $[a, c]$ and $g(x) \geq f(x)$ on $[c, b]$. The area A is given by

$$A = A_1 + A_2 = \int_a^c [f(x) - g(x)]\, dx + \int_c^b [g(x) - f(x)]\, dx.$$

If the graphs cross *several* times, then several integrals may be necessary. Problems in which graphs cross one or more times appear in Exercises 31–36.

FIGURE 6.17
Stress-strain diagram for an elastic material

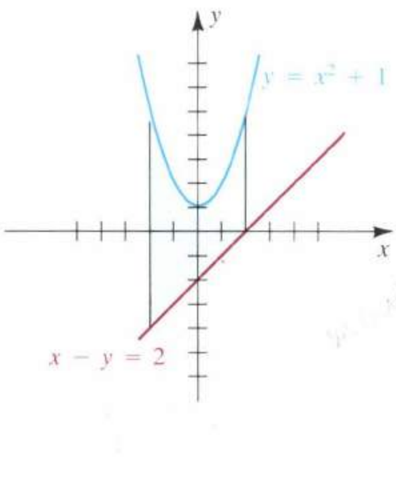


In scientific investigations, a physical quantity is often interpreted as an area. One illustration of this occurs in the *theory of elasticity*. To test the strength of a material, an investigator records values of strain that correspond to different loads (stresses). The sketch in Figure 6.17 is a typical stress-strain diagram for a sample of an elastic material, such as vulcanized rubber. (Note that stress values are assigned in the vertical direction.) Referring to the figure, we see that as the load applied to the material (the stress) increases, the strain (indicated by the arrows on the red graph) increases until the material is stretched to six times its original length. As the load decreases, the elastic material returns to its original length; however, the same graph is not retraced. Instead, the graph shown in blue is obtained. This phenomenon is called *elastic hysteresis*. (A similar occurrence takes place in the study of magnetic materials, where it is called *magnetic hysteresis*.) The two curves in the figure make up a *hysteresis loop* for the material. The area of the region enclosed by this loop is numerically equal to the energy dissipated within the elastic (or magnetic) material during the test. In the case of vulcanized rubber, the larger the area, the better the material is for absorbing vibrations.

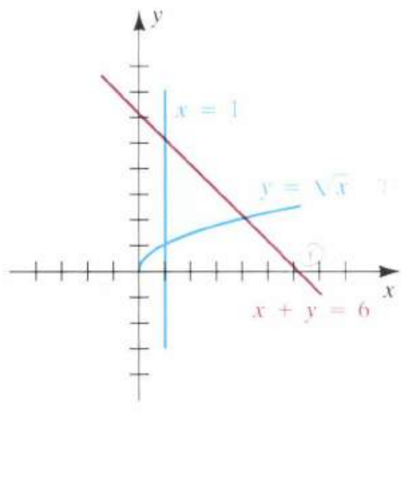
EXERCISES 6.1

Exer. 1–4: Set up an integral that can be used to find the area of the shaded region.

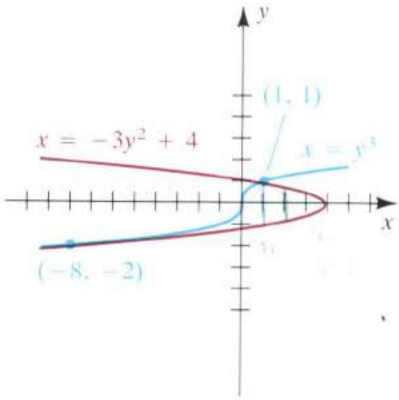
1



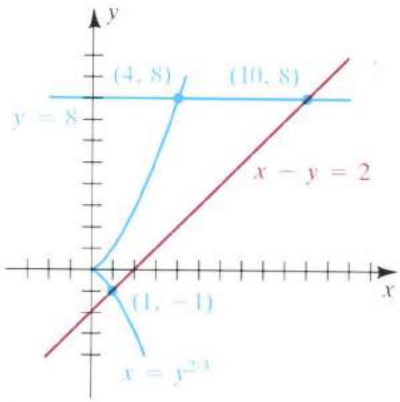
2



3



4



Exer. 5–22: Sketch the region bounded by the graphs of the equations and find its area.

5 $y = x^2$; $y = 4x$

6 $x + y = 3$; $y + x^2 = 3$

7 $y = x^2 + 1$; $y = 5$

8 $y = 4 - x^2$; $y = -4$

9 $y = 1/x^2$; $y = -x^2$; $x = 1$; $x = 2$

10 $y = x^3$; $y = x^2$

11 $y^2 = -x$; $x - y = 4$; $y = -1$; $y = 2$

12 $x = y^2$; $y - x = 2$; $y = -2$; $y = 3$

13 $y^2 = 4 + x$; $y^2 + x = 2$

14 $x = y^2$; $x - y = 2$

15 $x = 4y - y^3$; $x = 0$

16 $x = y^{2/3}$; $x = y^2$

17 $y = x^3 - x$; $y = 0$

18 $y = x^3 - x^2 - 6x$; $y = 0$

19 $x = y^3 + 2y^2 - 3y$; $x = 0$

20 $x = 9y - y^3$; $x = 0$

21 $y = x\sqrt{4 - x^2}$; $y = 0$

22 $y = x\sqrt{x^2 - 9}$; $y = 0$; $x = 5$

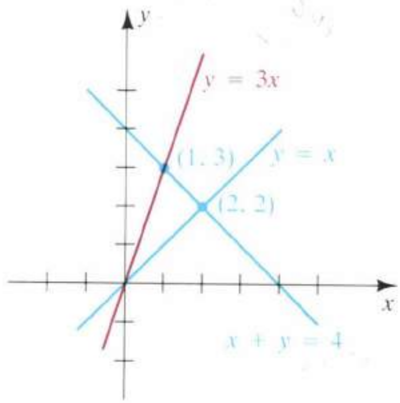
Exer. 23–24: Find the area of the region between the graphs of the two equations from $x = 0$ to $x = \pi$.

23 $y = \sin 4x$; $y = 1 + \cos \frac{1}{3}x$

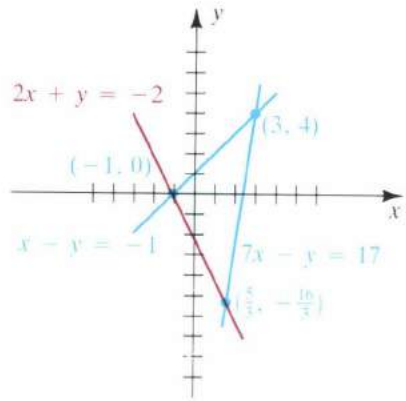
24 $y = 4 + \cos 2x$; $y = 3 \sin \frac{1}{2}x$

Exer. 25–26: Set up sums of integrals that can be used to find the area of the shaded region by integrating with respect to (a) x and (b) y.

25



26



Exer. 27–30: Set up sums of integrals that can be used to find the area of the region bounded by the graphs of the equations by integrating with respect to (a) x and (b) y.

27 $y = \sqrt{x}$; $y = -x$; $x = 1$; $x = 4$

28 $y = 1 - x^2$; $y = x - 1$

29 $y = x + 3$; $x = -y^2 + 3$

30 $x = y^2$; $x = 2y^2 - 4$

Exer. 31–36: Find the area of the region between the graphs of f and g if x is restricted to the given interval.

31 $f(x) = 6 - 3x^2$; $g(x) = 3x$; $[0, 2]$

32 $f(x) = x^2 - 4$; $g(x) = x + 2$; $[1, 4]$

33 $f(x) = x^3 - 4x + 2$; $g(x) = 2$; $[-1, 3]$

34 $f(x) = x^2$; $g(x) = x^3$; $[-1, 2]$

35 $f(x) = \sin x$; $g(x) = \cos x$; $[0, 2\pi]$

36 $f(x) = \sin x$; $g(x) = \frac{1}{2}$; $[0, \pi/2]$

Exer. 37–38: Let R be the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. Set up a sum of integrals, not containing the absolute value symbol, that can be used to find the area of R.

37 $f(x) = |x^2 - 6x + 5|$; $a = 0$, $b = 7$

38 $f(x) = |-x^2 + 2x + 3|$; $a = -3$, $b = 4$

39 The shape of a particular stress-strain diagram is shown in the figure (see the last paragraph of this section).

EXERCISE 39

Estimate y-coordinates and approximate the area of the region enclosed by the hysteresis loop using, with $n = 6$,

(a) the trapezoidal rule

(b) Simpson's rule

40 Suppose the function values of f and g in the table below were obtained empirically. Assuming that f and g are continuous, approximate the area between their graphs from $x = 1$ to $x = 5$ using, with $n = 8$,

(a) the trapezoidal rule

(b) Simpson's rule

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$f(x)$	3.5	2.5	3	4	3.5	2.5	2	2	3
$g(x)$	1.5	2	2	1.5	1	0.5	1	1.5	1

c 41 Graph $f(x) = |x^3 - 0.7x^2 - 0.8x + 1/3|$ on $[-1.5, 1.5]$. Set up a sum of integrals not containing the absolute value symbol that can be used to approximate the area of the region bounded by the graph of f, the x-axis, and the lines $x = -1.5$ and $x = 1.5$.

c 42 Graph, on the same coordinate axes, $f(x) = \sin x$ and $g(x) = x^3 - x + 0.2$ for $-2 \le x \le 2$. Set up a sum of integrals that can be used to approximate the area of the region bounded by the graphs.

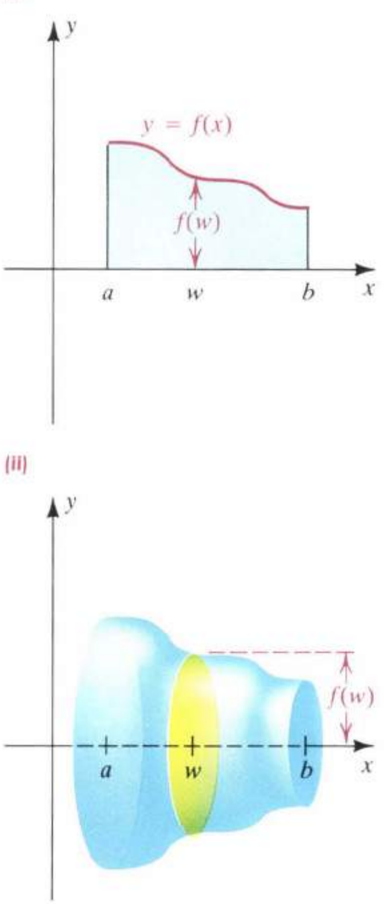
6.2 SOLIDS OF REVOLUTION

The volume of an object plays an important role in many problems in the physical sciences, such as finding centers of mass and moments of inertia. (These concepts will be discussed later in the text.) Since it is difficult to determine the volume of an irregularly shaped object, we shall begin with objects that have simple shapes. Included in this category are the solids of revolution discussed here and in the next section.

If a region in a plane is revolved about a line in the plane, the resulting solid is a **solid of revolution**, and we say that the solid is **generated** by the region. The line is an **axis of revolution**. In particular, if the R_x region shown in Figure 6.18(i), on the following page, is revolved about the x-axis, we obtain the solid illustrated in (ii) of the figure. As a special case, if f is a constant function, say $f(x) = k$, then the region is rectangular and the solid generated is a right circular cylinder. If the graph of f is a semicircle with endpoints of a diameter at the points $(a, 0)$ and $(b, 0)$, then the solid of revolution is a sphere. If the region is a right triangle with base on the x-axis and two vertices at the points $(a, 0)$ and $(b, 0)$ with the right angle at one of these points, then a right circular cone is generated.

If a plane perpendicular to the x-axis intersects the solid shown in Figure 6.18(ii), a circular cross section is obtained. If, as indicated in the figure, the plane passes through the point on the axis with x-coordinate w, then the radius of the circle is $f(w)$ and hence its area is $\pi[f(w)]^2$. We shall arrive at a definition for the volume of such a solid of revolution by using Riemann sums.

FIGURE 6.18



Let us partition the interval $[a, b]$, as we did for areas in the preceding section, and consider the rectangles in Figure 6.19(i). The solid of revolution generated by these rectangles has the shape indicated in (ii) of the figure. Beginning with Figure 6.23, we shall remove, or cut out, parts of solids of revolution to help us visualize portions generated by typical rectangles. When referring to such figures, remember that the entire solid is obtained by one *complete* revolution about an axis, not a partial one.

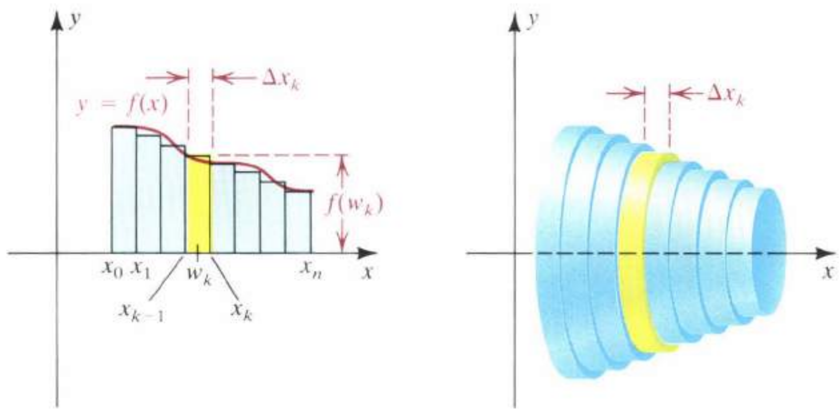
Observe that the kth rectangle generates a **circular disk** (a flat right circular cylinder) of base radius $f(w_k)$ and altitude (thickness) $\Delta x_k = x_k - x_{k-1}$. The volume of this disk is the area of the base times the altitude—that is, $\pi[f(w_k)]^2 \Delta x_k$. The volume of the solid shown in Figure 6.19(ii) is the sum of the volumes of all such disks:

$$\sum_k \pi[f(w_k)]^2 \Delta x_k$$

This sum may be regarded as a Riemann sum for $\pi[f(x)]^2$. If the norm $\|P\|$ of the partition is close to zero, then the sum should be close to the volume of the solid. Hence we define the volume of the solid of revolution as a limit of these sums.

FIGURE 6.19

(i) (ii)



Definition **(6.5)**

Let f be continuous on $[a, b]$, and let R be the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$. The **volume** V of the solid of revolution generated by revolving R about the x-axis is

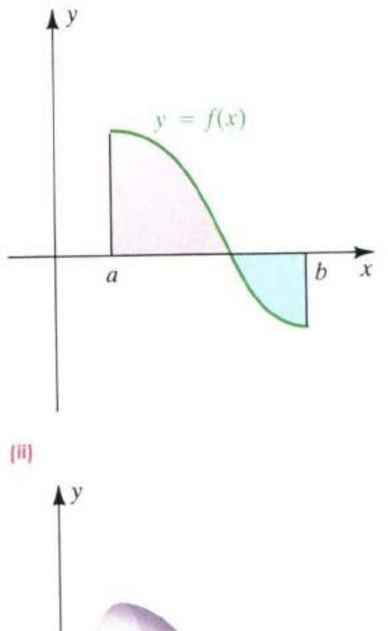
$$V = \lim_{\|P\| \to 0} \sum_k \pi[f(w_k)]^2 \Delta x_k = \int_a^b \pi[f(x)]^2\, dx.$$

The fact that the limit of sums in this definition equals $\int_a^b \pi[f(x)]^2\, dx$ follows from the definition of the definite integral. We shall not ordinarily specify the units of measure for volume. If the linear measurement is inches, the volume is in cubic inches (in.^{3}). If x is measured in centimeters, then V is in cubic centimeters (cm^3), and so on.

FIGURE 6.20

(i)

(ii)

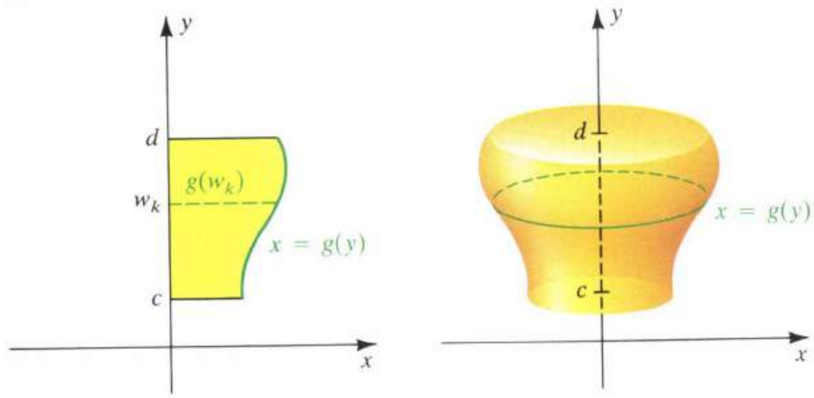


The requirement that $f(x) \geq 0$ was omitted intentionally in Definition (6.5). If f is negative for some x, as in Figure 6.20(i), and if the region bounded by the graphs of f, $x = a$, $x = b$, and the x-axis is revolved about the x-axis, we obtain the solid shown in (ii) of the figure. This solid is the same as that generated by revolving the region under the graph of $y = |f(x)|$ from a to b about the x-axis. Since $|f(x)|^2 = [f(x)]^2$, the limit in Definition (6.5) gives us the volume.

Let us interchange the roles of x and y and revolve the R_y region in Figure 6.21(i) about the y-axis, obtaining the solid illustrated in (ii) of the figure. If we partition the y-interval $[c, d]$ and use *horizontal* rectangles of width Δy_k and length $g(w_k)$, the same type of reasoning that gave us (6.5) leads to the following definition.

FIGURE 6.21

(i) (ii)



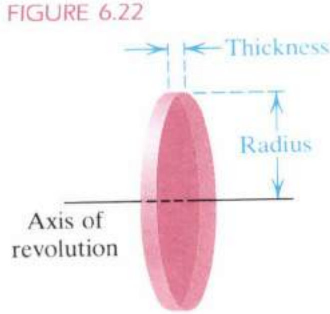
Definition **(6.6)**

x width

$$V = \lim_{\|P\| \to 0} \sum_k \pi[g(w_k)]^2 \, \Delta y_k = \int_c^d \pi[g(y)]^2 \, dy$$

Since we may revolve a region about the x-axis, the y-axis, or some other line, *it is not advisable to merely memorize the formulas in* (6.5) *and* (6.6). It is better to remember the following general rule for finding the volume of a circular disk (see Figure 6.22).

FIGURE 6.22



Volume V of a circular disk **(6.7)**

$$V = \pi(\text{radius})^2 \cdot (\text{thickness})$$

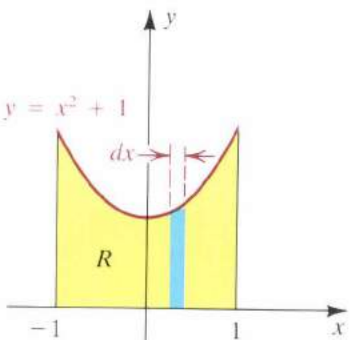
When working problems we shall use the intuitive method developed in Section 6.1, replacing Δx_k or Δy_k by dx or dy, and so on. The following guidelines may be helpful.

Guidelines for finding the volume of a solid of revolution using disks **(6.8)**

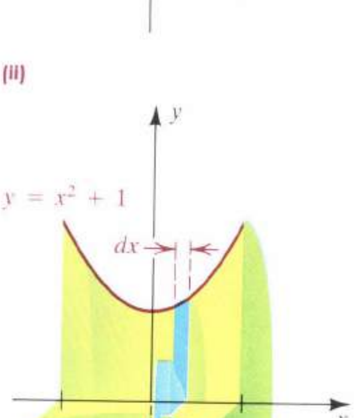
1. Sketch the region R to be revolved, and label the boundaries. Show a typical vertical rectangle of width dx or a horizontal rectangle of width dy.
2. Sketch the solid generated by R and the disk generated by the rectangle in guideline 1.
3. Express the radius of the disk in terms of x or y, depending on whether its thickness is dx or dy.
4. Use (6.7) to find a formula for the volume of the disk.
5. Apply the limit of sums operator $\int_a^b$ or $\int_c^d$ to the expression in guideline 4 and evaluate the integral.

FIGURE 6.23

(i)



(ii)



EXAMPLE 1 The region bounded by the x-axis, the graph of the equation $y = x^2 + 1$, and the lines $x = -1$ and $x = 1$ is revolved about the x-axis. Find the volume of the resulting solid.

SOLUTION As specified in guideline 1, we sketch the region and show a vertical rectangle of width dx (see Figure 6.23(i)). Following guideline 2, we sketch the solid generated by R and the disk generated by the rectangle (see Figure 6.23(ii)). As specified in guidelines 3 and 4, we note the following:

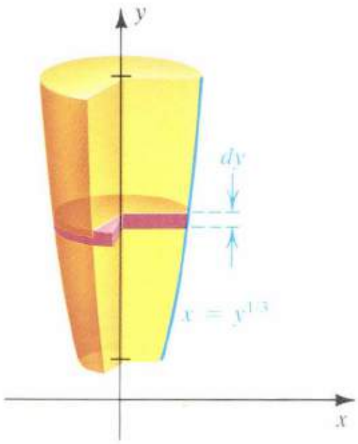
$$\begin{aligned} \text{thickness of disk:}\quad & dx \\ \text{radius of disk:}\quad & x^2 + 1 \\ \text{volume of disk:}\quad & \pi(x^2 + 1)^2\, dx \end{aligned}$$

We could next apply guideline 5 with $a = -1$ and $b = 1$, finding the volume V by regarding $\int_{-1}^{1}$ as an operator that takes a limit of sums of volumes of disks. Another method is to use the symmetry of the region with respect to the y-axis and find V by applying $\int_0^1$ to $\pi(x^2 + 1)^2\, dx$ and doubling the result. Thus,

$$\begin{aligned} V &= \int_{-1}^{1} \pi(x^2 + 1)^2\, dx \\ &= 2\int_0^1 \pi(x^4 + 2x^2 + 1)\, dx \\ &= 2\pi\left[\frac{x^5}{5} + 2\left(\frac{x^3}{3}\right) + x\right]_0^1 \\ &= 2\pi\left(\tfrac{1}{5} + \tfrac{2}{3} + 1\right) = \tfrac{56}{15}\pi \approx 11.7. \end{aligned}$$

EXAMPLE 2 The region bounded by the y-axis and the graphs of $y = x^3$, $y = 1$, and $y = 8$ is revolved about the y-axis. Find the volume of the resulting solid.

FIGURE 6.24



SOLUTION The region and the solid are sketched in Figure 6.24, together with a disk generated by a typical horizontal rectangle. Since we plan to integrate with respect to y, we solve the equation $y = x^3$ for x in terms of y, obtaining $x = y^{1/3}$. We note the following facts (see guidelines 3 and 4):

thickness of disk: dy

radius of disk: $y^{1/3}$

volume of disk: $\pi(y^{1/3})^2\, dy$

Finally, we apply guideline 5, with $c = 1$ and $d = 8$, regarding $\int_1^8$ as an operator that takes a limit of sums of disks:

$$V = \int_1^8 \pi(y^{1/3})^2\, dy = \pi \int_1^8 y^{2/3}\, dy = \pi\left[\frac{y^{5/3}}{\frac{5}{3}}\right]_1^8$$

$$= \tfrac{3}{5}\pi\Big[y^{5/3}\Big]_1^8 = \tfrac{3}{5}\pi[32 - 1] = \tfrac{93}{5}\pi \approx 58.4$$

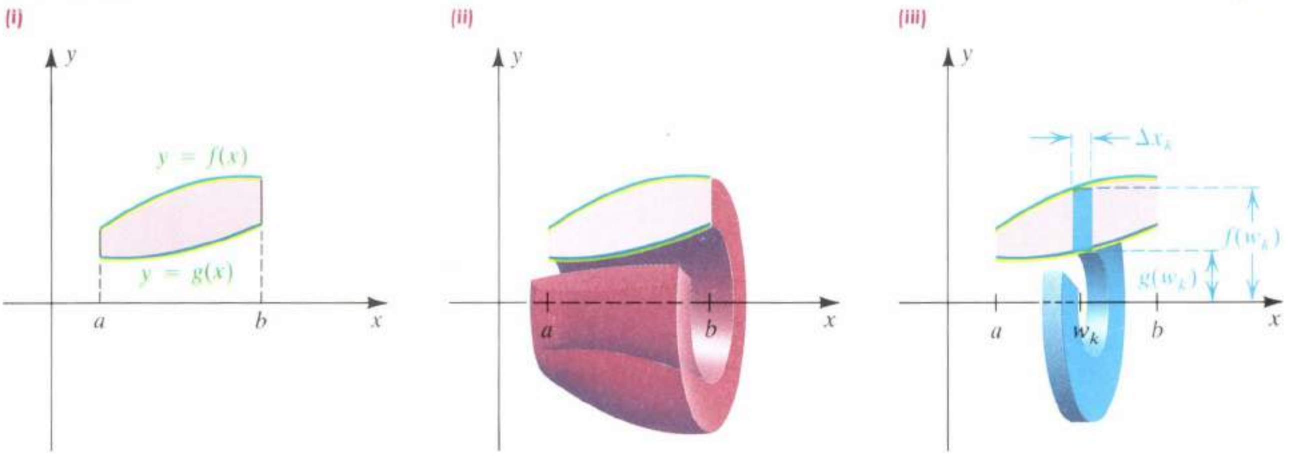
Let us next consider an R_x region of the type illustrated in Figure 6.25(i). If this region is revolved about the x-axis, we obtain the solid illustrated in (ii) of the figure. Note that if $g(x) > 0$ for every x in $[a, b]$, there is a hole through the solid.

The volume V of the solid may be found by subtracting the volume of the solid generated by the smaller region from the volume of the solid generated by the larger region. Using Definition (6.5) gives us

$$V = \int_a^b \pi[f(x)]^2\, dx - \int_a^b \pi[g(x)]^2\, dx = \int_a^b \pi\{[f(x)]^2 - [g(x)]^2\}\, dx.$$

The last integral has an interesting interpretation as a limit of sums. As illustrated in Figure 6.25(iii), a vertical rectangle extending from the graph

FIGURE 6.25



of g to the graph of f, through the points with x-coordinate w_k, generates a washer-shaped solid whose volume is

$$\pi[f(w_k)]^2\,\Delta x_k - \pi[g(w_k)]^2\,\Delta x_k = \pi\{[f(w_k)]^2 - [g(w_k)]^2\}\,\Delta x_k.$$

Summing the volumes of all such washers and taking the limit gives us the desired definite integral. When working problems of this type it is convenient to use the following general rule.

Volume V of a washer **(6.9)**

$$\boxed{V = \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\cdot(\text{thickness})}$$

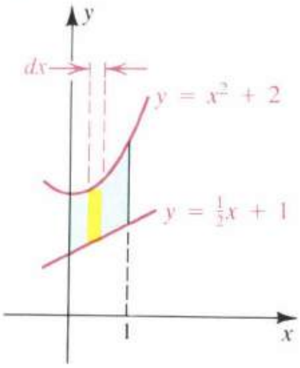
In applying (6.9), a common error is to use the square of the difference of the radii instead of the difference of the squares. Note that

$$\text{volume of a washer} \neq \pi[(\text{outer radius}) - (\text{inner radius})]^2\cdot(\text{thickness}).$$

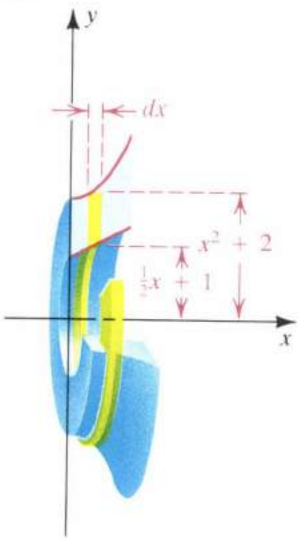
Guidelines similar to (6.8) can be stated for problems involving washers. The principal differences are that in guideline 3 we find expressions for the outer radius and inner radius of a typical washer, and in guideline 4 we use (6.9) to find a formula for the volume of the washer.

FIGURE 6.26

(i)



(ii)



EXAMPLE 3 The region bounded by the graphs of the equations $x^2 = y - 2$ and $2y - x - 2 = 0$ and by the vertical lines $x = 0$ and $x = 1$ is revolved about the x-axis. Find the volume of the resulting solid.

SOLUTION The region and a typical vertical rectangle are sketched in Figure 6.26(i). Since we wish to integrate with respect to x, we solve the first two equations for y in terms of x, obtaining $y = x^2 + 2$ and $y = \frac{1}{2}x + 1$. The solid and a washer generated by the rectangle are illustrated in (ii) of the figure. Using (6.9), we obtain the following:

thickness of washer: dx

outer radius: $x^2 + 2$

inner radius: $\frac{1}{2}x + 1$

volume: $\pi[(x^2 + 2)^2 - (\frac{1}{2}x + 1)^2]\,dx$

We take a limit of sums of volumes of washers by applying $\int_0^1$:

$$\begin{aligned} V &= \int_0^1 \pi[(x^2 + 2)^2 - (\tfrac{1}{2}x + 1)^2]\,dx \\ &= \pi\int_0^1 (x^4 + \tfrac{15}{4}x^2 - x + 3)\,dx \\ &= \pi\left[\frac{x^5}{5} + \frac{15}{4}\left(\frac{x^3}{3}\right) - \frac{x^2}{2} + 3x\right]_0^1 = \frac{79\pi}{20} \approx 12.4 \end{aligned}$$

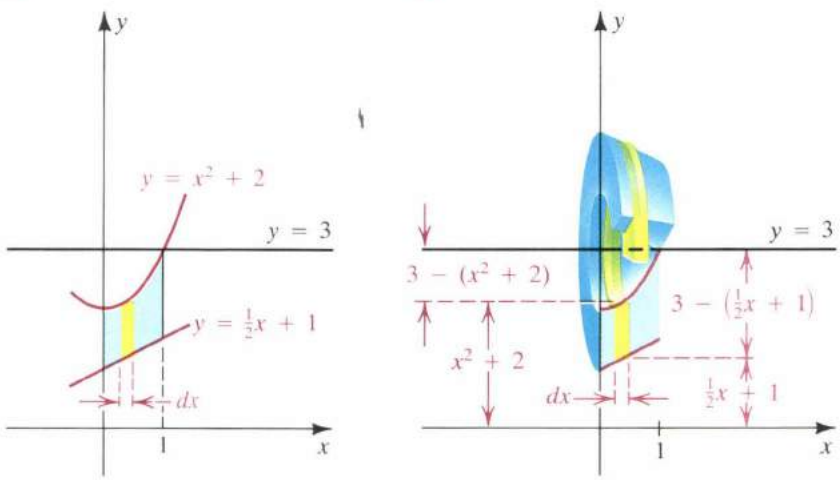
EXAMPLE 4 Find the volume of the solid generated by revolving the region described in Example 3 about the line $y = 3$.

SOLUTION The region and a typical vertical rectangle are resketched in Figure 6.27(i), together with the axis of revolution $y = 3$. The solid and a washer generated by the rectangle are illustrated in (ii) of the figure. We note the following:

$$\begin{aligned} \text{thickness of washer:} &\quad dx \\ \text{outer radius:} &\quad 3 - (\tfrac{1}{2}x + 1) = 2 - \tfrac{1}{2}x \\ \text{inner radius:} &\quad 3 - (x^2 + 2) = 1 - x^2 \\ \text{volume:} &\quad \pi[(2 - \tfrac{1}{2}x)^2 - (1 - x^2)^2]\, dx \end{aligned}$$

FIGURE 6.27

(i) (ii)



Applying the limit of sums operator $\int_0^1$ gives us the volume:

$$\begin{aligned} V &= \int_0^1 \pi[(2 - \tfrac{1}{2}x)^2 - (1 - x^2)^2]\, dx \\ &= \pi \int_0^1 (3 - 2x + \tfrac{9}{4}x^2 - x^4)\, dx \\ &= \pi\left[3x - x^2 + \frac{9}{4}\left(\frac{x^3}{3}\right) - \frac{x^5}{5}\right]_0^1 = \frac{51\pi}{20} \approx 8.01 \end{aligned}$$

FIGURE 6.28

(i)

EXAMPLE 5 The region in the first quadrant bounded by the graphs of $y = \frac{1}{8}x^3$ and $y = 2x$ is revolved about the y-axis. Find the volume of the resulting solid.

SOLUTION The region and a typical horizontal rectangle are shown in Figure 6.28(i). We wish to integrate with respect to y, so we solve the given equations for x in terms of y, obtaining

$$x = \tfrac{1}{2}y \quad \text{and} \quad x = 2y^{1/3}.$$

FIGURE 6.28
(ii)

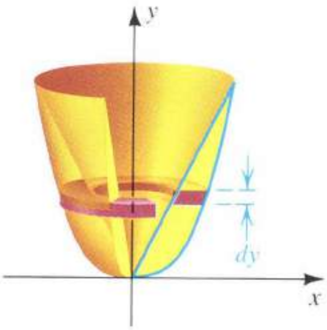


Figure 6.28(ii) illustrates the volume generated by the region and the washer generated by the rectangle. We note the following:

thickness of washer: dy

outer radius: $2y^{1/3}$

inner radius: $\frac{1}{2}y$

volume: $\pi[(2y^{1/3})^2 - (\frac{1}{2}y)^2]\,dy = \pi(4y^{2/3} - \frac{1}{4}y^2)\,dy$

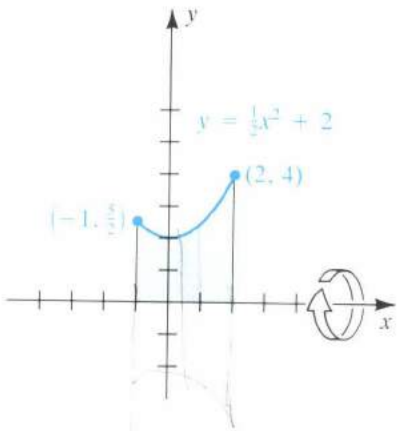
Applying the limit of sums operator $\int_0^8$ gives us the volume:

$$V = \int_0^8 \pi(4y^{2/3} - \tfrac{1}{4}y^2)\,dy$$
$$= \pi\left[\tfrac{12}{5}y^{5/3} - \tfrac{1}{12}y^3\right]_0^8 = \tfrac{512}{15}\pi \approx 107.2$$

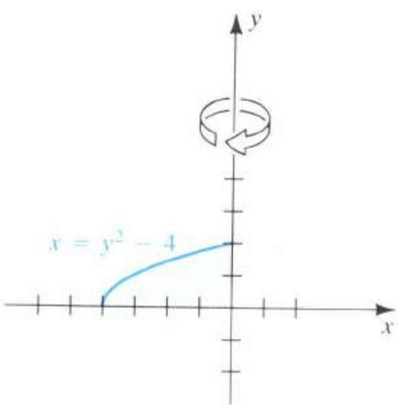
EXERCISES 6.2

Exer. 1–4: Set up an integral that can be used to find the volume of the solid obtained by revolving the shaded region about the indicated axis.

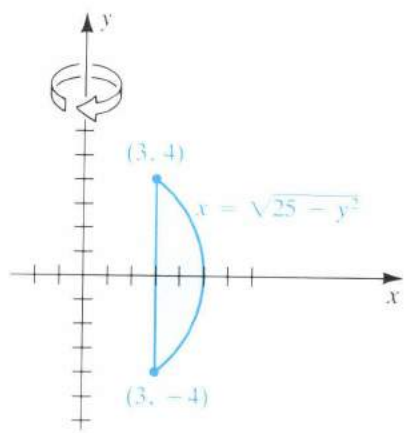
1



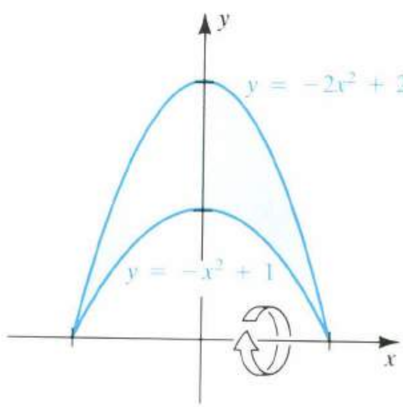
2



3



4



Exer. 5–24: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated if R is revolved about the indicated axis.

5 $y = 1/x$,	$x = 1$,	$x = 3$, $y = 0$;	x-axis
6 $y = \sqrt{x}$,	$x = 4$,	$y = 0$;	x-axis
7 $y = x^2 - 4x$,	$y = 0$;		x-axis
8 $y = x^3$,	$x = -2$,	$y = 0$;	x-axis
9 $y = x^2$,	$y = 2$;		y-axis
10 $y = 1/x$,	$y = 1$,	$y = 3$, $x = 0$;	y-axis
11 $x = 4y - y^2$,	$x = 0$;		y-axis
12 $y = x$,	$y = 3$,	$x = 0$;	y-axis
13 $y = x^2$,	$y = 4 - x^2$;		x-axis
14 $x = y^3$,	$x^2 + y = 0$;		x-axis
15 $y = x$,	$x + y = 4$,	$x = 0$;	x-axis
16 $y = (x-1)^2 + 1$,	$y = -(x-1)^2 + 3$;		x-axis
17 $y^2 = x$,	$2y = x$;		y-axis
18 $y = 2x$,	$y = 4x^2$;		y-axis
19 $x = y^2$,	$x - y = 2$;		y-axis
20 $x + y = 1$,	$x - y = -1$, $x = 2$;		y-axis
21 $y = \sin 2x$,	$x = 0$,	$x = \pi$, $y = 0$;	x-axis

(*Hint:* Use a half-angle formula.)

22 $y = 1 + \cos 3x$, $x = 0$, $x = 2\pi$, $y = 0$; x-axis
(*Hint:* Use a half-angle formula.)

23 $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$; x-axis
(*Hint:* Use a double angle formula.)

24 $y = \sec x$, $y = \sin x$, $x = 0$, $x = \pi/4$; x-axis

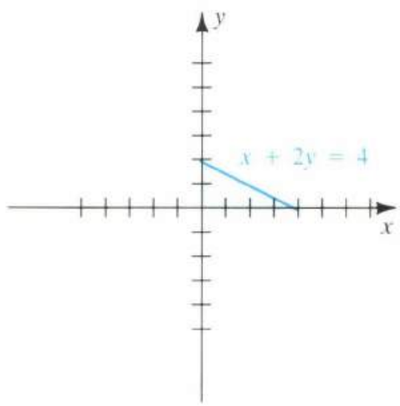
Exer. 25–26: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated if R is revolved about the given line.

25 $y = x^2$, $y = 4$
(a) $y = 4$ (b) $y = 5$
(c) $x = 2$ (d) $x = 3$

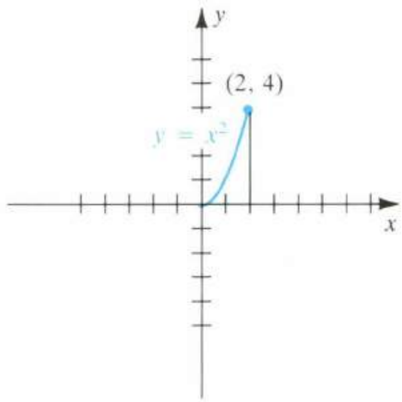
26 $y = \sqrt{x}$, $y = 0$, $x = 4$
(a) $x = 4$ (b) $x = 6$
(c) $y = 2$ (d) $y = 4$

Exer. 27–28: Set up an integral that can be used to find the volume of the solid generated by revolving the shaded region about the line (a) $y = -2$, (b) $y = 5$, (c) $x = 7$, and (d) $x = -4$.

27



28



Exer. 29–34: Sketch the region R bounded by the graphs of the equations, and set up integrals that can be used to find the volume of the solid generated if R is revolved about the given line.

29 $y = x^3$,	$y = 4x$;	$y = 8$
30 $y = x^3$,	$y = 4x$;	$x = 4$
31 $x + y = 3$,	$y + x^2 = 3$;	$x = 2$
32 $y = 1 - x^2$,	$x - y = 1$;	$y = 3$
33 $x^2 + y^2 = 1$;		$x = 5$
34 $y = x^{2/3}$,	$y = x^2$;	$y = -1$

Exer. 35–40: Use a definite integral to derive a formula for the volume of the indicated solid.

35 A right circular cylinder of altitude h and radius r

36 A cylindrical shell of altitude h, outer radius R, and inner radius r

37 A right circular cone of altitude h and base radius r

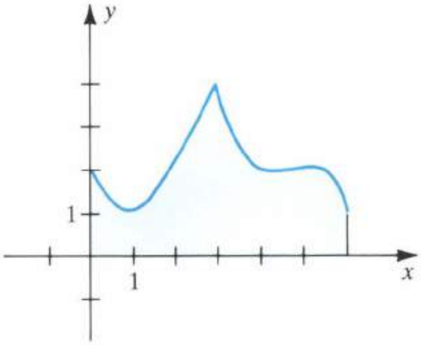
38 A sphere of radius r

39 A frustum of a right circular cone of altitude h, lower base radius R, and upper base radius r

40 A spherical segment of altitude h in a sphere of radius r

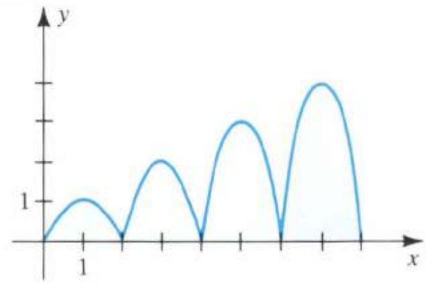
41 If the region shown in the figure is revolved about the x-axis, use the trapezoidal rule with $n = 6$ to approximate the volume of the resulting solid.

EXERCISE 41



42 If the region shown in the figure is revolved about the x-axis, use Simpson's rule with $n = 8$ to approximate the volume of the resulting solid.

EXERCISE 42



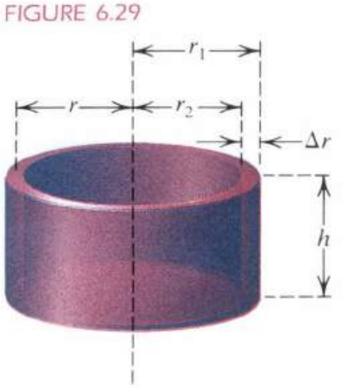
c **Exer. 43–44: Graph f and g on the same coordinate axes for $0 \le x \le \pi$. (a) Estimate the x-coordinates of the points of intersection of the graphs. (b) If the region bounded by the graphs of f and g is revolved about the x-axis, use Simpson's rule with $n = 4$ to approximate the volume of the resulting solid.**

43 $f(x) = \dfrac{\sin x}{1 + x}$; $\quad g(x) = 0.3$

44 $f(x) = \sqrt[4]{|\sin x|}$; $\quad g(x) = 0.2x + 0.7$

6.3 VOLUMES BY CYLINDRICAL SHELLS

FIGURE 6.29



In the preceding section we found volumes of solids of revolution by using circular disks or washers. For certain types of solids it is convenient to use hollow circular cylinders—that is, thin **cylindrical shells** of the type illustrated in Figure 6.29, where r_1 is the *outer radius*, r_2 is the *inner radius*, h is the *altitude*, and $\Delta r = r_1 - r_2$ is the *thickness* of the shell. The **average radius** of the shell is $r = \frac{1}{2}(r_1 + r_2)$. We can find the volume of the shell by subtracting the volume $\pi r_2^2 h$ of the inner cylinder from the volume $\pi r_1^2 h$ of the outer cylinder. If we do this and change the form of the resulting expression, we obtain

$$\begin{aligned} \pi r_1^2 h - \pi r_2^2 h &= \pi(r_1^2 - r_2^2)h \\ &= \pi(r_1 + r_2)(r_1 - r_2)h \\ &= 2\pi \cdot \tfrac{1}{2}(r_1 + r_2)h(r_1 - r_2) \\ &= 2\pi r h\, \Delta r, \end{aligned}$$

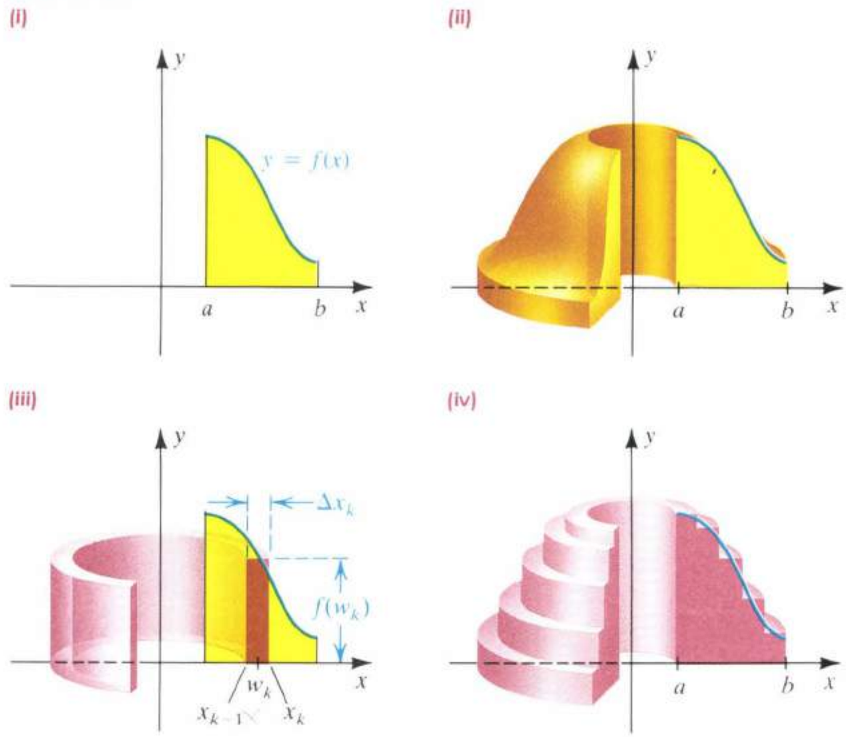
which gives us the following general rule.

Volume V of a cylindrical shell **(6.10)**

$$\boxed{V = 2\pi(\text{average radius})(\text{altitude})(\text{thickness})}$$

If the R_x region in Figure 6.30(i) is revolved about the y-axis, we obtain the solid illustrated in (ii) of the figure.

FIGURE 6.30



Let P be a partition of $[a, b]$, and consider the typical vertical rectangle in Figure 6.30(iii), where w_k is the midpoint of $[x_{k-1}, x_k]$. If we revolve this rectangle about the y-axis, we obtain a cylindrical shell of average radius w_k, altitude $f(w_k)$, and thickness Δx_k. Hence, by (6.10), the volume of the shell is

$$2\pi w_k f(w_k)\,\Delta x_k.$$

Revolving the rectangular polygon formed by *all* the rectangles determined by P gives us the solid illustrated in Figure 6.30(iv). The volume of this solid is a Riemann sum:

$$\sum_k 2\pi w_k f(w_k)\,\Delta x_k$$

The smaller the norm $\|P\|$ of the partition, the better the sum approximates the volume V of the solid shown in (ii) of the figure. This is the motivation for the following definition.

Definition **(6.11)**

Let f be continuous and suppose $f(x) \geq 0$ on $[a, b]$, where $0 \leq a \leq b$. Let R be the region under the graph of f from a to b. The volume V of the solid of revolution generated by revolving R about the y-axis is

$$V = \lim_{\|P\| \to 0} \sum_k 2\pi w_k f(w_k)\,\Delta x_k = \int_a^b 2\pi x f(x)\,dx.$$

It can be proved that if the methods of Section 6.2 are also applicable, then both methods lead to the same answer.

We may also consider solids obtained by revolving a region about the y-axis or some other line. The following guidelines may be useful.

Guidelines for finding the volume of a solid of revolution using cylindrical shells **(6.12)**

1 Sketch the region R to be revolved, and label the boundaries. Show a typical vertical rectangle of width dx or a horizontal rectangle of width dy.

2 Sketch the cylindrical shell generated by the rectangle in guideline 1.

3 Express the average radius of the shell in terms of x or y, depending on whether its thickness is dx or dy. *Remember that x represents a distance from the y-axis to a vertical rectangle, and y represents a distance from the x-axis to a horizontal rectangle.*

4 Express the altitude of the shell in terms of x or y, depending on whether its thickness is dx or dy.

5 Use (6.10) to find a formula for the volume of the shell.

6 Apply the limit of sums operator $\int_a^b$ or $\int_c^d$ to the expression in guideline 5 and evaluate the integral.

EXAMPLE 1 The region bounded by the graph of $y = 2x - x^2$ and the x-axis is revolved about the y-axis. Find the volume of the resulting solid.

SOLUTION The region to be revolved is sketched in Figure 6.31(i), together with a typical vertical rectangle of width dx. Figure 6.31(ii) shows the cylindrical shell generated by revolving the rectangle about the y-axis. Note that x represents the distance from the y-axis to the midpoint of the rectangle (the average radius of the shell). Referring to the figure and using (6.10) gives us the following:

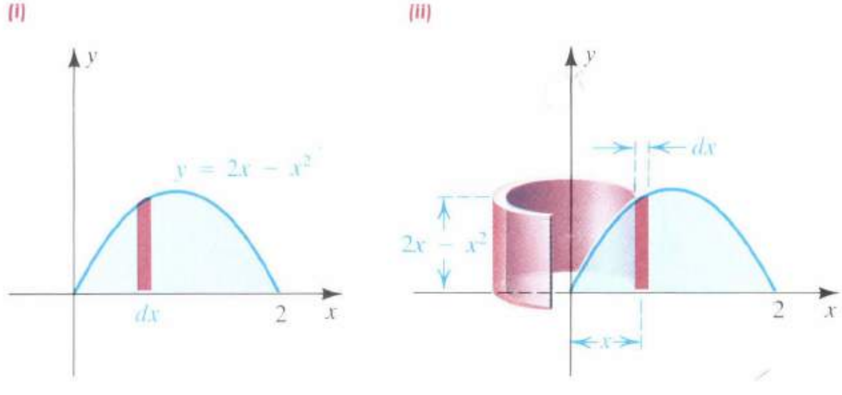
thickness of shell: dx

average radius: x

altitude: $2x - x^2$

volume: $2\pi x(2x - x^2)\,dx$

FIGURE 6.31



To sum all such shells, we move from left to right through the region from $a = 0$ to $b = 2$ (do *not* sum from -2 to 2). Hence the limit of sums is

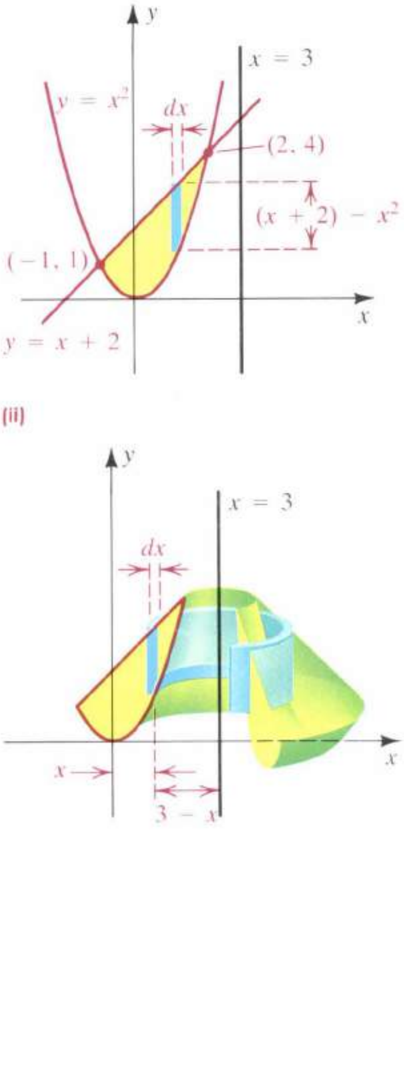
$$V = \int_0^2 2\pi x(2x - x^2)\,dx = 2\pi \int_0^2 (2x^2 - x^3)\,dx$$
$$= 2\pi\left[2\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_0^2 = \frac{8\pi}{3} \approx 8.4.$$

The volume V can also be found using washers; however, the calculations would be much more involved, since the equation $y = 2x - x^2$ would have to be solved for x in terms of y.

FIGURE 6.32

(i)

(ii)



EXAMPLE 2 The region bounded by the graphs of $y = x^2$ and $y = x + 2$ is revolved about the line $x = 3$. Set up the integral for the volume of the resulting solid.

SOLUTION The region is sketched in Figure 6.32(i), together with a typical vertical rectangle extending from the lower boundary $y = x^2$ to the upper boundary $y = x + 2$. Also shown is the axis of revolution $x = 3$. In (ii) of the figure we have illustrated both the cylindrical shell and the solid that are generated by revolving the rectangle and the region about the line $x = 3$. It is important to note that since x is the distance from the y-axis to the rectangle, the radius of the shell is $3 - x$. Referring to Figure 6.32 and using (6.10) gives us the following:

thickness of shell: dx

average radius: $3 - x$

altitude: $(x + 2) - x^2$

volume: $2\pi(3 - x)(x + 2 - x^2)\,dx$

To sum all such shells, we move from left to right through the region from $a = -1$ to $b = 2$. Hence the limit of sums is

$$V = \int_{-1}^{2} 2\pi(3 - x)(x + 2 - x^2)\,dx.$$

EXAMPLE 3 The region in the first quadrant bounded by the graph of the equation $x = 2y^3 - y^4$ and the y-axis is revolved about the x-axis. Set up the integral for the volume of the resulting solid.

SOLUTION The region is sketched in Figure 6.33(i) on the following page, together with a typical horizontal rectangle. Part (ii) of the figure shows the cylindrical shell and the solid that are generated by the revolution about the x-axis. Referring to the figure and using (6.10) gives the following:

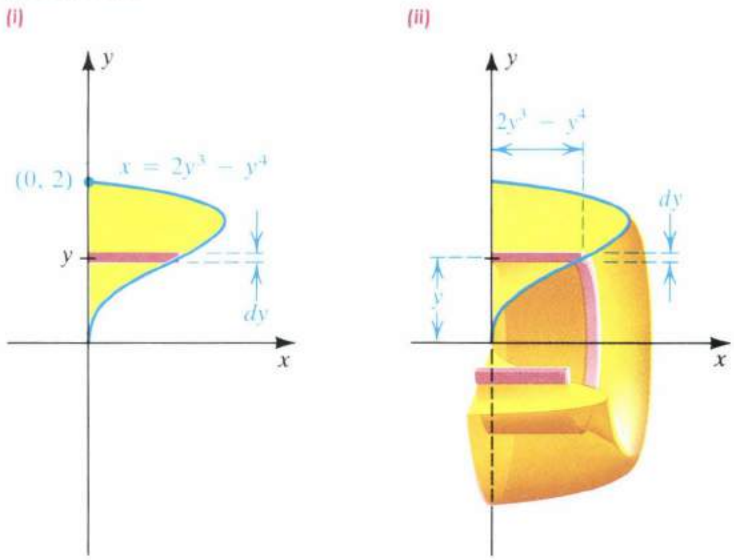
thickness of shell: dy

average radius: y

altitude: $2y^3 - y^4$

volume: $2\pi y(2y^3 - y^4)\,dy$

FIGURE 6.33



To sum all such shells, we move upward through the region from $c = 0$ to $d = 2$. Hence the limit of sums is

$$V = \int_0^2 2\pi y(2y^3 - y^4)\, dy.$$

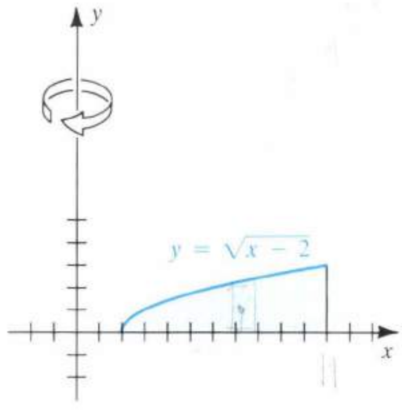
It is worth noting that in the preceding example we were forced to use shells and to integrate with respect to y, since the use of washers and integration with respect to x would require that we solve the equation $x = 2y^3 - y^4$ for y in terms of x, a rather formidable task.

EXERCISES 6.3

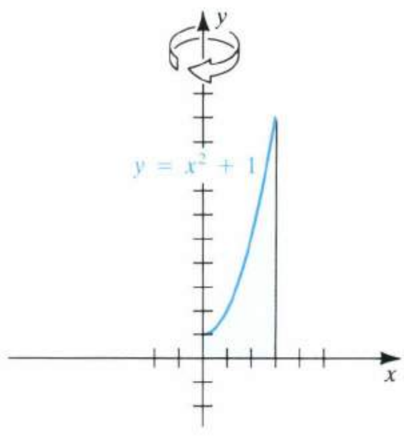
Use cylindrical shells for each exercise.

Exer. 1–4: Set up an integral that can be used to find the volume of the solid obtained by revolving the shaded region about the indicated axis.

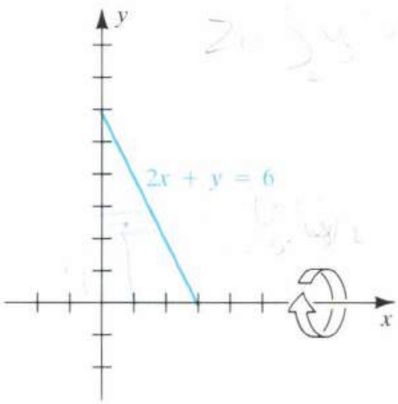
1



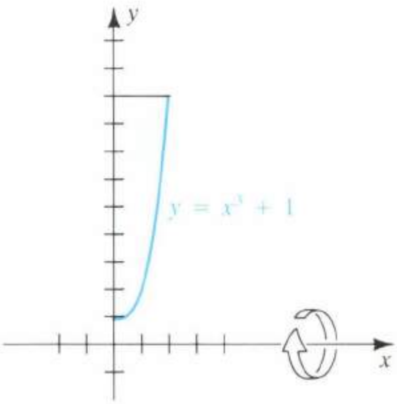
2



3



4



Exer. 5–18: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated if R is revolved about the indicated axis.

5 $y = \sqrt{x}$, $x = 4$, $y = 0$; y-axis

6 $y = 1/x$, $x = 1$, $x = 2$, $y = 0$; y-axis

7 $y = x^2$, $y^2 = 8x$; y-axis

8 $16y = x^2$, $y^2 = 2x$; y-axis

9 $2x - y = 12$, $x - 2y = 3$, $x = 4$; y-axis

10 $y = x^3 + 1$, $x + 2y = 2$, $x = 1$; y-axis

11 $2x - y = 4$, $x = 0$, $y = 0$; y-axis

12 $y = x^2 - 5x$, $y = 0$; y-axis

13 $x^2 = 4y$, $y = 4$; x-axis

14 $y^3 = x$, $y = 3$, $x = 0$; x-axis

15 $y = 2x$, $y = 6$, $x = 0$; x-axis

16 $2y = x$, $y = 4$, $x = 1$; x-axis

17 $y = \sqrt{x + 4}$, $y = 0$, $x = 0$; x-axis

18 $y = -x$, $x - y = -4$, $y = 0$; x-axis

Exer. 19–26: Let R be the region bounded by the graphs of the equations. Set up an integral that can be used to find the volume of the solid generated if R is revolved about the given line.

19 $y = x^2 + 1$, $x = 0$, $x = 2$, $y = 0$
(a) $x = 3$ (b) $x = -1$

20 $y = 4 - x^2$, $y = 0$
(a) $x = 2$ (b) $x = -3$

21 $y = x^2$, $y = 4$
(a) $y = 4$ (b) $y = 5$ (c) $x = 2$ (d) $x = -3$

22 $y = \sqrt{x}$, $y = 0$, $x = 4$
(a) $x = 4$ (b) $x = 6$ (c) $y = 2$ (d) $y = -4$

23 $x + y = 3$, $y + x^2 = 3$; $x = 2$

24 $y = 1 - x^2$, $x - y = 1$; $y = 3$

25 $x^2 + y^2 = 1$; $x = 5$

26 $y = x^{2/3}$, $y = x^2$; $y = -1$

Exer. 27–30: Let R be the region bounded by the graphs of the equations. Set up integrals that can be used to find the volume of the solid generated if R is revolved about the given axis using (a) cylindrical shells and (b) disks or washers.

27 $y = 1/\sqrt{x}$, $x = 1$, $x = 4$, $y = 0$; x-axis

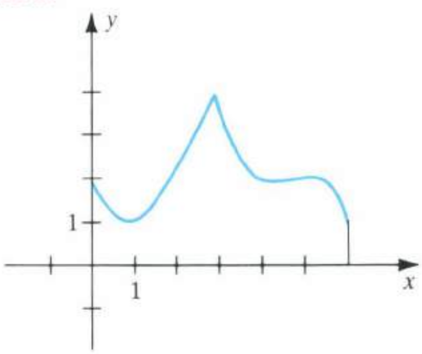
28 $y = 9 - x^2$, $x = 0$, $x = 2$, $y = 0$; x-axis

29 $y = x^2 + 2$, $x = 0$, $x = 1$, $y = 0$; y-axis

30 $y = x + 1$, $x = 0$, $x = 1$, $y = 0$; y-axis

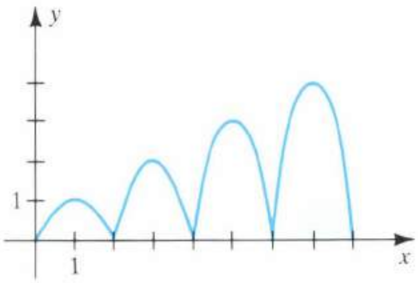
31 If the region shown in the figure is revolved about the y-axis, use the trapezoidal rule, with $n = 6$, to approximate the volume of the resulting solid.

EXERCISE 31



32 If the region shown in the figure on the following page is revolved about the y-axis, use Simpson's rule, with $n = 8$, to approximate the volume of the resulting solid.

EXERCISE 32



c 33 Graph $f(x) = -x^4 + 2.21x^3 - 3.21x^2 + 4.42x - 2$.

(a) Estimate the x-intercepts of the graph.

(b) If the region bounded by the graph of f and the x-axis is revolved about the y-axis, set up an integral that can be used to approximate the volume of the resulting solid.

c 34 Graph, on the same coordinate axes, $f(x) = \csc x$ and $g(x) = x + 1$ for $0 < x < \pi$.

(a) Use Newton's method to approximate, to two decimal places, the x-coordinates of the points of intersection of the graphs.

(b) If the region bounded by the graphs is revolved about the y-axis, use the trapezoidal rule with $n = 6$ to approximate the volume of the resulting solid.

6.4 VOLUMES BY CROSS SECTIONS

If a plane intersects a solid, then the region common to the plane and the solid is a **cross section** of the solid. In Section 6.2 we used circular and washer-shaped cross sections to find volumes of solids of revolution. Let us now consider a solid that has the following property (see Figure 6.34): For every x in $[a, b]$, the plane perpendicular to the x-axis at x intersects the solid in a cross section whose area is $A(x)$, where A is a continuous function on $[a, b]$.

FIGURE 6.34

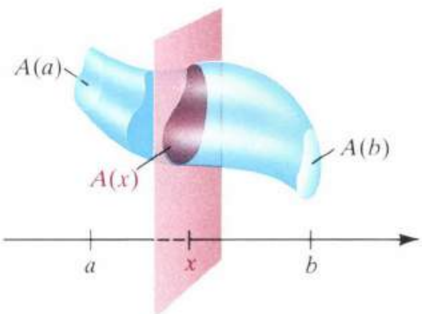
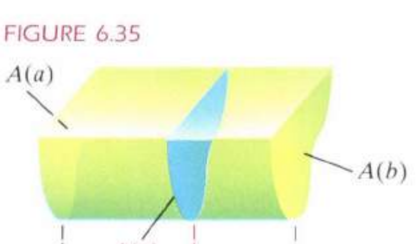


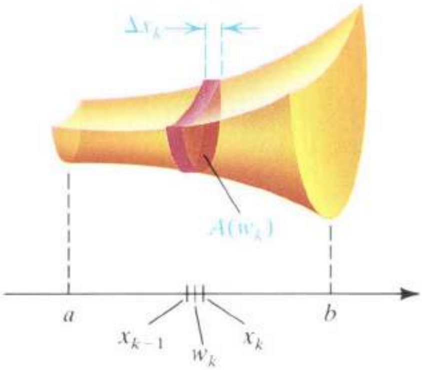
FIGURE 6.35



The solid is called a **cylinder** if, as illustrated in Figure 6.35, a line parallel to the x-axis that traces the boundary of the cross section corresponding to a also traces the boundary of the cross section corresponding to every x in $[a, b]$. The cross sections determined by the planes through $x = a$ and $x = b$ are the **bases** of the cylinder. The distance between the bases is the **altitude** of the cylinder. By definition, the volume of the cylinder is the area of a base multiplied by the altitude. Thus, the volume of the solid in Figure 6.35 is $A(a) \cdot (b - a)$.

To find the volume of a noncylindrical solid of the type illustrated in Figure 6.36, we begin with a partition P of $[a, b]$. Planes perpendicular

FIGURE 6.36



to the x-axis at each x_k in the partition slice the solid into smaller pieces. If we choose any number w_k in $[x_{k-1}, x_k]$, the volume of a typical slice can be approximated by the volume $A(w_k)\,\Delta x_k$ of the red cylinder shown in Figure 6.36. If V is the volume of the solid and if the norm $\|P\|$ is small, then

$$V \approx \sum_k A(w_k)\,\Delta x_k.$$

Since this approximation improves as $\|P\|$ gets smaller, we define the volume of the solid by

$$V = \lim_{\|P\| \to 0} \sum_k A(w_k)\,\Delta x_k = \int_a^b A(x)\,dx,$$

where the last equality follows from the definition of the definite integral. We may summarize our discussion as follows.

Volumes by cross sections **(6.13)**

Let S be a solid bounded by planes that are perpendicular to the x-axis at a and b. If, for every x in $[a, b]$, the cross-sectional area of S is given by $A(x)$, where A is continuous on $[a, b]$, then the volume of S is

$$V = \int_a^b A(x)\,dx.$$

An analogous result can be stated for a y-interval $[c, d]$ and a cross-sectional area $A(y)$.

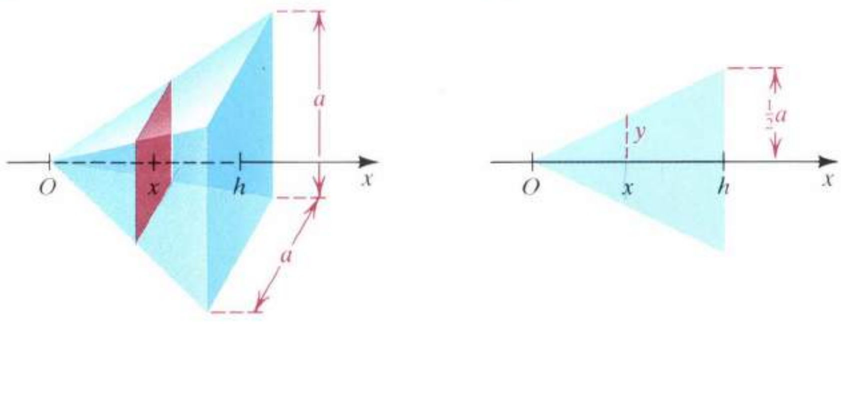
EXAMPLE 1 Find the volume of a right pyramid with a square base of side a and altitude h.

SOLUTION As in Figure 6.37(i), let us take the vertex of the pyramid at the origin, with the x-axis passing through the center of the square base, a distance h from O. Cross sections by planes perpendicular to the x-axis are squares. Figure 6.37(ii) is a side view of the pyramid. Since $2y$ is the length of the side of the square cross section corresponding to x, the cross-sectional area $A(x)$ is

$$A(x) = (2y)^2 = 4y^2.$$

FIGURE 6.37

(i) (ii)



Using similar triangles in Figure 6.37(ii), we have

$$\frac{y}{x} = \frac{\frac{1}{2}a}{h}, \quad \text{or} \quad y = \frac{ax}{2h}.$$

Hence
$$A(x) = 4y^2 = \frac{4a^2x^2}{4h^2} = \frac{a^2}{h^2}x^2.$$

Applying (6.13) yields

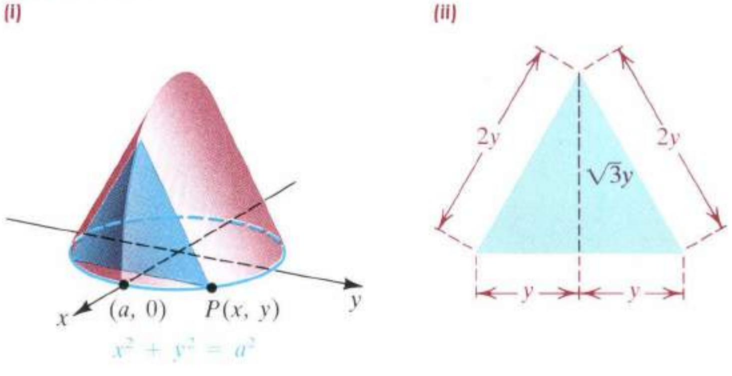
$$V = \int_0^h A(x)\,dx = \int_0^h \left(\frac{a^2}{h^2}\right)x^2\,dx$$
$$= \left(\frac{a^2}{h^2}\right)\left[\frac{x^3}{3}\right]_0^h = \frac{a^2}{h^2}\frac{h^3}{3} = \frac{1}{3}a^2h.$$

EXAMPLE 2 A solid has, as its base, the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$ with $a > 0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an equilateral triangle with one side in the base.

SOLUTION A triangular cross section by a plane x units from the origin is illustrated in Figure 6.38(i). If the point $P(x, y)$ is on the circle and $y > 0$, then the lengths of the sides of this equilateral triangle are $2y$. Referring to (ii) of the figure, we see, by the Pythagorean theorem, that the altitude of the triangle is

$$\sqrt{(2y)^2 - y^2} = \sqrt{3y^2} = \sqrt{3}\,y.$$

FIGURE 6.38



Hence the area $A(x)$ of the cross section is

$$A(x) = \tfrac{1}{2}(2y)(\sqrt{3}\,y) = \sqrt{3}\,y^2 = \sqrt{3}(a^2 - x^2).$$

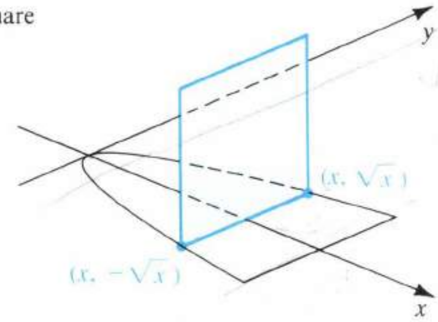
Applying (6.13) gives us

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \sqrt{3}(a^2 - x^2)\,dx$$
$$= \sqrt{3}\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4\sqrt{3}}{3}a^3.$$

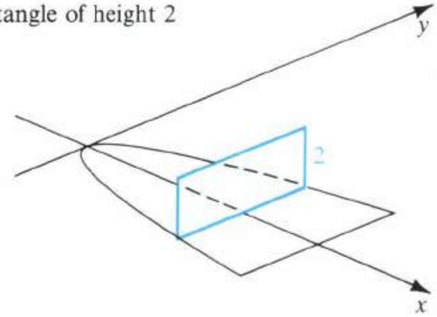
EXERCISES 6.4

Exer. 1–8: Let R be the region bounded by the graphs of $x = y^2$ and $x = 9$. Find the volume of the solid that has R as its base if every cross section by a plane perpendicular to the x-axis has the given shape.

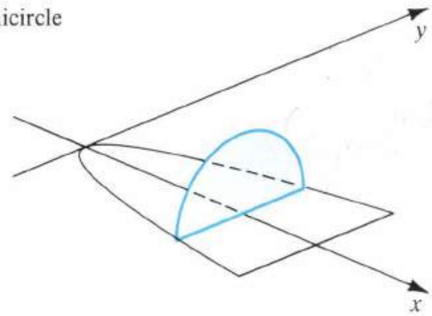
1 A square



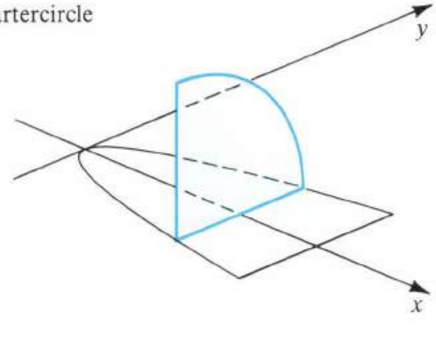
2 A rectangle of height 2



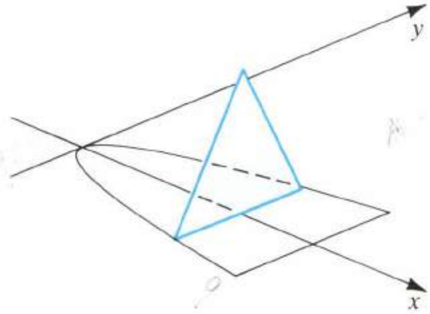
3 A semicircle



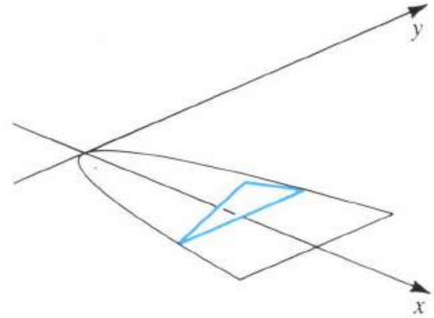
4 A quartercircle



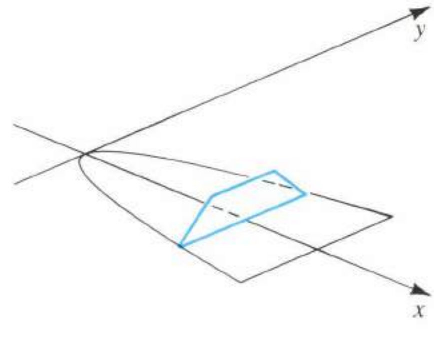
5 An equilateral triangle



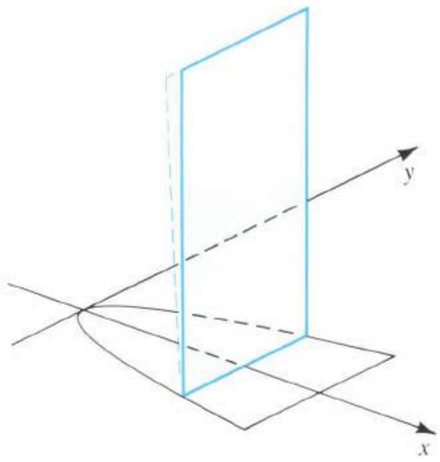
6 A triangle with height equal to $\frac{1}{4}$ the length of the base



7 A trapezoid with lower base in the xy-plane, upper base equal to $\frac{1}{2}$ the length of the lower base, and height equal to $\frac{1}{4}$ the length of the lower base



8 A parallelogram with base in the xy-plane and height equal to twice the length of the base



9 A solid has as its base the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$ with $a > 0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a square.

10 Work Exercise 9 if every cross section is an isosceles triangle with base on the xy-plane and altitude equal to the length of the base.

11 A solid has as its base the region in the xy-plane bounded by the graphs of $y = 4$ and $y = x^2$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an isosceles right triangle with hypotenuse on the xy-plane.

12 Work Exercise 11 if every cross section is a square.

13 Find the volume of a pyramid of the type illustrated in Figure 6.37 if the altitude is h and the base is a rectangle of dimensions a and $2a$.

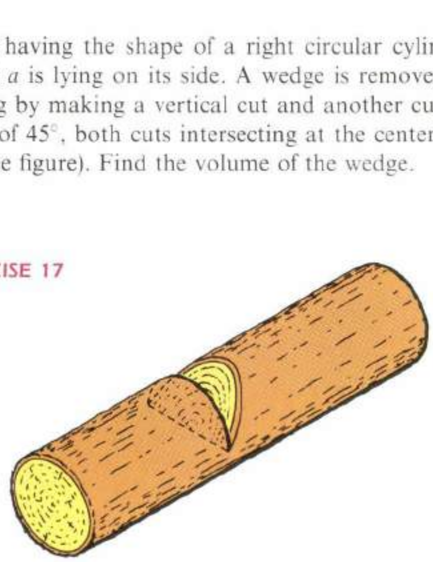
14 A solid has as its base the region in the xy-plane bounded by the graphs of $y = x$ and $y^2 = x$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a semicircle with diameter in the xy-plane.

15 A solid has as its base the region in the xy-plane bounded by the graphs of $y^2 = 4x$ and $x = 4$. If every cross section by a plane perpendicular to the y-axis is a semicircle, find the volume of the solid.

16 A solid has as its base the region in the xy-plane bounded by the graphs of $x^2 = 16y$ and $y = 2$. Every cross section by a plane perpendicular to the y-axis is a rectangle whose height is twice that of the side in the xy-plane. Find the volume of the solid.

17 A log having the shape of a right circular cylinder of radius a is lying on its side. A wedge is removed from the log by making a vertical cut and another cut at an angle of 45°, both cuts intersecting at the center of the log (see figure). Find the volume of the wedge.

EXERCISE 17



18 The axes of two right circular cylinders of radius a intersect at right angles. Find the volume of the solid bounded by the cylinders.

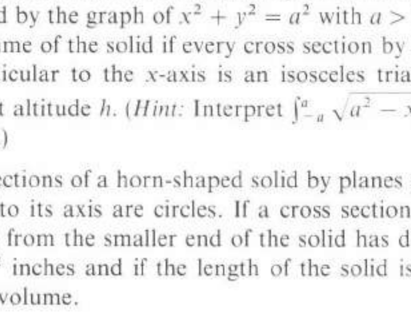
19 The base of a solid is the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$ with $a > 0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an isosceles triangle of constant altitude h. (*Hint:* Interpret $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$ as an area.)

20 Cross sections of a horn-shaped solid by planes perpendicular to its axis are circles. If a cross section that is s inches from the smaller end of the solid has diameter $6 + \frac{1}{36}s^2$ inches and if the length of the solid is 2 feet, find its volume.

21 A tetrahedron has three mutually perpendicular faces and three mutually perpendicular edges of lengths 2, 3, and 4 centimeters, respectively. Find its volume.

22 *Cavalieri's theorem* states that if two solids have equal altitudes and if all cross sections by planes parallel to their bases and at the same distances from their bases have equal areas, then the solids have the same volume (see figure). Prove Cavalieri's theorem.

EXERCISE 22



23 The base of a solid is an isosceles right triangle whose equal sides have length a. Find the volume if cross sections that are perpendicular to the base and to one of the equal sides are semicircular.

24 Work Exercise 23 if the cross sections are regular hexagons with one side in the base.

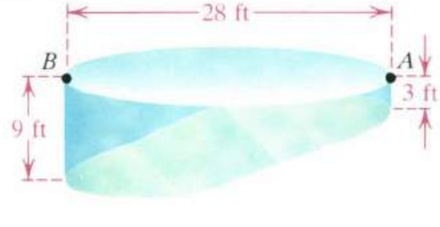
25 Show that the disk and washer methods discussed in Section 6.2 are special cases of (6.13).

c 26 A circular swimming pool has diameter 28 feet. The depth of the water changes slowly from 3 feet at a point A on one side of the pool to 9 feet at a point B diametrically opposite A (see figure). Depth readings $h(x)$ (in feet) taken along the diameter AB are given in the following table, where x is the distance (in feet) from A.

x	0	4	8	12	16	20	24	28
$h(x)$	3	3.5	4	5	6.5	8	8.5	9

Use the trapezoidal rule, with $n = 7$, to estimate the volume of water in the pool. Approximate the number of gallons of water contained in the pool (1 gal $\approx 0.134\ \text{ft}^3$).

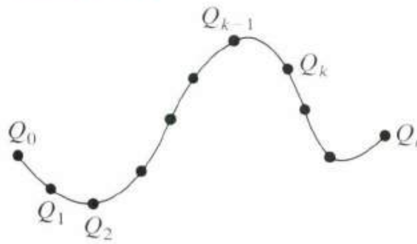
EXERCISE 26



6.5 ARC LENGTH AND SURFACES OF REVOLUTION

For some applications we must determine the *length* of the graph of a function. To obtain a suitable formula, we shall employ a process similar to one that could be used to approximate the length of a twisted wire. Let us imagine dividing the wire into many small pieces by placing dots at $Q_0, Q_1, Q_2, \ldots, Q_n$, as illustrated in Figure 6.39. We may then approximate the length of the wire between Q_{k-1} and Q_k (for each k) by measuring the distance $d(Q_{k-1}, Q_k)$ with a ruler. The sum of all these distances is an approximation for the total length of the wire. Evidently, the closer together we place the dots, the better the approximation. The process we shall use for the graph of a function is similar; however, we shall find the *exact* length by taking a *limit of sums* of lengths of line segments. This process leads to a definite integral. To guarantee that the integral exists, we must place restrictions on the function, as indicated in the following discussion.

FIGURE 6.39 Twisted wire



A function f is **smooth** on an interval if it has a derivative f' that is continuous throughout the interval. Intuitively, this means that a small change in x produces a small change in the slope $f'(x)$ of the tangent line to the graph of f. Thus, the graph has no corners or cusps. We shall define the **length of arc** between two points A and B on the graph of a smooth function.

If f is smooth on a closed interval $[a, b]$, the points $A(a, f(a))$ and $B(b, f(b))$ are called the **endpoints** of the graph of f. Let P be the partition of $[a, b]$ determined by $a = x_0, x_1, x_2, \ldots, x_n = b$, and let Q_k denote the point with coordinates $(x_k, f(x_k))$ on the graph of f, as illustrated in

FIGURE 6.40

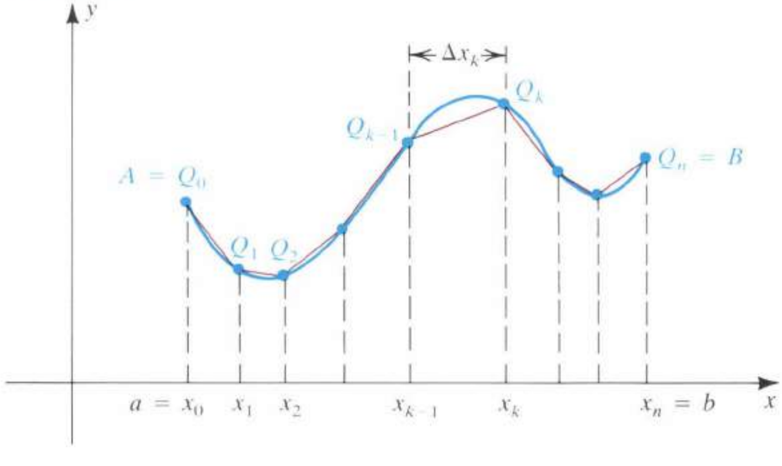


Figure 6.40. If we connect each Q_{k-1} to Q_k by a line segment of length $d(Q_{k-1}, Q_k)$, the length L_P of the resulting broken line is

$$L_P = \sum_{k=1}^{n} d(Q_{k-1}, Q_k).$$

Using the distance formula, we get

$$d(Q_{k-1}, Q_k) = \sqrt{(x_k - x_{k-1})^2 + [f(x_k) - f(x_{k-1})]^2}.$$

By the mean value theorem (4.12),

$$f(x_k) - f(x_{k-1}) = f'(w_k)(x_k - x_{k-1})$$

for some number w_k in the open interval (x_{k-1}, x_k). Substituting for $f(x_k) - f(x_{k-1})$ in the preceding formula and letting $\Delta x_k = x_k - x_{k-1}$, we obtain

$$\begin{aligned} d(Q_{k-1}, Q_k) &= \sqrt{(\Delta x_k)^2 + [f'(w_k)\,\Delta x_k]^2} \\ &= \sqrt{1 + [f'(w_k)]^2}\,\Delta x_k. \end{aligned}$$

Consequently,

$$L_P = \sum_{k=1}^{n} \sqrt{1 + [f'(w_k)]^2}\,\Delta x_k.$$

Observe that L_P is a Riemann sum for $g(x) = \sqrt{1 + [f'(x)]^2}$. Moreover, g is continuous on $[a, b]$, since f' is continuous. If the norm $\|P\|$ is small, then the length L_P of the broken line approximates the length of the graph of f from A to B. This approximation should improve as $\|P\|$ decreases, so we define the *length* (also called the *arc length*) of the graph of f from A to B as the limit of sums L_P. Since $g = \sqrt{1 + (f')^2}$ is a continuous function, the limit exists and equals the definite integral $\int_a^b \sqrt{1 + [f'(x)]^2}\,dx$. This arc length will be denoted by L_a^b.

Definition (6.14)

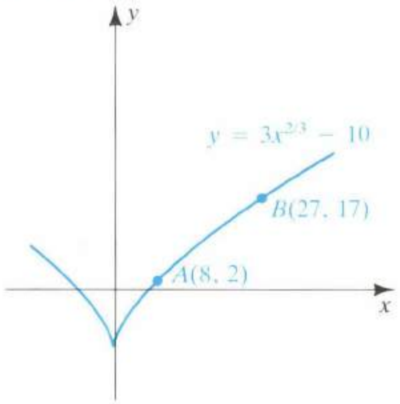
Let f be smooth on $[a, b]$. The **arc length of the graph** of f from $A(a, f(a))$ to $B(b, f(b))$ is

$$L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

Definition (6.14) will be extended to more general graphs in Chapter 13. If a function f is defined implicitly by an equation in x and y, then we shall also refer to the *arc length of the graph of the equation*.

EXAMPLE 1 If $f(x) = 3x^{2/3} - 10$, find the arc length of the graph of f from the point $A(8, 2)$ to $B(27, 17)$.

FIGURE 6.41



SOLUTION The graph of f is sketched in Figure 6.41. Since

$$f'(x) = 2x^{-1/3} = \frac{2}{x^{1/3}},$$

we have, by Definition (6.14),

$$\begin{aligned} L_8^{27} &= \int_8^{27} \sqrt{1 + \left(\frac{2}{x^{1/3}}\right)^2}\, dx = \int_8^{27} \sqrt{1 + \frac{4}{x^{2/3}}}\, dx \\ &= \int_8^{27} \sqrt{\frac{x^{2/3} + 4}{x^{2/3}}}\, dx \\ &= \int_8^{27} \sqrt{x^{2/3} + 4}\, \frac{1}{x^{1/3}}\, dx. \end{aligned}$$

To evaluate this integral, we make the substitution

$$u = x^{2/3} + 4, \qquad du = \frac{2}{3} x^{-1/3}\, dx = \frac{2}{3} \frac{1}{x^{1/3}}\, dx.$$

The integral can be expressed in a suitable form for integration by introducing the factor $\frac{2}{3}$ in the integrand and compensating by multiplying the integral by $\frac{3}{2}$:

$$L_8^{27} = \frac{3}{2} \int_8^{27} \sqrt{x^{2/3} + 4} \left(\frac{2}{3} \frac{1}{x^{1/3}}\right) dx$$

We next calculate the values of $u = x^{2/3} + 4$ that correspond to the limits of integration $x = 8$ and $x = 27$:

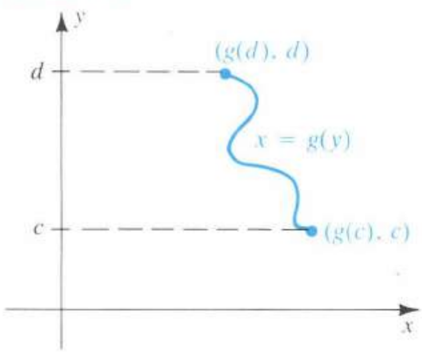
(i) If $x = 8$, then $u = 8^{2/3} + 4 = 8$.

(ii) If $x = 27$, then $u = 27^{2/3} + 4 = 13$.

Substituting in the integrand and changing the limits of integration gives us the arc length:

$$L_8^{27} = \tfrac{3}{2} \int_8^{13} \sqrt{u}\, du = u^{3/2}\Big]_8^{13} = 13^{3/2} - 8^{3/2} \approx 24.2$$

FIGURE 6.42



Interchanging the roles of x and y in Definition (6.14) gives us the following formula for integration with respect to y.

Definition **(6.15)**

Let $x = g(y)$ with g smooth on the interval $[c, d]$. The arc length of the graph of g from $(g(c), c)$ to $(g(d), d)$ (see Figure 6.42) is

$$L_c^d = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

The integrands $\sqrt{1 + [f'(x)]^2}$ and $\sqrt{1 + [g'(y)]^2}$ in formulas (6.14) and (6.15) often result in expressions that have no obvious antiderivatives. In such cases, numerical integration may be used to approximate arc length, as illustrated in the next example.

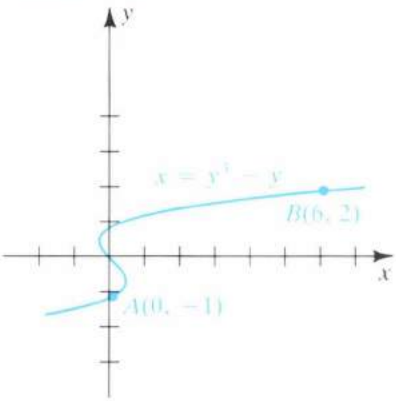
EXAMPLE 2

(a) Set up an integral for finding the arc length of the graph of the equation $y^3 - y - x = 0$ from $A(0, -1)$ to $B(6, 2)$.

(b) Approximate the integral in (a) by using Simpson's rule (5.38), with $n = 6$, and round the answer to one decimal place.

SOLUTION

FIGURE 6.43



(a) Since the equation is not of the form $y = f(x)$, Definition (6.14) cannot be applied directly. However, if we write $x = y^3 - y$, then we can employ (6.15) with $g(y) = y^3 - y$. The graph of the equation is sketched in Figure 6.43. Using (6.15) with $c = -1$ and $d = 2$ yields

$$L_{-1}^{2} = \int_{-1}^{2} \sqrt{1 + (3y^2 - 1)^2}\, dy$$
$$= \int_{-1}^{2} \sqrt{9y^4 - 6y^2 + 2}\, dy.$$

(b) To use Simpson's rule, we let $f(y) = \sqrt{9y^4 - 6y^2 + 2}$ and arrange our work as we did in Section 5.6, obtaining the following table.

k	y_k	$f(y_k)$	m	$mf(y_k)$
0	−1.0	2.2361	1	2.2361
1	−0.5	1.0308	4	4.1232
2	0.0	1.4142	2	2.8284
3	0.5	1.0308	4	4.1232
4	1.0	2.2361	2	4.4722
5	1.5	5.8363	4	23.3452
6	2.0	11.0454	1	11.0454

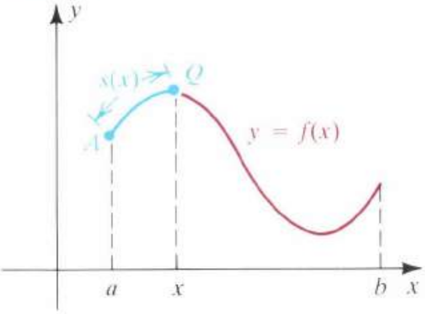
The sum of the numbers in the last column is 52.1737. Applying Simpson's rule with $a = -1$, $b = 2$, and $n = 6$ gives us

$$\int_{-1}^{2} \sqrt{9y^4 - 6y^2 + 2}\, dy \approx \frac{2 - (-1)}{3(6)}(52.1737) \approx 8.7.$$

A function f is **piecewise smooth** on its domain if the graph of f can be decomposed into a finite number of parts, each of which is the graph of a smooth function. The arc length of the graph is defined as the sum of the arc lengths of the individual graphs.

To avoid any misunderstanding in the following discussion, we shall denote the variable of integration by t. In this case the arc length formula

FIGURE 6.44



in Definition (6.14) is written

$$L_a^b = \int_a^b \sqrt{1 + [f'(t)]^2}\, dt.$$

If f is smooth on $[a, b]$, then f is smooth on $[a, x]$ for every number x in $[a, b]$, and the length of the graph from the point $A(a, f(a))$ to the point $Q(x, f(x))$ is

$$L_a^x = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt.$$

If we change the notation and use the symbol $s(x)$ in place of L_a^x, then s may be regarded as a function with domain $[a, b]$, since to each x in $[a, b]$ there corresponds a unique number $s(x)$. As shown in Figure 6.44, $s(x)$ is the length of arc of the graph of f from $A(a, f(a))$ to $Q(x, f(x))$. We shall call s the *arc length function* for the graph of f, as in the next definition.

Definition **(6.16)**

Let f be smooth on $[a, b]$. The **arc length function** s for the graph of f on $[a, b]$ is defined by

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

for $a \le x \le b$.

If s is the arc length function, the differential $ds = s'(x)\, dx$ is called the **differential of arc length**. The next theorem specifies formulas for finding ds.

Theorem **(6.17)**

Let f be smooth on $[a, b]$, and let s be the arc length function for the graph of $y = f(x)$ on $[a, b]$. If dx and dy are the differentials of x and y, then

(i) $ds = \sqrt{1 + [f'(x)]^2}\, dx$

(ii) $(ds)^2 = (dx)^2 + (dy)^2$

PROOF By Definition (6.16) and Theorem (5.35),

$$s'(x) = D_x\,[s(x)] = D_x\left[\int_a^x \sqrt{1 + [f'(t)]^2}\, dt\right] = \sqrt{1 + [f'(x)]^2}$$

Applying Definition (3.28) yields

$$ds = s'(x)\, dx = \sqrt{1 + [f'(x)]^2}\, dx.$$

This proves (i).

To prove (ii), we square both sides of (i), obtaining

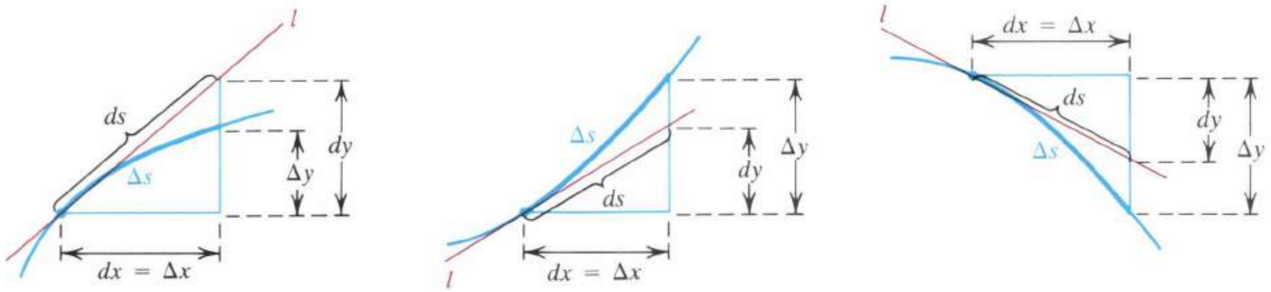
$$\begin{aligned}(ds)^2 &= \{1 + [f'(x)]^2\}(dx)^2 \\ &= (dx)^2 + [f'(x)\, dx]^2 \\ &= (dx)^2 + (dy)^2.\end{aligned}$$

The last equality follows from Definition (3.28).

Theorem (6.17)(ii) has an interesting and useful geometric interpretation. Consider $y = f(x)$, and let Δx be an increment of x. Let Δy denote the change in y, and let Δs denote the change in arc length corresponding to Δx. Typical increments are illustrated in Figure 6.45, where l is the tangent line at (x, y) (compare with Figure 3.27). Since $(ds)^2 = (dx)^2 + (dy)^2$, we may regard $|ds|$ as the length of the hypotenuse of a right triangle that has sides $|dx|$ and $|dy|$, as illustrated in the figure. Note that if Δx is small, then ds may be used to approximate the increment Δs of arc length.

FIGURE 6.45

(I) (II) (III)



EXAMPLE 3 Use differentials to approximate the arc length of the graph of $y = x^3 + 2x$ from $A(1, 3)$ to $B(1.2, 4.128)$.

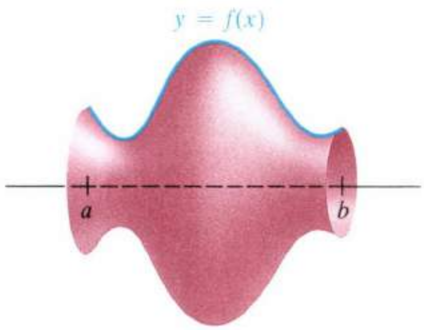
SOLUTION If we let $f(x) = x^3 + 2x$, then, by Theorem (6.17)(i),

$$ds = \sqrt{1 + (3x^2 + 2)^2}\, dx.$$

An approximation may be obtained by letting $x = 1$ and $dx = 0.2$:

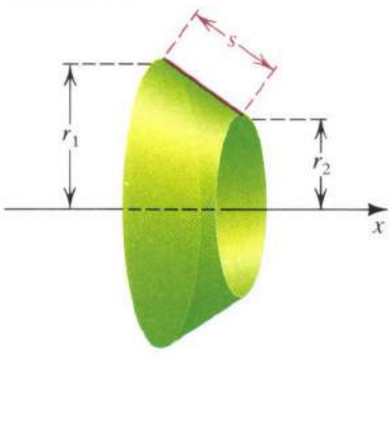
$$ds = \sqrt{1 + 5^2}\,(0.2) = \sqrt{26}(0.2) \approx 1.02$$

FIGURE 6.46



Let f be a function that is nonnegative throughout a closed interval $[a, b]$. If the graph of f is revolved about the x-axis, a **surface of revolution** is generated (see Figure 6.46). For example, if $f(x) = \sqrt{r^2 - x^2}$ for a positive constant r, the graph of f on $[-r, r]$ is the upper half of the circle $x^2 + y^2 = r^2$, and a revolution about the x-axis produces a sphere of radius r having surface area $4\pi r^2$.

FIGURE 6.47



If the graph of f is the line segment shown in Figure 6.47, then the surface generated is a frustum of a cone having base radii r_1 and r_2 and slant height s. It can be shown that the surface area is

$$\pi(r_1 + r_2)s = 2\pi\left(\frac{r_1 + r_2}{2}\right)s.$$

You may remember this formula as follows.

Surface area S of a frustum of a cone **(6.18)**

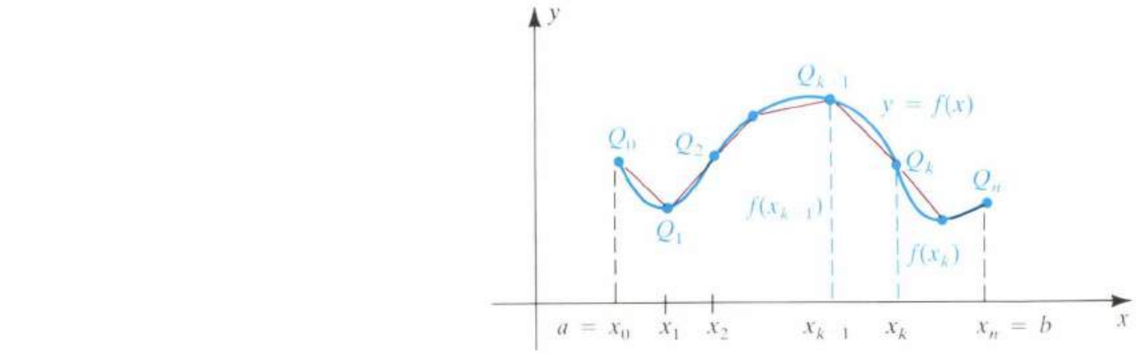
$$S = 2\pi \text{ (average radius)(slant height)}$$

We shall use this fact in the following discussion.

Let f be a smooth function that is nonnegative on $[a, b]$, and consider the surface generated by revolving the graph of f about the x-axis (see Figure 6.46). We wish to find a formula for the area S of this surface. Let P be a partition of $[a, b]$ determined by $a = x_0, x_1, \ldots, x_n = b$, and, for each k, let Q_k denote the point $(x_k, f(x_k))$ on the graph of f (see Figure 6.48). If the norm $\|P\|$ is close to zero, then the broken line l_P obtained by connecting Q_{k-1} to Q_k for each k is an approximation to the graph of f, and hence the area of the surface generated by revolving l_P about the x-axis should approximate S. The line segment $Q_{k-1}Q_k$ generates a frustum of a cone having base radii $f(x_{k-1})$ and $f(x_k)$ and slant height $d(Q_{k-1}, Q_k)$. By (6.18), its surface area is

$$2\pi \frac{f(x_{k-1}) + f(x_k)}{2} d(Q_{k-1}, Q_k).$$

FIGURE 6.48



Summing terms of this form from $k = 1$ to $k = n$ gives us the area S_P of the surface generated by the broken line l_P. If we use the expression for $d(Q_{k-1}, Q_k)$ on page 334, then

$$S_P = \sum_{k=1}^{n} 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{1 + [f'(w_k)]^2}\, \Delta x_k,$$

where $x_{k-1} < w_k < x_k$. We define the area S of the surface of revolution as

$$S = \lim_{\|P\| \to 0} S_P.$$

From the form of S_P, it is reasonable to expect that the limit is given by

$$\int_a^b 2\pi \frac{f(x) + f(x)}{2} \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

The proof of this fact requires results from advanced calculus and is omitted. The following definition summarizes our discussion.

Definition (6.19)

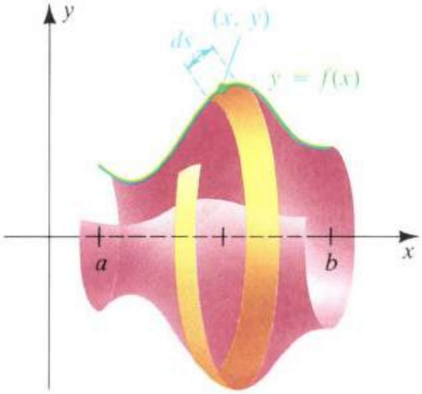
If f is smooth and $f(x) \geq 0$ on $[a, b]$, then the **area** S of the surface generated by revolving the graph of f about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

If f is negative for some x in $[a, b]$, then the following extension of Definition (6.19) can be used to find the surface area S:

$$S = \int_a^b 2\pi\,|f(x)|\sqrt{1 + [f'(x)]^2}\, dx$$

FIGURE 6.49



We can use (6.18) to remember the formula for S in Definition (6.19). As in Figure 6.49, let (x, y) denote an arbitrary point on the graph of f and, as in Theorem (6.17)(i), consider the differential of arc length

$$ds = \sqrt{1 + [f'(x)]^2}\, dx.$$

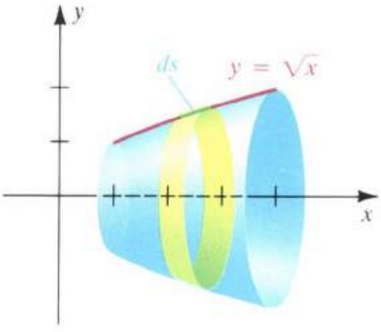
Next, regard ds as the slant height of the frustum of a cone that has average radius $y = f(x)$ (see Figure 6.49). Applying (6.18), the surface area of this frustum is given by

$$2\pi f(x)\, ds = 2\pi y\, ds.$$

As with our work in Sections 6.1 through 6.3, applying $\int_a^b$ may be regarded as taking a limit of sums of these areas of frustums. Thus,

$$S = \int_a^b 2\pi f(x)\, ds = \int_a^b 2\pi y\, ds.$$

FIGURE 6.50

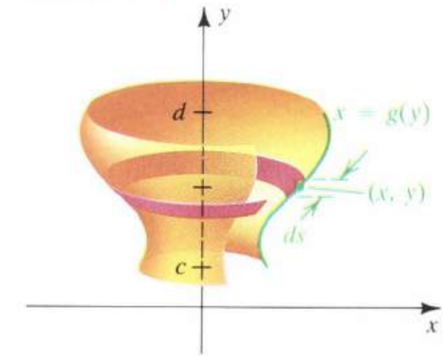


EXAMPLE 4 The graph of $y = \sqrt{x}$ from $(1, 1)$ to $(4, 2)$ is revolved about the x-axis. Find the area of the resulting surface.

SOLUTION The surface is illustrated in Figure 6.50. Using Definition (6.19) or the previous discussion, we have

$$\begin{aligned} S &= \int_1^4 2\pi y\, ds \\ &= \int_1^4 2\pi x^{1/2}\sqrt{1 + \left(\frac{1}{2x^{1/2}}\right)^2}\, dx \\ &= \int_1^4 2\pi x^{1/2}\sqrt{\frac{4x+1}{4x}}\, dx = \pi\int_1^4 \sqrt{4x+1}\, dx \\ &= \frac{\pi}{6}\left[(4x+1)^{3/2}\right]_1^4 = \frac{\pi}{6}(17^{3/2} - 5^{3/2}) \approx 30.85 \text{ square units.} \end{aligned}$$

FIGURE 6.51



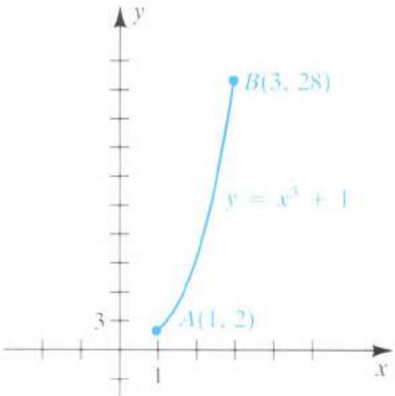
If we interchange the roles of x and y in the preceding discussion, then a formula analogous to (6.19) can be obtained for integration with respect to y. Thus, if $x = g(y)$ and g is smooth and nonnegative on $[c, d]$, then the area S of the surface generated by revolving the graph of g about the y-axis (see Figure 6.51) is

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy.$$

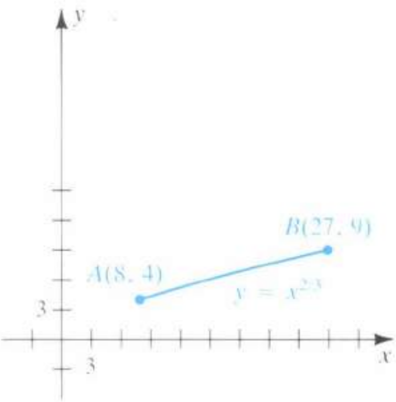
EXERCISES 6.5

Exer. 1–4: Set up an integral that can be used to find the arc length of the graph from A to B by integrating with respect to (a) x and (b) y.

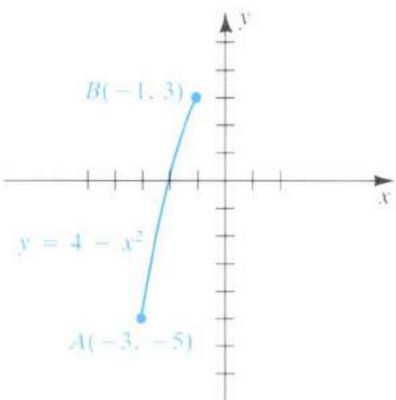
1



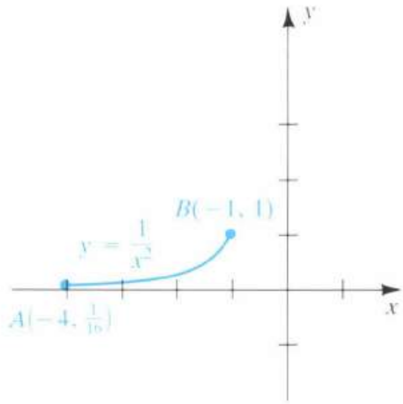
2



3



4



Exer. 5–12: Find the arc length of the graph of the equation from A to B.

5 $y = \frac{2}{3}x^{2/3}$; $A(1, \frac{2}{3})$, $B(8, \frac{8}{3})$

6 $(y+1)^2 = (x-4)^3$; $A(5, 0)$, $B(8, 7)$

7 $y = 5 - \sqrt{x^3}$; $A(1, 4)$, $B(4, -3)$

8 $y = 6\sqrt[3]{x^2} + 1$; $A(-1, 7)$, $B(-8, 25)$

9 $y = \dfrac{x^3}{12} + \dfrac{1}{x}$; $A(1, \frac{13}{12})$, $B(2, \frac{7}{6})$

10 $y + \dfrac{1}{4x} + \dfrac{x^3}{3} = 0$; $A(2, \frac{67}{24})$, $B(3, \frac{109}{12})$

11 $30xy^3 - y^8 = 15$; $A(\frac{8}{15}, 1)$, $B(\frac{271}{240}, 2)$

12 $x = \dfrac{y^4}{16} + \dfrac{1}{2y^2}$; $A(\frac{9}{8}, -2)$, $B(\frac{9}{16}, -1)$

Exer. 13–14: Set up an integral for finding the arc length of the graph of the equation from A to B.

13 $2y^3 - 7y + 2x = 8$; $A(3, 2)$, $B(4, 0)$

14 $11x - 4x^3 - 7y = -7$; $A(1, 2)$, $B(0, 1)$

15 Find the arc length of the graph of $x^{2/3} + y^{2/3} = 1$. (*Hint:* Use symmetry with respect to the line $y = x$.)

16 Find the arc length of the graph of $y = \dfrac{3x^8 + 5}{30x^3}$ from $(1, \frac{4}{5})$ to $(2, \frac{773}{240})$.

Exer. 17–18: (a) Find $s(x)$, where s is an arc length function for the graph of f. (b) If x increases from 1 to 1.1, find Δs and ds.

17 $f(x) = \sqrt[3]{x^2}$

18 $f(x) = \sqrt{x^3}$

Exer. 19–20: Use differentials to approximate the arc length of the graph of the equation from A to B.

19 $y = x^2$; $A(2, 4)$, $B(2.1, 4.41)$

20 $y + x^3 = 0$; $A(1, -1)$, $B(1.1, -1.331)$

Exer. 21–22: Use differentials to approximate the arc length of the graph of the equation between the points with x-coordinates a and b.

21 $y = \cos x$; $a = \pi/6$, $b = 31\pi/180$

22 $y = \sin x$; $a = 0$, $b = \pi/90$

c **Exer. 23–26: Use Simpson's rule with $n = 4$ to approximate the arc length of the graph of the equation from A to B. (Round the answer to two decimal places.)**

23 $y = x^2 + x + 3$; $A(-2, 5)$, $B(2, 9)$

24 $y = x^3$; $A(0, 0)$, $B(2, 8)$

25 $y = \sin x$; $A(0, 0)$, $B(\pi/2, 1)$

26 $y = \tan x$; $A(0, 0)$, $B(\pi/4, 1)$

c 27 (a) Approximate the arc length of the graph of $f(x) = \sin x$ from $(0, 0)$ to $(\pi, 0)$ by using $\sum_{k=1}^{4} d_k(Q_{k-1}, Q_k)$, where Q_k is the point $(\frac{1}{4}\pi k, f(\frac{1}{4}\pi k))$.

(b) If n is any positive integer, how does $\sum_{k=1}^{n} d_k(Q_{k-1}, Q_k)$ compare to the exact arc length?

c 28 (a) Set up an integral for the arc length in Exercise 27(a).

(b) Use the trapezoidal rule with $n = 4$ to approximate the integral in (a).

Exer. 29–32: The graph of the equation from A to B is revolved about the x-axis. Find the area of the resulting surface.

29 $4x = y^2$; $A(0, 0)$, $B(1, 2)$

30 $y = x^3$; $A(1, 1)$, $B(2, 8)$

31 $8y = 2x^4 + x^{-2}$; $A(1, \frac{3}{8})$, $B(2, \frac{129}{32})$

32 $y = 2\sqrt{x + 1}$; $A(0, 2)$, $B(3, 4)$

Exer. 33–34: The graph of the equation from A to B is revolved about the y-axis. Find the area of the resulting surface.

33 $y = 2\sqrt[3]{x}$; $A(1, 2)$, $B(8, 4)$

34 $x = 4\sqrt{y}$; $A(4, 1)$, $B(12, 9)$

Exer. 35–36: If the smaller arc of the circle $x^2 + y^2 = 25$ between the points $(-3, 4)$ and $(3, 4)$ is revolved about the given axis, find the area of the resulting surface.

35 The y-axis

36 The x-axis

Exer. 37–39: Use a definite integral to derive a formula for the surface area of the indicated solid.

37 A right circular cone of altitude h and base radius r

38 A spherical segment of altitude h in a sphere of radius r

39 A sphere of radius r

40 Show that the area of the surface of a sphere of radius a between two parallel planes depends only on the distance between the planes. (*Hint:* Use Exercise 38.)

41 If the graph in Figure 6.49 is revolved about the y-axis, show that the area of the resulting surface is given by

$$\int_a^b 2\pi x\sqrt{1 + [f'(x)]^2}\, dx.$$

42 Use Exercise 41 to find the area of the surface generated by revolving the graph of $y = 3\sqrt[3]{x}$ from $A(1, 3)$ to $B(8, 6)$ about the y-axis.

c 43 The graph of $f(x) = 1 - x^3$ from $(0, 1)$ to $(1, 0)$ is revolved about the x-axis. Approximate the area of the resulting surface by using

$$\sum_{k=1}^{4} 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} d(Q_{k-1}, Q_k),$$

where Q_k is the point $(\frac{1}{4}k, f(\frac{1}{4}k))$.

c 44 (a) Set up an integral for the area of the surface generated in Exercise 43.

(b) Use Simpson's rule with $n = 4$ to approximate the integral in (a).

6.6 WORK

The concept of force may be considered as a push or a pull on an object. For example, a force is needed to push or pull furniture across a floor, to lift an object off the ground, to stretch or compress a spring, or to move a charged particle through an electromagnetic field.

If an object weighs 10 pounds, then by definition the force required to lift it (or hold it off the ground) is 10 pounds. A force of this type is a **constant force**, since its magnitude does not change while it is applied to the object.

The concept of *work* is used when a force acts through a distance. The following definition covers the simplest case, in which the object moves along a line in the same direction as the applied force.

Definition (6.20)

> If a constant force F acts on an object, moving it a distance d in the direction of the force, the **work** W done is
>
> $$W = Fd.$$

The following table lists units of force and work in the British system and the International System (abbreviated SI, for the French *Système International*). In SI units, 1 Newton is the force required to impart an acceleration of 1 m/sec^2 to a mass of 1 kilogram.

System	Unit of force	Unit of distance	Unit of work
British	pound (lb)	foot (ft) inch (in.)	foot-pound (ft-lb) inch-pound (in.-lb)
International (SI)	Newton (N)	meter (m)	Newton-meter (N-m)

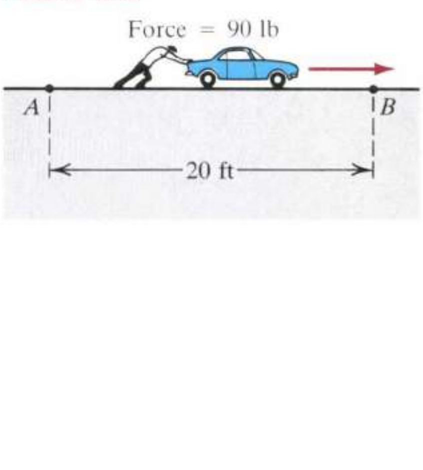
$\mathrm{kg} \cdot \mathrm{m/s^2}$

A Newton-meter is also called a *joule* (J). It can be shown that

$$1\ \mathrm{N} \approx 0.225\ \mathrm{lb} \quad \text{and} \quad 1\ \text{N-m} \approx 0.74\ \text{ft-lb}.$$

For simplicity, in examples and most exercises we will use the British system, in which the magnitude of the force is the same as the weight, in pounds, of the object. In using SI units it is often necessary to consider a gravitational constant a (9.81 m/sec^2) and use Newton's second law of motion, $F = ma$, to change a mass m (in kilograms) to a force F (in Newtons).

FIGURE 6.52



EXAMPLE 1 Find the work done in pushing an automobile a distance of 20 feet along a level road while exerting a constant force of 90 pounds.

SOLUTION The problem is illustrated in Figure 6.52. Since the constant force is $F = 90$ lb and the distance the automobile moves is $d = 20$ ft, it follows from Definition (6.20) that the work done is

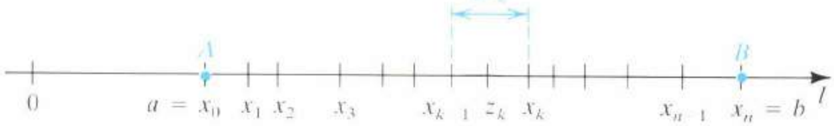
$$W = (90)(20) = 1800\ \text{ft-lb}.$$

Anyone who has pushed an automobile (or some other object) is aware that the force applied is seldom constant. Thus, if an automobile is stalled, a larger force may be required to get it moving than to keep it in motion. The force may also vary because of friction, since part of the road may be smooth and another part rough. A force that is not constant is a **variable force**. We shall next develop a method for determining the work done by a variable force in moving an object rectilinearly in the same direction as the force.

Suppose a force moves an object along the x-axis from $x = a$ to $x = b$ and that the force at x is given by $f(x)$, where f is continuous on $[a, b]$. (The phrase *force at x* means the force acting at the point with coordinate x.) As shown in Figure 6.53, we begin by considering a partition P of $[a, b]$ determined by

$$a = x_0, x_1, x_2, \ldots, x_n = b \quad \text{with} \quad \Delta x_k = x_k - x_{k-1}.$$

FIGURE 6.53



If ΔW_k is the **increment of work**—that is, the amount of work done from x_{k-1} to x_k—then the work W done from a to b is the sum

$$W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_n = \sum_{k=1}^{n} \Delta W_k.$$

To approximate ΔW_k, choose any number z_k in $[x_{k-1}, x_k]$ and consider the force $f(z_k)$ at z_k. If the norm $\|P\|$ is small, then intuitively we know that the function values change very little on $[x_{k-1}, x_k]$; that is, f is *almost constant* on this interval. Applying Definition (6.20) gives us

$$\Delta W_k \approx f(z_k)\, \Delta x_k$$

and hence

$$W = \sum_{k=1}^{n} \Delta W_k \approx \sum_{k=1}^{n} f(z_k)\, \Delta x_k.$$

Since this approximation should improve as $\|P\| \to 0$, we define W as a limit of such sums. This limit leads to a definite integral.

Definition **(6.21)**

> If $f(x)$ is the force at x and if f is continuous on $[a, b]$, then the **work** W done in moving an object along the x-axis from $x = a$ to $x = b$ is
>
> $$W = \lim_{\|P\| \to 0} \sum_{k} f(z_k)\, \Delta x_k = \int_a^b f(x)\, dx.$$

An analogous definition can be stated for an interval on the y-axis by replacing x with y throughout our discussion.

Definition (6.21) can be used to find the work done in stretching or compressing a spring. To solve problems of this type, it is necessary to use the following law from physics:

Hooke's Law: The force $f(x)$ required to stretch a spring x units beyond its natural length is given by $f(x) = kx$, where k is a constant called the **spring constant**.

EXAMPLE 2 A force of 9 pounds is required to stretch a spring from its natural length of 6 inches to a length of 8 inches. Find the work done

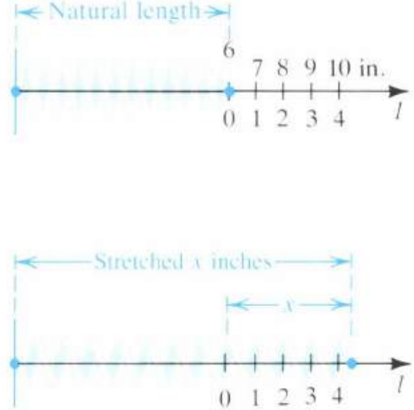
in stretching the spring

(a) from its natural length to a length of 10 inches

(b) from a length of 7 inches to a length of 9 inches

SOLUTION

FIGURE 6.54



(a) Let us introduce an x-axis as shown in Figure 6.54, with one end of the spring attached to a point to the left of the origin and the end to be pulled located at the origin. According to Hooke's law, the force $f(x)$ required to stretch the spring x units beyond its natural length is $f(x) = kx$ for some constant k. Since a 9-pound force is required to stretch the spring 2 inches beyond its natural length, we have $f(2) = 9$. We let $x = 2$ in $f(x) = kx$:

$$9 = k \cdot 2, \quad \text{or} \quad k = \tfrac{9}{2}$$

Consequently, for this spring, Hooke's law has the form

$$f(x) = \tfrac{9}{2}x.$$

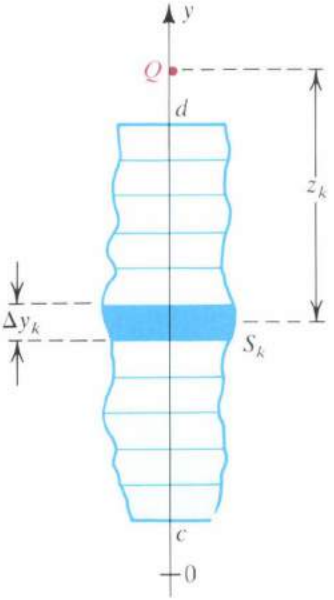
Applying Definition (6.21) with $a = 0$ and $b = 4$, we can determine the work done in stretching the spring 4 inches:

$$W = \int_0^4 \frac{9}{2}x\,dx = \frac{9}{2}\left[\frac{x^2}{2}\right]_0^4 = 36 \text{ in.-lb}$$

(b) We again use the force $f(x) = \frac{9}{2}x$ obtained in part (a). By Definition (6.21), the work done in stretching the spring from $x = 1$ to $x = 3$ is

$$W = \int_1^3 \frac{9}{2}x\,dx = \frac{9}{2}\left[\frac{x^2}{2}\right]_1^3 = 18 \text{ in.-lb.}$$

FIGURE 6.55



In some applications we wish to determine the work done in pumping out a tank containing a fluid or in lifting an object, such as a chain or a cable, that extends vertically between two points. A general situation is illustrated in Figure 6.55, which shows a solid that extends along the y-axis from $y = c$ to $y = d$. We wish to vertically lift all particles contained in the solid to the level of point Q. Let us consider a partition P of $[c, d]$ and imagine slicing the solid by means of planes perpendicular to the y-axis at each number y_k in the partition. As shown in the figure, $\Delta y_k = y_k - y_{k-1}$, and S_k represents the kth slice. We next introduce the following notation:

$$z_k = \text{the (approximate) distance } S_k \text{ is lifted}$$

$$\Delta F_k = \text{the (approximate) force required to lift } S_k$$

If ΔW_k is the work done in lifting S_k, then, by Definition (6.20),

$$\Delta W_k \approx \Delta F_k \cdot z_k = z_k \cdot \Delta F_k.$$

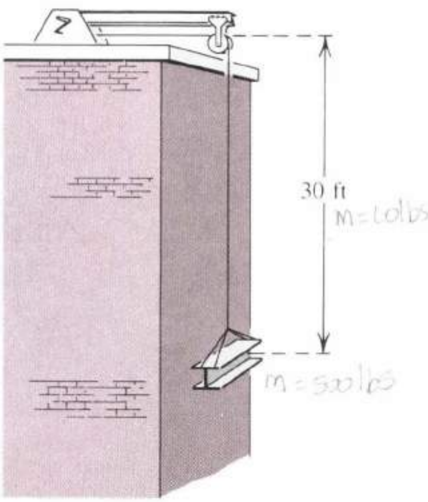
We define the work W done in lifting the entire solid as a limit of sums.

Definition **(6.22)**

$$\boxed{W = \lim_{\|P\| \to 0} \sum_k z_k \cdot \Delta F_k}$$

The limit leads to a definite integral. Notice the difference between this type of problem and that in our earlier discussion. To obtain (6.21), we considered *distance increments* Δx_k and the force $f(z_k)$ that acts through Δx_k. In the present situation we consider *force increments* ΔF_k and the distance z_k through which ΔF_k acts. The next two examples illustrate this technique. As in preceding sections, we shall use dy to represent a typical increment Δy_k and y to denote a number in $[c, d]$.

FIGURE 6.56



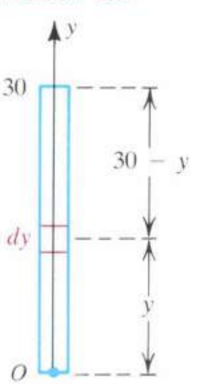
EXAMPLE 3 A uniform cable 30 feet long and weighing 60 pounds hangs vertically from a pulley system at the top of a building, as shown in Figure 6.56. A steel beam that weighs 500 pounds is attached to the end of the cable; find the work required to pull it to the top.

SOLUTION Let W_B denote the work required to pull the beam to the top, and let W_C denote the work required for the cable. Since the beam weighs 500 pounds and must move through a distance of 30 feet, we have, by Definition (6.20), Work of beam

$$W_B = 500 \cdot 30 = 15{,}000 \text{ ft-lb}.$$

The work required to pull the cable to the top may be found by the method used to obtain (6.22). Consider a y-axis with the lower end of the cable at the origin and the upper end at $y = 30$, as in Figure 6.57. Let dy denote an increment of length of the cable. Since each foot of cable weighs $60/30 = 2$ lb, the weight of the increment (and hence the force required to lift it) is $2\,dy$. If y denotes the distance from O to a point in the increment, then we have the following:

FIGURE 6.57



increment of force: $2\,dy$

distance lifted: $30 - y$

increment of work: $(30 - y)2\,dy$

Applying $\int_0^{30}$ takes a limit of sums of the increments of work. Hence

$$\begin{aligned} W_C &= \int_0^{30} (30 - y)2\,dy \\ &= 2\Big[30y - \tfrac{1}{2}y^2\Big]_0^{30} \\ &= 900 \text{ ft-lb}. \end{aligned}$$

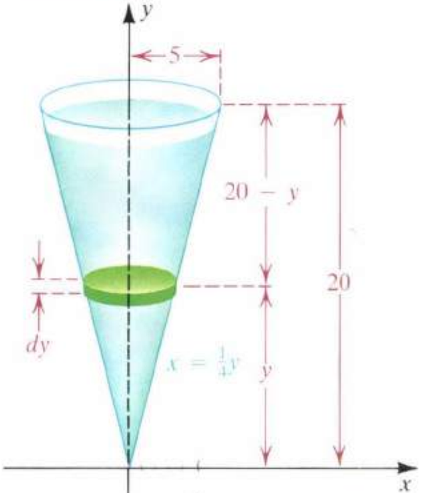
The total work required is

$$W = W_B + W_C = 15{,}000 + 900 = 15{,}900 \text{ ft-lb}.$$

EXAMPLE 4 A right circular conical tank of altitude 20 feet and radius of base 5 feet has its vertex at ground level and axis vertical. If the tank is full of water weighing 62.5 lb/ft^3, find the work done in pumping all the water over the top of the tank.

SOLUTION We begin by introducing a coordinate system, as shown in Figure 6.58. The cone intersects the xy-plane along the line of slope 4

FIGURE 6.58



through the origin. An equation for this line is

$$y = 4x, \quad \text{or} \quad x = \tfrac{1}{4}y.$$

Let us imagine subdividing the water into slices, using planes perpendicular to the y-axis, from $y = 0$ to $y = 20$. If dy represents the width of a typical slice, then its volume may be approximated by the circular disk shown in Figure 6.58. As we did in our work with volumes of revolution in Section 6.2, we obtain

$$\text{volume of disk} = \pi x^2\, dy = \pi(\tfrac{1}{4}y)^2\, dy.$$

Since water weighs 62.5 lb/ft^3, the weight of the disk, and hence the force required to lift it, is $62.5\pi(\frac{1}{4}y)^2\, dy$. Thus, we have

$$\text{increment of force:} \quad 62.5\pi(\tfrac{1}{16}y^2)\, dy$$

$$\text{distance lifted:} \quad 20 - y$$

$$\text{increment of work:} \quad (20 - y)62.5\pi(\tfrac{1}{16}y^2)\, dy$$

Applying $\int_0^{20}$ takes a limit of sums of the increments of work. Hence

$$\begin{aligned} W &= \int_0^{20} (20 - y)62.5\pi(\tfrac{1}{16}y^2)\, dy \\ &= \frac{62.5}{16}\pi \int_0^{20} (20y^2 - y^3)\, dy \\ &= \frac{62.5}{16}\pi \left[20\left(\frac{y^3}{3}\right) - \frac{y^4}{4} \right]_0^{20} \\ &= \frac{62.5}{16}\pi\left(\frac{40{,}000}{3}\right) \approx 163{,}625 \text{ ft-lb.} \end{aligned}$$

The next example is another illustration of how work may be calculated by means of a limit of sums—that is, by a definite integral.

EXAMPLE 5 A confined gas has pressure p (lb/in.^{2}) and volume v (in.^{3}). If the gas expands from $v = a$ to $v = b$, show that the work done (in.-lb) is given by

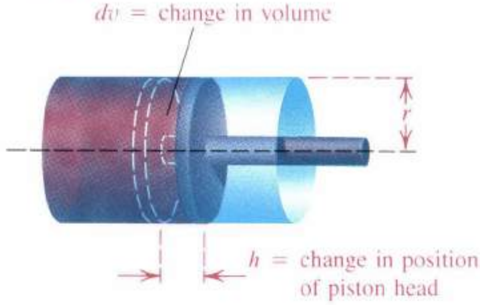
$$W = \int_a^b p\, dv.$$

SOLUTION Since the work done is independent of the shape of the container, we may assume that the gas is enclosed in a right circular cylinder of radius r and that the expansion takes place against a piston head, as illustrated in Figure 6.59 on the following page. As in the figure, let dv denote the change in volume that corresponds to a change of h inches in the position of the piston head. Thus,

$$dv = \pi r^2 h, \quad \text{or} \quad h = \frac{1}{\pi r^2}\, dv.$$

If p denotes the pressure at some point in the volume increment shown in Figure 6.59, then the force against the piston head is the product of p and the area πr^2 of the piston head. Thus, we have the following for the

FIGURE 6.59



indicated volume increment: dv

force against piston head: $p(\pi r^2)$

distance piston head moves: h

$$\text{increment of work:} \quad (p\pi r^2)h = (p\pi r^2)\frac{1}{\pi r^2}\,dv = p\,dv$$

Applying $\int_a^b$ to the increments of work gives us the work done as the gas expands from $v = a$ to $v = b$:

$$W = \int_a^b p\,dv$$

EXERCISES 6.6

1 A 400-pound gorilla climbs a vertical tree 15 feet high. Find the work done if the gorilla reaches the top in

(a) 10 seconds (b) 5 seconds

2 Find the work done in lifting an 80-pound sandbag a height of 4 feet.

3 A spring of natural length 10 inches stretches 1.5 inches under a weight of 8 pounds. Find the work done in stretching the spring

(a) from its natural length to a length of 14 inches

(b) from a length of 11 inches to a length of 13 inches

4 A force of 25 pounds is required to compress a spring of natural length 0.80 foot to a length of 0.75 foot. Find the work done in compressing the spring from its natural length to a length of 0.70 foot.

5 If a spring is 12 inches long, compare the work W_1 done in stretching it from 12 inches to 13 inches with the work W_2 done in stretching it from 13 inches to 14 inches.

6 It requires 60 in.-lb of work to stretch a certain spring from a length of 6 inches to 7 inches and another 120 in.-lb of work to stretch it from 7 inches to 8 inches. Find the spring constant and the natural length of the spring.

7 A freight elevator weighing 3000 pounds is supported by a 12-foot-long cable that weighs 14 pounds per linear foot. Approximate the work required to lift the elevator 9 feet by winding the cable onto a winch.

8 A construction worker pulls a 50-pound motor from ground level to the top of a 60-foot-high building using a rope that weighs $\frac{1}{4}$ lb/ft. Find the work done.

9 A bucket containing water is lifted vertically at a constant rate of 1.5 ft/sec by means of a rope of negligible weight. As the bucket rises, water leaks out at the rate of 0.25 lb/sec. If the bucket weighs 4 pounds when empty and if it contained 20 pounds of water at the instant that the lifting began, determine the work done in raising the bucket 12 feet.

10 In Exercise 9, find the work required to raise the bucket until half the water has leaked out.

11 A fishtank has a rectangular base of width 2 feet and length 4 feet, and rectangular sides of height 3 feet. If the tank is filled with water weighing 62.5 lb/ft^3, find the work required to pump all the water over the top of the tank.

12 Generalize Example 4 of this section to the case of a conical tank of altitude h feet and radius of base a feet that is filled with a liquid weighing ρ lb/ft^3.

13 A vertical cylindrical tank of diameter 3 feet and height 6 feet is full of water. Find the work required to pump all the water

(a) over the top of the tank

(b) through a pipe that rises to a height of 4 feet above the top of the tank

14 Work Exercise 13 if the tank is only half-full of water.

15 The ends of an 8-foot-long water trough are equilateral triangles having sides of length 2 feet. If the trough is full of water, find the work required to pump all of it over the top.

16 A cistern has the shape of the lower half of a sphere of radius 5 feet. If the cistern is full of water, find the work required to pump all the water to a point 4 feet above the top of the cistern.

17 Refer to Example 5 in this section. The volume and pressure of a certain gas vary in accordance with the law $pv^{1.2} = 115$, where the units of measurement are inches and pounds. Find the work done if the gas expands from 32 in.^{3} to 40 in.^{3}

18 Refer to Example 5. The pressure and volume of a quantity of enclosed steam are related by the formula $pv^{1.14} = c$, where c is a constant. If the initial pressure and volume are p_0 and v_0, respectively, find a formula for the work done if the steam expands to twice its volume.

19 Newton's law of gravitation states that the force F of attraction between two particles having masses m_1 and m_2 is given by $F = Gm_1m_2/s^2$, where G is a gravitational constant and s is the distance between the particles. If the mass m_1 of the earth is regarded as concentrated at the center of the earth and a rocket of mass m_2 is on the surface (a distance 4000 miles from the center), find a general formula for the work done in firing the rocket vertically upward to an altitude h (see figure).

EXERCISE 19

20 In the study of electricity, the formula $F = kq/r^2$, where k is a constant, is used to find the force (in Newtons) with which a positive charge Q of strength q units repels a unit positive charge located r meters from Q. Find the work done in moving a unit charge from a point d centimeters from Q to a point $\frac{1}{2}d$ centimeters from Q.

c **Exer. 21–22: Suppose the table was obtained experimentally for a force $f(x)$ acting at the point with coordinate x on a coordinate line. Use the trapezoidal rule to approximate the work done on the interval $[a, b]$, where a and b are the smallest and largest values of x, respectively.**

21

x (ft)	0	0.5	1.0	1.5	2.0	2.5
$f(x)$ (lb)	7.4	8.1	8.4	7.8	6.3	7.1

x (ft)	3.0	3.5	4.0	4.5	5.0
$f(x)$ (lb)	5.9	6.8	7.0	8.0	9.2

22

x (m)	1	2	3	4	5
$f(x)$ (N)	125	120	130	146	165

x (m)	6	7	8	9
$f(x)$ (N)	157	150	143	140

23 The force (in Newtons) with which two electrons repel each other is inversely proportional to the square of the distance (in meters) between them.

(a) If one electron is held fixed at the point (5, 0), find the work done in moving a second electron along the x-axis from the origin to the point (3, 0).

(b) If two electrons are held fixed at the points (5, 0) and (−5, 0), respectively, find the work done in moving a third electron from the origin to (3, 0).

24 If the force function is constant, show that Definition (6.21) reduces to Definition (6.20).

6.7 MOMENTS AND CENTERS OF MASS

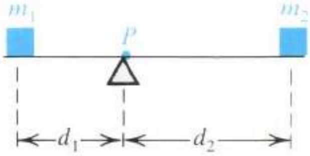
In this section we consider some topics involving the mass of an object. The terms *mass* and *weight* are sometimes confused with each other. Weight is determined by the force of gravity. For example, the weight of

an object on the moon is approximately one-sixth its weight on earth, because the force of gravity is weaker. However, the mass is the same. Newton used the term *mass* synonymously with *quantity of matter* and related it to force by his *second law of motion*, $F = ma$, where F denotes the force acting on an object of mass m that has acceleration a. In the British system, we often approximate a by 32 ft/sec^2 and use the **slug** as the unit of mass. In SI units, $a \approx 9.81$ m/sec^2, and the kilogram is the unit of mass. It can be shown that

$$1 \text{ slug} \approx 14.6 \text{ kg} \quad \text{and} \quad 1 \text{ kg} \approx 0.07 \text{ slug}.$$

In applications we usually assume that the mass of an object is concentrated at a point, and we refer to the object as a **point-mass**, regardless of its size. For example, using the earth as a frame of reference, we may regard a human being, an automobile, or a building as a point-mass.

FIGURE 6.60



In an elementary physics experiment we consider two point-masses m_1 and m_2 attached to the ends of a thin rod, as illustrated in Figure 6.60, and then locate the point P at which a fulcrum should be placed so that the rod balances. (This situation is similar to balancing a seesaw with a person sitting at each end.) If the distances from m_1 and m_2 to P are d_1 and d_2, respectively, then it can be shown experimentally that P is the balance point if

$$m_1 d_1 = m_2 d_2.$$

FIGURE 6.61

In order to generalize this concept, let us introduce an x-axis, as illustrated in Figure 6.61, with m_1 and m_2 located at points with coordinates x_1 and x_2. If the coordinate of the balance point P is $\bar{x}$, then using the formula $m_1 d_1 = m_2 d_2$ yields

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1 x_1 + m_2 x_2$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

This gives us a formula for locating the balance point P.

If a mass m is located at a point on the axis with coordinate x, then the product mx is called the *moment* M_0 *of the mass about the origin*. Our formula for $\bar{x}$ states that to find the coordinate of the balance point, we may divide the sum of the moments about the origin by the total mass. The point with coordinate $\bar{x}$ is called the *center of mass* (or *center of gravity*) of the two point-masses. The next definition extends this discussion to many point-masses located on an axis, as shown in Figure 6.62.

FIGURE 6.62

Definition (6.23)

Let S denote a system of point-masses $m_1, m_2, \ldots, m_n$ located at $x_1, x_2, \ldots, x_n$ on a coordinate line, and let $m = \sum_{k=1}^{n} m_k$ denote the total mass.

(i) The **moment of S about the origin** is $M_0 = \sum_{k=1}^{n} m_k x_k$.

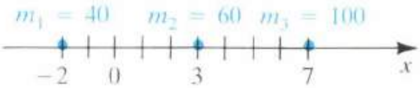
(ii) The **center of mass** (or **center of gravity**) of S is given by $\bar{x} = \dfrac{M_0}{m}$.

The point with coordinate $\bar{x}$ is the balance point of the system S in the same sense as in our seesaw illustration.

EXAMPLE 1 Three point-masses of 40, 60, and 100 kilograms are located at -2, 3, and 7, respectively, on an x-axis. Find the center of mass.

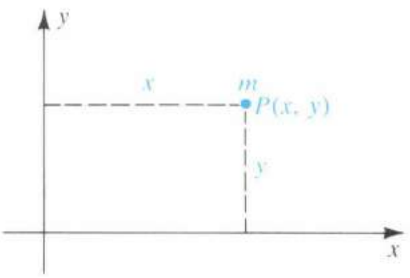
SOLUTION If we denote the three masses by m_1, m_2, and m_3, we have the situation illustrated in Figure 6.63, with $x_1 = -2$, $x_2 = 3$, and $x_3 = 7$. Applying Definition (6.23) gives us the coordinate $\bar{x}$ of the center of mass:

FIGURE 6.63



$$\bar{x} = \frac{40(-2) + 60(3) + 100(7)}{40 + 60 + 100} = \frac{800}{200} = 4$$

FIGURE 6.64



Let us next consider a point-mass m located at $P(x, y)$ in a coordinate plane (see Figure 6.64). We define the moments M_x and M_y of m about the coordinate axes as follows:

moment about the x-axis: $M_x = my$

moment about the y-axis: $M_y = mx$

In words, to find M_x we multiply m by the y-coordinate of P, and to find M_y we multiply m by the x-coordinate. To find M_x and M_y for a *system* of point-masses, we add the individual moments, as in (i) and (ii) of the next definition.

Definition **(6.24)**

Let S denote a system of point-masses $m_1, m_2, \ldots, m_n$ located at (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$ in a coordinate plane, and let $m = \sum_{k=1}^{n} m_k$ denote the total mass.

(i) The **moment of S about the x-axis** is $M_x = \sum_{k=1}^{n} m_k y_k$.

(ii) The **moment of S about the y-axis** is $M_y = \sum_{k=1}^{n} m_k x_k$.

(iii) The **center of mass** (or **center of gravity**) of S is the point $(\bar{x}, \bar{y})$ such that

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}.$$

From (iii) of this definition,

$$m\bar{x} = M_y \quad \text{and} \quad m\bar{y} = M_x.$$

Since $m\bar{x}$ and $m\bar{y}$ are the moments about the y-axis and x-axis, respectively, of a single point-mass m located at $(\bar{x}, \bar{y})$, we may interpret the center of mass as the point at which the total mass can be concentrated to obtain the moments M_y and M_x of S.

FIGURE 6.65

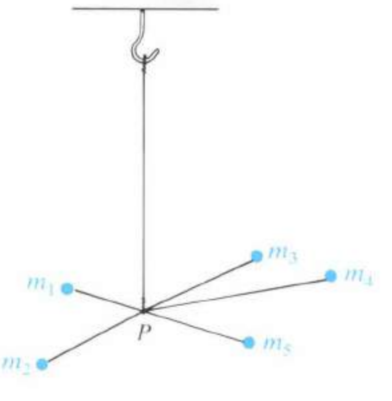
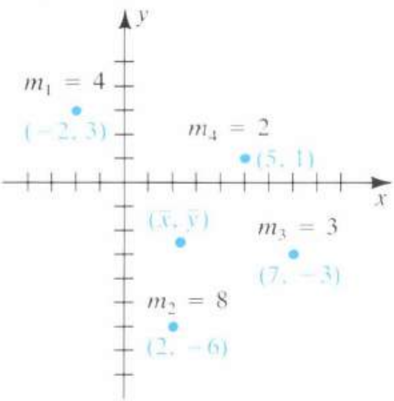


FIGURE 6.66



We might think of the n point-masses in (6.24) as being fastened to the center of mass P by weightless rods, as spokes of a wheel are attached to the center of the wheel. The system S would balance if supported by a cord attached to P, as illustrated in Figure 6.65. The appearance would be similar to that of a mobile having all its objects in the same horizontal plane.

EXAMPLE 2 Point-masses of 4, 8, 3, and 2 kilograms are located at $(-2, 3)$, $(2, -6)$, $(7, -3)$, and $(5, 1)$, respectively. Find M_x, M_y, and the center of mass of the system.

SOLUTION The masses are illustrated in Figure 6.66, in which we have also anticipated the position of $(\bar{x}, \bar{y})$. Applying Definition (6.24) gives us

$$M_x = (4)(3) + (8)(-6) + (3)(-3) + (2)(1) = -43$$

$$M_y = (4)(-2) + (8)(2) + (3)(7) + (2)(5) = 39.$$

Since $m = 4 + 8 + 3 + 2 = 17$,

$$\bar{x} = \frac{M_y}{m} = \frac{39}{17} \approx 2.3 \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = -\frac{43}{17} \approx -2.5.$$

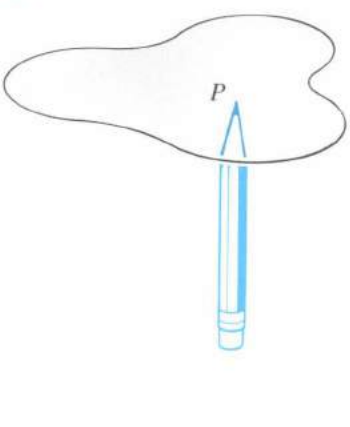
Thus, the center of mass is $(\frac{39}{17}, -\frac{43}{17})$.

Later in the text we shall consider solid objects that are **homogeneous** in the sense that the mass is uniformly distributed throughout the solid. In physics, the **density** ρ (rho) of a homogeneous solid of mass m and volume V is defined by $\rho = m/V$. Thus, *density is mass per unit volume*. The SI unit for density is kg/m^3; however, g/cm^3 is also used. The British unit is lb/ft^3 or $lb/in.^3$

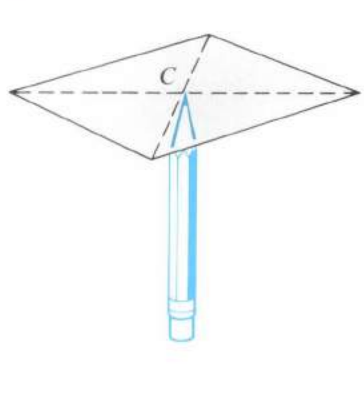
In this section we shall restrict our discussion to homogeneous **laminas** (thin flat plates) that have **area density** (mass per unit area) ρ. Area density is measured in kg/m^2, lb/ft^2, and so on. If the area of one face of a lamina is A and the area density is ρ, then its mass m is given by $m = \rho A$. We wish to define the center of mass P such that if the tip of a sharp pencil were placed at P, as illustrated in Figure 6.67, the lamina would balance in

FIGURE 6.67

(i)



(ii)



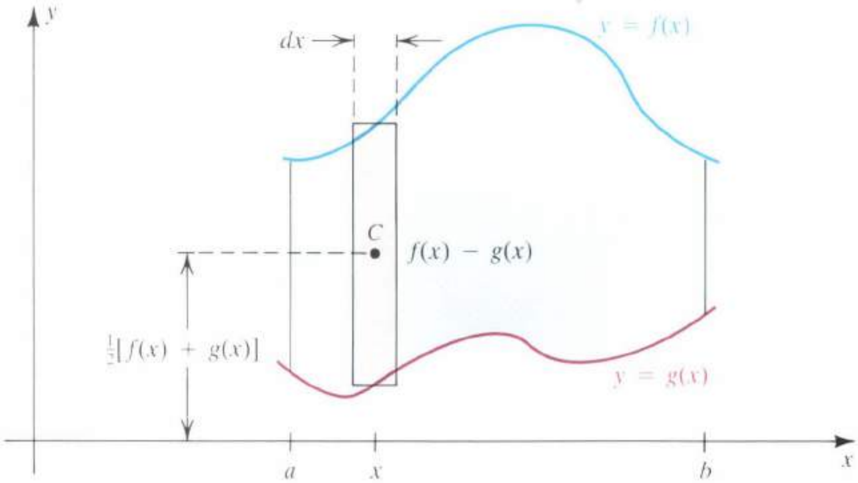
a horizontal position. As in (ii) of the figure, we shall assume that *the center of mass of a rectangular lamina is the point C at which the diagonals intersect*. We call C the *center* of the rectangle. Thus, for problems involving mass, we may assume that a rectangular lamina is a point-mass located at the center of the rectangle. This assumption is the key to our definition of the center of mass of a lamina.

Consider a lamina that has area density ρ and the shape of the R_x region in Figure 6.68. Since we have had ample experience using limits of Riemann sums for definitions in Sections 6.1 through 6.6, let us proceed directly to the method of representing the width of the rectangle in the figure by dx (instead of Δx_k), obtaining

$$\text{area of rectangle:}\quad [f(x) - g(x)]\,dx.$$

length width

FIGURE 6.68



Since the area density of the lamina is ρ, we may write

$$\text{mass of rectangular lamina:}\quad \rho[f(x) - g(x)]\,dx.$$

If, as in previous sections, we regard $\int_a^b$ as an operator that takes limits of sums, we arrive at the following definition for the mass m of the lamina:

$$m = \int_a^b \rho[f(x) - g(x)]\,dx$$

We next assume that the rectangular lamina in Figure 6.68 is a point-mass located at the center C of the rectangle. Since, by the midpoint formula (1.5), the distance from the x-axis to C is $\frac{1}{2}[f(x) + g(x)]$, we obtain the following result for the rectangular lamina:

$$\textit{moment about the x-axis:}\quad \tfrac{1}{2}[f(x) + g(x)] \cdot \rho[f(x) - g(x)]\,dx$$

Similarly, since the distance from the y-axis to C is x,

$$\textit{moment about the y-axis:}\quad x \cdot \rho[f(x) - g(x)]\,dx.$$

Taking limits of sums by applying $\int_a^b$ leads to the next definition.

Definition (6.25)

Let a lamina L of area density ρ have the shape of the R_x region in Figure 6.68.

(i) The **mass** of L is $m = \int_a^b \rho[f(x) - g(x)]\,dx$.

(ii) The **moments of L about the x-axis and y-axis** are

$$M_x = \int_a^b \tfrac{1}{2}[f(x) + g(x)] \cdot \rho[f(x) - g(x)]\,dx$$

and
$$M_y = \int_a^b x \cdot \rho[f(x) - g(x)]\,dx.$$

(iii) The **center of mass** (or **center of gravity**) of L is the point $(\bar{x}, \bar{y})$ such that

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}.$$

An analogous definition can be stated if L has the shape of an R_y region and the integrations are with respect to y. We could also obtain formulas for moments with respect to lines other than the x-axis or y-axis; however, it is advisable to remember the *technique* for finding moments—multiplying a mass by a distance from an axis—instead of memorizing formulas that cover all possible cases.

EXAMPLE 3 A lamina of area density ρ has the shape of the region bounded by the graphs of $y = x^2 + 1$, $x = 0$, $x = 1$, and $y = 0$. Find the center of mass.

FIGURE 6.69

SOLUTION The region and a typical rectangle of width dx and height y are sketched in Figure 6.69. As indicated in the figure, the distance from the x-axis to the center C of the rectangle is $\frac{1}{2}y$, and the distance from the y-axis to C is x. Hence, *for the rectangular lamina*, we have the following:

$$\begin{aligned} \text{mass:} &\quad \rho y\,dx = \rho(x^2 + 1)\,dx \\ \text{moment about } x\text{-axis:} &\quad \tfrac{1}{2}y \cdot \rho y\,dx = \tfrac{1}{2}\rho(x^2 + 1)^2\,dx \\ \text{moment about } y\text{-axis:} &\quad x \cdot \rho y\,dx = \rho x(x^2 + 1)\,dx \end{aligned}$$

We now take a limit of sums of these expressions by applying the operator $\int_0^1$:

$$m = \int_0^1 \rho(x^2 + 1)\,dx = \rho\left[\tfrac{1}{3}x^3 + x\right]_0^1 = \tfrac{4}{3}\rho$$

$$\begin{aligned} M_x &= \int_0^1 \tfrac{1}{2}\rho(x^2 + 1)^2\,dx = \tfrac{1}{2}\rho \int_0^1 (x^4 + 2x^2 + 1)\,dx \\ &= \tfrac{1}{2}\rho\left[\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x\right]_0^1 = \tfrac{14}{15}\rho \end{aligned}$$

$$\begin{aligned} M_y &= \int_0^1 \rho x(x^2 + 1)\,dx = \rho \int_0^1 (x^3 + x)\,dx \\ &= \rho\left[\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2\right]_0^1 = \tfrac{3}{4}\rho \end{aligned}$$

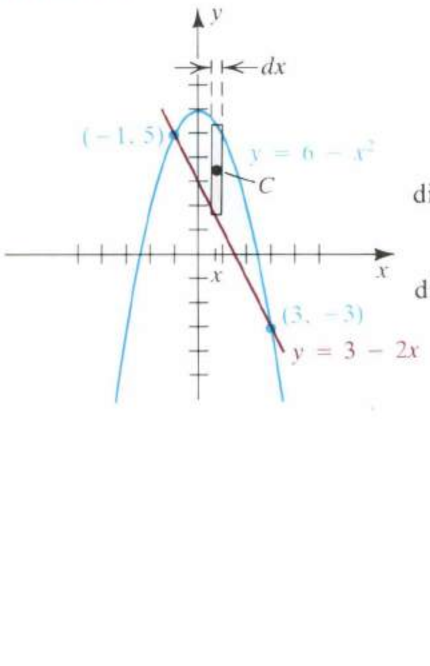
To find the center of mass $(\bar{x}, \bar{y})$, we use (iii) of Definition (6.25):

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{3}{4}\rho}{\frac{4}{3}\rho} = \frac{9}{16} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{14}{15}\rho}{\frac{4}{3}\rho} = \frac{7}{10}$$

When we found $(\bar{x}, \bar{y})$ in Example 3, the constant ρ in the numerator and denominator canceled. This will always be the case for a homogeneous lamina. Hence the center of mass is independent of the area density ρ; that is, $\bar{x}$ and $\bar{y}$ depend only on the shape of the lamina. For this reason, the point $(\bar{x}, \bar{y})$ is sometimes referred to as the center of mass of a *region* in the plane, or as the **centroid** of the region. We can obtain formulas for moments of centroids by letting $\rho = 1$ and $m = A$ (the area of the region) in our previous work.

EXAMPLE 4 Find the centroid of the region bounded by the graphs of $y = 6 - x^2$ and $y = 3 - 2x$.

FIGURE 6.70



SOLUTION The region is the same as that considered in Example 2 of Section 6.1 and is resketched in Figure 6.70. To find the moments and the centroid, we take $\rho = 1$ and $m = A$. Referring to the typical rectangle with center C shown in Figure 6.70, we obtain the following:

area of rectangle: $[(6 - x^2) - (3 - 2x)]\, dx$

distance from x-axis to C: $\frac{1}{2}[(6 - x^2) + (3 - 2x)]$

moment about x-axis: $\frac{1}{2}[(6 - x^2) + (3 - 2x)] \cdot [(6 - x^2) - (3 - 2x)]\, dx$

distance from y-axis to C: x

moment about y-axis: $x[(6 - x^2) - (3 - 2x)]\, dx$

We now take a limit of sums by applying the operator $\int_{-1}^{3}$:

$$\begin{aligned} M_x &= \int_{-1}^{3} \tfrac{1}{2}[(6 - x^2) + (3 - 2x)] \cdot [(6 - x^2) - (3 - 2x)]\, dx \\ &= \tfrac{1}{2} \int_{-1}^{3} [(6 - x^2)^2 - (3 - 2x)^2]\, dx \\ &= \tfrac{1}{2} \int_{-1}^{3} (x^4 - 16x^2 + 12x + 27)\, dx = \tfrac{416}{15} \\ M_y &= \int_{-1}^{3} x[(6 - x^2) - (3 - 2x)]\, dx \\ &= \int_{-1}^{3} (3x + 2x^2 - x^3)\, dx = \tfrac{32}{3} \end{aligned}$$

Using $A = \frac{32}{3}$ and (iii) of Definition (6.25), we determine the centroid:

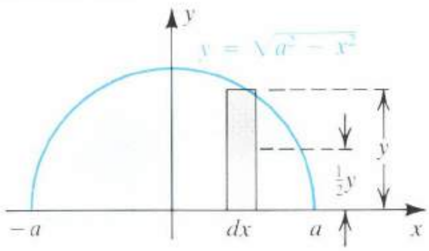
$$\bar{x} = \frac{M_y}{m} = \frac{\frac{32}{3}}{\frac{32}{3}} = 1 \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{\frac{416}{15}}{\frac{32}{3}} = \frac{13}{5}$$

We could have found the centroid by using Definition (6.25) with $f(x) = 6 - x^2$, $g(x) = 3 - 2x$, $a = -1$, and $b = 3$, but that would merely teach you how to substitute and not how to think.

If a homogeneous lamina has the shape of a region that has an axis of symmetry, then the center of mass must lie on that axis. This fact is used in the next example.

EXAMPLE 5 Find the centroid of the semicircular region bounded by the x-axis and the graph of $y = \sqrt{a^2 - x^2}$ with $a > 0$.

FIGURE 6.71



SOLUTION The region is sketched in Figure 6.71. By symmetry, the centroid is on the y-axis; that is, $\bar{x} = 0$. Hence we need find only $\bar{y}$. Referring to the rectangle in the figure and using $\rho = 1$ gives us the following result.

$$\text{moment about } x\text{-axis:} \quad \tfrac{1}{2}y \cdot y\, dx = \tfrac{1}{2}y^2\, dx = \tfrac{1}{2}(a^2 - x^2)\, dx$$

We now take a limit of sums by applying the operator $\int_{-a}^{a}$:

$$\begin{aligned} M_x &= \int_{-a}^{a} \tfrac{1}{2}(a^2 - x^2)\, dx = 2\int_0^a \tfrac{1}{2}(a^2 - x^2)\, dx \\ &= \Big[a^2 x - \tfrac{1}{3}x^3\Big]_0^a = \tfrac{2}{3}a^3 \end{aligned}$$

Using $m = A = \frac{1}{2}\pi a^2$ gives us

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{2}{3}a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi} \approx 0.42a.$$

Thus, the centroid is the point $\left(0, \dfrac{4}{3\pi}a\right)$.

We shall conclude this section by stating a useful theorem about solids of revolution. To illustrate a special case of the theorem, consider an R_x region R of the type shown in Figure 6.68. Using $\rho = 1$ and $m = A$ (the area of R), we find that the moment of R about the y-axis is given by

$$M_y = \int_a^b x[f(x) - g(x)]\, dx.$$

If R is revolved about the y-axis, then using cylindrical shells, we find that the volume V of the resulting solid is given by

$$V = \int_a^b 2\pi x[f(x) - g(x)]\, dx.$$

Comparing these two equations, we see that

$$M_y = \frac{V}{2\pi}.$$

If $(\bar{x}, \bar{y})$ is the centroid of R, then, by Definition (6.25)(iii),

$$\bar{x} = \frac{M_y}{m} = \frac{(V/2\pi)}{A} = \frac{V}{2\pi A}$$

and hence

$$V = 2\pi\bar{x}A.$$

Since $\bar{x}$ is the distance from the y-axis to the centroid of R, the last formula states that the volume V of the solid of revolution may be found by multiplying the area A of R by the distance $2\pi\bar{x}$ that the centroid travels when R is revolved once about the y-axis. A similar statement is true if R is revolved about the x-axis. In Chapter 17 we shall prove the following more general theorem, named after the mathematician Pappus of Alexandria (ca. 300 A.D.).

Theorem of Pappus **(6.26)**

> Let R be a region in a plane that lies entirely on one side of a line l in the plane. If R is revolved once about l, the volume of the resulting solid is the product of the area of R and the distance traveled by the centroid of R.

FIGURE 6.72

EXAMPLE 6 The region bounded by a circle of radius a is revolved about a line l, in the plane of the circle, that is a distance b from the center of the circle, where $b > a$ (see Figure 6.72). Find the volume V of the resulting solid. (The surface of this doughnut-shaped solid is called a **torus**.)

SOLUTION The region bounded by the circle has area πa^2, and the distance traveled by the centroid is $2\pi b$. Hence, by the theorem of Pappus,

$$V = (2\pi b)(\pi a^2) = 2\pi^2 a^2 b.$$

EXERCISES 6.7

Exer. 1–2: The table lists point-masses (in kilograms) and their coordinates (in meters) on an x-axis. Find m, M_0, and the center of mass.

1

mass	100	80	70
coordinate	-3	2	4

2

mass	50	100	50
coordinate	-10	2	3

Exer. 3–4: The table lists point-masses (in kilograms) and their locations (in meters) in an xy-plane. Find m, M_x, M_y, and the center of mass of the system.

3

mass	2	7	5
location	$(4, -1)$	$(-2, 0)$	$(-8, -5)$

4

mass	10	3	4	1	8
location	$(-5, -2)$	$(3, 7)$	$(0, -3)$	$(-8, -3)$	$(0, 0)$

Exer. 5–14: Sketch the region bounded by the graphs of the equations, and find m, M_x, M_y, and the centroid.

5 $y = x^3$, $y = 0$, $x = 1$

6 $y = \sqrt{x}$, $y = 0$, $x = 9$

7 $y = 4 - x^2$, $y = 0$

8 $2x + 3y = 6$, $y = 0$, $x = 0$

9 $y^2 = x$, $2y = x$

10 $y = x^2$, $y = x^3$

11 $y = 1 - x^2$, $x - y = 1$

12 $y = x^2$, $x + y = 2$

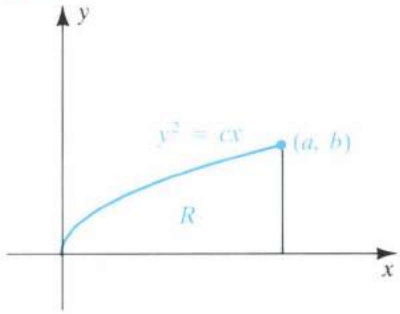
13 $x = y^2$, $x - y = 2$

14 $x = 9 - y^2$, $x + y = 3$

15 Find the centroid of the region in the first quadrant bounded by the circle $x^2 + y^2 = a^2$ and the coordinate axes.

16 Let R be the region in the first quadrant bounded by part of the parabola $y^2 = cx$ with $c > 0$, the x-axis, and the vertical line through the point (a, b) on the parabola, as shown in the figure on the following page. Find the centroid of R.

EXERCISE 16



17 A region has the shape of a square of side $2a$ surmounted by a semicircle of radius a. Find the centroid. (*Hint:* Use Example 5 and the fact that the moment of the region is the sum of the moments of the square and the semicircle.)

18 Let the points P, Q, R, and S have coordinates $(-b, 0)$, $(-a, 0)$, $(a, 0)$, and $(b, 0)$, respectively, with $0 < a < b$. Find the centroid of the region bounded by the graphs of $y = \sqrt{b^2 - x^2}$, $y = \sqrt{a^2 - x^2}$, and the line segments PQ and RS. (*Hint:* Use Example 5.)

19 Prove that the centroid of a triangle coincides with the intersection of the medians. (*Hint:* Take the vertices at the points $(0, 0)$, (a, b), and $(0, c)$, with a, b, and c positive.)

20 A region has the shape of a square of side a surmounted by an equilateral triangle of side a. Find the centroid. (*Hint:* See Exercise 19 and the hint given for Exercise 17.)

Exer. 21–24: Use the theorem of Pappus.

21 Let R be the rectangular region with vertices $(1, 2)$, $(2, 1)$, $(5, 4)$, and $(4, 5)$. Find the volume of the solid generated by revolving R about the y-axis.

22 Let R be the triangular region with vertices $(1, 1)$, $(2, 2)$, and $(3, 1)$. Find the volume of the solid generated by revolving R about the y-axis.

23 Find the centroid of the region in the first quadrant bounded by the graph of $y = \sqrt{a^2 - x^2}$ and the coordinate axes.

24 Find the centroid of the triangular region with vertices $O(0, 0)$, $A(0, a)$, and $B(b, 0)$ for positive numbers a and b.

c 25 A lamina of area density ρ has the shape of the region bounded by the graphs of $f(x) = \sqrt{|\cos x|}$ and $g(x) = x^2$. Graph f and g on the same coordinate axes.

(a) Set up an integral that can be used to approximate the mass of the lamina.

(b) Use Simpson's rule with $n = 4$ to approximate the integral in (a).

c 26 Use Simpson's rule with $n = 4$ to approximate the centroid of the region bounded by the graphs of $y = 0$, $y = (\sin x)/x$, $x = 1$, and $x = 2$.

6.8 OTHER APPLICATIONS

It should be evident from our work in this chapter that if a quantity can be approximated by a sum of many terms, then it is a candidate for representation as a definite integral. The main requirement is that as the number of terms increases, the sums approach a limit. In this section we consider several miscellaneous applications of the definite integral. Let us begin with the force exerted by a liquid on a submerged object.

In physics, the **pressure** p at a depth h in a fluid is defined as the weight of fluid contained in a column that has a cross-sectional area of one square unit and an altitude h. Pressure may also be regarded as the force per unit area exerted by the fluid. If a fluid has density ρ, then the pressure p at depth h is given by

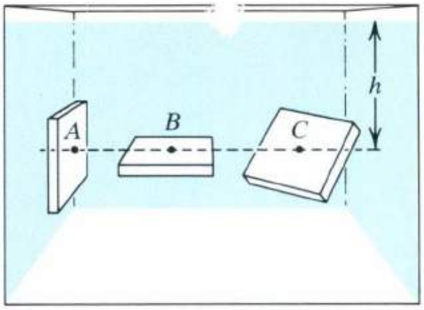
$$p = \rho h.$$

The following illustration is for water, with $\rho = 62.5 \text{ lb/ft}^3$.

ILLUSTRATION

DENSITY ρ (lb/ft^3)	DEPTH h (ft)	PRESSURE $p = \rho h$ (lb/ft^2)
62.5	2	125
62.5	4	250
62.5	6	375

FIGURE 6.73

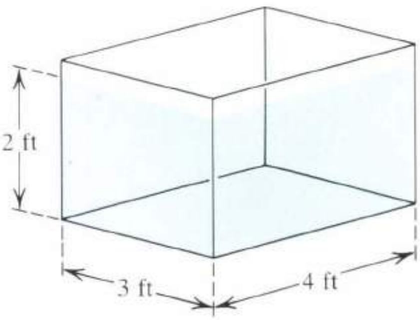


Pascal's principle in physics states that the pressure at a depth h in a fluid is the same in every direction. Thus, if a flat plate is submerged in a fluid, then the pressure on one side of the plate at a point that is h units below the surface is ρh, regardless of whether the plate is submerged vertically, horizontally, or obliquely (see Figure 6.73, where the pressure at points A, B, and C is ρh).

If a rectangular tank, such as a fish aquarium, is filled with water (see Figure 6.74), the total force exerted by the water on the base may be calculated as follows:

$$\text{force on base} = (\text{pressure at base}) \cdot (\text{area of base})$$

FIGURE 6.74



For the tank in Figure 6.74, we use $\rho = 62.5$ lb/ft^3 and $h = 2$ ft to obtain

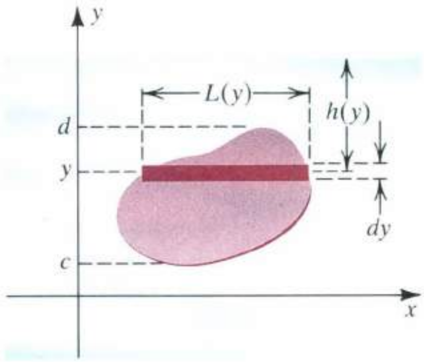
$$\text{force on base} = (125 \text{ lb/ft}^2) \cdot (12 \text{ ft}^2) = 1500 \text{ lb}.$$

This corresponds to 12 columns of water, each having cross-sectional area 1 ft^2 and each weighing 125 pounds.

It is more complicated to find the force exerted on one of the sides of the aquarium, since the pressure is not constant there but increases as the depth increases. Instead of investigating this particular problem, let us consider the following more general situation.

Suppose a flat plate is submerged in a fluid of density ρ such that the face of the plate is perpendicular to the surface of the fluid. Let us introduce a coordinate system as shown in Figure 6.75, where the width of the plate extends over the interval $[c, d]$ on the y-axis. Assume that for each y in $[c, d]$, the corresponding depth of the fluid is $h(y)$ and the length of the plate is $L(y)$, where h and L are continuous functions.

FIGURE 6.75



We shall employ our standard technique of considering a typical horizontal rectangle of width dy and length $L(y)$, as illustrated in Figure 6.75. If dy is small, then the pressure at any point in the rectangle is approximately $\rho h(y)$. Thus, the force on one side of the rectangle can be approximated by

$$\text{force on rectangle} \approx (\text{pressure}) \cdot (\text{area of rectangle}),$$

or

$$\text{force on rectangle} \approx \rho h(y) \cdot L(y)\, dy.$$

Taking a limit of sums of these forces by applying the operator $\int_c^d$ leads to the following definition.

Definition (6.27)

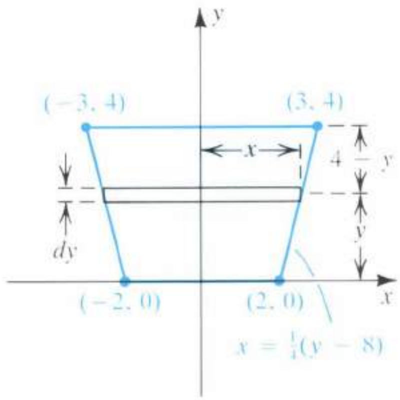
The **force F exerted by a fluid** of constant density ρ on one side of a submerged region of the type illustrated in Figure 6.75 is

$$F = \int_c^d \rho h(y) \cdot L(y)\, dy.$$

If a more complicated region is divided into subregions of the type illustrated in Figure 6.75, we apply Definition (6.27) to each subregion and add the resulting forces. The coordinate system may be introduced in various ways: In Example 2 we choose the x-axis along the surface of the liquid and the positive direction of the y-axis downward.

EXAMPLE 1 The ends of a water trough 8 feet long have the shape of isosceles trapezoids of lower base 4 feet, upper base 6 feet, and height 4 feet. Find the total force on one end if the trough is full of water.

FIGURE 6.76



SOLUTION Figure 6.76 illustrates one end of the trough superimposed on a rectangular coordinate system. An equation of the line through the points (2, 0) and (3, 4) is $y = 4x - 8$, or, equivalently, $x = \frac{1}{4}(y + 8)$. Referring to Figure 6.76 gives us the following, for a horizontal rectangle of width dy:

$$\begin{aligned} \text{length:}&\quad 2x = 2 \cdot \tfrac{1}{4}(y + 8) = \tfrac{1}{2}(y + 8) \\ \text{area:}&\quad \tfrac{1}{2}(y + 8)\,dy \\ \text{depth:}&\quad 4 - y \\ \text{pressure:}&\quad 62.5(4 - y) \\ \text{force:}&\quad 62.5(4 - y) \cdot \tfrac{1}{2}(y + 8)\,dy \end{aligned}$$

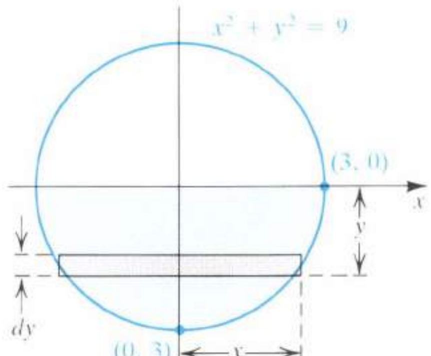
Taking a limit of sums by applying the operator $\int_0^4$, we obtain, as in Definition (6.27),

$$\begin{aligned} F &= \int_0^4 62.5(4 - y) \cdot \tfrac{1}{2}(y + 8)\,dy \\ &= 31.25 \int_0^4 (32 - 4y - y^2)\,dy = \frac{7000}{3} \approx 2333 \text{ lb.} \end{aligned}$$

In the preceding example, the *length* of the water trough was irrelevant when we considered the force on one end. The same is true for the oil tank in the next example.

EXAMPLE 2 A cylindrical oil storage tank 6 feet in diameter and 10 feet long is lying on its side. If the tank is half full of oil that weighs 58 lb/ft^3, set up an integral for the force exerted by the oil on one end of the tank.

FIGURE 6.77



SOLUTION Let us introduce a coordinate system such that the end of the tank is a circle of radius 3 feet with the center at the origin. The equation of the circle is $x^2 + y^2 = 9$. If we choose the positive direction of the y-axis *downward*, then referring to the horizontal rectangle in Figure 6.77 gives us the following:

$$\begin{aligned} \text{length:}&\quad 2x = 2\sqrt{9 - y^2} \\ \text{area:}&\quad 2\sqrt{9 - y^2}\,dy \\ \text{depth:}&\quad y \\ \text{pressure:}&\quad 58y \\ \text{force:}&\quad 58y \cdot 2\sqrt{9 - y^2}\,dy \end{aligned}$$

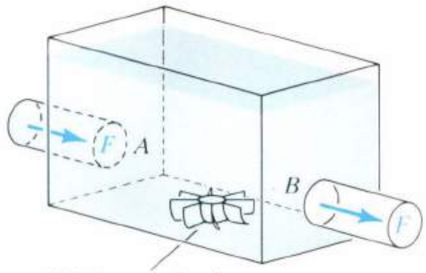
Taking a limit of sums by applying $\int_0^3$ we obtain

$$F = \int_0^3 116y\sqrt{9 - y^2}\, dy.$$

Evaluating the integral by using the method of substitution would give us

$$F = 1044 \text{ lb}.$$

FIGURE 6.78



Definite integrals can be applied to dye-dilution or tracer methods used in physiological tests and elsewhere. One example involves the measurement of cardiac output—that is, the rate at which blood flows through the aorta. A simple model for tracer experiments is sketched in Figure 6.78, where a liquid (or gas) flows into a tank at A and exits at B, with a constant flow rate F (in L/sec). Suppose that at time $t = 0$, Q_0 grams of tracer (or dye) are introduced into the tank at A and that a stirring mechanism thoroughly mixes the solution at all times. The concentration $c(t)$ (in g/L) of tracer at time t is monitored at B. Thus the amount of tracer passing B at time t is given by

$$(\text{flow rate}) \cdot (\text{concentration}) = F \cdot c(t) \text{ g/sec}.$$

If the amount of tracer in the tank at time t is $Q(t)$, where Q is a differentiable function, then the rate of change $Q'(t)$ of Q is given by

$$Q'(t) = -F \cdot c(t)$$

(the negative sign indicates that Q is decreasing).

If T is a time at which all the tracer has left the tank, then $Q(T) = 0$ and, by the fundamental theorem of calculus,

$$\begin{aligned}\int_0^T Q'(t)\, dt = Q(t)\Big]_0^T &= Q(T) - Q(0)\\ &= 0 - Q_0 = -Q_0.\end{aligned}$$

We may also write

$$\int_0^T Q'(t)\, dt = \int_0^T [-F \cdot c(t)]\, dt = -F \int_0^T c(t)\, dt.$$

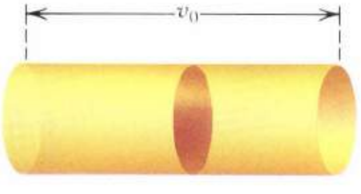
Equating the two forms for the integral gives us the following formula.

Flow concentration formula **(6.28)**

$$\boxed{Q_0 = F \int_0^T c(t)\, dt}$$

Usually an explicit form for $c(t)$ will not be known, but, instead, a table of function values will be given. By employing numerical integration, we may find an approximation to the flow rate F (see Exercises 11 and 12).

FIGURE 6.79



Let us next consider another aspect of the flow of liquids. If a liquid flows through a cylindrical tube and if the velocity is a constant v_0, then the volume of liquid passing a fixed point per unit time is given by v_0A, where A is the area of a cross section of the tube (see Figure 6.79).

A more complicated formula is required to study the flow of blood in an arteriole. In this case the flow is in layers, as illustrated in Figure 6.80, on the following page. In the layer closest to the wall of the arteriole, the

blood tends to stick to the wall, and its velocity may be considered zero. The velocity increases as the layers approach the center of the arteriole.

For computational purposes, we may regard the blood flow as consisting of thin cylindrical shells that slide along, with the outer shell fixed and the velocity of the shells increasing as the radii of the shells decrease (see Figure 6.80). If the velocity in each shell is considered constant, then from the theory of liquids in motion, the velocity $v(r)$ in a shell having average radius r is

FIGURE 6.80

$$v(r) = \frac{P}{4\nu l}(R^2 - r^2),$$

where R is the radius of the arteriole (in centimeters), l is the length of the arteriole (in centimeters), P is the pressure difference between the two ends of the arteriole (in dyn/cm^2), and ν is the viscosity of the blood (in dyn-sec/cm^2). Note that the formula gives zero velocity if $r = R$ and maximum velocity $PR^2/(4\nu l)$ as r approaches 0. If the radius of the kth shell is r_k and the thickness of the shell is Δr_k, then, by (6.10), the volume of blood in this shell is

$$2\pi r_k v(r_k)\,\Delta r_k = \frac{2\pi r_k P}{4\nu l}(R^2 - r_k^2)\,\Delta r_k.$$

If there are n shells, then the total flow in the arteriole per unit time may be approximated by

$$\sum_{k=1}^{n} \frac{2\pi r_k P}{4\nu l}(R^2 - r_k^2)\,\Delta r_k.$$

To estimate the total flow F (the volume of blood per unit time), we consider the limit of these sums as n increases without bound. This leads to the following definite integral:

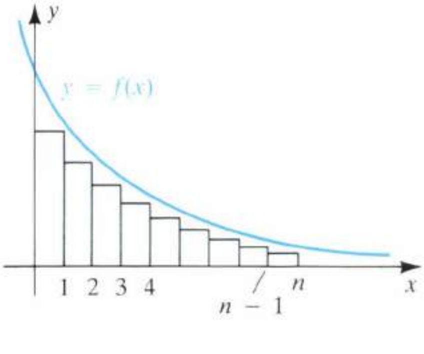
$$\begin{aligned}
F &= \int_0^R \frac{2\pi r P}{4\nu l}(R^2 - r^2)\,dr \\
&= \frac{2\pi P}{4\nu l}\int_0^R (R^2 r - r^3)\,dr \\
&= \frac{\pi P}{2\nu l}\left[\frac{1}{2}R^2 r^2 - \frac{1}{4}r^4\right]_0^R \\
&= \frac{\pi P R^4}{8\nu l}\ \text{cm}^3
\end{aligned}$$

This formula for F is not exact, because the thickness of the shells cannot be made arbitrarily small. The lower limit is the width of a red blood cell, or approximately 2×10^{-4} centimeter. We may assume, however, that the formula gives a reasonable estimate. It is interesting to observe that a small change in the radius of an arteriole produces a large change in the flow, since F is directly proportional to the fourth power of R. A small change in pressure difference has a lesser effect, since P appears to the first power.

In many types of employment, a worker must perform the same assignment repeatedly. For example, a bicycle shop employee may be asked to assemble new bicycles. As more and more bicycles are assembled, the time required for each assembly should decrease until a certain minimum assembly time is reached. Another example of this process of learning by repetition is that of a data processor who must keyboard information from written forms into a computer system. The time required to process each entry should decrease as the number of entries increases. As a final illustration, the time required for a person to trace a path through a maze should improve with practice.

Let us consider a general situation in which a certain task is to be repeated many times. Suppose experience has shown that the time required to perform the task for the kth time can be approximated by $f(k)$ for a continuous decreasing function f on a suitable interval. The total time required to perform the task n times is given by the sum

FIGURE 6.81



$$\sum_{k=1}^{n} f(k) = f(1) + f(2) + \cdots + f(n).$$

If we consider the graph of f, then, as illustrated in Figure 6.81, the preceding sum equals the area of the pictured inscribed rectangular polygon and, therefore, may be approximated by the definite integral $\int_0^n f(x)\,dx$. Evidently, the approximation will be close to the actual sum if f decreases slowly on $[0, n]$. If f changes rapidly per unit change in x, then an integral should not be used as an approximation.

EXAMPLE 3 A company that conducts polls via telephone interviews finds that the time required by an employee to complete one interview depends on the number of interviews that the employee has completed previously. Suppose it is estimated that, for a certain survey, the number of minutes required to complete the kth interview is given by $f(k) = 6(1 + k)^{-1/5}$ for $0 \leq k \leq 500$. Use a definite integral to approximate the time required for an employee to complete 100 interviews and 200 interviews. If an interviewer receives \$4.80 per hour, estimate how much more expensive it is to have two employees each conduct 100 interviews than to have one employee conduct 200 interviews.

SOLUTION From the preceding discussion, the time required for 100 interviews is approximately

$$\int_0^{100} 6(1 + x)^{-1/5}\,dx = 6 \cdot \tfrac{5}{4}(1 + x)^{4/5}\Big]_0^{100} \approx 293.5 \text{ min}.$$

The time required for 200 interviews is approximately

$$\int_0^{200} 6(1 + x)^{-1/5}\,dx \approx 514.4 \text{ min}.$$

Since an interviewer receives \$0.08 per minute, the cost for one employee to conduct 200 interviews is roughly (\$0.08)(514.4), or \$41.15. If two employees each conduct 100 interviews, the cost is about 2(\$0.08)(293.5), or \$46.96, which is \$5.81 more than the cost of one employee. Note, however, that the time saved in using two people is approximately 221 minutes.

Using a computer, we have

$$\sum_{k=1}^{100} 6(1+k)^{-1/5} \approx 291.75$$

and

$$\sum_{k=1}^{200} 6(1+k)^{-1/5} \approx 512.57.$$

Hence the results obtained by integration (the area under the graph of f) are roughly 2 minutes more than the value of the corresponding sum (the area of the inscribed rectangular polygon).

In economics, the process that a corporation uses to increase its accumulated wealth is called **capital formation**. If the amount K of capital at time t can be approximated by $K = f(t)$ for a differentiable function f, the rate of change of K with respect to t is called the **net investment flow**. Hence, if I denotes the investment flow, then

$$I = \frac{dK}{dt} = f'(t).$$

Conversely, if I is given by $g(t)$ for a function g that is continuous on an interval $[a, b]$, then the increase in capital over this time interval is

$$\int_a^b g(t)\,dt = f(b) - f(a).$$

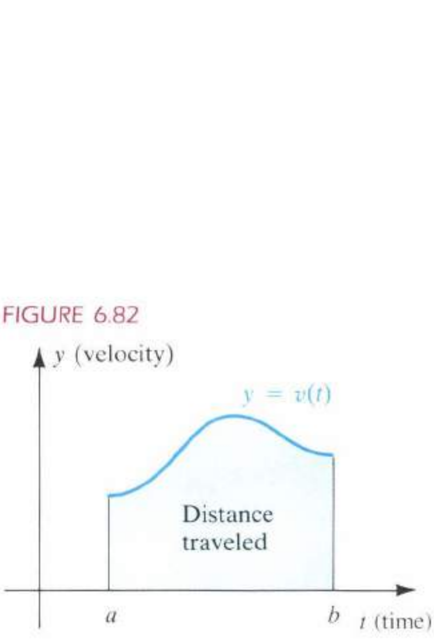
EXAMPLE 4 Suppose a corporation wishes to have its net investment flow approximated by $g(t) = t^{1/3}$ for t in years and $g(t)$ in millions of dollars per year. If $t = 0$ corresponds to the present time, estimate the amount of capital formation over the next eight years.

SOLUTION From the preceding discussion, the increase in capital over the next eight years is

$$\int_0^8 g(t)\,dt = \int_0^8 t^{1/3}\,dt = \tfrac{3}{4}t^{4/3}\Big]_0^8 = 12.$$

Consequently, the amount of capital formation is \$12,000,000.

FIGURE 6.82

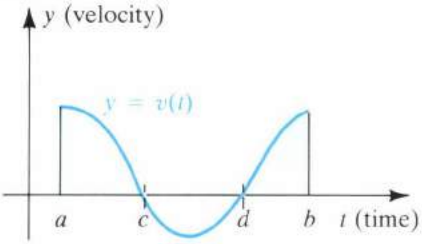


Any quantity that can be interpreted as an area of a region in a plane may be investigated by means of a definite integral. (See, for example, the discussion of hysteresis at the end of Section 6.1.) Conversely, definite integrals allow us to represent physical quantities as areas. In the following illustrations, a quantity is *numerically equal* to an area of a region; that is, we *disregard units of measurement*, such as centimeter, foot-pound, and so on.

Suppose $v(t)$ is the velocity, at time t, of an object that is moving on a coordinate line. If s is the position function, then $s'(t) = v(t)$ and

$$\int_a^b v(t)\,dt = \int_a^b s'(t)\,dt = s(t)\Big]_a^b = s(b) - s(a).$$

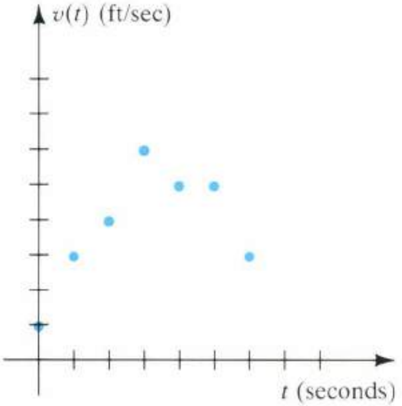
FIGURE 6.83



If $v(t) > 0$ throughout the time interval $[a, b]$, this tells us that the area under the graph of the function v from a to b represents the distance the object travels, as illustrated in Figure 6.82. This observation is useful to an engineer or physicist, who may not have an explicit form for $v(t)$ but merely a graph (or table) indicating the velocity at various times. The distance traveled may then be estimated by approximating the area under the graph.

If $v(t) < 0$ at certain times in $[a, b]$, the graph of v may resemble that in Figure 6.83. The figure indicates that the object moved in the negative direction from $t = c$ to $t = d$. The distance it traveled during that time is given by $\int_c^d |v(t)|\, dt$. It follows that $\int_a^b |v(t)|\, dt$ is the *total* distance traveled in $[a, b]$, whether $v(t)$ is positive or negative.

FIGURE 6.84



EXAMPLE 5 As an object moves along a straight path, its velocity $v(t)$ (in ft/sec) at time t is recorded each second for 6 seconds. The results are given in the following table.

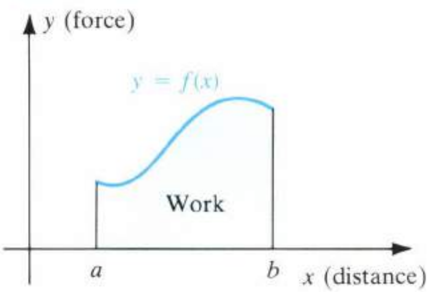
t	0	1	2	3	4	5	6
$v(t)$	1	3	4	6	5	5	3

Approximate the distance traveled by the object.

SOLUTION The points $(t, v(t))$ are plotted in Figure 6.84. If we assume that v is a continuous function, then, as in the preceding discussion, the distance traveled during the time interval $[0, 6]$ is $\int_0^6 v(t)\, dt$. Let us approximate this definite integral by means of Simpson's rule, with $n = 6$:

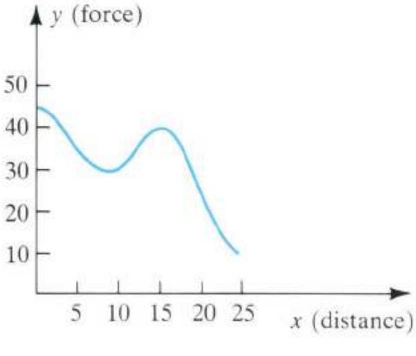
$$\int_0^6 v(t)\, dt \approx \frac{6-0}{3 \cdot 6}\,[v(0) + 4v(1) + 2v(2) + 4v(3) + 2v(4) + 4v(5) + v(6)]$$
$$= \tfrac{1}{3}[1 + 4 \cdot 3 + 2 \cdot 4 + 4 \cdot 6 + 2 \cdot 5 + 4 \cdot 5 + 3] = 26 \text{ ft}$$

FIGURE 6.85



In (6.21) we defined the work W done by a variable force $f(x)$ that acts along a coordinate line from $x = a$ to $x = b$ by $W = \int_a^b f(x)\, dx$. Suppose $f(x) \geq 0$ throughout $[a, b]$. If we sketch the graph of f as illustrated in Figure 6.85, then the work W is numerically equal to the area under the graph from a to b.

FIGURE 6.86



EXAMPLE 6 An engineer obtains the graph in Figure 6.86, which shows the force (in pounds) acting on a small cart as it moves 25 feet along horizontal ground. Estimate the work done.

SOLUTION If we assume that the force is a continuous function f for $0 \leq x \leq 25$, the work done is

$$W = \int_0^{25} f(x)\, dx.$$

We do not have an explicit form for $f(x)$; however, we may estimate function values from the graph and approximate W by means of numerical integration.

Let us apply the trapezoidal rule with $a = 0$, $b = 25$, and $n = 5$. Referring to the graph to estimate function values gives us the following table.

k	x_k	$f(x_k)$	m	$mf(x_k)$
0	0	45	1	45
1	5	35	2	70
2	10	30	2	60
3	15	40	2	80
4	20	25	2	50
5	25	10	1	10

The sum of the numbers in the last column is 315. Since

$$(b - a)/(2n) = (25 - 0)/10 = 2.5,$$

it follows from (5.36) that

$$W = \int_0^{25} f(x)\, dx \approx 2.5(315) \approx 790 \text{ ft-lb}.$$

For greater accuracy we could use a larger value of n or Simpson's rule.

Suppose that the amount of a physical entity, such as oil, water, electric power, money supply, bacteria count, or blood flow, is increasing or decreasing in some manner, and that $R(t)$ is the rate at which it is changing at time t. If $Q(t)$ is the amount of the entity present at time t and if Q is differentiable, then $Q'(t) = R(t)$. If $R(t) > 0$ (or $R(t) < 0$) in a time interval $[a, b]$, then the amount that the entity increases (or decreases) between $t = a$ and $t = b$ is

$$Q(b) - Q(a) = \int_a^b Q'(t)\, dt = \int_a^b R(t)\, dt.$$

This number may be represented as the area of the region in a ty-plane bounded by the graphs of R, $t = a$, $t = b$, and $y = 0$.

EXAMPLE 7 Starting at 9:00 A.M., oil is pumped into a storage tank at a rate of $(150t^{1/2} + 25)$ gal/hr, for time t (in hours) after 9:00 A.M. How many gallons will have been pumped into the tank at 1:00 P.M.?

SOLUTION Letting $R(t) = 150t^{1/2} + 25$ in the preceding discussion, we obtain the following:

$$\int_0^4 (150t^{1/2} + 25)\, dt = \Big[100t^{3/2} + 25t\Big]_0^4 = 900 \text{ gal}$$

We have given only a few illustrations of the use of definite integrals. The interested reader may find many more in books on the physical and biological sciences, economics, and business, and even such areas as political science and sociology.

EXERCISES 6.8

1 A glass aquarium tank is 3 feet long and has square ends of width 1 foot. If the tank is filled with water, find the force exerted by the water on

(a) one end (b) one side

2 If one of the square ends of the tank in Exercise 1 is divided into two parts by means of a diagonal, find the force exerted on each part.

3 The ends of a water trough 6 feet long have the shape of isosceles triangles with equal sides of length 2 feet and the third side of length $2\sqrt{3}$ feet at the top of the trough. Find the force exerted by the water on one end of the trough if the trough is

(a) full of water (b) half full of water

4 The ends of a water trough have the shape of the region bounded by the graphs of $y = x^2$ and $y = 4$, with x and y measured in feet. If the trough is full of water, find the force on one end.

5 A cylindrical oil storage tank 4 feet in diameter and 5 feet long is lying on its side. If the tank is half full of oil weighing 60 lb/ft^3, find the force exerted by the oil on one end of the tank.

6 A rectangular gate in a dam is 5 feet long and 3 feet high. If the gate is vertical, with the top of the gate parallel to the surface of the water and 6 feet below it, find the force of the water against the gate.

7 A plate having the shape of an isosceles trapezoid with upper base 4 feet long and lower base 8 feet long is submerged vertically in water such that the bases are parallel to the surface. If the distances from the surface of the water to the lower and upper bases are 10 feet and 6 feet, respectively, find the force exerted by the water on one side of the plate.

8 A circular plate of radius 2 feet is submerged vertically in water. If the distance from the surface of the water to the center of the plate is 6 feet, find the force exerted by the water on one side of the plate.

9 A rectangular plate 3 feet wide and 6 feet long is submerged vertically in oil weighing 50 lb/ft^3, with its short side parallel to, and 2 feet below, the surface.

(a) Find the total force exerted on one side of the plate.

(b) If the plate is divided into two parts by means of a diagonal, find the force exerted on each part.

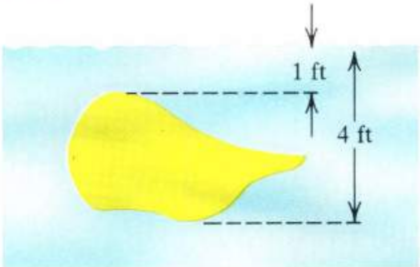
c 10 A flat, irregularly shaped plate is submerged vertically in water (see figure). Measurements of its width, taken at successive depths at intervals of 0.5 foot, are compiled in the following table.

Water depth (ft)	1	1.5	2	2.5	3	3.5	4
Width of plate (ft)	0	2	3	5.5	4.5	3.5	0

Estimate the force of the water on one side of the plate by using, with $n = 6$,

(a) the trapezoidal rule (b) Simpson's rule

EXERCISE 10



c 11 Refer to (6.28). To estimate cardiac output F (the number of liters of blood per minute that the heart pumps through the aorta), a 5-milligram dose of the tracer indocyanine-green is injected into a pulmonary artery and dye concentration measurements $c(t)$ are taken every minute from a peripheral artery near the aorta. The results are given in the following table. Use Simpson's rule, with $n = 12$, to estimate the cardiac output.

t (min)	$c(t)$ (mg/L)
0	0
1	0
2	0.15
3	0.48
4	0.86
5	0.72
6	0.48
7	0.26
8	0.15
9	0.09
10	0.05
11	0.01
12	0

c 12 Refer to (6.28). Suppose 1200 kilograms of sodium dichromate are mixed into a river at point A and sodium dichromate samples are taken every 30 seconds at a point B downstream. The concentration $c(t)$ at time t is recorded in the following table. Use the trapezoidal rule, with $n = 12$, to estimate the river flow rate F.

t (sec)	$c(t)$ (mg/L or g/m^3)
0	0
30	2.14
60	3.89
90	5.81
120	8.95
150	7.31
180	6.15
210	4.89
240	2.98
270	1.42
300	0.89
330	0.29
360	0

13 A manufacturer estimates that the time required for a worker to assemble a certain item depends on the number of this item the worker has previously assembled. If the time (in minutes) required to assemble the kth item is given by $f(k) = 20(k+1)^{-0.4} + 3$, use a definite integral to approximate the time, to the nearest minute, required to assemble

(a) 1 item (b) 4 items (c) 8 items (d) 16 items

14 The number of minutes needed for a person to trace a path through a certain maze without error is estimated to be $f(k) = 5k^{-1/2}$, where k is the number of trials previously completed. Use a definite integral to approximate the time required to complete 10 trials.

15 A data processor keyboards registration data for college students from written forms to electronic files. The number of minutes required to process the kth registration is estimated to be approximately $f(k) = 6(1+k)^{-1/3}$. Use a definite integral to estimate the time required for

(a) one person to keyboard 600 registrations

(b) two people to keyboard 300 each

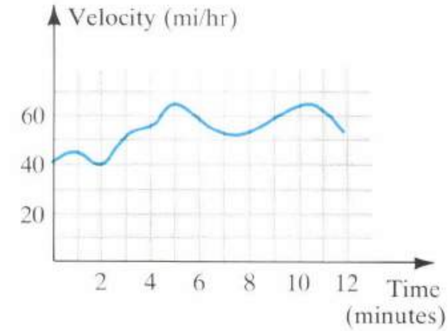
16 If, in Example 4, the rate of investment is approximated by $g(t) = 2t(3t+1)$, with $g(t)$ in thousands of dollars, use a definite integral to approximate the amount of capital formation over the intervals $[0, 5]$ and $[5, 10]$.

Exer. 17–18: Use a definite integral to approximate the sum, and round your answer to the nearest integer.

17 $\sum_{k=1}^{100} k(k^2+1)^{-1/4}$ 18 $\sum_{k=1}^{200} 5k(k^2+10)^{-1/3}$

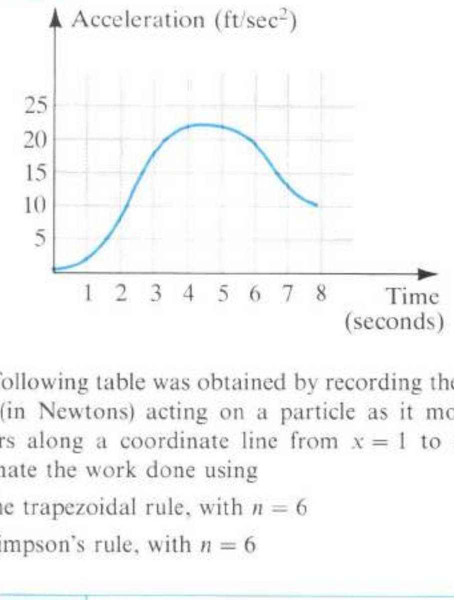
c 19 The velocity (in mi/hr) of an automobile as it traveled along a freeway over a 12-minute interval is indicated in the figure. Use the trapezoidal rule to approximate the distance traveled to the nearest mile.

EXERCISE 19



c 20 The acceleration (in ft/sec^2) of an automobile over a period of 8 seconds is indicated in the figure. Use the trapezoidal rule to approximate the net change in velocity in this time period.

EXERCISE 20



21 The following table was obtained by recording the force $f(x)$ (in Newtons) acting on a particle as it moved 6 meters along a coordinate line from $x = 1$ to $x = 7$. Estimate the work done using

(a) the trapezoidal rule, with $n = 6$

(b) Simpson's rule, with $n = 6$

x	1	2	3	4	5	6	7
$f(x)$	20	23	25	22	26	30	28

22 A bicyclist pedals directly up a hill, recording the velocity $v(t)$ (in ft/sec) at the end of every two seconds. Referring to the results recorded in the following table, use the trapezoidal rule to approximate the distance traveled.

t	0	2	4	6	8	10
$v(t)$	24	22	16	10	2	0

23 A motorboat uses gasoline at the rate of $t\sqrt{9-t^2}$ gal/hr. If the motor is started at $t = 0$, how much gasoline is used in 2 hours?

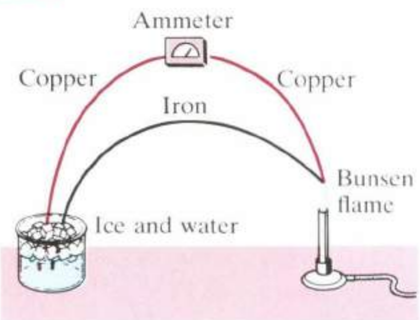
24 The population of a city has increased since 1985 at a rate of $1.5 + 0.3\sqrt{t} + 0.006t^2$ thousand people per year, where t is the number of years after 1985. Assuming that this rate continues and that the population was 50,000 in 1985, estimate the population in 1994.

25 A simple thermocouple, in which heat is transformed into electrical energy, is shown in the figure. To determine the total charge Q (in coulombs) transferred to the copper wire, current readings (in amperes) are recorded every $\frac{1}{2}$ second, and the results are shown in the following table.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0
I (amp)	0	0.2	0.6	0.7	0.8	0.5	0.2

Use the fact that $I = dQ/dt$ and the trapezoidal rule, with $n = 6$, to estimate the total charge transferred to the copper wire during the first three seconds.

EXERCISE 25



c 26 Suppose $\rho(x)$ is the density (in cm/km) of ozone in the atmosphere at a height of x kilometers above the ground. For example, if $\rho(6) = 0.0052$, then at a height of 6 kilometers there is effectively a thickness of 0.0052 centimeter of ozone for each kilometer of atmosphere. If ρ is a continuous function, the thickness of the ozone layer between heights a and b can be found by evaluating $\int_a^b \rho(x)\,dx$. Values for $\rho(x)$ found experimentally are shown in the following table.

x (km)	$\rho(x)$ (spring)	$\rho(x)$ (autumn)
0	0.0034	0.0038
6	0.0052	0.0043
12	0.0124	0.0076
18	0.0132	0.0104
24	0.0136	0.0109
30	0.0084	0.0072
36	0.0034	0.0034
42	0.0017	0.0016

(a) Use the trapezoidal rule to estimate the thickness of the ozone layer between the altitudes of 6 and 42 kilometers during both spring and autumn.

(b) Work part (a) using Simpson's rule.

c 27 Radon gas can pose a serious health hazard if inhaled. If $V(t)$ is the volume of air (in cm^3) in an adult's lungs at time t (in minutes), then the rate of change of V can often be approximated by $V'(t) = 12{,}450\pi \sin(30\pi t)$. Inhaling and exhaling correspond to $V'(t) > 0$ and $V'(t) < 0$, respectively. Suppose an adult lives in a home that has a radioactive energy concentration due to radon of 4.1×10^{-12} joule/cm^3.

(a) Approximate the volume of air inhaled by the adult with each breath.

(b) If inhaling more than 0.02 joule of radioactive energy in one year is considered hazardous, is it safe for the adult to remain at home?

28 A stationary exercise bicycle is programmed so that it can be set for different intensity levels L and workout times T. It displays the elapsed time t (in minutes), for $0 \le t \le T$, and the number of calories $C(t)$ that are being burned per minute at time t, where

$$C(t) = 5 + 3L - 6\frac{L}{T}\left|t - \frac{1}{2}T\right|.$$

Suppose an individual exercises 16 minutes, with $L = 3$ for $0 \le t \le 8$ and with $L = 2$ for $8 \le t \le 16$. Find the total number of calories burned during the workout.

29 The rate of growth R (in cm/yr) of an average boy who is t years old is shown in the following table for $10 \le t \le 15$.

t (yr)	10	11	12	13	14	15
R (cm/yr)	5.3	5.2	4.9	6.5	9.3	7.0

Use the trapezoidal rule, with $n = 5$, to approximate the number of centimeters the boy grows between his tenth and fifteenth birthdays.

30 To determine the number of zooplankton in a portion of an ocean 80 meters deep, marine biologists take samples at successive depths of 10 meters, obtaining the following table, where $\rho(x)$ is the density (in number/m^3) of zooplankton at a depth of x meters.

x	0	10	20	30	40	50	60	70	80
$\rho(x)$	0	10	25	30	20	15	10	5	0

Use Simpson's rule, with $n = 8$, to estimate the total number of zooplankton in a *water column* (a column of water) having a cross section 1 meter square extending from the surface to the ocean floor.

6.9 REVIEW EXERCISES

Exer. 1–2: Sketch the region bounded by the graphs of the equations, and find the area by integrating with respect to (a) x and (b) y.

1 $y = -x^2, \quad y = x^2 - 8$

2 $y^2 = 4 - x, \quad x + 2y = 1$

Exer. 3–4: Find the area of the region bounded by the graphs of the equations.

3 $x = y^2, \quad x + y = 1$

4 $y = -x^3, \quad y = \sqrt{x}, \quad 7x + 3y = 10$

5 Find the area of the region between the graphs of the equations $y = \cos \frac{1}{2}x$ and $y = \sin x$, from $x = \pi/3$ to $x = \pi$.

6 The region bounded by the graph of $y = \sqrt{1 + \cos 2x}$ and the x-axis, from $x = 0$ to $x = \pi/2$, is revolved about the x-axis. Find the volume of the resulting solid.

Exer. 7–10: Sketch the region R bounded by the graphs of the equations, and find the volume of the solid generated by revolving R about the indicated axis.

7 $y = \sqrt{4x + 1}, \quad y = 0, \quad x = 0, \quad x = 2;$ x-axis

8 $y = x^4, \quad y = 0, \quad x = 1;$ y-axis

9 $y = x^3 + 1, \quad x = 0, \quad y = 2;$ y-axis

10 $y = \sqrt[3]{x}, \quad y = \sqrt{x};$ x-axis

Exer. 11–12: The region bounded by the x-axis and the graph of the given equation, from $x = 0$ to $x = b$, is revolved about the y-axis. Find the volume of the resulting solid.

11 $y = \cos x^2; \quad b = \sqrt{\pi/2}$

12 $y = x \sin x^3; \quad b = 1$

13 Find the volume of the solid generated by revolving the region bounded by the graphs of $y = 4x^2$ and $4x + y = 8$ about

(a) the x-axis (b) $x = 1$ (c) $y = 16$

14 Find the volume of the solid generated by revolving the region bounded by the graphs of $y = x^3$, $x = 2$, and $y = 0$ about

(a) the x-axis (b) the y-axis (c) $x = 2$

(d) $x = 3$ (e) $y = 8$ (f) $y = -1$

15 Find the arc length of the graph of $(x + 3)^2 = 8(y - 1)^3$ from $A(-2, \frac{3}{2})$ to $B(5, 3)$.

16 A solid has for its base the region in the xy-plane bounded by the graphs of $y^2 = 4x$ and $x = 4$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an isosceles right triangle with one of its equal sides on the base of the solid.

17 An above-ground swimming pool has the shape of a right circular cylinder of diameter 12 feet and height 5 feet. If the depth of the water in the pool is 4 feet, find the work required to empty the pool by pumping the water out over the top.

18 As a bucket is raised a distance of 30 feet from the bottom of a well, water leaks out at a uniform rate. Find the work done if the bucket originally contains 24 pounds of water and one-third leaks out. Assume that the weight of the empty bucket is 4 pounds, and disregard the weight of the rope.

19 A square plate of side 4 feet is submerged vertically in water such that one of the diagonals is parallel to the surface. If the distance from the surface of the water to the center of the plate is 6 feet, find the force exerted by the water on one side of the plate.

20 Use differentials to approximate the arc length of the graph of $y = 2 \sin \frac{1}{3}x$ between the points with x-coordinates π and $91\pi/90$.

Exer. 21–22: Sketch the region bounded by the graphs of the equations, and find m, M_x, M_y, and the centroid.

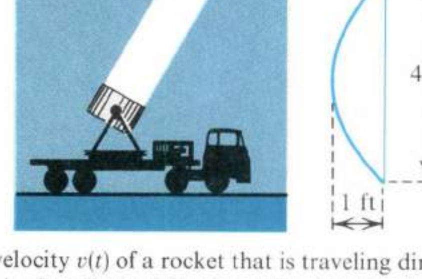
21 $y = x^3 + 1, \quad x + y = -1, \quad x = 1$

22 $y = x^2 + 1, \quad y = x, \quad x = -1, \quad x = 2$

23 The graph of the equation $12y = 4x^3 + (3/x)$ from $A(1, \frac{7}{12})$ to $B(2, \frac{67}{24})$ is revolved about the x-axis. Find the area of the resulting surface.

24 The shape of a reflector in a searchlight is obtained by revolving a parabola about its axis. If, as shown in the figure, the reflector is 4 feet across at the opening and 1 foot deep, find its surface area.

EXERCISE 24



25 The velocity $v(t)$ of a rocket that is traveling directly upward is given in the following table. Use the trapezoidal rule to approximate the distance the rocket travels from $t = 0$ to $t = 5$.

t (sec)	0	1	2	3	4	5
$v(t)$ (ft/sec)	100	120	150	190	240	300

c 26 An electrician suspects that a meter showing the total consumption Q in kilowatt hours (kwh) of electricity is not functioning properly. To check the accuracy, the electrician measures the consumption rate R directly every 10 minutes, obtaining the results in the following table.

t (min)	0	10	20	30	40	50	60
R (kwh/min)	1.31	1.43	1.45	1.39	1.36	1.47	1.29

(a) Use Simpson's rule to estimate the total consumption during this one-hour period.

(b) If the meter read 48,792 kwh at the beginning of the experiment and 48,953 kwh at the end, what should the electrician conclude?

27 Interpret $\int_0^1 2\pi x^4\,dx$ in the following ways:

(a) as the area of a region in the xy-plane

(b) as the volume of a solid obtained by revolving a region in the xy-plane about
  (i) the x-axis
  (ii) the y-axis

(c) as the work done by a force

28 Let R be the semicircular region in the xy-plane with endpoints of its diameter at $(4, 0)$ and $(10, 0)$. Use the theorem of Pappus to find the volume of the solid obtained by revolving R about the y-axis.

CHAPTER

7

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

INTRODUCTION

In precalculus mathematics we sketch graphs of equations such as $y = a^x$ for $a > 0$ without defining a^x if x is irrational. Instead, we *assume* that real numbers such as a^π and $a^{\sqrt{3}}$ exist, and that the graph rises if $a > 1$ or falls if $0 < a < 1$. We *further* assume that the laws of exponents are true for all real exponents. Next, we define the logarithm $\log_a x$ using our *undefined* exponential expressions and "prove" properties of logarithms by applying the *unproved* laws of exponents! Although this development is acceptable in elementary algebra, it is unsatisfactory in calculus, where the standards of mathematical rigor are higher.

Our approach in this chapter is to employ a definite integral to introduce the *natural logarithmic function*. We then use this function to define the *natural exponential function*. Finally, we give precise meanings to a^x and $\log_a x$. You may think this is an awkward procedure; however, it is the simplest way to treat these topics *rigorously*. Moreover, our approach enables us to establish results on continuity, derivatives, and integrals in a simple manner and to *prove* the laws of exponents that are assumed in precalculus mathematics.

Many applications of logarithmic and exponential functions are included throughout the chapter. Since these functions are inverses of each other, we shall begin by discussing the general concept of inverse functions.

7.1 INVERSE FUNCTIONS

A function f may have the same value for different numbers in its domain. For example, if $f(x) = x^2$, then $f(2) = 4 = f(-2)$ but $2 \neq -2$. In order to define the *inverse of a function*, it is essential that different numbers in the domain *always* give different values of f. Such functions are called *one-to-one functions*.

Definition **(7.1)**

> A function f with domain D and range R is a **one-to-one function** if whenever $a \neq b$ in D, then $f(a) \neq f(b)$ in R.

FIGURE 7.1

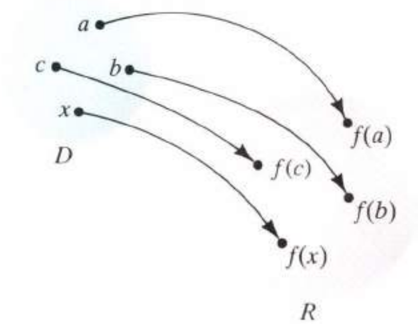
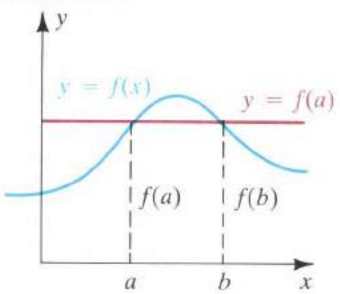


FIGURE 7.2

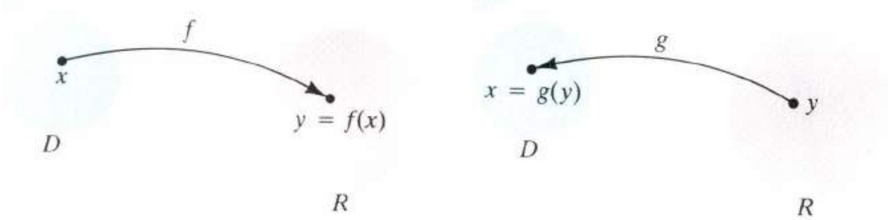


The diagram in Figure 7.1 illustrates a one-to-one function, because each function value in the range R corresponds to *exactly one* element in the domain D. The function whose graph is illustrated in Figure 7.2 is not one-to-one, because $a \neq b$ but $f(a) = f(b)$. Note that the horizontal line $y = f(a)$ (or $y = f(b)$) intersects the graph in more than one point. Thus, *if any horizontal line intersects the graph of a function f in more than one point, then f is not one-to-one*. Every increasing function is one-to-one, because if $a < b$, then $f(a) < f(b)$, and if $b < a$, then $f(b) < f(a)$. Thus, if $a \neq b$, then $f(a) \neq f(b)$. Similarly, every decreasing function is one-to-one.

If f is a one-to-one function with domain D and range R, then for each number y in R, there is *exactly one* number x in D such that $y = f(x)$, as illustrated by the arrow in Figure 7.3(i). Since x is *unique*, we may define a function g from R to D by means of the rule $x = g(y)$. As in Figure 7.3(ii), *g reverses the correspondence given by f*. We call g the *inverse function* of f, as in the following definition.

FIGURE 7.3

(i) (ii)



Definition **(7.2)**

> Let f be one-to-one function with domain D and range R. A function g with domain R and range D is the **inverse function** of f, provided the following condition is true for every x in D and every y in R:
>
> $$y = f(x) \quad \text{if and only if} \quad x = g(y)$$

The following theorem can be used to verify that a function g is the inverse of f.

Theorem (7.3)

Let f be a one-to-one function with domain D and range R. If g is a function with domain R and range D, then g is the inverse function of f if and only if both of the following conditions are true:

(i) $g(f(x)) = x$ for every x in D

(ii) $f(g(y)) = y$ for every y in R

PROOF Let us first prove that if g is the inverse function of f, then conditions (i) and (ii) are true. By the definition of an inverse function,

$$y = f(x) \quad \text{if and only if} \quad x = g(y)$$

for every x in D and every y in R. If we substitute $f(x)$ for y in the equation $x = g(y)$, we obtain condition (i): $x = g(f(x))$. Similarly, if we substitute $g(y)$ for x in the equation $y = f(x)$, we obtain condition (ii): $y = f(g(y))$. Thus, if g is the inverse function of f, then conditions (i) and (ii) are true.

Conversely, let g be a function with domain R and range D, and suppose that conditions (i) and (ii) are true. To show that g is the inverse function of f, we must prove that

$$y = f(x) \quad \text{if and only if} \quad x = g(y)$$

for every x in D and every y in R.

First suppose that $y = f(x)$. Since (i) is true, $g(f(x)) = x$; that is, $g(y) = x$. This shows that if $y = f(x)$, then $x = g(y)$.

Next suppose that $x = g(y)$. Since (ii) is true, $f(g(y)) = y$; that is, $f(x) = y$. This shows that if $x = g(y)$, then $y = f(x)$, which completes the proof. ■

A one-to-one function f can have only one inverse function. Conditions (i) and (ii) of Theorem (7.3) imply that if g is the inverse function of f, then f is the inverse function of g. We say that *f and g are inverse functions of each other*.

If a function f has an inverse function g, we often denote g by f^{-1}. The -1 used in this notation should not be mistaken for an exponent; that is, $f^{-1}(y)$ *does not mean* $1/[f(y)]$. The reciprocal $1/[f(y)]$ may be denoted by $[f(y)]^{-1}$. It is important to remember the following relationships.

Domains and ranges of f and f^{-1} (7.4)

domain of f^{-1} = range of f

range of f^{-1} = domain of f

When we discuss functions, we often let x denote an arbitrary number in the domain. Thus, for the inverse function f^{-1}, we may consider $f^{-1}(x)$, *where x is in the domain of f^{-1}*. In this case the two conditions in Theorem (7.3) are written as follows:

(i) $f^{-1}(f(x)) = x$ for every x in the domain of f

(ii) $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}

In some cases we can find the inverse of a one-to-one function by solving the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = g(y)$. If the two conditions $g(f(x)) = x$ and $f(g(x)) = x$ are true for every x in the domains of f and g, respectively, then g is the required inverse function f^{-1}. The following guidelines summarize this procedure. In guideline 2, in anticipation of finding f^{-1}, we shall write $x = f^{-1}(y)$ instead of $x = g(y)$.

Guidelines for finding f^{-1} in simple cases (7.5)

1. Verify that f is a one-to-one function (or that f is increasing or is decreasing) throughout its domain.
2. Solve the equation $y = f(x)$ for x in terms of y, obtaining an equation of the form $x = f^{-1}(y)$.
3. Verify the two conditions

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

for every x in the domains of f and f^{-1}, respectively.

The success of this method depends on the nature of the equation $y = f(x)$, since we must be able to solve for x in terms of y. For this reason, we include *simple cases* in the title of the guidelines.

EXAMPLE 1 Let $f(x) = 3x - 5$. Find the inverse function of f.

SOLUTION We shall follow the three guidelines. First, we note that the graph of the linear function f is a line of slope 3. Since f is increasing throughout $\mathbb{R}$, f is one-to-one, and hence the inverse function f^{-1} exists. Moreover, since the domain and range of f are $\mathbb{R}$, the same is true for f^{-1}.

As in guideline 2, we consider the equation

$$y = 3x - 5$$

and then solve for x in terms of y, obtaining

$$x = \frac{y+5}{3}.$$

We now let

$$f^{-1}(y) = \frac{y+5}{3}.$$

Since the symbol used for the variable is immaterial, we may also write

$$f^{-1}(x) = \frac{x+5}{3}.$$

We next verify the conditions (i) $f^{-1}(f(x)) = x$ and (ii) $f(f^{-1}(x)) = x$:

(i)
$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(3x-5) && \text{(definition of } f\text{)} \\ &= \frac{(3x-5)+5}{3} && \text{(definition of } f^{-1}\text{)} \\ &= x && \text{(simplifying)} \end{aligned}$$

(ii) $f(f^{-1}(x)) = f\left(\frac{x+5}{3}\right)$ (definition of f^{-1})

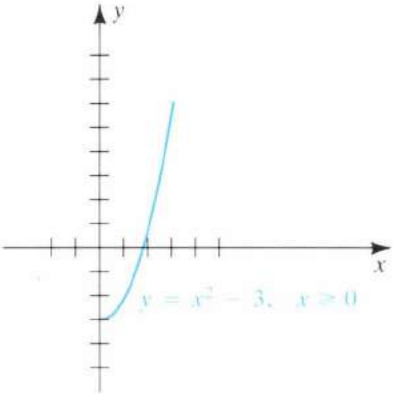
$= 3\left(\frac{x+5}{3}\right) - 5$ (definition of f)

$= x$ (simplifying)

Thus, by Theorem (7.3), the inverse function of f is given by $f^{-1}(x) = (x+5)/3$.

EXAMPLE 2 Let $f(x) = x^2 - 3$ for $x \geq 0$. Find the inverse function of f.

FIGURE 7.4



SOLUTION The graph of f is sketched in Figure 7.4. The domain of f is $[0, \infty)$, and the range is $[-3, \infty)$. Since f is increasing, it is one-to-one and hence has an inverse function f^{-1} that has domain $[-3, \infty)$ and range $[0, \infty)$.

As in guideline 2, we consider the equation

$$y = x^2 - 3$$

and solve for x, obtaining

$$x = \pm\sqrt{y+3}.$$

Since x is nonnegative, we reject $x = -\sqrt{y+3}$ and let

$$f^{-1}(y) = \sqrt{y+3}, \quad \text{or, equivalently,} \quad f^{-1}(x) = \sqrt{x+3}.$$

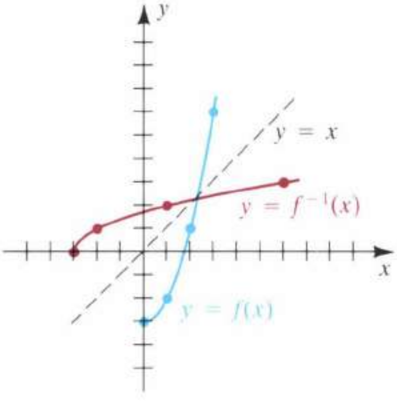
Finally, we verify that (i) $f^{-1}(f(x)) = x$ for x in $[0, \infty)$ and (ii) $f(f^{-1}(x)) = x$ for x in $[-3, \infty)$:

(i) $f^{-1}(f(x)) = f^{-1}(x^2 - 3) = \sqrt{(x^2-3)+3} = \sqrt{x^2} = x$ if $x \geq 0$

(ii) $f(f^{-1}(x)) = f(\sqrt{x+3}) = (\sqrt{x+3})^2 - 3 = (x+3) - 3 = x$ if $x \geq -3$

Thus, the inverse function is given by $f^{-1}(x) = \sqrt{x+3}$ for $x \geq -3$.

FIGURE 7.5



There is an interesting relationship between the graph of a function f and the graph of its inverse function f^{-1}. We first note that $b = f(a)$ is equivalent to $a = f^{-1}(b)$. These equations imply that *the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}*.

As an illustration, in Example 2 we found that the functions f and f^{-1} given by

$$f(x) = x^2 - 3 \quad \text{and} \quad f^{-1}(x) = \sqrt{x+3}$$

are inverse functions of each other, provided that x is suitably restricted. Some points on the graph of f are $(0, -3)$, $(1, -2)$, $(2, 1)$, and $(3, 6)$. Corresponding points on the graph of f^{-1} are $(-3, 0)$, $(-2, 1)$, $(1, 2)$, and $(6, 3)$. The graphs of f and f^{-1} are sketched on the same coordinate plane in Figure 7.5. If the page is folded along the line $y = x$ that bisects

quadrants I and III (as indicated by the dashes in the figure), then the graphs of f and f^{-1} coincide. The two graphs are *reflections* of each other through the line $y = x$. This is typical of the graph of every function f that has an inverse function f^{-1} (see Exercise 14).

FIGURE 7.6

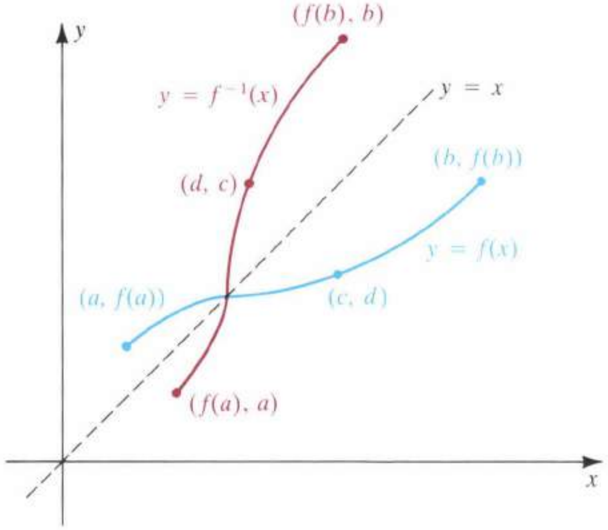


Figure 7.6 illustrates the graphs of an arbitrary one-to-one function f and its inverse function f^{-1}. As indicated in the figure, (c, d) is on the graph of f if and only if (d, c) is on the graph of f^{-1}. Thus, if we restrict the domain of f to the interval $[a, b]$, then the domain of f^{-1} is restricted to $[f(a), f(b)]$. If f is continuous, then the graph of f has no breaks or holes, and hence the same is true for the (reflected) graph of f^{-1}. Thus, we see intuitively that if f is continuous on $[a, b]$, then f^{-1} is continuous on $[f(a), f(b)]$. We can also show that if f is increasing, then so is f^{-1}. These facts are stated in the next theorem. A proof is given in Appendix II.

Theorem **(7.6)**

If f is continuous and increasing on $[a, b]$, then f has an inverse function f^{-1} that is continuous and increasing on $[f(a), f(b)]$.

We can also prove the analogous result obtained by replacing the word *increasing* in Theorem (7.6) by *decreasing*.

The next theorem provides a method for finding the derivative of an inverse function.

Theorem **(7.7)**

If a differentiable function f has an inverse function $g = f^{-1}$ and if $f'(g(c)) \neq 0$, then g is differentiable at c and

$$g'(c) = \frac{1}{f'(g(c))}.$$

PROOF By Definition (3.6),

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}.$$

Let $y = g(x)$ and $a = g(c)$. Since f and g are inverse functions of each other,

$$g(x) = y \quad \text{if and only if} \quad f(y) = x$$

and

$$g(c) = a \quad \text{if and only if} \quad f(a) = c.$$

Since f is differentiable, it is continuous and hence, by Theorem (7.6), so is the inverse function $g = f^{-1}$. Thus, if $x \to c$, then $g(x) \to g(c)$; that is, $y \to a$. If $y \to a$, then $f(y) \to f(a)$. Thus, we may write

$$\begin{aligned} g'(c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= \lim_{y \to a} \frac{y - a}{f(y) - f(a)} \\ &= \lim_{y \to a} \frac{1}{\dfrac{f(y) - f(a)}{y - a}} \\ &= \frac{1}{\displaystyle\lim_{y \to a} \frac{f(y) - f(a)}{y - a}} \\ &= \frac{1}{f'(a)} = \frac{1}{f'(g(c))}. \end{aligned}$$

It is convenient to restate Theorem (7.7) as follows.

Corollary (7.8)

If g is the inverse function of a differentiable function f and if $f'(g(x)) \neq 0$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

EXAMPLE 3 If $f(x) = x^3 + 2x - 1$, prove that f has an inverse function g, and find the slope of the tangent line to the graph of g at the point $P(2, 1)$.

SOLUTION Since $f'(x) = 3x^2 + 2 > 0$ for every x, f is increasing and hence is one-to-one. Thus, f has an inverse function g. Since $f(1) = 2$, it follows that $g(2) = 1$, and consequently the point $P(2, 1)$ is on the graph of g. It would be difficult to find g using Guidelines (7.5), because we would have to solve the equation $y = x^3 + 2x - 1$ for x in terms of y. However, even if we cannot find g explicitly, we *can* find the slope $g'(2)$

of the tangent line to the graph of g at $P(2, 1)$. Thus, by Theorem (7.7),

$$g'(2) = \frac{1}{f'(g(2))} = \frac{1}{f(1)} = \frac{1}{5}.$$

An easy way to remember Corollary (7.8) is to let $y = f(x)$. If g is the inverse function of f, then $g(y) = g(f(x)) = x$. From (7.8),

$$g'(y) = \frac{1}{f'(g(y))} = \frac{1}{f'(x)},$$

or, in differential notation,

$$\frac{dx}{dy} = \frac{1}{\left(\dfrac{dy}{dx}\right)}.$$

This shows that, in a sense, the derivative of the inverse function g is the reciprocal of the derivative of f. A disadvantage of using the last two formulas is that neither is stated in terms of the independent variable for the inverse function. To illustrate, in Example 3 let $y = x^3 + 2x - 1$ and $x = g(y)$. Then

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{3x^2 + 2};$$

that is,

$$g'(y) = \frac{1}{3x^2 + 2} = \frac{1}{3(g(y))^2 + 2}.$$

This may also be written in the form

$$g'(x) = \frac{1}{3(g(x))^2 + 2}.$$

Consequently, to find $g'(x)$ it is necessary to know $g(x)$, just as in Corollary (7.8).

EXERCISES 7.1

Exer. 1–12: Find $f^{-1}(x)$.

1 $f(x) = 3x + 5$

2 $f(x) = 7 - 2x$

3 $f(x) = \dfrac{1}{3x - 2}$

4 $f(x) = \dfrac{1}{x + 3}$

5 $f(x) = \dfrac{3x + 2}{2x - 5}$

6 $f(x) = \dfrac{4x}{x - 2}$

7 $f(x) = 2 - 3x^2, \quad x \le 0$

8 $f(x) = 5x^2 + 2, \quad x \ge 0$

9 $f(x) = \sqrt{3 - x}$

10 $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

11 $f(x) = \sqrt[3]{x} + 1$

12 $f(x) = (x^3 + 1)^5$

13 (a) Prove that the linear function defined by $f(x) = ax + b$ with $a \ne 0$ has an inverse function, and find $f^{-1}(x)$.

(b) Does a constant function have an inverse? Explain.

14 Show that the graph of f^{-1} is the reflection of the graph of f through the line $y = x$ by verifying the following conditions:

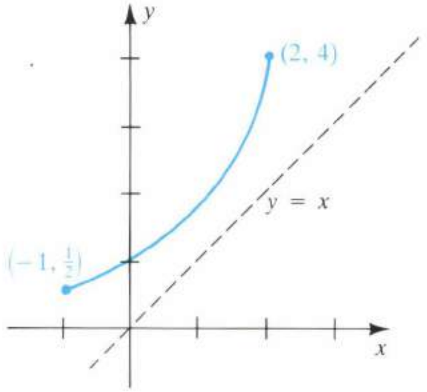
(i) If $P(a, b)$ is on the graph of f, then $Q(b, a)$ is on the graph of f^{-1}.

(ii) The midpoint of line segment PQ is on the line $y = x$.

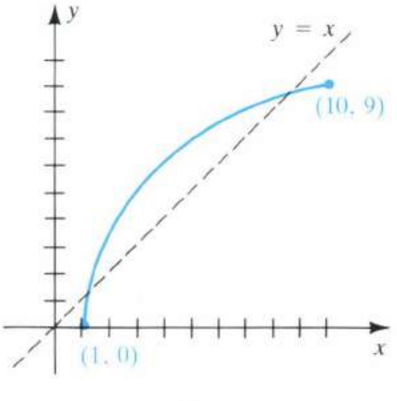
(iii) The line PQ is perpendicular to the line $y = x$.

Exer. 15–18: The graph of a one-to-one function f is shown in the figure. (a) Use a reflection to sketch the graph of f^{-1}. (b) Find the domain and range of f. (c) Find the domain and range of f^{-1}.

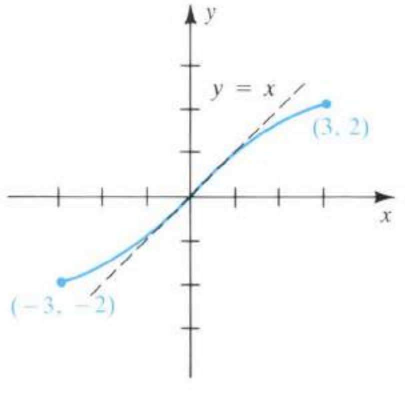
15



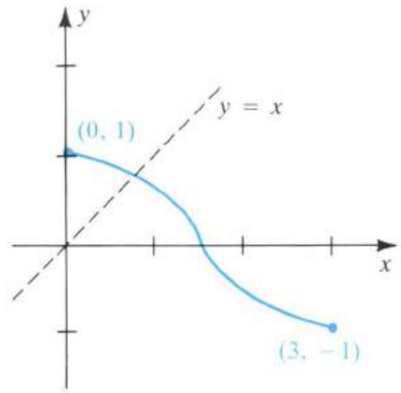
16



17



18



c **Exer. 19–20: Graph f on the given interval. (a) Estimate the largest interval $[a, b]$ with $a < 0 < b$ on which f is one-to-one. (b) If g is the function with domain $[a, b]$ such that $g(x) = f(x)$ for $a \leq x \leq b$, estimate the domain and range of g^{-1}.**

19 $f(x) = 2.1x^3 - 2.98x^2 - 2.11x + 3$; $[-1, 2]$

20 $f(x) = \sin(\sin(1.1x))$; $[-2, 2]$

Exer. 21–26: (a) Prove that f has an inverse function. (b) State the domain of f^{-1}. (c) Use Corollary (7.8) to find $D_x f^{-1}(x)$.

21 $f(x) = \sqrt{2x + 3}$

22 $f(x) = \sqrt[3]{5x + 2}$

23 $f(x) = 4 - x^2$, $x \geq 0$

24 $f(x) = x^2 - 4x + 5$, $x \geq 2$

25 $f(x) = \dfrac{1}{x}$, $x \neq 0$

26 $f(x) = \sqrt{9 - x^2}$, $0 \leq x \leq 3$

Exer. 27–32: (a) Use f' to prove that f has an inverse function. (b) Find the slope of the tangent line at the point P on the graph of f^{-1}.

27 $f(x) = x^5 + 3x^3 + 2x - 1$; $P(5, 1)$

28 $f(x) = 2 - x - x^3$; $P(-8, 2)$

29 $f(x) = -2x + (8/x^3)$, $x > 0$; $P(-3, 2)$

30 $f(x) = 4x^5 - (1/x^3)$, $x > 0$; $P(3, 1)$

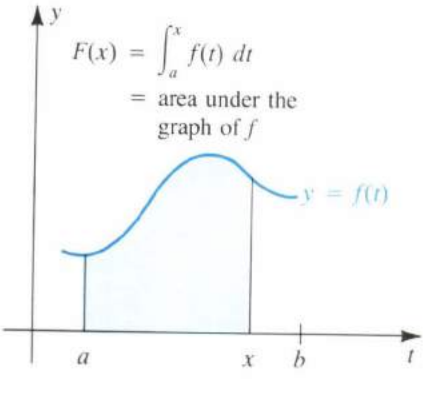
31 $f(x) = x^3 + 4x - 1$; $P(15, 2)$

32 $f(x) = x^5 + x$; $P(2, 1)$

7.2 THE NATURAL LOGARITHMIC FUNCTION

In the chapter introduction we explained why it is necessary to take a different approach to logarithmic and exponential functions than that used in precalculus mathematics. At first you may think it strange to define a logarithmic function as a definite integral; however, later you will see

FIGURE 7.7



that the function we obtain obeys the familiar laws of logarithms considered in precalculus courses.

Let f be a function that is continuous on a closed interval $[a, b]$. As in the proof of Part I of the fundamental theorem of calculus (5.30), we can define a function F by

$$F(x) = \int_a^x f(t)\,dt$$

for x in $[a, b]$. If $f(t) \geq 0$ throughout $[a, b]$, then $F(x)$ is the area under the graph of f from a to x, as illustrated in Figure 7.7. For the special case $f(t) = t^n$, where n is a rational number and $n \neq -1$, we can find an explicit form for F. Thus, by the power rule for integrals,

$$F(x) = \int_a^x t^n\,dt = \left[\frac{t^{n+1}}{n+1}\right]_a^x$$
$$= \frac{1}{n+1}(x^{n+1} - a^{n+1}) \quad \text{if} \quad n \neq -1.$$

As indicated, we cannot use $t^{-1} = 1/t$ for the integrand, since $1/(n + 1)$ is undefined if $n = -1$. Up to this point in our work we have been unable to determine an antiderivative of $1/x$—that is, a function F such that $F'(x) = 1/x$. The next definition will remedy this situation.

Definition **(7.9)**

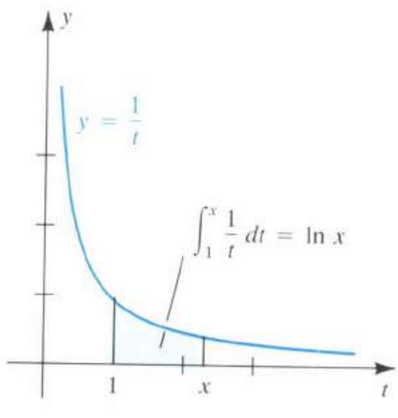
> The **natural logarithmic function**, denoted by **ln**, is defined by
>
> $$\ln x = \int_1^x \frac{1}{t}\,dt$$
>
> for every $x > 0$.

The expression $\ln x$ (read *ell-en of* x) is called the **natural logarithm of** $\boldsymbol{x}$. We use this terminology because, as we shall see, ln has the same properties as the logarithmic functions studied in precalculus courses. The restriction $x > 0$ is necessary because if $x \leq 0$, the integrand $1/t$ has an infinite discontinuity between x and 1 and hence $\int_1^x (1/t)\,dt$ does not exist.

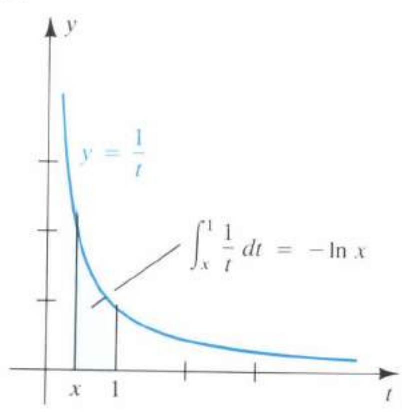
If $x > 1$, the definite integral $\int_1^x (1/t)\,dt$ may be interpreted as the area of the region the graph of $y = 1/t$ from $t = 1$ to $t = x$ (see Figure 7.8(i)).

FIGURE 7.8

(i)



(ii)



If $0 < x < 1$, then, since

$$\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt,$$

the integral is the *negative* of the area of the region under the graph of $y = 1/t$ from $t = x$ to $t = 1$ (see Figure 7.8(ii)). This shows that *ln x is negative for* $0 < x < 1$ *and positive for* $x > 1$. Also note that, by Definition (5.18),

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$$

Applying Theorem (5.35) yields

$$D_x \int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

for every $x > 0$. Substituting $\ln x$ for $\int_1^x (1/t)\,dt$ gives us the following theorem.

Theorem **(7.10)**

$$\boxed{D_x \ln x = \frac{1}{x}}$$

By Theorem (7.10), $\ln x$ *is an antiderivative of* $1/x$. Since $\ln x$ is differentiable and its derivative $1/x$ is positive for every $x > 0$, it follows from Theorems (3.11) and (4.13) that *the natural logarithmic function is continuous and increasing throughout its domain*. Also note that

$$D_x^2 \ln x = D_x (D_x \ln x) = D_x\left(\frac{1}{x}\right) = -\frac{1}{x^2},$$

which is negative for every $x > 0$. Hence, by (4.16), the graph of the natural logarithmic function is concave downward on $(0, \infty)$.

Let us sketch the graph of $y = \ln x$. If $0 < x < 1$, then $\ln x < 0$ and the graph is below the x-axis. If $x > 1$, the graph is above the x-axis. Since $\ln 1 = 0$, the x-intercept is 1. We may approximate y-coordinates of points on the graph by applying the trapezoidal rule or Simpson's rule. If $x = 2$, then, by Example 2 in Section 5.7,

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx 0.693.$$

We will show in Theorem (7.12) that if $a > 0$, then $\ln a^r = r \ln a$ for every rational number r. Using this result yields the following:

$$\ln 4 = \ln 2^2 = 2 \ln 2 \approx 2(0.693) \approx 1.386$$

$$\ln 8 = \ln 2^3 = 3 \ln 2 \approx 2.079$$

$$\ln \tfrac{1}{2} = \ln 2^{-1} = -\ln 2 \approx -0.693$$

$$\ln \tfrac{1}{4} = \ln 2^{-2} = -2 \ln 2 \approx -1.386$$

$$\ln \tfrac{1}{8} = \ln 2^{-3} = -3 \ln 2 \approx -2.079$$

FIGURE 7.9

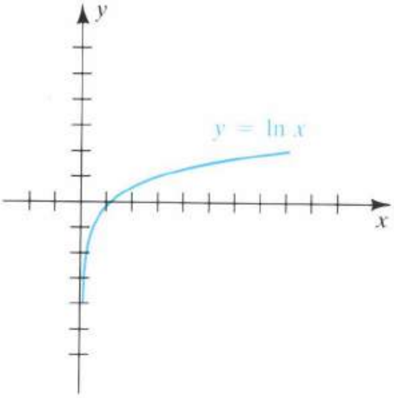


Table C in Appendix III provides a list of natural logarithms of many other numbers, correct to three decimal places. Calculators may also be used to estimate values of ln.

Plotting the points that correspond to the y-coordinates we have calculated and using the fact that ln is continuous and increasing gives us the sketch in Figure 7.9.

At the end of this section we prove that

$$\lim_{x \to \infty} \ln x = \infty \quad \text{and} \quad \lim_{x \to 0^+} \ln x = -\infty.$$

The first of these results tells us that $y = \ln x$ increases without bound as $x \to \infty$. Note, however, that the *rate of change* of y with respect to x is very small if x is large. For example, if $x = 10^6$, then

$$\left.\frac{dy}{dx}\right]_{10^6} = \left.\frac{1}{x}\right]_{10^6} = \frac{1}{10^6} = 0.000001.$$

Thus, the tangent line is *almost* horizontal at the point on the graph with x-coordinate 10^6, and hence the graph is very flat near that point. The fact that $\lim_{x \to 0^+} \ln x = -\infty$ tells us that the line $x = 0$ (the y-axis) is a vertical asymptote for the graph (see Figure 7.9).

The next result generalizes Theorem (7.10).

Theorem **(7.11)**

If $u = g(x)$ and g is differentiable, then

(i) $D_x \ln u = \dfrac{1}{u} D_x u \quad \text{if} \quad g(x) > 0$

(ii) $D_x \ln |u| = \dfrac{1}{u} D_x u \quad \text{if} \quad g(x) \neq 0$

PROOF

(i) If we let $y = \ln u$ and $u = g(x)$, then, by the chain rule and Theorem (7.10),

$$D_x \ln u = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u} D_x u.$$

(ii) If $u > 0$, then $|u| = u$ and, by part (i),

$$D_x \ln |u| = D_x \ln u = \frac{1}{u} D_x u.$$

If $u < 0$, then $|u| = -u > 0$ and, by part (i),

$$\begin{aligned} D_x \ln |u| &= D_x \ln (-u) = \frac{1}{-u} D_x(-u) \\ &= -\frac{1}{u}(-1) D_x u = \frac{1}{u} D_x u. \end{aligned}$$

In examples and exercises, if a function is defined in terms of the natural logarithmic function, its domain will not usually be stated explicitly. Instead *we shall tacitly assume that x is restricted to values for which*

the logarithmic expression has meaning. Thus, in Example 1, we assume $x^2 - 6 > 0$; that is, $|x| > \sqrt{6}$. In Example 2, we assume $x + 1 > 0$.

EXAMPLE 1 If $f(x) = \ln(x^2 - 6)$, find $f'(x)$.

SOLUTION Letting $u = x^2 - 6$ in Theorem (7.11)(i) yields

$$f'(x) = D_x \ln(x^2 - 6) = \frac{1}{x^2 - 6} D_x(x^2 - 6) = \frac{2x}{x^2 - 6}.$$

EXAMPLE 2 If $y = \ln\sqrt{x + 1}$, find $\dfrac{dy}{dx}$.

SOLUTION Letting $u = \sqrt{x + 1}$ in Theorem (7.11)(i) gives us

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\ln\sqrt{x+1} \\ &= \frac{1}{\sqrt{x+1}}\frac{d}{dx}\sqrt{x+1} = \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2}(x+1)^{-1/2} \\ &= \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2}\frac{1}{\sqrt{x+1}} = \frac{1}{2(x+1)}.\end{aligned}$$

EXAMPLE 3 If $f(x) = \ln|4 + 5x - 2x^3|$, find $f'(x)$.

SOLUTION Using Theorem (7.11)(ii) with $u = 4 + 5x - 2x^3$ yields

$$\begin{aligned}f'(x) &= D_x \ln|4 + 5x - 2x^3| \\ &= \frac{1}{4 + 5x - 2x^3} D_x(4 + 5x - 2x^3) = \frac{5 - 6x^2}{4 + 5x - 2x^3}.\end{aligned}$$

The next result states that natural logarithms satisfy the laws of logarithms studied in precalculus mathematics courses.

Laws of natural logarithms **(7.12)**

If $p > 0$ and $q > 0$, then

(i) $\ln pq = \ln p + \ln q$

(ii) $\ln \dfrac{p}{q} = \ln p - \ln q$

(iii) $\ln p^r = r \ln p$ for every rational number r

PROOF

(i) If $p > 0$, then using Theorem (7.11) with $u = px$ gives us

$$D_x \ln px = \frac{1}{px} D_x(px) = \frac{1}{px} p = \frac{1}{x}.$$

Thus, $\ln px$ and $\ln x$ are both antiderivatives of $1/x$, and hence, by Theorem (5.2),

$$\ln px = \ln x + C$$

for some constant C. Letting $x = 1$, we obtain

$$\ln p = \ln 1 + C.$$

Since $\ln 1 = 0$, we see that $C = \ln p$, and therefore

$$\ln px = \ln x + \ln p.$$

Substituting q for x in the last equation gives us

$$\ln pq = \ln q + \ln p,$$

which is what we wished to prove.

(ii) Using the formula $\ln p + \ln q = \ln pq$ with $p = 1/q$, we see that

$$\ln \frac{1}{q} + \ln q = \ln \left(\frac{1}{q} \cdot q\right) = \ln 1 = 0$$

and hence $$\ln \frac{1}{q} = -\ln q.$$

Consequently,

$$\ln \frac{p}{q} = \ln \left(p \cdot \frac{1}{q}\right) = \ln p + \ln \frac{1}{q} = \ln p - \ln q.$$

(iii) If r is a rational number and $x > 0$, then, by Theorem (7.11) with $u = x^r$,

$$D_x(\ln x^r) = \frac{1}{x^r} D_x(x^r) = \frac{1}{x^r} rx^{r-1} = r\left(\frac{1}{x}\right) = \frac{r}{x}.$$

By Theorems (3.18)(iv) and (7.7), we may also write

$$D_x(r \ln x) = r\, D_x(\ln x) = r\left(\frac{1}{x}\right) = \frac{r}{x}.$$

Since $\ln x^r$ and $r \ln x$ are both antiderivatives of r/x, it follows from Theorem (5.2) that

$$\ln x^r = r \ln x + C$$

for some constant C. If we let $x = 1$ in the last formula, we obtain

$$\ln 1 = r \ln 1 + C.$$

Since $\ln 1 = 0$, this implies that $C = 0$ and, therefore,

$$\ln x^r = r \ln x.$$

In Section 7.5 we shall extend this law to irrational exponents. ■

As shown in the following illustration, sometimes it is convenient to use laws of natural logarithms *before* differentiating.

ILLUSTRATION

$f(x)$	$f(x)$ AFTER USING LAWS OF LOGARITHMS	$f'(x)$
$\ln [(x+2)(3x-5)]$	$\ln (x+2) + \ln (3x-5)$	$\dfrac{1}{x+2} + \dfrac{1}{3x-5} \cdot 3 = \dfrac{6x+1}{(x+2)(3x-5)}$
$\ln \dfrac{x+2}{3x-5}$	$\ln (x+2) - \ln (3x-5)$	$\dfrac{1}{x+2} - \dfrac{1}{3x-5} \cdot 3 = \dfrac{-11}{(x+2)(3x-5)}$
$\ln (x^2+1)^5$	$5 \ln (x^2+1)$	$5 \cdot \dfrac{1}{x^2+1} \cdot 2x = \dfrac{10x}{x^2+1}$
$\ln \sqrt{x+1}$	$\dfrac{1}{2} \ln (x+1)$	$\dfrac{1}{2} \cdot \dfrac{1}{x+1} = \dfrac{1}{2(x+1)}$

An advantage of using laws of logarithms before differentiating may be seen by comparing the method of finding $D_x \ln \sqrt{x+1}$ in the preceding illustration with the solution of Example 2.

In the next two examples we apply laws of logarithms to complicated expressions before differentiating.

EXAMPLE 4 If $f(x) = \ln [\sqrt{6x-1}(4x+5)^3]$, find $f'(x)$.

SOLUTION We first write $\sqrt{6x-1} = (6x-1)^{1/2}$ and then use laws of logarithms (i) and (iii):

$$\begin{aligned} f(x) &= \ln [(6x-1)^{1/2}(4x+5)^3] \\ &= \ln (6x-1)^{1/2} + \ln (4x+5)^3 \\ &= \tfrac{1}{2} \ln (6x-1) + 3 \ln (4x+5) \end{aligned}$$

By Theorem (7.11),

$$\begin{aligned} f'(x) &= \left(\frac{1}{2} \cdot \frac{1}{6x-1} \cdot 6\right) + \left(3 \cdot \frac{1}{4x+5} \cdot 4\right) \\ &= \frac{3}{6x-1} + \frac{12}{4x+5} \\ &= \frac{84x+3}{(6x-1)(4x+5)}. \end{aligned}$$

EXAMPLE 5 If $y = \ln \sqrt[3]{\dfrac{x^2-1}{x^2+1}}$, find $\dfrac{dy}{dx}$.

SOLUTION We first use laws of logarithms to change the form of y as follows:

$$\begin{aligned} y = \ln \left(\frac{x^2-1}{x^2+1}\right)^{1/3} &= \frac{1}{3} \ln \left(\frac{x^2-1}{x^2+1}\right) \\ &= \tfrac{1}{3}[\ln (x^2-1) - \ln (x^2+1)] \end{aligned}$$

Next we use Theorem (7.11) to obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3}\left(\frac{1}{x^2-1}\cdot 2x - \frac{1}{x^2+1}\cdot 2x\right)\\ &= \frac{2x}{3}\left(\frac{1}{x^2-1} - \frac{1}{x^2+1}\right)\\ &= \frac{2x}{3}\left[\frac{2}{(x^2-1)(x^2+1)}\right] = \frac{4x}{3(x^2-1)(x^2+1)}.\end{aligned}$$

Given $y = f(x)$, we may sometimes find $D_x\, y$ by **logarithmic differentiation**. This method is especially useful if $f(x)$ involves complicated products, quotients, or powers. In the following guidelines it is assumed that $f(x) > 0$; however, we shall show that the same steps can be used if $f(x) < 0$.

Guidelines for logarithmic differentiation **(7.13)**

1	$y = f(x)$	(given)
2	$\ln y = \ln f(x)$	(take natural logarithms and simplify)
3	$D_x\,[\ln y] = D_x\,[\ln f(x)]$	(differentiate implicitly)
4	$\frac{1}{y}D_x\, y = D_x\,[\ln f(x)]$	(by Theorem (7.11))
5	$D_x\, y = f(x)\, D_x\,[\ln f(x)]$	(multiply by $y = f(x)$)

Of course, to complete the solution we must differentiate $\ln f(x)$ at some stage after guideline 3. If $f(x) < 0$ for some x, then guideline 2 is invalid, since $\ln f(x)$ is undefined. In this event we can replace guideline 1 by $|y| = |f(x)|$ and take natural logarithms, obtaining $\ln|y| = \ln|f(x)|$. If we now differentiate implicitly and use Theorem (7.11)(ii), we again arrive at guideline 4. Thus, negative values of $f(x)$ do not change the outcome, and we are not concerned whether $f(x)$ is positive or negative. The method should not be used to find $f'(a)$ if $f(a) = 0$, since $\ln 0$ is undefined.

EXAMPLE 6 If $y = \dfrac{(5x-4)^3}{\sqrt{2x+1}}$, use logarithmic differentiation to find $D_x\, y$.

SOLUTION As in guideline 2, we begin by taking the natural logarithm of each side, obtaining

$$\begin{aligned}\ln y &= \ln (5x-4)^3 - \ln\sqrt{2x+1}\\ &= 3\ln(5x-4) - \tfrac{1}{2}\ln(2x+1).\end{aligned}$$

Applying guidelines 3 and 4, we differentiate implicitly with respect to x and then use Theorem (7.8) to obtain

$$\begin{aligned}\frac{1}{y}D_x\, y &= \left(3\cdot\frac{1}{5x-4}\cdot 5\right) - \left(\frac{1}{2}\cdot\frac{1}{2x+1}\cdot 2\right)\\ &= \frac{25x+19}{(5x-4)(2x+1)}.\end{aligned}$$

Finally, as in guideline 5, we multiply both sides of the last equation by y (that is, by $(5x-4)^3/\sqrt{2x+1}$) to get

$$D_x y = \frac{25x+19}{(5x-4)(2x+1)} \cdot \frac{(5x-4)^3}{\sqrt{2x+1}}$$
$$= \frac{(25x+19)(5x-4)^2}{(2x+1)^{3/2}}.$$

We could check this result by applying the quotient rule to y.

An application of natural logarithms to growth processes is given in the next example. Many additional applied problems involving ln appear in other examples and exercises of this chapter.

EXAMPLE 7 The *Count model* is an empirically based formula that can be used to predict the height of a preschooler. If $h(x)$ denotes the height (in centimeters) at age x (in years) for $\frac{1}{4} \le x \le 6$, then $h(x)$ can be approximated by

$$h(x) = 70.228 + 5.104x + 9.222 \ln x.$$

(a) Predict the height and rate of growth when a child reaches age 2.

(b) When is the rate of growth largest?

SOLUTION

(a) The height at age 2 is approximately

$$h(2) = 70.228 + 5.104(2) + 9.222 \ln 2 \approx 86.8 \text{ cm}.$$

The rate of change of h with respect to x is

$$h'(x) = 5.104 + 9.222\left(\frac{1}{x}\right).$$

Letting $x = 2$ gives us

$$h'(2) = 5.104 + 9.222(\tfrac{1}{2}) = 9.715.$$

Hence the rate of growth on the child's second birthday is about 9.7 cm/yr.

(b) To determine the maximum value of the rate of growth $h'(x)$, we first find the critical numbers of h'. Differentiating $h'(x)$, we obtain

$$h''(x) = 9.222\left(-\frac{1}{x^2}\right) = -\frac{9.222}{x^2}.$$

Since $h''(x)$ is negative for every x in $[\frac{1}{4}, 6]$, h' has no critical numbers in $(\frac{1}{4}, 6)$. It follows from Theorem (4.13) that h' is decreasing on $[\frac{1}{4}, 6]$. Consequently, the maximum value of $h'(x)$ occurs at $x = \frac{1}{4}$; that is, the rate of growth is largest at the age of 3 months.

We shall conclude this section by investigating $\ln x$ as $x \to \infty$ and as $x \to 0^+$. If $x > 1$, we may interpret the integral $\int_1^x (1/t)\,dt = \ln x$ as the

FIGURE 7.10

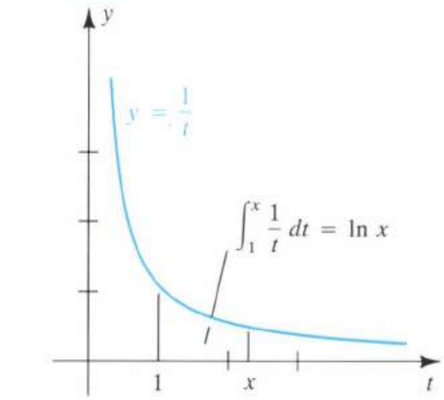
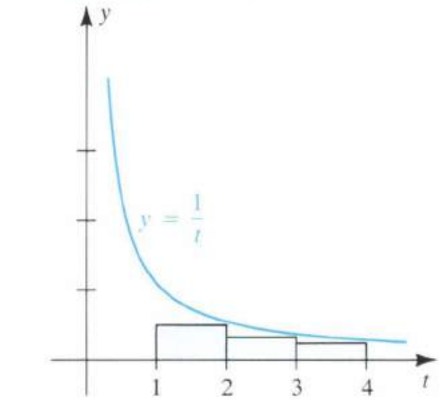


FIGURE 7.11



area of the region shown in Figure 7.10. The sum of the areas of the three rectangles shown in Figure 7.11 is

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{13}{12}.$$

Since the area under the graph of $y = 1/t$ from $t = 1$ to $t = 4$ is $\ln 4$, we see that

$$\ln 4 > \tfrac{13}{12} > 1.$$

It follows that if M is any positive rational number, then

$$M \ln 4 > M, \quad \text{or} \quad \ln 4^M > M.$$

If $x > 4^M$, then since ln is an increasing function,

$$\ln x > \ln 4^M > M.$$

This proves that $\ln x$ can be made as large as desired by choosing x sufficiently large; that is,

$$\lim_{x \to \infty} \ln x = \infty.$$

To investigate the case $x \to 0^+$, we first note that

$$\ln \frac{1}{x} = \ln 1 - \ln x = 0 - \ln x = -\ln x.$$

Hence
$$\lim_{x \to 0^+} \ln x = \lim_{x \to 0^+} \left(-\ln \frac{1}{x} \right).$$

As x approaches zero through positive values, $1/x$ increases without bound and, therefore, so does $\ln (1/x)$. Consequently, $-\ln (1/x)$ *decreases* without bound; that is,

$$\lim_{x \to 0^+} \ln x = -\infty.$$

EXERCISES 7.2

Exer. 1–34: Find $f'(x)$ if $f(x)$ is the given expression.

1 $\ln (9x + 4)$

2 $\ln (x^4 + 1)$

3 $\ln (3x^2 - 2x + 1)$

4 $\ln (4x^3 - x^2 + 2)$

5 $\ln |3 - 2x|$

6 $\ln |4 - 3x|$

7 $\ln |2 - 3x|^5$

8 $\ln |5x^2 - 1|^3$

9 $\ln \sqrt{7 - 2x^3}$

10 $\ln \sqrt[3]{6x + 7}$

11 $x \ln x$

12 $\ln (\ln x)$

13 $\ln \sqrt{x} + \sqrt{\ln x}$

14 $\ln x^3 + (\ln x)^3$

15 $\dfrac{1}{\ln x} + \ln \dfrac{1}{x}$

16 $\dfrac{x^2}{\ln x}$

17 $\ln [(5x - 7)^4(2x + 3)^3]$

18 $\ln [\sqrt[3]{4x - 5}\,(3x + 8)^2]$

19 $\ln \dfrac{\sqrt{x^2 + 1}}{(9x - 4)^2}$

20 $\ln \dfrac{x^2(2x - 1)^3}{(x + 5)^2}$

21 $\ln \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

22 $\ln \sqrt{\dfrac{4 + x^2}{4 - x^2}}$

23 $\ln (x + \sqrt{x^2 - 1})$

24 $\ln (x + \sqrt{x^2 + 1})$

25 $\ln \cos 2x$

26 $\cos (\ln 2x)$

27 $\ln \tan^3 3x$

28 $\ln \cot (x^2)$

29 $\ln \ln \sec 2x$

30 $\ln \csc^2 4x$

31 $\ln |\sec x|$

32 $\ln |\sin x|$

33 $\ln |\sec x + \tan x|$

34 $\ln |\csc x - \cot x|$

Exer. 35–38: Use implicit differentiation to find y'.

35 $3y - x^2 + \ln xy = 2$

36 $y^2 + \ln (x/y) - 4x = -3$

37 $x \ln y - y \ln x = 1$

38 $y^3 + x^2 \ln y = 5x + 3$

Exer. 39–44: Use logarithmic differentiation to find dy/dx.

39 $y = (5x + 2)^3(6x + 1)^2$

40 $y = (x+1)^2(x+2)^3(x+3)^4$

41 $y = \sqrt{4x+7}(x-5)^3$

42 $y = \sqrt{(3x^2+2)\sqrt{6x-7}}$

43 $y = \dfrac{(x^2+3)^5}{\sqrt{x+1}}$

44 $y = \dfrac{(x^2+3)^{2/3}(3x-4)^4}{\sqrt{x}}$

45 Find an equation of the tangent line to the graph of $y = x^2 + \ln(2x-5)$ at the point $P(3, 9)$.

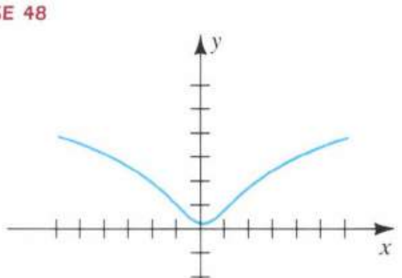
46 Find an equation of the tangent line to the graph of $y = x + \ln x$ that is perpendicular to the line whose equation is $2x + 6y = 5$.

47 Shown in the figure is a graph of $y = 5\ln x - \frac{1}{2}x$. Find the coordinates of the highest point, and show that the graph is concave downward for $x > 0$.

EXERCISE 47

48 Shown in the figure is a graph of $y = \ln(x^2+1)$. Find the points of inflection.

EXERCISE 48



49 An approximation to the age T (in years) of a female blue whale can be obtained from a length measurement L (in feet) using the formula $T = -2.57\ln[(87-L)/63]$. A blue whale has been spotted by a research vessel, and her length is estimated to be 80 feet. If the maximum error in estimating L is ± 2 feet, use differentials to approximate the maximum error in T.

50 The *Ehrenberg relation*, $\ln W = \ln 2.4 + 0.0184h$, is an empirically based formula relating the height h (in centimeters) to the weight W (in kilograms) for children aged 5 through 13. The formula, with minor changes in the constants, has been verified in many different countries. Find the relationship between the rates of change dW/dt and dh/dt, for time t (in years).

51 A rocket of mass m_1 is filled with fuel of mass m_2, which will be burned at a constant rate of b kg/sec. If the fuel is expelled from the rocket at a constant rate, the distance $s(t)$ (in meters) that the rocket has traveled after t seconds is

$$s(t) = ct + \frac{c}{b}(m_1 + m_2 - bt)\ln\left(\frac{m_1 + m_2 - bt}{m_1 + m_2}\right)$$

for some constant $c > 0$.

(a) Find the initial velocity and initial acceleration of the rocket.

(b) Burnout occurs when $t = m_2/b$. Find the velocity and acceleration at burnout.

52 One method of estimating the thickness of the ozone layer is to use the formula $\ln(I/I_0) = -\beta T$, where I_0 is the intensity of a particular wavelength of light from the sun before it reaches the atmosphere, I is the intensity of the same wavelength after passing through a layer of ozone T centimeters thick, and β is the absorption coefficient for that wavelength. Suppose that for a wavelength of 3055×10^{-8} centimeter with $\beta \approx 2.7$, I_0/I is measured as 2.3.

(a) Approximate the thickness of the ozone layer to the nearest 0.01 centimeter.

(b) If the maximum error in the measured value of I_0/I is ± 0.1, use differentials to approximate the maximum error in the approximation obtained in (a).

53 Describe the difference between the graphs of $y = \ln(x^2)$ and $y = 2\ln x$.

54 Sketch the graphs of

(a) $y = \ln|x|$ (b) $y = |\ln x|$

c 55 Use Newton's method to approximate the real root of $\ln x + x = 0$ to three decimal places.

56 Show that $x > \ln x$ for every $x > 0$.

c 57 Let $f(x) = \ln(x^4)$ and $g(x) = \sqrt{x}$.

(a) Is $f(x) \ge g(x)$ on $[1, 2]$?

(b) Is $f(x) \ge g(x)$ on $[3, 64]$?

(c) Is there a positive integer M such that $f(x) \ge g(x)$ on $[M, \infty)$? (*Hint:* Estimate $\lim_{x\to\infty}[f(x)/g(x)]$.)

c 58 (a) Use the trapezoidal rule, with $n = 4$, to approximate $\int_1^2 \ln(x^2)\,dx$.

(b) Estimate the error in (a) using (5.37).

7.3 THE NATURAL EXPONENTIAL FUNCTION

In Section 7.2 we saw that

$$\lim_{x \to \infty} \ln x = \infty \quad \text{and} \quad \lim_{x \to 0+} \ln x = -\infty.$$

These facts are used in the proof of the following result.

Theorem **(7.14)**

> To every real number x there corresponds exactly one positive real number y such that $\ln y = x$.

PROOF First note that if $x = 0$, then $\ln 1 = 0$. Moreover, since ln is an increasing function, 1 is the only value of y such that $\ln y = 0$.

If x is positive, then we may choose a number b such that

$$\ln 1 < x < \ln b.$$

Since ln is continuous, it takes on every value between ln 1 and ln b (see the intermediate value theorem (2.26)). Thus, there is a number y between 1 and b such that $\ln y = x$. Since ln is an increasing function, there is only one such number.

Finally, if x is negative, then there is a number $b > 0$ such that

$$\ln b < x < \ln 1,$$

and, as before, there is exactly one number y between b and 1 such that $\ln y = x$. ■

It follows from Theorem (7.14) that the range of the natural logarithms is $\mathbb{R}$. Since ln is an increasing function, it is one-to-one and therefore has an inverse function, to which we give the following special name.

Definition **(7.15)**

> The **natural exponential function**, denoted by **exp**, is the inverse of the natural logarithmic function.

The reason for the term *exponential* in this definition will become clear shortly. Since exp is the inverse of ln, its domain is $\mathbb{R}$ and its range is $(0, \infty)$. Moreover, as in (7.2),

$$y = \exp x \quad \text{if and only if} \quad x = \ln y,$$

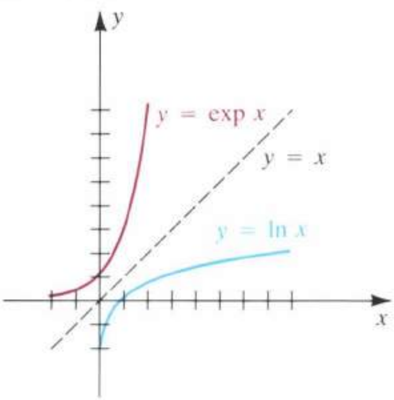
where x is any real number and $y > 0$. By Theorem (7.3), we may also write

$$\ln(\exp x) = x \quad \text{and} \quad \exp(\ln y) = y.$$

As we observed in Section 7.1, if two functions are inverses of each other, then their graphs are reflections through the line $y = x$. Hence the graph of $y = \exp x$ can be obtained by reflecting the graph of $y = \ln x$ through this line, as illustrated in Figure 7.12. Note that

$$\lim_{x \to \infty} \exp x = \infty \quad \text{and} \quad \lim_{x \to -\infty} \exp x = 0.$$

FIGURE 7.12



By Theorem (7.14), there exists exactly one positive real number whose natural logarithm is 1. This number is denoted by e. The great Swiss mathematician Leonhard Euler (1707–1783) was among the first to study its properties extensively.

Definition of e **(7.16)**

> The letter e denotes the positive real number such that $\ln e = 1$.

Several values of ln were calculated in Section 7.2. We can show, by means of the trapezoidal rule, that

$$\int_1^{2.7} \frac{1}{t}\,dt < 1 < \int_1^{2.8} \frac{1}{t}\,dt.$$

(See Exercise 25 in Section 5.7.) Applying Definitions (7.9) and (7.16) yields

$$\ln 2.7 < \ln e < \ln 2.8$$

and hence

$$2.7 < e < 2.8.$$

Later, in Theorem (7.32), we show that

$$e = \lim_{h \to 0} (1 + h)^{1/h}.$$

This limit formula can be used to approximate e to any degree of accuracy. In Section 7.5 we shall obtain the following approximation to five decimal places.

Approximation to e **(7.17)**

$$\boxed{e \approx 2.71828}$$

It can be shown that e is an irrational number.

If r is any *rational* number, then

$$\ln e^r = r \ln e = r(1) = r.$$

The formula $\ln e^r = r$ may be used to motivate a definition of e^x for every *real* number x. Specifically, we shall *define* e^x as the real number y such that $\ln y = x$. The following statement is a convenient way to remember this definition.

Definition of e^x **(7.18)**

> If x is any real number, then
>
> $$e^x = y \quad \text{if and only if} \quad \ln y = x.$$

Since exp is the inverse function of ln,

$$\exp x = y \quad \text{if and only if} \quad \ln y = x.$$

Comparing this relationship with Definition (7.18), we see that

$$e^x = \exp x \quad \text{for every } x.$$

This is the reason for calling exp an *exponential* function and referring to it as the **exponential function with base e**. The graph of $y = e^x$ is the same as that of $y = \exp x$, illustrated in Figure 7.12. Hereafter *we shall use e^x instead of* $\exp x$ *to denote values of the natural exponential function.*

The fact that $\ln(\exp x) = x$ for every x and $\exp(\ln x) = x$ for every $x > 0$ may now be written as follows:

Theorem **(7.19)**

(i) $\ln e^x = x$ for every x

(ii) $e^{\ln x} = x$ for every $x > 0$

Some special cases of this theorem are given in the following illustration.

ILLUSTRATION

- $\ln e^5 = 5$
- $e^{\ln 5} = 5$
- $e^{3 \ln x} = (e^{\ln x})^3 = x^3$
- $\ln e^{\sqrt{x+1}} = \sqrt{x+1}$
- $e^{\ln \sqrt{x+1}} = \sqrt{x+1}$
- $e^{k \ln x} = (e^{\ln x})^k = x^k$

A brief table of values of e^x and e^{-x} is given in Table B of Appendix III. To approximate values of e^x with a scientific calculator, we employ either an $\boxed{e^x}$ key or the succession $\boxed{\text{INV}}$ $\boxed{\ln x}$. Still another method is to use the $\boxed{y^x}$ key with $y = e \approx 2.71828$.

The next theorem states that the laws of exponents are true for powers of e.

Theorem **(7.20)**

If p and q are real numbers and r is a rational number, then

(i) $e^p e^q = e^{p+q}$ **(ii)** $\dfrac{e^p}{e^q} = e^{p-q}$ **(iii)** $(e^p)^r = e^{pr}$

PROOF Using Theorems (7.12) and (7.19), we obtain

$$\ln e^p e^q = \ln e^p + \ln e^q = p + q = \ln e^{p+q}.$$

Since the natural logarithmic function is one-to-one,

$$e^p e^q = e^{p+q}.$$

This proves (i). The proofs for (ii) and (iii) are similar. We show in Section 7.5 that (iii) is also true if r is irrational.

By Theorem (7.7), the inverse function of a differentiable function is differentiable, and hence $D_x e^x$ exists. The next theorem states that e^x *is its own derivative.*

Theorem **(7.21)**

$$D_x e^x = e^x$$

PROOF By (i) of Theorem (7.19),

$$\ln e^x = x.$$

Differentiating each side of this equation and using Theorem (7.11)(i) with $u = e^x$ gives us the following:

$$D_x(\ln e^x) = D_x(x)$$

$$\frac{1}{e^x} D_x e^x = 1$$

$$D_x e^x = e^x$$

EXAMPLE 1 If $f(x) = x^2e^x$, find $f'(x)$.

SOLUTION By the product rule and Theorem 7.21,

$$\begin{aligned} f'(x) &= x^2(D_x e^x) + e^x(D_x x^2) \\ &= x^2e^x + e^x(2x) = xe^x(x + 2). \end{aligned}$$

The next result is a generalization of Theorem (7.21).

Theorem **(7.22)**

If $u = g(x)$ and g is differentiable, then

$$D_x e^u = e^u D_x u.$$

PROOF Letting $y = e^u$ with $u = g(x)$, and using the chain rule and Theorem (7.21), we have

$$D_x e^u = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u D_x u.$$

If $u = x$, then Theorem (7.22) reduces to (7.21).

EXAMPLE 2 If $y = e^{\sqrt{x^2+1}}$, find dy/dx.

SOLUTION By Theorem (7.22),

$$\begin{aligned} \frac{dy}{dx} = \frac{d}{dx} e^{\sqrt{x^2+1}} &= e^{\sqrt{x^2+1}} \frac{d}{dx} \sqrt{x^2 + 1} \\ &= e^{\sqrt{x^2+1}} \frac{d}{dx} (x^2 + 1)^{1/2} \\ &= e^{\sqrt{x^2+1}} (\tfrac{1}{2})(x^2 + 1)^{-1/2}(2x) \\ &= e^{\sqrt{x^2+1}} \cdot \frac{x}{\sqrt{x^2 + 1}} \\ &= \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2 + 1}}. \end{aligned}$$

EXAMPLE 3 The function f defined by $f(x) = e^{-x^2/2}$ occurs in the branch of mathematics called *probability*. Find the local extrema of f, discuss concavity, find the points of inflection, and sketch the graph of f.

SOLUTION By Theorem (7.22),

$$f'(x) = e^{-x^2/2}\, D_x\left(-\frac{x^2}{2}\right) = e^{-x^2/2}\left(-\frac{2x}{2}\right) = -xe^{-x^2/2}.$$

Since $e^{-x^2/2}$ is always positive, the only critical number of f is 0. If $x < 0$, then $f'(x) > 0$, and if $x > 0$, then $f'(x) < 0$. It follows from the first derivative test that f has a local maximum at 0. The maximum value is $f(0) = e^{-0} = 1$.

Applying the product rule to $f'(x)$ yields

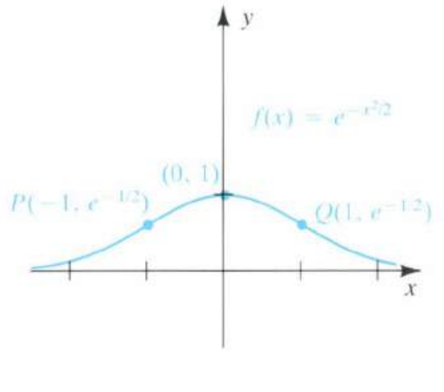
$$\begin{aligned} f''(x) &= -x\, D_x\, e^{-x^2/2} + e^{-x^2/2}\, D_x(-x) \\ &= -xe^{-x^2/2}(-2x/2) - e^{-x^2/2} \\ &= e^{-x^2/2}(x^2 - 1), \end{aligned}$$

and hence the second derivative is zero at -1 and 1. If $-1 < x < 1$, then $f''(x) < 0$ and, by (4.16), the graph of f is concave downward in the open interval $(-1, 1)$. If $x < -1$ or $x > 1$, then $f''(x) > 0$ and, therefore, the graph is concave upward throughout the infinite intervals $(-\infty, -1)$ and $(1, \infty)$. Consequently, $P(-1, e^{-1/2})$ and $Q(1, e^{-1/2})$ are points of inflection. From the expression

$$f(x) = \frac{1}{e^{x^2/2}}$$

it is evident that as x increases numerically, $f(x)$ approaches 0. We can prove that $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$; that is, the x-axis is a horizontal asymptote. The graph of f is sketched in Figure 7.13.

FIGURE 7.13



Exponential functions play an important role in the field of *radiotherapy*, the treatment of tumors by radiation. The fraction of cells in a tumor that survive a treatment, called the *surviving fraction*, depends not only on the energy and nature of the radiation, but also on the depth, size, and characteristics of the tumor itself. The exposure to radiation may be thought of as a number of potentially damaging events, where only one *hit* is required to kill a tumor cell. Suppose that each cell has exactly one *target* that must be hit. If k denotes the average target size of a tumor cell and if x is the number of damaging events (the *dose*), then the surviving fraction $f(x)$ is given by

$$f(x) = e^{-kx}.$$

This is called the *one-target–one hit surviving fraction*.

Suppose next that each cell has n targets and that hitting each target once results in the death of a cell. In this case, the *n target–one hit surviving fraction* is given by

$$f(x) = 1 - (1 - e^{-kx})^n.$$

In the next example we examine the case where $n = 2$.

EXAMPLE 4 If each cell of a tumor has two targets, then the two target–one hit surviving fraction is given by

$$f(x) = 1 - (1 - e^{-kx})^2,$$

where k is the average size of a cell. Analyze the graph of f to determine what effect increasing the dosage x has on decreasing the surviving fraction of tumor cells.

SOLUTION First note that if $x = 0$, then $f(0) = 1$; that is, if there is no dose, then all cells survive. Differentiating, we obtain

$$\begin{aligned} f'(x) &= 0 - 2(1 - e^{-kx})\, D_x (1 - e^{-kx}) \\ &= -2(1 - e^{-kx})(ke^{-kx}) \\ &= -2ke^{-kx}(1 - e^{-kx}). \end{aligned}$$

Since $f'(x) < 0$ for every $x > 0$ and $f'(0) = 0$, the function f is decreasing and the graph has a horizontal tangent line at the point $(0, 1)$. We may verify that the second derivative is

$$f''(x) = 2k^2e^{-kx}(1 - 2e^{-kx}).$$

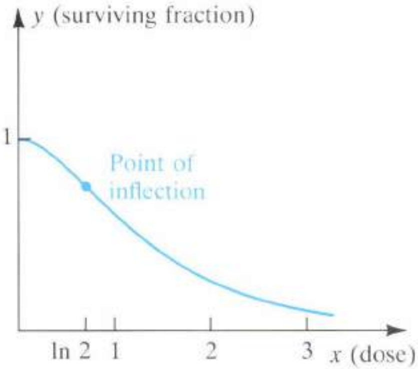
We see that $f''(x) = 0$ if $1 - 2e^{-kx} = 0$ (that is, if $e^{-kx} = \frac{1}{2}$, or, equivalently, $-kx = \ln \frac{1}{2} = -\ln 2$). This gives us

$$x = \frac{1}{k} \ln 2.$$

It can be verified that if $0 \le x < (1/k) \ln 2$, then $f''(x) < 0$, and hence the graph is concave downward. If $x > (1/k) \ln 2$, then $f''(x) > 0$, and the graph is concave upward. This implies that there is a point of inflection with x-coordinate $(1/k) \ln 2$. The y-coordinate of this point is

$$\begin{aligned} f\left(\frac{1}{k} \ln 2\right) &= 1 - (1 - e^{-\ln 2})^2 \\ &= 1 - (1 - \tfrac{1}{2})^2 = \tfrac{3}{4}. \end{aligned}$$

FIGURE 7.14
Surviving fraction of tumor cells after a radiation treatment



The graph is sketched in Figure 7.14 for the case $k = 1$. The *shoulder* on the curve near the point $(0, 1)$ represents the threshold nature of the treatment; that is, a small dose results in very little tumor elimination. Note that if x is large, then an increase in dosage has little effect on the surviving fraction. To determine the ideal dose that should be administered to a given patient, specialists in radiation therapy must also take into account the number of healthy cells that are killed during a treatment.

EXERCISES 7.3

Exer. 1–30: Find $f'(x)$ if $f(x)$ equals the given expression.

1 e^{-5x}

2 e^{3x}

3 e^{3x^2}

4 e^{1-x^3}

5 $\sqrt{1 + e^{2x}}$

6 $1/(e^x + 1)$

7 $e^{\sqrt{x+1}}$

8 xe^{-x}

9 x^2e^{-2x}

10 $\sqrt{e^{2x} + 2x}$

11 $e^x/(x^2 + 1)$

12 $x/e^{(x^2)}$

13 $(e^{4x} - 5)^3$

14 $(e^{3x} - e^{-3x})^4$

15 $e^{1/x} + (1/e^x)$

16 $e^{\sqrt{x}} + \sqrt{e^x}$

17 $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

18 $e^{x \ln x}$

19 $e^{-2x} \ln x$

20 $\ln e^x$

21 $\sin e^{5x}$

22 $e^{\sin 5x}$

23 $\ln \cos e^{-x}$

24 $e^{-3x} \cos 3x$

25 $e^{3x} \tan \sqrt{x}$

26 $\sec e^{-2x}$

27 $\sec^2 (e^{-4x})$

28 $e^{-x} \tan^2 x$

29 $xe^{\cot x}$

30 $\ln (\csc e^{3x})$

Exer. 31–34: Use implicit differentiation to find y'.

31 $e^{xy} - x^3 + 3y^2 = 11$

32 $xe^y + 2x - \ln (y + 1) = 3$

33 $e^x \cot y = xe^{2y}$

34 $e^x \cos y = xe^y$

35 Find an equation of the tangent line to the graph of $y = (x - 1)e^x + 3 \ln x + 2$ at the point $P(1, 2)$.

36 Find an equation of the tangent line to the graph of $y = x - e^{-x}$ that is parallel to the line $6x - 2y = 7$.

Exer. 37–42: Find the local extrema of f. Determine where f is increasing or is decreasing, discuss concavity, find the points of inflection, and sketch the graph of f.

37 $f(x) = xe^x$

38 $f(x) = x^2e^{-2x}$

39 $f(x) = e^{1/x}$

40 $f(x) = xe^{-x}$

41 $f(x) = x \ln x$

42 $f(x) = (1 - \ln x)^2$

43 A radioactive substance decays according to the formula $q(t) = q_0e^{-ct}$, where q_0 is the initial amount of the substance, c is a positive constant, and $q(t)$ is the amount remaining after time t. Show that the rate at which the substance decays is proportional to $q(t)$.

44 The current $I(t)$ at time t in an electrical circuit is given by $I(t) = I_0e^{-Rt/L}$, where R is the resistance, L is the inductance, and I_0 is the current at time $t = 0$. Show that the rate of change of the current at any time t is proportional to $I(t)$.

45 If a drug is injected into the bloodstream, then its concentration t minutes later is given by

$$C(t) = \frac{k}{a - b}(e^{-bt} - e^{-at})$$

for positive constants a, b, and k.

(a) At what time does the maximum concentration occur?

(b) What can be said about the concentration after a long period of time?

46 If a beam of light that has intensity k is projected vertically downward into water, then its intensity $I(x)$ at a depth of x meters is $I(x) = ke^{-1.4x}$.

(a) At what rate is the intensity changing with respect to depth at 1 meter? 5 meters? 10 meters?

(b) At what depth is the intensity one-half its value at the surface? One-tenth its value?

47 The *Jenss model* is generally regarded as the most accurate formula for predicting the height of a preschooler. If $h(x)$ denotes the height (in centimeters) at age x (in years) for $\frac{1}{4} \le x \le 6$, then $h(x)$ can be approximated by

$$h(x) = 79.041 + 6.39x - e^{3.261 - 0.993x}.$$

(Compare with Example 7 of Section 7.2.)

(a) Predict the height and rate of growth when a child reaches the age of 1.

(b) When is the rate of growth largest, and when is it smallest?

48 For a population of female African elephants, the weight $W(t)$ (in kilograms) at age t (in years) may be approximated by a *von Bertanlanffy growth function* W such that

$$W(t) = 2600(1 - 0.51e^{-0.075t})^3.$$

(a) Approximate the weight and the rate of growth of a newborn.

(b) Assuming that an adult female weighs 1800 kilograms, estimate her age and her rate of growth at present.

(c) Find and interpret $\lim_{t \to \infty} W(t)$.

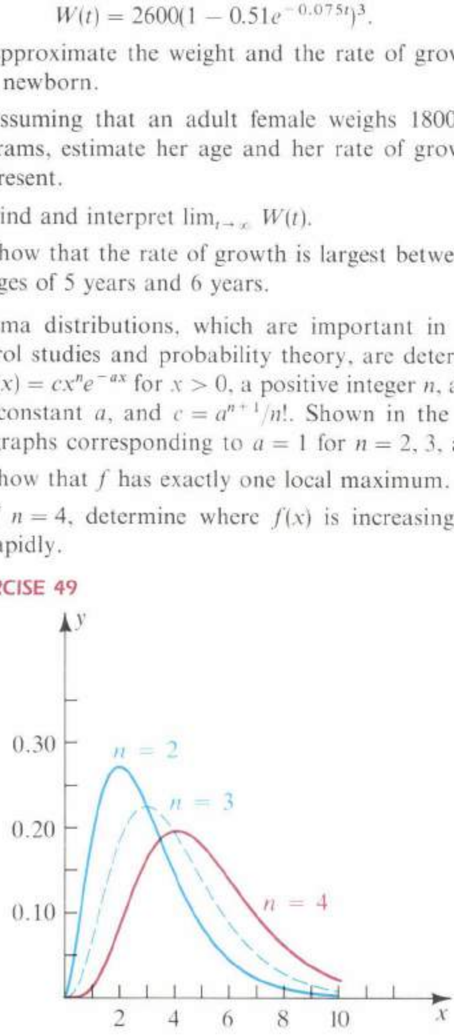
(d) Show that the rate of growth is largest between the ages of 5 years and 6 years.

49 Gamma distributions, which are important in traffic control studies and probability theory, are determined by $f(x) = cx^ne^{-ax}$ for $x > 0$, a positive integer n, a positive constant a, and $c = a^{n+1}/n!$. Shown in the figure are graphs corresponding to $a = 1$ for $n = 2$, 3, and 4.

(a) Show that f has exactly one local maximum.

(b) If $n = 4$, determine where $f(x)$ is increasing most rapidly.

EXERCISE 49



50 The relative number of gas molecules in a container that travel at a velocity of v cm/sec can be computed by means of the *Maxwell-Boltzmann speed distribution*,

$F(v) = cv^2e^{-mv^2/(2kT)}$, where T is the temperature (in °K), m is the mass of a molecule, and c and k are positive constants. Show that the maximum value of F occurs when $v = \sqrt{2kT/m}$.

51 An *urban density model* is a formula that relates the population density (in number/mi²) to the distance r (in miles) from the center of the city. The formula $D = ae^{-br+cr^2}$, where a, b, and c are positive constants, has been found to be appropriate for certain cities. Determine the shape of the graph for $r \geq 0$.

52 The effect of light on the rate of photosynthesis can be described by

$$f(x) = x^a e^{(a/b)(1-x^b)}$$

for $x > 0$ and positive constants a and b.

(a) Show that f has a maximum at $x = 1$.

(b) Conclude that if $x_0 > 0$ and $y_0 > 0$, then $g(x) = y_0 f(x/x_0)$ has a maximum $g(x_0) = y_0$.

53 The rate R at which a tumor grows is related to its size x by the equation $R = rx \ln (K/x)$, where r and K are positive constants. Show that the tumor is growing most rapidly when $x = e^{-1}K$.

54 If p denotes the selling price (in dollars) of a commodity and x is the corresponding demand (in number sold per day), then the relationship between p and x may be given by $p = p_0e^{-ax}$ for positive constants p_0 and a. Suppose $p = 300e^{-0.02x}$. Find the selling price that will maximize daily revenues (see page 228).

55 In statistics the probability density function for the normal distribution is defined by

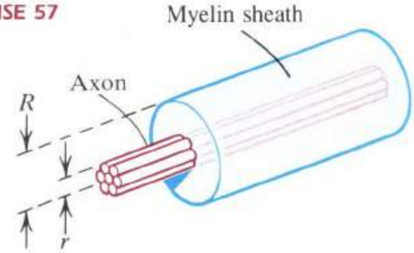
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-z^2/2} \quad \text{with} \quad z = \frac{x-\mu}{\sigma}$$

for real numbers μ and $\sigma > 0$ (μ is the *mean* and σ^2 is the *variance* of the distribution). Find the local extrema of f, and determine where f is increasing or is decreasing. Discuss concavity, find points of inflection, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, and sketch the graph of f (see Example 3).

c 56 The integral $\int_a^b e^{-x^2}\,dx$ has applications in statistics. Use the trapezoidal rule, with $n = 10$, to approximate this integral if $a = 0$ and $b = 1$.

57 Nerve impulses in the human body travel along nerve fibers that consist of an *axon*, which transports the impulse, and an insulating coating surrounding the axon, called the *myelin sheath* (see figure). The nerve fiber is similar to an insulated cylindrical cable, for which the velocity v of an impulse is given by $v = -k(r/R)^2 \ln (r/R)$, where r is the radius of the cable and R is the insulation radius. Find the value of r/R that maximizes v. (In most nerve fibers $r/R \approx 0.6$.)

EXERCISE 57



58 Sketch the graph of the *three target–one hit surviving fraction* (with $k = 1$) given by $f(x) = 1 - (1 - e^{-x})^3$ (see Example 4).

c **Exer. 59–60: Use Newton's method to approximate the real root of the equation to two decimal places.**

59 $e^{-x} = x$ 60 $x \ln x = 1$

7.4 INTEGRATION

We may use differentiation formulas for ln to obtain formulas for integration. In particular, by Theorem (7.11),

$$D_x \ln |g(x)| = \frac{1}{g(x)}g'(x),$$

which gives us the integration formula

$$\int \frac{1}{g(x)}g'(x)\,dx = \ln |g(x)| + C.$$

This is restated in the next theorem in terms of the variable u.

Theorem (7.23)

If $u = g(x) \neq 0$ and g is differentiable, then

$$\int \frac{1}{u}\, du = \ln |u| + C.$$

Of course, if $u > 0$, then the absolute value sign may be deleted. A special case of Theorem (7.23) is

$$\int \frac{1}{x}\, dx = \ln |x| + C.$$

EXAMPLE 1 Evaluate $\int \frac{x}{3x^2 - 5}\, dx$.

SOLUTION Rewriting the integral as

$$\int \frac{x}{3x^2 - 5}\, dx = \int \frac{1}{3x^2 - 5}\, x\, dx$$

suggests that we use Theorem(7.23)with $u = 3x^2 - 5$. Thus, we make the substitution

$$u = 3x^2 - 5, \qquad du = 6x\, dx.$$

Introducing a factor 6 in the integrand and using Theorem (7.23) yields

$$\begin{aligned}\int \frac{x}{3x^2 - 5}\, dx &= \frac{1}{6}\int \frac{1}{3x^2 - 5}\, 6x\, dx = \frac{1}{6}\int \frac{1}{u}\, du \\ &= \tfrac{1}{6} \ln |u| + C = \tfrac{1}{6} \ln |3x^2 - 5| + C.\end{aligned}$$

Another technique is to replace the expression $x\, dx$ in the integral by $\frac{1}{6}\, du$ and then integrate.

EXAMPLE 2 Evaluate $\int_2^4 \frac{1}{9 - 2x}\, dx$.

SOLUTION Since $1/(9 - 2x)$ is continuous on $[2, 4]$, the definite integral exists. One method of evaluation consists of using an indefinite integral to find an antiderivative of $1/(9 - 2x)$. We let

$$u = 9 - 2x, \qquad du = -2\, dx$$

and proceed as follows:

$$\begin{aligned}\int \frac{1}{9 - 2x}\, dx &= -\frac{1}{2}\int \frac{1}{9 - 2x}\,(-2)\, dx \\ &= -\frac{1}{2}\int \frac{1}{u}\, du = -\frac{1}{2} \ln |u| + C \\ &= -\tfrac{1}{2} \ln |9 - 2x| + C\end{aligned}$$

Applying the fundamental theorem of calculus yields

$$\int_2^4 \frac{1}{9-2x}\,dx = -\frac{1}{2}\Big[\ln|9-2x|\Big]_2^4$$
$$= -\tfrac{1}{2}(\ln 1 - \ln 5) = \tfrac{1}{2}\ln 5.$$

Another method is to use the same substitution in the *definite* integral and change the limits of integration. Since $u = 9 - 2x$, we obtain the following:

(i) If $x = 2$, then $u = 5$.

(ii) If $x = 4$, then $u = 1$.

Thus,

$$\int_2^4 \frac{1}{9-2x}\,dx = -\frac{1}{2}\int_2^4 \frac{1}{9-2x}(-2)\,dx$$
$$= -\frac{1}{2}\int_5^1 \frac{1}{u}\,du = -\frac{1}{2}\Big[\ln|u|\Big]_5^1$$
$$= -\tfrac{1}{2}(\ln 1 - \ln 5) = \tfrac{1}{2}\ln 5.$$

EXAMPLE 3 Evaluate $\int \frac{\sqrt{\ln x}}{x}\,dx$.

SOLUTION Two possible substitutions are $u = \sqrt{\ln x}$ and $u = \ln x$. If we use

$$u = \ln x, \qquad du = \frac{1}{x}\,dx,$$

then

$$\int \frac{\sqrt{\ln x}}{x}\,dx = \int \sqrt{\ln x}\cdot\frac{1}{x}\,dx = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C$$
$$= \tfrac{2}{3}(\ln x)^{3/2} + C.$$

The substitution $u = \sqrt{\ln x}$ could also be used; however, the algebraic manipulations would be somewhat more involved.

The derivative formula $D_x\, e^{g(x)} = e^{g(x)}g'(x)$ gives us the following integration formula for the natural exponential function:

$$\int e^{g(x)}g'(x)\,dx = e^{g(x)} + C$$

This is restated in the next theorem in terms of the variable u.

Theorem **(7.24)**

If $u = g(x)$ and g is differentiable, then

$$\int e^u\,du = e^u + C.$$

As a special case of Theorem (7.24), if $u = x$, then

$$\int e^x \, dx = e^x + C.$$

EXAMPLE 4 Evaluate

(a) $\displaystyle\int \frac{e^{3/x}}{x^2} \, dx$ **(b)** $\displaystyle\int_1^2 \frac{e^{3/x}}{x^2} \, dx$

SOLUTION

(a) Rewriting the integral as

$$\int \frac{e^{3/x}}{x^2} \, dx = \int e^{3/x} \frac{1}{x^2} \, dx$$

suggests that we use Theorem (7.24) with $u = 3/x$. Thus, we make the substitution

$$u = \frac{3}{x}, \qquad du = -\frac{3}{x^2} \, dx.$$

The integrand may be written in the form of (7.24) by introducing the factor -3. Doing this and compensating by multiplying the integral by $-\frac{1}{3}$, we obtain

$$\begin{aligned} \int \frac{e^{3/x}}{x^2} \, dx &= -\frac{1}{3} \int e^{3/x} \left(-\frac{3}{x^2} \right) dx \\ &= -\tfrac{1}{3} \int e^u \, du \\ &= -\tfrac{1}{3} e^u + C \\ &= -\tfrac{1}{3} e^{3/x} + C. \end{aligned}$$

(b) Using the antiderivative found in (a) and applying the fundamental theorem of calculus yields

$$\begin{aligned} \int_1^2 \frac{e^{3/x}}{x^2} \, dx &= -\frac{1}{3} \Big[e^{3/x} \Big]_1^2 \\ &= -\tfrac{1}{3}(e^{3/2} - e^3) \approx 5.2. \end{aligned}$$

We can also evaluate the integral by using the method of substitution. As in (a), we let $u = 3/x$, $du = (-3/x^2) \, dx$, and we note that if $x = 1$, then $u = 3$, and if $x = 2$, then $u = \frac{3}{2}$. Consequently,

$$\begin{aligned} \int_1^2 \frac{e^{3/x}}{x^2} \, dx &= -\frac{1}{3} \int_1^2 e^{3/x} \left(-\frac{3}{x^2} \right) dx \\ &= -\tfrac{1}{3} \int_3^{3/2} e^u \, du \\ &= -\tfrac{1}{3} \Big[e^u \Big]_3^{3/2} = -\tfrac{1}{3}(e^{3/2} - e^3) \approx 5.2. \end{aligned}$$

The integral $\int e^{ax} \, dx$, with $a \neq 0$, occurs frequently. We can show that

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$$

either by using Theorem (7.24) or by showing that $(1/a)e^{ax}$ is an antiderivative of e^{ax}.

ILLUSTRATION

- $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$
- $\int e^{-5x}\,dx = -\frac{1}{5}e^{-5x} + C$
- $\int e^{-x}\,dx = -e^{-x} + C$

In the next example we solve a differential equation that contains exponential expressions.

EXAMPLE 5 Solve the differential equation

$$\frac{dy}{dx} = 3e^{2x} + 6e^{-3x}$$

subject to the initial condition $y = 4$ if $x = 0$.

SOLUTION As in Example 6 of Section 5.1, we may multiply both sides of the equation by dx and then integrate as follows:

$$\begin{aligned} dy &= (3e^{2x} + 6e^{-3x})\,dx \\ \int dy &= \int (3e^{2x} + 6e^{-3x})\,dx = 3\int e^{2x}\,dx + 6\int e^{-3x}\,dx \\ y &= 3(\tfrac{1}{2})e^{2x} + 6(-\tfrac{1}{3})e^{-3x} + C \\ &= \tfrac{3}{2}e^{2x} - 2e^{-3x} + C \end{aligned}$$

Using the initial condition $y = 4$ if $x = 0$ gives us

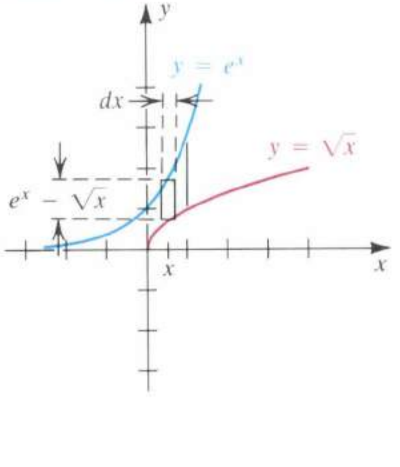
$$4 = \tfrac{3}{2}e^0 - 2e^0 + C = \tfrac{3}{2} - 2 + C.$$

Hence $C = 4 - \frac{3}{2} + 2 = \frac{9}{2}$, and the solution of the differential equation is

$$y = \tfrac{3}{2}e^{2x} - 2e^{-3x} + \tfrac{9}{2}.$$

EXAMPLE 6 Find the area of the region bounded by the graphs of the equations $y = e^x$, $y = \sqrt{x}$, $x = 0$, and $x = 1$.

FIGURE 7.15



SOLUTION The region and a typical rectangle of the type considered in Chapter 6 are shown in Figure 7.15. As usual, we list the following:

width of rectangle: dx

length of rectangle: $e^x - \sqrt{x}$

area of rectangle: $(e^x - \sqrt{x})\,dx$

We next take a limit of sums of these rectangular areas by applying the operator $\int_0^1$:

$$\begin{aligned} \int_0^1 (e^x - \sqrt{x})\,dx &= \int_0^1 (e^x - x^{1/2})\,dx \\ &= \Big[e^x - \tfrac{2}{3}x^{3/2}\Big]_0^1 = e - \tfrac{5}{3} \approx 1.05 \end{aligned}$$

In Chapter 5 we obtained integration formulas for the sine and cosine functions. We were unable to consider the remaining four trigonometric functions at that time because, as indicated in the next theorem, their integrals are logarithmic functions. In the theorem we assume that $u = g(x)$, with g differentiable whenever the function is defined.

Theorem **(7.25)**

(i) $\int \tan u \, du = -\ln |\cos u| + C$

(ii) $\int \cot u \, du = \ln |\sin u| + C$

(iii) $\int \sec u \, du = \ln |\sec u + \tan u| + C$

(iv) $\int \csc u \, du = \ln |\csc u - \cot u| + C$

PROOF It is sufficient to consider the case $u = x$, since the formulas for $u = g(x)$ then follow from Theorem (5.7).

To find $\int \tan x \, dx$, we first use a trigonometric identity to express $\tan x$ in terms of $\sin x$ and $\cos x$ as follows:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{\cos x} \sin x \, dx$$

The form of the integrand on the right suggests that we make the substitution

$$v = \cos x, \qquad dv = -\sin x \, dx.$$

This gives us

$$\int \tan x \, dx = -\int \frac{1}{v} \, dv.$$

If $\cos x \neq 0$, then, by Theorem (7.11)(ii),

$$\int \tan x \, dx = -\ln |v| + C = -\ln |\cos x| + C.$$

A formula for $\int \cot x \, dx$ may be obtained in similar fashion by first writing $\cot x = (\cos x)/(\sin x)$.

To find a formula for $\int \sec x \, dx$, we begin as follows:

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x) \, dx \end{aligned}$$

Using the substitution

$$v = \sec x + \tan x, \qquad dv = (\sec x \tan x + \sec^2 x) \, dx$$

gives us

$$\begin{aligned}\int \sec x\, dx &= \int \frac{1}{v}\, dv \\ &= \ln |v| + C \\ &= \ln |\sec x + \tan x| + C.\end{aligned}$$

A similar proof can be given for (iv). ■

If we use $\cos u = 1/\sec u$, $\sin u = 1/\csc u$, and $\ln (1/v) = -\ln v$, then formulas (i) and (ii) of Theorem (7.25) can be written as follows:

$$\int \tan u\, du = \ln |\sec u| + C$$

$$\int \cot u\, du = -\ln |\csc u| + C$$

EXAMPLE 7 Evaluate $\int x \cot x^2\, dx$.

SOLUTION To obtain the form $\int \cot u\, du$, we make the substitution

$$u = x^2, \qquad du = 2x\, dx.$$

We next introduce the factor 2 in the integrand as follows:

$$\int x \cot x^2\, dx = \tfrac{1}{2} \int (\cot x^2) 2x\, dx$$

Since $u = x^2$ and $du = 2x\, dx$,

$$\begin{aligned}\int x \cot x^2\, dx &= \tfrac{1}{2} \int \cot u\, du = \tfrac{1}{2} \ln |\sin u| + C \\ &= \tfrac{1}{2} \ln |\sin x^2| + C.\end{aligned}$$

EXAMPLE 8 Evaluate $\int_0^{\pi/2} \tan \frac{x}{2}\, dx$.

SOLUTION We make the substitution

$$u = \frac{x}{2}, \qquad du = \frac{1}{2}\, dx$$

and note that $u = 0$ if $x = 0$, and $u = \pi/4$ if $x = \pi/2$. Thus,

$$\begin{aligned}\int_0^{\pi/2} \tan \frac{x}{2}\, dx &= 2 \int_0^{\pi/2} \tan \frac{x}{2} \cdot \frac{1}{2}\, dx \\ &= 2 \int_0^{\pi/4} \tan u\, du = 2\Big[\ln \sec u\Big]_0^{\pi/4}.\end{aligned}$$

In this case we may drop the absolute value sign given in Theorem (7.25)(iii), since $\sec u$ is positive if u is between 0 and $\pi/4$. Since $\ln \sec (\pi/4) = \ln \sqrt{2} = \frac{1}{2} \ln 2$ and $\ln \sec 0 = \ln 1 = 0$, it follows that

$$\int_0^{\pi/2} \tan \frac{x}{2}\, dx = 2 \cdot \frac{1}{2} \ln 2 = \ln 2 \approx 0.69.$$

EXAMPLE 9 Evaluate $\int e^{2x} \sec e^{2x}\, dx$.

SOLUTION We let

$$u = e^{2x}, \qquad du = 2e^{2x}\, dx$$

and proceed as follows:

$$\begin{aligned}\int e^{2x} \sec e^{2x}\, dx &= \tfrac{1}{2}\int (\sec e^{2x}) 2e^{2x}\, dx\\ &= \tfrac{1}{2}\int \sec u\, du\\ &= \tfrac{1}{2} \ln |\sec u + \tan u| + C\\ &= \tfrac{1}{2} \ln |\sec e^{2x} + \tan e^{2x}| + C\end{aligned}$$

EXAMPLE 10 Evaluate $\int (\csc x - 1)^2\, dx$.

SOLUTION

$$\begin{aligned}\int (\csc x - 1)^2\, dx &= \int (\csc^2 x - 2 \csc x + 1)\, dx\\ &= \int \csc^2 x\, dx - 2\int \csc x\, dx + \int dx\\ &= -\cot x - 2 \ln |\csc x - \cot x| + x + C.\end{aligned}$$

We shall discuss additional methods for integrating trigonometric expressions in Chapter 9.

EXERCISES 7.4

Exer. 1–36: Evaluate the integral. ✓

1 (a) $\int \frac{1}{2x+7}\, dx$ (b) $\int_{-2}^{1} \frac{1}{2x+7}\, dx$

2 (a) $\int \frac{1}{4-5x}\, dx$ (b) $\int_{-1}^{0} \frac{1}{4-5x}\, dx$

3 (a) $\int \frac{4x}{x^2-9}\, dx$ (b) $\int_{1}^{2} \frac{4x}{x^2-9}\, dx$

4 (a) $\int \frac{3x}{x^2+4}\, dx$ (b) $\int_{1}^{2} \frac{3x}{x^2+4}\, dx$

5 (a) $\int e^{-4x}\, dx$ (b) $\int_{1}^{3} e^{-4x}\, dx$

6 (a) $\int x^2 e^{3x^3}\, dx$ (b) $\int_{1}^{2} x^2 e^{3x^3}\, dx$

7 (a) $\int \tan 2x\, dx$ (b) $\int_{0}^{\pi/8} \tan 2x\, dx$

8 (a) $\int \cot \tfrac{1}{3}x\, dx$ (b) $\int_{3\pi/2}^{9\pi/4} \cot \tfrac{1}{3}x\, dx$

9 (a) $\int \csc \tfrac{1}{2}x\, dx$ (b) $\int_{\pi}^{5\pi/3} \csc \tfrac{1}{2}x\, dx$

10 (a) $\int \sec 3x\, dx$ (b) $\int_{0}^{\pi/12} \sec 3x\, dx$

11 $\int \frac{x-2}{x^2-4x+9}\, dx$

12 $\int \frac{x^3}{x^4-5}\, dx$

13 $\int \frac{(x+2)^2}{x}\, dx$

14 $\int \frac{(2+\ln x)^{10}}{x}\, dx$

15 $\int \frac{\ln x}{x}\, dx$

16 $\int \frac{1}{x(\ln x)^2}\, dx$

17 $\int (x + e^{5x})\, dx$

18 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

19 $\int \frac{3 \sin x}{1 + 2 \cos x}\, dx$

20 $\int \frac{\sec^2 x}{1 + \tan x}\, dx$

21 $\int \frac{(e^x + 1)^2}{e^x}\, dx$

22 $\int \frac{e^x}{(e^x + 1)^2}\, dx$

23 $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

24 $\int \frac{e^x}{e^x + 1}\, dx$

25 $\int \frac{\cot \sqrt[3]{x}}{\sqrt[3]{x^2}}\, dx$

26 $\int e^x(1 + \tan e^x)\, dx$

27 $\int \frac{1}{\cos 2x}\, dx$

28 $\int (x + \csc 8x)\, dx$

29 $\int \frac{\tan e^{-3x}}{e^{3x}}\, dx$

30 $\int e^{\cos x} \sin x\, dx$

31 $\int \frac{\cos^2 x}{\sin x}\, dx$

32 $\int \frac{\tan^2 2x}{\sec 2x}\, dx$

33 $\int \frac{\cos x \sin x}{\cos^2 x - 1}\, dx$

34 $\int (\tan 3x + \sec 3x)\, dx$

35 $\int (1 + \sec x)^2\, dx$

36 $\int \csc x\,(1 - \csc x)\, dx$

Exer. 37–38: Find the area of the region bounded by the graphs of the given equations.

37 $y = e^{2x}$, $y = 0$, $x = 0$, $x = \ln 3$

38 $y = 2 \tan x$, $y = 0$, $x = 0$, $x = \pi/4$

Exer. 39–40: Find the volume of the solid generated if the region bounded by the graphs of the equations is revolved about the indicated axis.

39 $y = e^{-x^2}$, $x = 0$, $x = 1$, $y = 0$; y-axis

40 $y = \sec x$, $x = -\pi/3$, $x = \pi/3$, $y = 0$; x-axis

Exer. 41–44: Solve the differential equation subject to the given conditions.

41 $y' = 4e^{2x} + 3e^{-2x}$; $y = 4$ if $x = 0$

42 $y' = 3e^{4x} - 8e^{-2x}$; $y = -2$ if $x = 0$

43 $y'' = 3e^{-x}$; $y = -1$ and $y' = 1$ if $x = 0$

44 $y'' = 6e^{2x}$; $y = -3$ and $y' = 2$ if $x = 0$

Exer. 45–46: A nonnegative function f defined on a closed interval $[a, b]$ is called a *probability density function* if $\int_a^b f(x)\, dx = 1$. Determine c so that the resulting function is a probability density function.

45 $f(x) = \frac{cx}{x^2 + 4}$ for $0 \le x \le 3$

46 $f(x) = cxe^{-x^2}$ for $0 \le x \le 10$

47 A culture of bacteria is growing at a rate of $3e^{0.2t}$ per hour, with t in hours and $0 \le t \le 20$.

(a) How many new bacteria will be in the culture after the first five hours?

(b) How many new bacteria are introduced in the sixth through the fourteenth hours?

(c) For approximately what value of t will the culture contain 150 new bacteria?

48 If a savings bond is purchased for \$500 with interest compounded continuously at 7% per year, then after t years the bond will be worth $500e^{0.07t}$ dollars.

(a) Approximately when will the bond be worth \$1000?

(b) Approximately when will the value of the bond be growing at a rate of \$50 per year?

49 The specific heat c of a metal such as silver is constant at temperatures T above 200°K. If the temperature of the metal increases from T_1 to T_2, the area under the curve $y = c/T$ from T_1 to T_2 is called the *change in entropy* ΔS, a measurement of the increased molecular disorder of the system. Express ΔS in terms of T_1 and T_2.

50 The 1952 earthquake in Assam had a magnitude of 8.7 on the Richter scale—the largest ever recorded. (The October 1989 San Francisco earthquake had a magnitude of 7.1.) Seismologists have determined that if the largest earthquake in a given year has magnitude R, then the energy E (in joules) released by all earthquakes in that year can be estimated by using the formula

$$E = 9.13 \times 10^{12} \int_0^R e^{1.25x}\, dx.$$

Find E if $R = 8$.

51 In a circuit containing a 12-volt battery, a resistor, and a capacitor, the current $I(t)$ at time t is predicted to be $I(t) = 10e^{-4t}$ amperes. If $Q(t)$ is the charge (in coulombs) on the capacitor, then $I = dQ/dt$.

(a) If $Q(0) = 0$, find $Q(t)$.

(b) Find the charge on the capacitor after a long period of time.

52 A country that presently has coal reserves of 50 million tons used 6.5 million tons last year. Based on population projections, the rate of consumption R (in million tons/year) is expected to increase according to the formula $R = 6.5e^{0.02t}$, where t is the time in years. If the country uses only its own resources, estimate how many years the coal reserves will last.

53 A very small spherical particle (on the order of 5 microns in diameter) is projected into still air with an initial velocity of v_0 m/sec, but its velocity decreases because of drag forces. Its velocity after t seconds is given by $v(t) = v_0 e^{-t/k}$ for some positive constant k.

(a) Express the distance that the particle travels as a function of t.

(b) The *stopping distance* is the distance traveled by the particle before it comes to rest. Express the stopping distance in terms of v_0 and k.

54 If the temperature remains constant, the pressure p and volume v of an expanding gas are related by the equation $pv = k$ for some constant k. Show that the work done if the gas expands from v_0 to v_1 is $k \ln (v_1/v_0)$. (*Hint:* See Example 5 of Section 6.6.)

c **Exer. 55–56: If T_2 and T_4 are approximations of a definite integral obtained by using the trapezoidal rule with $n = 2$ and $n = 4$, respectively, then it can be shown that $R = \frac{1}{3}(4T_4 - T_2)$ is usually a better approximation. Find T_2, T_4, T_{10}, and R for the given integral, and decide whether R or T_{10} is the better approximation.**

55 $\int_0^1 e^{-x^2}\,dx \approx 0.746824$

56 $\int_1^2 (\ln x)^2\,dx \approx 0.188317$

7.5 GENERAL EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Throughout this section a will denote a positive real number. Let us begin by defining a^x for every real number x. If the exponent is a *rational* number r, then applying Theorems (7.19)(ii) and (7.12)(iii) yields

$$a^r = e^{\ln a^r} = e^{r \ln a}.$$

This formula is the motivation for the following definition of a^x.

Definition of a^x **(7.26)**

$$a^x = e^{x \ln a}$$

for every $a > 0$ and every real number x

ILLUSTRATION

- $2^\pi = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$
- $(\frac{1}{2})^{\sqrt{3}} = e^{\sqrt{3} \ln (1/2)} \approx e^{-1.20} \approx 0.3$

If $f(x) = a^x$, then f is the **exponential function with base a**. Since e^x is positive for every x, so is a^x. To approximate values of a^x, we may use a calculator or refer to tables of logarithmic and exponential functions.

It is now possible to prove that the law of logarithms stated in Theorem (7.12)(iii) is also true for irrational exponents. Thus, if u is any *real* number, then, by Definition (7.26) and Theorem (7.19)(i),

$$\ln a^u = \ln e^{u \ln a} = u \ln a.$$

The next theorem states that properties of rational exponents from elementary algebra are also true for real exponents.

Laws of exponents **(7.27)**

Let $a > 0$ and $b > 0$. If u and v are any real numbers, then

$$a^u a^v = a^{u+v} \qquad (a^u)^v = a^{uv} \qquad (ab)^u = a^u b^u$$

$$\frac{a^u}{a^v} = a^{u-v} \qquad \left(\frac{a}{b}\right)^u = \frac{a^u}{b^u}.$$

PROOF To show that $a^u a^v = a^{u+v}$, we use Definition (7.26) and Theorem (7.20)(i) as follows:

$$\begin{aligned} a^u a^v &= e^{u \ln a} e^{v \ln a} \\ &= e^{u \ln a + v \ln a} \\ &= e^{(u+v) \ln a} \\ &= a^{u+v} \end{aligned}$$

To prove that $(a^u)^v = a^{uv}$, we first use Definition (7.26) with a^u in place of a and $v = x$ to write

$$(a^u)^v = e^{v \ln a^u}.$$

Using the fact that $\ln a^u = u \ln a$ and then applying Definition (7.26), we obtain

$$(a^u)^v = e^{vu \ln a} = a^{vu} = a^{uv}.$$

The proofs of the remaining laws are similar.

As usual, in part (ii) of the next theorem, $u = g(x)$, where g is differentiable.

Theorem **(7.28)**

(i) $D_x\, a^x = a^x \ln a$ **(ii)** $D_x\, a^u = (a^u \ln a)\, D_x\, u$

PROOF Applying Definition (7.26) and Theorem (7.22), we obtain

$$D_x\, a^x = D_x\, e^{x \ln a} = e^{x \ln a}\, D_x\, (x \ln a) = e^{x \ln a}(\ln a).$$

Since $e^{x \ln a} = a^x$, this gives us formula (i):

$$D_x\, a^x = a^x \ln a$$

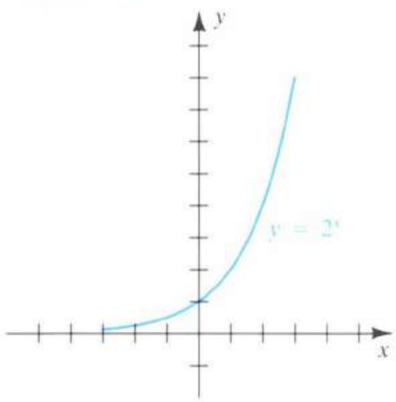
Formula (ii) follows from the chain rule.

Note that if $a = e$, then Theorem (7.28)(i) reduces to (7.21), since $\ln e = 1$.

ILLUSTRATION

- $D_x\, 3^x = 3^x \ln 3$
- $D_x\, 10^x = 10^x \ln 10$
- $D_x\, 3^{\sqrt{x}} = (3^{\sqrt{x}} \ln 3)\, D_x \sqrt{x} = (3^{\sqrt{x}} \ln 3)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{3^{\sqrt{x}} \ln 3}{2\sqrt{x}}$
- $D_x\, 10^{\sin x} = (10^{\sin x} \ln 10)\, D_x \sin x = (10^{\sin x} \ln 10) \cos x$

FIGURE 7.16

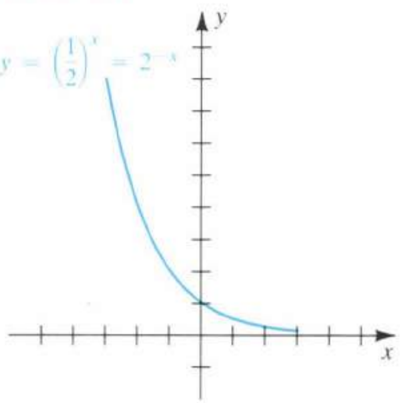


If $a > 1$, then $\ln a > 0$ and, therefore, $D_x\, a^x = a^x \ln a > 0$. Hence a^x is increasing on the interval $(-\infty, \infty)$ if $a > 1$.

If $0 < a < 1$, then $\ln a < 0$ and $D_x\, a^x = a^x \ln a < 0$. Thus, a^x is decreasing for every x if $0 < a < 1$.

The graphs of $y = 2^x$ and $y = (\frac{1}{2})^x = 2^{-x}$ are sketched in Figures 7.16 and 7.17 (on the following page). The graph of $y = a^x$ has the general shape illustrated in Figure 7.16 or 7.17 if $a > 1$ or $0 < a < 1$, respectively.

FIGURE 7.17



If $u = g(x)$, *it is important to distinguish between expressions of the form* a^u *and* u^a. To differentiate a^u, we use (7.28); for u^a the power rule must be employed, as illustrated in the next example.

EXAMPLE 1 Find y' if $y = (x^2 + 1)^{10} + 10^{x^2+1}$.

SOLUTION Using the power rule for functions and Theorem (7.28), we obtain

$$\begin{aligned} y' &= 10(x^2 + 1)^9(2x) + (10^{x^2+1} \ln 10)(2x) \\ &= 20x[(x^2 + 1)^9 + 10^{x^2} \ln 10]. \end{aligned}$$

The integration formula in (i) of the next theorem may be verified by showing that the integrand is the derivative of the expression on the right side of the equation. Formula (ii) follows from Theorem (7.28)(ii), where $u = g(x)$.

Theorem **(7.29)**

(i) $\int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + C$ **(ii)** $\int a^u \, du = \left(\frac{1}{\ln a}\right) a^u + C$

EXAMPLE 2 Evaluate

(a) $\int 3^x \, dx$ **(b)** $\int x 3^{(x^2)} \, dx$

SOLUTION

(a) Using (i) of Theorem (7.29) yields

$$\int 3^x \, dx = \left(\frac{1}{\ln 3}\right) 3^x + C.$$

(b) To use (ii) of Theorem (7.29), we make the substitution

$$u = x^2, \qquad du = 2x \, dx$$

and proceed as follows:

$$\begin{aligned} \int x 3^{(x^2)} \, dx &= \tfrac{1}{2} \int 3^{(x^2)}(2x) \, dx = \tfrac{1}{2} \int 3^u \, du \\ &= \frac{1}{2}\left(\frac{1}{\ln 3}\right) 3^u + C = \left(\frac{1}{2 \ln 3}\right) 3^{(x^2)} + C \end{aligned}$$

EXAMPLE 3 An important problem in oceanography is determining the light intensity at different ocean depths. The *Beer-Lambert law* states that at a depth x (in meters), the light intensity $I(x)$ (in calories/cm^2/sec) is given by $I(x) = I_0 a^x$, where I_0 and a are positive constants.

(a) What is the light intensity at the surface?

(b) Find the rate of change of the light intensity with respect to depth at a depth x.

(c) If $a = 0.4$ and $I_0 = 10$, find the average light intensity between the surface and a depth of x meters.

(d) Show that $I(x) = I_0 e^{kx}$ for some constant k.

SOLUTION

(a) At the surface, $x = 0$ and

$$I(0) = I_0 a^0 = I_0.$$

Hence the light intensity at the surface is I_0.

(b) The rate of change of $I(x)$ with respect to x is $I'(x)$. Thus,

$$I'(x) = I_0(a^x \ln a) = (\ln a)(I_0 a^x) = (\ln a)I(x).$$

Hence the rate of change $I'(x)$ at depth x is directly proportional to $I(x)$, and the constant of proportionality is $\ln a$.

(c) If $I(x) = 10(0.4)^x$, then, by Definition (5.29), the average value of I on the interval $[0, 5]$ is

$$\begin{aligned} I_{\text{av}} &= \frac{1}{5-0}\int_0^5 10(0.4)^x \, dx = 2\int_0^5 (0.4)^x \, dx \\ &= 2\left[\frac{1}{\ln (0.4)}(0.4)^x\right]_0^5 = \frac{2}{\ln (0.4)}\left[(0.4)^5 - (0.4)^0\right] \\ &= \frac{-1.97952}{\ln (0.4)} \approx 2.16. \end{aligned}$$

(d) Using Definition (7.26) yields

$$I(x) = I_0 a^x = I_0 e^{x \ln a} = I_0 e^{kx},$$

where $k = \ln a$.

If $a \neq 1$ and $f(x) = a^x$, then f is a one-to-one function. Its inverse function is denoted by $\log_a$ and is called the **logarithmic function with base a**. Another way of stating this relationship is as follows.

Definition of $\log_a x$ **(7.30)**

$$\boxed{y = \log_a x \quad \text{if and only if} \quad x = a^y}$$

The expression $\log_a x$ is called **the logarithm of x with base a**. In this terminology, natural logarithms are logarithms with base e; that is,

$$\ln x = \log_e x.$$

Laws of logarithms similar to Theorem (7.12) are true for logarithms with base a.

To obtain the relationship between $\log_a$ and ln, consider $y = \log_a x$, or, equivalently, $x = a^y$. Taking the natural logarithm of both sides of the last equation gives us $\ln x = y \ln a$, or $y = (\ln x)/(\ln a)$. This proves that

$$\log_a x = \frac{\ln x}{\ln a}.$$

Differentiating both sides of the last equation leads to (i) of the next theorem. Using the chain rule and generalizing to absolute values as in Theorem (7.11) gives us (ii), where $u = g(x)$.

Theorem **(7.31)**

(i) $D_x \log_a x = D_x \left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{\ln a} \cdot \dfrac{1}{x}$

(ii) $D_x \log_a |u| = D_x \left(\dfrac{\ln |u|}{\ln a}\right) = \dfrac{1}{\ln a} \cdot \dfrac{1}{u} D_x u$

ILLUSTRATION

- $D_x \log_2 x = D_x \left(\dfrac{\ln x}{\ln 2}\right) = \dfrac{1}{\ln 2} \cdot \dfrac{1}{x} = \dfrac{1}{(\ln 2)x}$
- $D_x \log_2 |x^2 - 9| = D_x \left(\dfrac{\ln |x^2 - 9|}{\ln 2}\right) = \dfrac{1}{\ln 2} \cdot \dfrac{1}{x^2 - 9} \cdot 2x = \dfrac{2x}{(\ln 2)(x^2 - 9)}$

Logarithms with base 10 are useful for certain applications (see Exercises 48–51 and 53). We refer to such logarithms as **common logarithms** and use the symbol **log *x*** as an abbreviation for $\log_{10} x$. This notation is used in the next example.

EXAMPLE 4 If $f(x) = \log \sqrt[3]{(2x + 5)^2}$, find $f'(x)$.

SOLUTION We first write $f(x) = \log (2x + 5)^{2/3}$. The law $\log u^r = r \log u$ is true only if $u > 0$; however, since $(2x + 5)^{2/3} = |2x + 5|^{2/3}$, we may proceed as follows:

$$\begin{aligned} f(x) &= \log (2x + 5)^{2/3} \\ &= \log |2x + 5|^{2/3} \\ &= \tfrac{2}{3} \log |2x + 5| \\ &= \frac{2}{3} \frac{\ln |2x + 5|}{\ln 10} \end{aligned}$$

Differentiating yields

$$f'(x) = \frac{2}{3} \cdot \frac{1}{\ln 10} \cdot \frac{1}{2x + 5} (2) = \frac{4}{3(2x + 5) \ln 10}.$$

Now that we have defined irrational exponents, we may consider the **general power function** f given by $f(x) = x^c$ for any real number c. If c is irrational, then, by definition, the domain of f is the set of positive real numbers. Using Definition (7.26) and Theorems (7.22) and (7.11)(i), we have

$$\begin{aligned} D_x x^c &= D_x e^{c \ln x} = e^{c \ln x} D_x (c \ln x) \\ &= e^{c \ln x}\left(\frac{c}{x}\right) = x^c\left(\frac{c}{x}\right) = cx^{c-1}. \end{aligned}$$

This proves that the power rule is true for irrational as well as rational exponents. The power rule for functions may also be extended to irrational exponents.

ILLUSTRATION

- $D_x(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$
- $D_x(1+e^{2x})^\pi = \pi(1+e^{2x})^{\pi-1}D_x(1+e^{2x})$
 $= \pi(1+e^{2x})^{\pi-1}(2e^{2x}) = 2\pi e^{2x}(1+e^{2x})^{\pi-1}$

EXAMPLE 5 If $y = x^x$ and $x > 0$, find $D_x\, y$.

SOLUTION Since the exponent in x^x is a variable, the power rule may not be used. Similarly, Theorem (7.28) is not applicable, since the base a is not a fixed real number. However, by Definition (7.26), $x^x = e^{x\ln x}$ for every $x > 0$, and hence

$$\begin{aligned} D_x(x^x) &= D_x(e^{x\ln x}) \\ &= e^{x\ln x}D_x(x\ln x) \\ &= e^{x\ln x}\left[x\left(\frac{1}{x}\right) + (1)\ln x\right] \\ &= x^x(1+\ln x). \end{aligned}$$

Another way of solving this problem is to use the method of logarithmic differentiation introduced in the preceding section. In this case we take the natural logarithm of both sides of the equation $y = x^x$ and then differentiate implicitly as follows:

$$\begin{aligned} \ln y &= \ln x^x = x\ln x \\ D_x(\ln y) &= D_x(x\ln x) \\ \frac{1}{y}D_x\, y &= 1 + \ln x \\ D_x\, y &= y(1+\ln x) = x^x(1+\ln x) \end{aligned}$$

We shall conclude this section by expressing the number e as a limit.

Theorem **(7.32)**

(i) $\lim_{h\to 0}(1+h)^{1/h} = e$ **(ii)** $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

PROOF Applying the definition of derivative (3.5) to $f(x) = \ln x$ and using laws of logarithms yields

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0}\frac{1}{h}\ln\frac{x+h}{x} \\ &= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right) = \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{1/h}. \end{aligned}$$

Since $f'(x) = 1/x$, we have, for $x = 1$,

$$1 = \lim_{h \to 0} \ln (1 + h)^{1/h}.$$

We next observe, from Theorem (7.19), that

$$(1 + h)^{1/h} = e^{\ln (1+h)^{1/h}}.$$

Since the natural exponential function is continuous at 1, it follows from Theorem (2.25) that

$$\begin{aligned}\lim_{h \to 0} (1 + h)^{1/h} &= \lim_{h \to 0} \left[e^{\ln (1+h)^{1/h}}\right] \\ &= e^{\left[\lim_{h \to 0} \ln (1+h)^{1/h}\right]} = e^1 = e.\end{aligned}$$

This establishes part (i) of the theorem. The limit in part (ii) may be obtained by introducing the change of variable $n = 1/h$ with $h > 0$.

The formulas in Theorem (7.32) are sometimes used to *define* the number e. You may find it instructive to calculate $(1 + h)^{1/h}$ for numerically small values of h. Some approximate values are given in the following table.

h	$(1 + h)^{1/h}$	h	$(1 + h)^{1/h}$
0.01	2.704814	−0.01	2.731999
0.001	2.716924	−0.001	2.719642
0.0001	2.718146	−0.0001	2.718418
0.00001	2.718268	−0.00001	2.718295
0.000001	2.718280	−0.000001	2.718283

To five decimal places, $e \approx 2.71828$.

EXERCISES 7.5

Exer. 1–26: Find $f'(x)$ if $f(x)$ is the given expression.

1 7^x

2 5^{-x}

3 8^{x^2+1}

4 $9^{\sqrt{x}}$

5 $\log (x^4 + 3x^2 + 1)$

6 $\log_3 |6x - 7|$

7 5^{3x-4}

8 3^{2-x^2}

9 $(x^2 + 1)10^{1/x}$

10 $(10^x + 10^{-x})^{10}$

11 $\log (3x^2 + 2)^5$

12 $\log \sqrt{x^2 + 1}$

13 $\log_5 \left|\dfrac{6x + 4}{2x - 3}\right|$

14 $\log \left|\dfrac{1 - x^2}{2 - 5x^3}\right|$

15 $\log \ln x$

16 $\ln \log x$

17 $x^e + e^x$

18 $x^\pi \pi^x$

19 $(x + 1)^x$

20 x^{4+x^2}

21 $2^{\sin^2 x}$

22 $4^{\sec 3x}$

23 $x^{\tan x}$

24 $(\cos 2x)^x$

25 (a) e^e (b) $x^{\sqrt{5}}$ (c) $x^{\sqrt{5}}$ (d) $(\sqrt{5})^x$ (e) $x^{(x^2)}$

26 (a) π^π (b) x^4 (c) x^π (d) π^x (e) x^{2x}

Exer. 27–42: Evaluate the integral.

27 (a) $\int 7^x \, dx$ (b) $\int_{-2}^{1} 7^x \, dx$

28 (a) $\int 3^x \, dx$ (b) $\int_{-1}^{0} 3^x \, dx$

29 (a) $\int 5^{-2x} \, dx$ (b) $\int_1^2 5^{-2x} \, dx$

30 (a) $\int 2^{3x-1} \, dx$ (b) $\int_{-1}^{1} 2^{3x-1} \, dx$

31 $\int 10^{3x} \, dx$

32 $\int 5^{-5x} \, dx$

33 $\int x(3^{-x^2}) \, dx$

34 $\int \dfrac{(2^x + 1)^2}{2^x} \, dx$

35 $\int \frac{2^x}{2^x + 1}\,dx$

36 $\int \frac{3^x}{\sqrt{3^x + 4}}\,dx$

37 $\int \frac{1}{x \log x}\,dx$

38 $\int \frac{10^{\sqrt{x}}}{\sqrt{x}}\,dx$

39 $\int 3^{\cos x} \sin x\,dx$

40 $\int \frac{5^{\tan x}}{\cos^2 x}\,dx$

41 (a) $\int \pi^\pi\,dx$ (b) $\int x^4\,dx$

(c) $\int x^\pi\,dx$ (d) $\int \pi^x\,dx$

42 (a) $\int e^e\,dx$ (b) $\int x^5\,dx$

(c) $\int x^{\sqrt{5}}\,dx$ (d) $\int (\sqrt{5})^x\,dx$

43 Find the area of the region bounded by the graphs of $y = 2^x$, $x + y = 1$, and $x = 1$.

44 The region under the graph of $y = 3^{-x}$ from $x = 1$ to $x = 2$ is revolved about the x-axis. Find the volume of the resulting solid.

45 An economist predicts that the buying power $B(t)$ of a dollar t years from now will decrease according to the formula $B(t) = (0.95)^t$.

(a) At approximately what rate will the buying power be decreasing two years from now?

(b) Estimate the *average buying power* of the dollar over the next two years.

46 When a person takes a 100-milligram tablet of an asthma drug orally, the rate R at which the drug enters the bloodstream is predicted to be $R = 5(0.95)^t$ mg/min. If the bloodstream does not contain any trace of the drug when the tablet is taken, determine the number of minutes needed for 50 milligrams to enter the bloodstream.

47 One thousand trout, each one year old, are introduced into a large pond. The number still alive after t years is predicted to be $N(t) = 1000(0.9)^t$.

(a) Approximate the death rate dN/dt at times $t = 1$ and $t = 5$. At what rate is the population decreasing when $N = 500$?

(b) The weight $W(t)$ (in pounds) of an individual trout is expected to increase according to the formula $W(t) = 0.2 + 1.5t$. After approximately how many years is the total number of pounds of trout in the pond a maximum?

48 The vapor pressure P (in psi), a measure of the volatility of a liquid, is related to its temperature T (in °F) by the *Antoine equation*: $\log P = a + [b/(c + T)]$, for constants a, b, and c. Vapor pressure increases rapidly with an increase in temperature. Find conditions on a, b, and c that guarantee that P is an increasing function of T.

49 Chemists use a number denoted by pH to describe quantitatively the acidity or basicity of solutions. By definition, $\text{pH} = -\log [\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. For a certain brand of vinegar it is estimated (with a maximum percentage error of $\pm 0.5\%$) that $[\text{H}^+] \approx 6.3 \times 10^{-3}$. Calculate the pH and use differentials to estimate the maximum percentage error in the calculation.

50 The magnitude R (on the Richter scale) of an earthquake of intensity I may be found by means of the formula $R = \log (I/I_0)$, where I_0 is a certain minimum intensity. Suppose the intensity of an earthquake is estimated to be 100 times I_0. If the maximum percentage error in the estimate is $\pm 1\%$, use differentials to approximate the maximum percentage error in the calculated value of R.

51 Let $R(x)$ be the reaction of a subject to a stimulus of strength x. For example, if the stimulus x is *saltiness* (in grams of salt/liter), $R(x)$ may be the subject's estimate of how salty the solution tasted on a scale from 0 to 10. A function that has been proposed to relate R to x is given by the *Weber-Fechner formula*: $R = a \log (x/x_0)$, where a is a positive constant.

(a) Show that $R = 0$ for the *threshold stimulus* $x = x_0$.

(b) The derivative $S = dR/dx$ is the *sensitivity* at stimulus level x and measures the ability to detect small changes in stimulus level. Show that S is inversely proportional to x, and compare $S(x)$ to $S(2x)$.

52 The loudness of sound, as experienced by the human ear, is based on intensity level. A formula used for finding the intensity level α that corresponds to a sound intensity I is $\alpha = 10 \log (I/I_0)$ decibels, where I_0 is a special value of I agreed to be the weakest sound that can be detected by the ear under certain conditions. Find the rate of change of α with respect to I if

(a) I is 10 times as great as I_0

(b) I is 1000 times as great as I_0

(c) I is 10,000 times as great as I_0 (This is the intensity level of the average voice.)

53 If a principal of P dollars is invested in a savings account for t years and the yearly interest rate r (expressed as a decimal) is compounded n times per year, then the amount A in the account after t years is given by the *compound interest formula*: $A = P[1 + (r/n)]^{nt}$.

(a) Let $h = r/n$ and show that

$$\ln A = \ln P + rt \ln (1 + h)^{1/h}.$$

(b) Let $n \to \infty$ and use the expression in part (a) to establish the formula $A = Pe^{rt}$ for interest *compounded continuously*.

54 Establish Theorem (7.32)(ii) by using the limit in part (i) and the change of variable $n = 1/h$.

55 Prove that $\lim_{n\to\infty} [1 + (x/n)]^n = e^x$ by letting $h = x/n$ and using Theorem (7.32)(i).

c 56 By letting $h = 0.1$, 0.01, and 0.001, predict which of the following expressions gives the best approximation of e for small values of h:

$$(1+h)^{1/h}, \quad (1+h+h^2)^{1/h}, \quad (1+h+\tfrac{1}{2}h^2)^{1/h}$$

c 57 Graph, on the same coordinate axes, $y = 2^{-x}$ and $y = \log_2 x$.

(a) Estimate the x-coordinate of the point of intersection of the graphs.

(b) If the region R bounded by the graphs and the line $x = 1$ is revolved about the x-axis, set up an integral that can be used to approximate the volume of the resulting solid.

(c) Use Simpson's rule, with $n = 4$, to approximate the integral in (b).

7.6 LAWS OF GROWTH AND DECAY

Suppose that a physical quantity varies with time and that the magnitude of the quantity at time t is given by $q(t)$, where q is differentiable and $q(t) > 0$ for every t. The derivative $q'(t)$ is the rate of change of $q(t)$ with respect to time. In many applications this rate of change is directly proportional to the magnitude of the quantity at time t; that is,

$$q'(t) = cq(t)$$

for some constant c. The number of bacteria in certain cultures behaves in this way. If the number of bacteria $q(t)$ is small, then the rate of increase $q'(t)$ is small; however, as the number of bacteria increases, the *rate of increase* also increases. The decay of a radioactive substance obeys a similar law: as the amount of matter decreases, the rate of decay—that is, the amount of radiation—also decreases. As a final illustration, suppose an electrical condenser is allowed to discharge. If the charge on the condenser is large at the outset, the rate of discharge is also large, but as the charge weakens, the condenser discharges less rapidly.

In applied problems the equation $q'(t) = cq(t)$ is often expressed in terms of differentials. Thus, if $y = q(t)$, we may write

$$\frac{dy}{dt} = cy, \quad \text{or} \quad dy = cy\,dt.$$

Dividing both sides of the last equation by y, we obtain

$$\frac{1}{y}\,dy = c\,dt.$$

Since it is possible to **separate the variables** y and t—in the sense that they can be placed on opposite sides of the equals sign—the differential equation $dy/dt = cy$ is a **separable differential equation**. We will study such equations in more detail later in the text and will show that solutions can be found by integrating both sides of the "separated" equation $(1/y)\,dy = c\,dt$. Thus,

$$\int \frac{1}{y}\,dy = \int c\,dt$$

and, assuming $y > 0$,

$$\ln y = ct + d$$

for some constant d. It follows that

$$y = e^{ct+d} = e^d e^{ct}.$$

If y_0 denotes the initial value of y (that is, the value corresponding to $t = 0$), then letting $t = 0$ in the last equation gives us

$$y_0 = e^d e^0 = e^d,$$

and hence the solution $y = e^d e^{ct}$ may be written

$$y = y_0 e^{ct}.$$

We have proved the following theorem.

Theorem **(7.33)**

Let y be a differentiable function of t such that $y > 0$ for every t, and let y_0 be the value of y at $t = 0$. If $dy/dt = cy$ for some constant c, then

$$y = y_0 e^{ct}.$$

The preceding theorem states that *if the rate of change of $y = q(t)$ with respect to t is directly proportional to y, then y may be expressed in terms of an exponential function.* If y increases with t, the formula $y = y_0 e^{ct}$ is a **law of growth**, and if y decreases, it is a **law of decay**.

EXAMPLE 1 The number of bacteria in a culture increases from 600 to 1800 in two hours. Assuming that the rate of increase is directly proportional to the number of bacteria present, find

(a) a formula for the number of bacteria at time t

(b) the number of bacteria at the end of four hours

SOLUTION

(a) Let $y = q(t)$ denote the number of bacteria after t hours. Thus, $y_0 = q(0) = 600$ and $q(2) = 1800$. By hypothesis,

$$\frac{dy}{dt} = cy.$$

Following exactly the same steps used in the proof of Theorem (7.33), we obtain

$$y = y_0 e^{ct} = 600e^{ct}.$$

Since $y = 1800$ when $t = 2$, we obtain the following equivalent equations:

$$1800 = 600e^{2c}, \quad 3 = e^{2c}, \quad e^c = 3^{1/2}.$$

Substituting for e^c in $y = 600e^{ct}$ gives us

$$y = 600(3^{1/2})^t, \quad \text{or} \quad y = 600(3)^{t/2}.$$

(b) Letting $t = 4$ in $y = 600(3)^{t/2}$ yields

$$y = 600(3)^{4/2} = 600(9) = 5400.$$

EXAMPLE 2 Radium decays exponentially and has a half-life of approximately 1600 years; that is, given any quantity, one-half of it will disintegrate in 1600 years.

(a) Find a formula for the amount y remaining from 50 milligrams of pure radium after t years.

(b) When will the amount remaining be 20 milligrams?

SOLUTION

(a) If we let $y = q(t)$, then

$$y_0 = q(0) = 50 \quad \text{and} \quad q(1600) = \tfrac{1}{2}(50) = 25.$$

Since $dy/dt = cy$ for some c, it follows from Theorem (7.33) that

$$y = 50e^{ct}.$$

Since $y = 25$ when $t = 1600$,

$$25 = 50e^{1600c}, \quad \text{or} \quad e^{1600c} = \tfrac{1}{2}.$$

Hence

$$e^c = (\tfrac{1}{2})^{1/1600} = 2^{-1/1600}.$$

Substituting for e^c in $y = 50e^{ct}$ gives us

$$y = 50(2^{-1/1600})^t, \quad \text{or} \quad y = 50(2)^{-t/1600}.$$

(b) Using $y = 50(2)^{-t/1600}$, we see that the value of t at which $y = 20$ is a solution of the equation

$$20 = 50(2)^{-t/1600}, \quad \text{or} \quad 2^{t/1600} = \tfrac{5}{2}.$$

Taking the natural logarithm of each side, we obtain

$$\frac{t}{1600} \ln 2 = \ln \frac{5}{2},$$

or

$$t = \frac{1600 \ln \frac{5}{2}}{\ln 2} \approx 2115 \text{ yr}.$$

EXAMPLE 3 Newton's law of cooling states that the rate at which an object cools is directly proportional to the difference in temperature between the object and the surrounding medium. If an object cools from 125 °F to 100 °F in half an hour when surrounded by air at a temperature of 75 °F, find its temperature at the end of the next half hour.

SOLUTION Let y denote the temperature of the object after t hours of cooling. Since the temperature of the surrounding medium is 75°, the difference in temperature is $y - 75$, and hence, by Newton's law of cooling,

$$\frac{dy}{dt} = c(y - 75)$$

for some constant c. We separate variables and integrate as follows:

$$\frac{1}{y-75}\,dy = c\,dt$$

$$\int \frac{1}{y-75}\,dy = \int c\,dt$$

$$\ln(y-75) = ct + b$$

for some constant b. The last equation is equivalent to

$$y - 75 = e^{ct+b} = e^{b}e^{ct}.$$

Since $y = 125$ when $t = 0$,

$$125 - 75 = e^{b}e^{0} = e^{b}, \quad \text{or} \quad e^{b} = 50.$$

Hence

$$y - 75 = 50e^{ct}, \quad \text{or} \quad y = 50e^{ct} + 75.$$

Using the fact that $y = 100$ when $t = \frac{1}{2}$ leads to the following equivalent equations:

$$100 = 50e^{c/2} + 75, \quad e^{c/2} = \tfrac{25}{50} = \tfrac{1}{2}, \quad e^{c} = \tfrac{1}{4}$$

Substituting $\frac{1}{4}$ for e^{c} in $y = 50e^{ct} + 75$ gives us a formula for the temperature after t hours:

$$y = 50(\tfrac{1}{4})^{t} + 75$$

In particular, if $t = 1$,

$$y = 50(\tfrac{1}{4}) + 75 = 87.5\,^{\circ}\mathrm{F}.$$

In biology, a function G is sometimes used as follows to estimate the size of a quantity at time t:

$$G(t) = ke^{(-Ae^{-Bt})}$$

for positive constants k, A, and B. The function G is called a **Gompertz growth function**. It is always positive and increasing, but has a limit as t increases without bound. The graph of G is called a **Gompertz growth curve**.

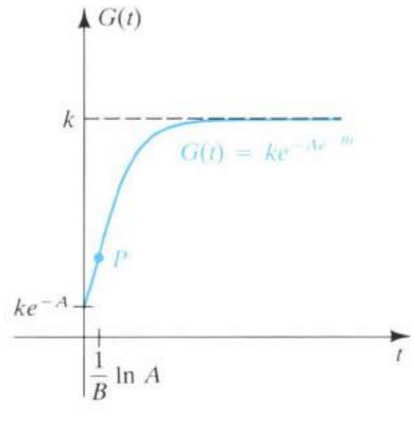
EXAMPLE 4 Discuss and sketch the graph of the Gompertz growth function G.

SOLUTION We first observe that the y-intercept is $G(0) = ke^{-A}$ and that $G(t) > 0$ for every t. Differentiating twice, we obtain

$$\begin{aligned} G'(t) &= ke^{(-Ae^{-Bt})}\,D_t\,(-Ae^{-Bt}) \\ &= ABke^{(-Bt-Ae^{-Bt})} \end{aligned}$$

$$\begin{aligned} G''(t) &= ABke^{(-Bt-Ae^{-Bt})}\,D_t\,(-Bt - Ae^{-Bt}) \\ &= ABk(-B + ABe^{-Bt})e^{-Bt-Ae^{-Bt}} \end{aligned}$$

FIGURE 7.18



Since $G'(t) > 0$ for every t, the function G is increasing on $[0, \infty)$. The second derivative $G''(t)$ is zero if

$$-B + ABe^{-Bt} = 0 \quad \text{or} \quad e^{Bt} = A.$$

Solving the last equation for t gives us $t = (1/B) \ln A$, which is a critical number for the function G'. We leave it as an exercise to show that at this time the rate of growth G' has a maximum value Bk/e. We can also show that

$$\lim_{t \to \infty} G'(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} G(t) = k.$$

Hence, as t increases without bound, the rate of growth approaches 0 and the graph of G has a horizontal asymptote $y = k$. A typical graph is sketched in Figure 7.18. The point P on the graph, corresponding to $t = (1/B) \ln A$, is a point of inflection, and the concavity changes from upward to downward at P.

In the next example we consider a physical quantity that increases to a maximum value and then decreases asymptotically to 0.

EXAMPLE 5 When uranium disintegrates into lead, one step in the process is the radioactive decay of radium into radon gas. Radon gas enters homes by diffusing through the soil into basements, where it presents a health hazard if inhaled. If a quantity Q of radium is present initially, then the amount of radon gas present after t years is given by

$$A(t) = \frac{c_1 Q}{c_2 - c_1}(e^{-c_1 t} - e^{-c_2 t}),$$

where $c_1 = \frac{1}{1600} \ln 2$ and $c_2 = \frac{1}{0.0105} \ln 2$ are the *decay constants* for radium and radon gas, respectively.

(a) Find the amount of radon gas present initially and after a long period of time.

(b) When is the amount of radon gas greatest?

SOLUTION

(a) The initial amount of radon gas is

$$A(0) = \frac{c_1 Q}{c_2 - c_1}(e^0 - e^0) = 0.$$

If we let t increase without bound, then

$$\begin{aligned}\lim_{t \to \infty} A(t) &= \frac{c_1 Q}{c_2 - c_1} \lim_{t \to \infty} (e^{-c_1 t} - e^{-c_2 t}) \\ &= \frac{c_1 Q}{c_2 - c_1}(0 - 0) = 0.\end{aligned}$$

Hence, over a long period of time, the amount of radon gas decreases to 0.

(b) To find the critical numbers of A, we differentiate, obtaining

$$A'(t) = \frac{c_1 Q}{c_2 - c_1}(-c_1 e^{-c_1 t} + c_2 e^{-c_2 t}).$$

Thus, $A'(t) = 0$ if

$$c_1 e^{-c_1 t} = c_2 e^{-c_2 t}, \quad \text{or} \quad e^{(c_2 - c_1)t} = \frac{c_2}{c_1}.$$

It follows that

$$(c_2 - c_1)t = \ln \frac{c_2}{c_1} = \ln c_2 - \ln c_1,$$

or

$$t = \frac{\ln c_2 - \ln c_1}{c_2 - c_1}.$$

This value of t yields the maximum value of A. Substituting the given values for c_1 and c_2 gives us the approximation

$$t \approx 0.181 \text{ year} \approx 66 \text{ days}.$$

EXERCISES 7.6

1 The number of bacteria in a culture increases from 5000 to 15,000 in 10 hours. Assuming that the rate of increase is proportional to the number of bacteria present, find a formula for the number of bacteria in the culture at any time t. Estimate the number at the end of 20 hours. When will the number be 50,000?

2 The polonium isotope ^{210}Po has a half-life of approximately 140 days. If a sample weighs 20 milligrams initially, how much remains after t days? Approximately how much will be left after two weeks?

3 If the temperature is constant, then the rate of change of barometric pressure p with respect to altitude h is proportional to p. If $p = 30$ in. at sea level and $p = 29$ in. when $h = 1000$ ft, find the pressure at an altitude of 5000 feet.

4 The population of a city is increasing at the rate of 5% per year. If the present population is 500,000 and the rate of increase is proportional to the number of people, what will the population be in 10 years?

5 Agronomists use the assumption that a quarter acre of land is required to provide food for one person and estimate that there are 10 billion acres of tillable land in the world. Hence a maximum population of 40 billion people can be sustained if no other food source is available. The world population at the beginning of 1980 was approximately 4.5 billion. Assuming that the population increases at a rate of 2% per year and the rate of increase is proportional to the number of people, when will the maximum population be reached?

6 A metal plate that has been heated cools from 180°F to 150°F in 20 minutes when surrounded by air at a temperature of 60°F. Use Newton's law of cooling (see Example 3) to approximate its temperature at the end of one hour of cooling. When will the temperature be 100°F?

7 An outdoor thermometer registers a temperature of 40°F. Five minutes after it is brought into a room where the temperature is 70°F, the thermometer registers 60°F. When will it register 65°F?

8 The rate at which salt dissolves in water is directly proportional to the amount that remains undissolved. If 10 pounds of salt are placed in a container of water and 4 pounds dissolve in 20 minutes, how long will it take for two more pounds to dissolve?

9 According to Kirchhoff's first law for electrical circuits, $V = RI + L(dI/dt)$, where the constants V, R, and L denote the electromotive force, the resistance, and the inductance, respectively, and I denotes the current at time t. If the electromotive force is terminated at time $t = 0$ and if the current is I_0 at the instant of removal, prove that $I = I_0 e^{-Rt/L}$.

10 A physicist finds that an unknown radioactive substance registers 2000 counts per minute on a Geiger counter. Ten days later the substance registers 1500 counts per minute. Approximate its half-life.

11 The air pressure P (in atmospheres) at an elevation of z meters above sea level is a solution of the differential equation $dP/dz = -9.81\rho(z)$, where $\rho(z)$ is the density of air at elevation z. Assuming that air obeys the ideal gas law, this differential equation can be rewritten as $dP/dz = -0.0342P/T$, where T is the temperature (in °K) at elevation z. If $T = 288 - 0.01z$ and if the pressure is 1 atmosphere at sea level, express P as a function of z.

12 During the first month of growth for crops such as maize, cotton, and soybeans, the rate of growth (in grams/day) is proportional to the present weight W. For a species of cotton, $dW/dt = 0.21W$. Predict the weight of a plant at the end of the month ($t = 30$) if the plant weighs 70 milligrams at the beginning of the month.

13 Radioactive strontium-90, ^{90}Sr, with a half-life of 29 years, can cause bone cancer in humans. The substance is carried by acid rain, soaks into the ground, and is passed through the food chain. The radioactivity level in a particular field is estimated to be 2.5 times the safe level S. For approximately how many years will this field be contaminated?

14 The radioactive tracer ^{51}Cr, with a half-life of 27.8 days, can be used in medical testing to locate the position of a placenta in a pregnant woman. Often the tracer must be ordered from a medical supply lab. If 35 units are needed for a test and delivery from the lab requires two days, estimate the minimum number of units that should be ordered.

15 Veterinarians use sodium pentobarbital to anesthetize animals. Suppose that to anesthetize a dog, 30 milligrams are required for each kilogram of body weight. If sodium pentobarbital is eliminated exponentially from the bloodstream and half is eliminated in four hours, approximate the single dose that will anesthetize a 20-kilogram dog for 45 minutes.

16 In the study of lung physiology, the following differential equation is used to describe the transport of a substance across a capillary wall:

$$\frac{dh}{dt} = -\frac{V}{Q}\left(\frac{h}{k+h}\right),$$

where h is the hormone concentration in the bloodstream, t is time, V is the maximum transport rate, Q is the volume of the capillary, and k is a constant that measures the affinity between the hormones and enzymes that assist with the transport process. Find the general solution of the differential equation.

17 A space probe is shot upward from the earth. If air resistance is disregarded, a differential equation for the velocity after burnout is $v\,(dv/dy) = -ky^{-2}$, where y is the distance from the center of the earth and k is a positive constant. If y_0 is the distance from the center of the earth at burnout and v_0 is the corresponding velocity, express v as a function of y.

18 At high temperatures, nitrogen dioxide, NO_2, decomposes into NO and O_2. If $y(t)$ is the concentration of NO_2 (in moles per liter), then, at 600°K, $y(t)$ changes according to the *second-order reaction law* $dy/dt = -0.05y^2$ for time t in seconds. Express y in terms of t and the initial concentration y_0.

19 The technique of carbon-14 dating is used to determine the age of archeological or geological specimens. This method is based on the fact that the unstable isotope carbon-14 (^{14}C) is present in the CO_2 in the atmosphere. Plants take in carbon from the atmosphere; when they die, the ^{14}C that has accumulated begins to decay, with a half-life of approximately 5700 years. By measuring the amount of ^{14}C that remains in a specimen, it is possible to approximate when the organism died. Suppose that a bone fossil contains 20% as much ^{14}C as an equal amount of carbon in present-day bone. Approximate the age of the bone.

20 Refer to Exercise 19. The hydrogen isotope ^3_1H, which has a half-life of 12.3 years, is produced in the atmosphere by cosmic rays and is brought to earth by rain. If the wood siding of an old house contains 10% as much ^3_1H as the siding on a similar new house, approximate the age of the old house.

21 The earth's atmosphere absorbs approximately 32% of the sun's incoming radiation. The earth also emits radiation (mostly in the form of heat), and the atmosphere absorbs approximately 93% of this outgoing radiation. This difference in absorption of incoming and outgoing radiation by the atmosphere is called the *greenhouse effect*. Changes in this balance will affect the earth's climate. Suppose I_0 is the intensity of the sun's radiation and I is the intensity of the radiation after traveling a distance x through the atmosphere. If $\rho(h)$ is the density of the atmosphere at height h, then the *optical thickness* is $f(x) = k\int_0^x \rho(h)\,dh$, where k is an absorption constant, and I is given by $I = I_0e^{-f(x)}$. Show that $dI/dx = -k\rho(x)I$.

22 Certain learning processes may be illustrated by the graph of $f(x) = a + b(1 - e^{-cx})$ for positive constants a, b, and c. Suppose a manufacturer estimates that a new employee can produce five items the first day on the job. As the employee becomes more proficient, the daily production increases until a certain maximum production is reached. Suppose that on the nth day on the job, the number $f(n)$ of items produced is approximated by the formula $f(n) = 3 + 20(1 - e^{-0.1n})$.

(a) Estimate the number of items produced on the fifth day, the ninth day, the twenty-fourth day, and the thirtieth day.

(b) Sketch the graph of f from $n = 0$ to $n = 30$. (Graphs of this type are called *learning curves* and are used frequently in education and psychology.)

(c) What happens as n increases without bound?

23 A spherical cell has volume V and surface area S. A simple model for cell growth before mitosis assumes that the rate of growth dV/dt is proportional to the surface area of the cell. Show that $dV/dt = kV^{2/3}$ for some $k > 0$, and express V as a function of t.

24 In Theorem (7.33) we assumed that the rate of change of a quantity $q(t)$ at time t is directly proportional to $q(t)$. Find $q(t)$ if its rate of change is directly proportional to $[q(t)]^2$.

25 Refer to Example 4.

(a) Verify that Bk/e is a maximum value for G'.

(b) Show that $\lim_{t\to\infty} G'(t) = 0$ and $\lim_{t\to\infty} G(t) = k$.

(c) Sketch the graph of G if $k = 10$, $A = 2$, and $B = 1$.

c 26 Graph the Gompertz growth function G on the interval $[0, 5]$ for $k = 1.1$, $A = 3.2$, and $B = 1.1$.

7.7 REVIEW EXERCISES

1–77 odd . 81-91 EOO

Exer. 1–2: Find $f^{-1}(x)$.

1 $f(x) = 10 - 15x$

2 $f(x) = 9 - 2x^2, \quad x \le 0$

Exer. 3–4: Show that the function f has an inverse function, and find $[D_x f^{-1}(x)]_{x=a}$ for the given number a.

3 $f(x) = 2x^3 - 8x + 5, \quad -1 \le x \le 1; \quad a = 5$

4 $f(x) = e^{3x} + 2e^x - 5, \quad x \ge 0; \quad a = -2$

Exer. 5–38: Find $f'(x)$ if $f(x)$ is the given expression.

5 $\ln|4 - 5x^3|^5$

6 $\ln|x^2 - 7|^3$

7 $(1 - 2x)\ln|1 - 2x|$

8 $\log\left|\dfrac{2 - 9x}{1 - x^2}\right|$

9 $\ln\dfrac{(3x + 2)^4\sqrt{6x - 5}}{8x - 7}$

10 $\ln\sqrt[4]{\dfrac{x}{3x + 5}}$

11 $\dfrac{1}{\ln(2x^2 + 3)}$

12 $\dfrac{\ln x}{e^{2x} + 1}$

13 $\dfrac{x}{\ln x}$

14 $\dfrac{\ln x}{x}$

15 $e^{\ln(x^2+1)}$

16 $\ln e^{\sqrt{x}}$

17 $\ln(e^{4x} + 9)$

18 $4^{\sqrt{2x+3}}$

19 $10^x \log x$

20 $5^{3x} + (3x)^5$

21 $\sqrt{\ln\sqrt{x}}$

22 $(1 + \sqrt{x})^e$

23 $x^2 e^{-x^2}$

24 $\dfrac{2^{-3x}}{x^3 + 4}$

25 $\sqrt{e^{3x} + e^{-3x}}$

26 $(x^2 + 1)^{2x}$

27 $10^{\ln x}$

28 $7^{\ln|x|}$

29 $x^{\ln x}$

30 $(\ln x)^{\ln x}$

31 $\ln|\tan x - \sec x|$

32 $\ln\csc\sqrt{x}$

33 $\csc e^{-2x}\cot e^{-2x}$

34 $x^2 e^{\tan 2x}$

35 $\ln\cos^4 4x$

36 $3^{\sin 3x}$

37 $(\sin x)^{\cos x}$

38 $\dfrac{1}{\sin^2 e^{-3x}}$

Exer. 39–40: Use implicit differentiation to find y'.

39 $1 + xy = e^{xy}$

40 $\ln(x + y) + x^2 - 2y^3 = 1$

Exer. 41–42: Use logarithmic differentiation to find dy/dx.

41 $y = (x + 2)^{4/3}(x - 3)^{3/2}$

42 $y = \sqrt[3]{(3x - 1)\sqrt{2x + 5}}$

Exer. 43–78: Evaluate the integral.

43 (a) $\displaystyle\int \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}\,dx$ (b) $\displaystyle\int_1^4 \frac{1}{\sqrt{x}\,e^{\sqrt{x}}}\,dx$

44 (a) $\displaystyle\int e^{-3x+2}\,dx$ (b) $\displaystyle\int_0^1 e^{-3x+2}\,dx$

45 (a) $\displaystyle\int x4^{-x^2}\,dx$ (b) $\displaystyle\int_0^1 x4^{-x^2}\,dx$

46 (a) $\displaystyle\int \frac{x^2 + 1}{x^3 + 3x}\,dx$ (b) $\displaystyle\int_1^2 \frac{x^2 + 1}{x^3 + 3x}\,dx$

47 $\displaystyle\int x\tan x^2\,dx$

48 $\displaystyle\int \cot\left(x + \frac{\pi}{6}\right)dx$

49 $\displaystyle\int x^e\,dx$

50 $\displaystyle\int \frac{1}{7 - 5x}\,dx$

51 $\displaystyle\int \frac{1}{x - x\ln x}\,dx$

52 $\displaystyle\int \frac{1}{x\ln x}\,dx$

53 $\displaystyle\int \frac{(1 + e^x)^2}{e^{2x}}\,dx$

54 $\displaystyle\int \frac{(e^{2x} + e^{3x})^2}{e^{5x}}\,dx$

55 $\displaystyle\int \frac{x^2}{x + 2}\,dx$

56 $\displaystyle\int \frac{x^2 + 1}{x + 1}\,dx$

57 $\displaystyle\int \frac{e^{4/x^2}}{x^3}\,dx$

58 $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

59 $\displaystyle\int \frac{x}{x^4 + 2x^2 + 1}\,dx$

60 $\displaystyle\int \frac{5x^3}{x^4 + 1}\,dx$

61 $\int \frac{e^x}{1+e^x}\,dx$

62 $\int (1+e^{-3x})^2\,dx$

63 $\int 5^x e^x\,dx$

64 $\int x10^{(x^2)}\,dx$

65 $\int \frac{1}{x\sqrt{\log x}}\,dx$

66 $\int 7^x\sqrt{1+7^x}\,dx$

67 $\int e^{-x}\sin e^{-x}\,dx$

68 $\int \tan x\, e^{\sec x}\sec x\,dx$

69 $\int \frac{\csc^2 x}{1+\cot x}\,dx$

70 $\int \frac{\cos x+\sin x}{\sin x-\cos x}\,dx$

71 $\int \frac{\cos 2x}{1-2\sin 2x}\,dx$

72 $\int 3^x(3+\sin 3^x)\,dx$

73 $\int e^x\tan e^x\,dx$

74 $\int \frac{\sec(1/x)}{x^2}\,dx$

75 $\int (\csc 3x+1)^2\,dx$

76 $\int \cos 2x\csc 2x\,dx$

77 $\int (\cot 9x+\csc 9x)\,dx$

78 $\int \frac{\sin x+1}{\cos x}\,dx$

79 Solve the differential equation $y''=-e^{-3x}$ subject to the conditions $y=-1$ and $y'=2$ if $x=0$.

80 In *seasonal population growth*, the population $q(t)$ at time t (in years) increases during the spring and summer but decreases during the fall and winter. A differential equation that is sometimes used to describe this type of growth is $q'(t)/q(t)=k\sin 2\pi t$, where $k>0$ and $t=0$ corresponds to the first day of spring.

(a) Show that the population $q(t)$ is seasonal.

(b) If $q_0=q(0)$, find a formula for $q(t)$.

81 A particle moves on a coordinate line with an acceleration at time t of $e^{t/2}$ cm/sec^2. At $t=0$ the particle is at the origin and its velocity is 6 cm/sec. How far does it travel during the time interval $[0,4]$?

82 Find the local extrema of $f(x)=x^2\ln x$ for $x>0$. Discuss concavity, find the points of inflection, and sketch the graph of f.

83 Find an equation of the tangent line to the graph of the equation $y=xe^{1/x^3}+\ln|2-x^2|$ at the point $P(1,e)$.

84 Find the area of the region bounded by the graphs of the equations $y=e^{2x}$, $y=x/(x^2+1)$, $x=0$, and $x=1$.

85 The region bounded by the graphs of $y=e^{4x}$, $x=-2$, $x=-3$, and $y=0$ is revolved about the x-axis. Find the volume of the resulting solid.

86 The 1980 population estimate for India was 651 million, and the population has been increasing at a rate of about 2% per year, with the rate of increase proportional to the number of people. If t denotes the time (in years) after 1980, find a formula for $N(t)$, the population (in millions) at time t. Assuming that this rapid growth rate continues, estimate the population and the rate of population growth in the year 2000.

87 A radioactive substance has a half-life of 5 days. How long will it take for an amount A to disintegrate to the extent that only 1% of A remains?

88 The carbon-14 dating equation $T=-8310\ln x$ is used to predict the age T (in years) of a fossil in terms of the percentage $100x$ of carbon still present in the specimen (see Exercise 19, Section 7.6).

(a) If $x=0.04$, estimate the age of the fossil to the nearest 1000 years.

(b) If the maximum error in estimating x in part (a) is ± 0.005, use differentials to approximate the maximum error in T.

89 The rate at which sugar dissolves in water is proportional to the amount that remains undissolved. Suppose that 10 pounds of sugar are placed in a container of water at 1:00 P.M., and one-half is dissolved at 4:00 P.M.

(a) How long will it take two more pounds to dissolve?

(b) How much of the 10 pounds will be dissolved at 8:00 P.M.?

90 According to Newton's law of cooling, the rate at which an object cools is directly proportional to the difference in temperature between the object and its surrounding medium. If $f(t)$ denotes the temperature at time t, show that $f(t)=T+[f(0)-T]e^{-kt}$, where T is the temperature of the surrounding medium and k is a positive constant.

91 The bacterium *E. coli* undergoes cell division approximately every 20 minutes. Starting with 100,000 cells, determine the number of cells after 2 hours.

92 The differential equation $p\,dv+cv\,dp=0$ describes the adiabatic change of state of air for pressure p, volume v, and a constant c. Solve for p as a function of v.