November 2012 MLC Solutions

Question # 1
Answer: E

Pr(last survivor dies in the third year)

$$= {}_2p_{\overline{80:90}} - {}_3p_{\overline{80:90}}$$

$$= \left({}_2p_{80} + {}_2p_{90} - {}_2p_{80:90}\right) - \left({}_3p_{80} + {}_3p_{90} - {}_3p_{80:90}\right)$$

$$= [0.9(0.8) + 0.6(0.5) - 0.9(0.8)(0.6)(0.5)]$$

$$\quad - [0.9(0.8)(0.7) + 0.6(0.5)(0.4) - 0.9(0.8)(0.7)(0.6)(0.5)(0.4)]$$

$$= (0.72 + 0.30 - 0.216) - (0.504 + 0.12 - 0.06048)$$

$$= 0.804 - 0.56352$$

$$= 0.24048$$

Question # 2
Answer: B

Under constant force over each year of age, $l_{x+k} = (l_x)^{1-k}(l_{x+1})^k$ for x an integer and $0 \le k \le 1$.

$${}_{2|3}q_{[60]+0.75} = \frac{l_{[60]+2.75} - l_{[60]+5.75}}{l_{[60]+0.75}}$$

$$l_{[60]+0.75} = \left(80,000^{0.25}\right)\left(79,000^{0.75}\right) = 79,249$$

$$l_{[60]+2.75} = \left(77,000^{0.25}\right)\left(74,000^{0.75}\right) = 74,739$$

$$l_{[60]+5.75} = \left(67,000^{0.25}\right)\left(65,000^{0.75}\right) = 65,494$$

$${}_{2|3}q_{[60]+0.75} = \frac{74,739 - 65,494}{79,249}$$

$$= 0.1167$$

$$1000\ {}_{2|3}q_{[60]+0.75} = 1000(0.11679) = 116.8$$

Question # 3
Answer: B

Since $S_0(t) = 1 - F_0(t) = \left(1 - \frac{t}{\omega}\right)^{\frac{1}{4}}$, we have $\ln S_0(t) = \frac{1}{4} \ln \frac{\omega - t}{\omega}$.

Then $\mu_t = -\frac{d}{dt} \ln S_0(t) = \frac{1}{4} \frac{1}{\omega - t}$ and $\mu_{65} = \frac{1}{180} = \frac{1}{4} \frac{1}{\omega - 65} \Rightarrow \omega = 110.$

$$e_{106} = \sum_{t=1}^{3} {}_t p_{106}, \text{ since } {}_4 p_{106} = 0$$

$$ {}_t p_{106} = \frac{S_0(106+t)}{S_0(106)} = \frac{\left(1 - \frac{106+t}{110}\right)^{1/4}}{\left(1 + \frac{106}{110}\right)^{1/4}} = \left(\frac{4-t}{4}\right)^{1/4}$$

$$\begin{aligned} e_{106} &= \sum_{t=1}^{3} \left(\frac{4-t}{4}\right)^{\frac{1}{4}} \\ &= \frac{1}{4^{0.25}} \left(3^{0.25} + 2^{0.25} + 1^{0.25}\right) \\ &= 2.4786 \end{aligned}$$

Question # 4
Answer: D

$$\begin{aligned} {}_{10}V &= 50,000\left(A_{50} + {}_{10}E_{50}\, A_{60}\right) - 1116\left[\ddot{a}_{50} - {}_{10}E_{50}\, \ddot{a}_{60}\right] \\ &= 50,000[0.24905 + 0.51081(0.36913)] - 1116[13.2668 - 0.51081(11.1454)] \\ &= 13,428 \end{aligned}$$

Question # 5
Answer: C

$${}_0V = 0$$

$${}_2V = 2000$$

Year 1: $\left({}_0V + P\right)\left(1+i\right) = q_x\left(2000 + {}_1V\right) + p_x\ {}_1V$

$$P(1.1) = 0.15(2000 + {}_1V) + 0.85({}_1V)$$

$$1.1P - 300 = {}_1V$$

Year 2: $\left({}_1V + P\right)\left(1+i\right) = q_{x+1}(2000 + {}_2V) + p_{x+1}(2000)$

$$(1.1P - 300 + P)(1.1) = 0.165(2000 + 2000) + 0.835(2000)$$

$$2.31P - 330 = 2330$$

$$P = \frac{2330 + 330}{2.31} = 1152$$

Question # 6
Answer: B

$\left.\frac{d}{dt}\left({}_tV\right)\right|_{t=9.6} = G - E - S\mu + {}_{9.6}V(\mu + \delta)$ where G, E, S and μ are evaluated at t = 9.6 and where S includes claims-related expenses. Then,

$$\left.\frac{d}{dt}\left({}_tV\right)\right|_{t=9.6} = 450 - 0.02(450) - (106,000 + 200)(0.01) + {}_{9.6}V(0.01 + 0.05)$$

$$= -621 + 0.06\,{}_{9.6}V$$

$${}_{9.6}V \approx {}_{9.8}V - 0.2\left.\frac{d}{dt}\left({}_tV\right)\right|_{t=9.6} = 126.68 - 0.2(-621 + 0.06\,{}_{9.6}V)$$

$$= 250.88 - 0.012\,{}_{9.6}V$$

$${}_{9.6}V \approx \frac{250.88}{1.012} = 247.91$$

Question # 7
Answer: C

Account Value at end of year 5 $= (30 + 20 - 9)1.06 = 43.46$
Surrender Value at end of year 5 $= 43.46 - 20 = 23.46$

Asset Share

$$= [(20 + (20 - 2))1.08 - 0.001(1000) - 0.05(23.46)]/(1 - 0.001 - 0.05)$$

$$= 38.867/0.949 = 40.96$$

Question # 8
Answer: A

Let CSV_k and SC_k denote the cash surrender value and surrender charge at time k.
In this solution, COI_k denotes the cost of insurance for month k, to be deducted from the account value at time $k - 1$, that is, the beginning of month k.

$$i^{(12)} = 0.054 \Rightarrow i \text{ per month} = 0.0045$$
$$CSV_{13} = AV_{13} - SC_{13}$$
$$1802.94 = AV_{13} - 125 \Rightarrow AV_{13} = 1927.94$$
$$AV_{12} = \left[AV_{11} + 300(1-W) - 10 - COI_{12}\right]1.0045$$
$$COI_{12} = 50,000(0.002)/1.0045 = 99.55$$
$$AV_{11} = CSV_{11} + SC_{11} = 1200 + 500 = 1700$$
$$AV_{12} = \left[1700 + 300(1-W) - 10 - 99.55\right]1.0045$$
$$AV_{12} = \left[1890.45 - 300W\right]1.0045 = 1898,96 - 301.35W$$
$$COI_{13} = 50,000(0.003)/1.0045 = 149.33$$
$$AV_{13} = [AV_{12} + 300(1-0.15) - 10 - 149.33](1.0045)$$
$$= (1898.96 - 301.35W + 95.67)(1.0045)$$
$$= 2003.61 - 302.71W = 1927.94$$
$$W = (2003.61 - 1927.94)/302.71 = 0.25$$

Question # 9
Answer: A

Year	Premium	Expense Charge	COI	Interest	EOY AV
1	5000	100	200(5.40/1.06) = 1019	0.06(5000–100 –1019) = 233	5000–100–1019+233 = 4114
2	5000	100	200(6.00/1.06) = 1132	0.06(4114+5000 –100–1132) = 473	4114+5000–100 –1132+473 = 8355

PV expected surrender cost in year 2

$$= 0.06(1-0.0034)(1-0.0038)(1-0.06)(8355)0.93/1.07^2 = 380$$

Question # 10
Answer: C

$p_{\overline{x+k:y+k}} = p_{x+k} + p_{y+k} - p_{x+k:y+k} = 0.84366 + 0.86936 - 0.77105 = 0.94197$ for $k = 0,1,2$.

$q_{\overline{x+k:y+k}} = 1 - p_{\overline{x+k:y+k}} = 1 - 0.94197 = 0.05803$ for $k = 0,1,2$.

${}_k p_{xy} = p_{xy} p_{x+1:y+1} \cdots p_{x+k-1:y+k-1} = \left(p_{xy}\right)^k = 0.77105^k$ for $k = 0,1,2$.

k	${}_k p_{xy}$	$q_{\overline{x+k:y+k}}$	Discount factor	Product
0	1	0.05803	1/1.03=0.97087	0.05634
1	0.77105	0.05803	$1/1.08^2$ = 0.85734	0.03836
2	0.59452	0.05803	$1/1.1^3$ = 0.75131	0.02592

EPV = (100,000)(0.05634 + 0.03836 + 0.02592) = 12,062

Question # 11
Answer: C

x	q_x	t	${}_t p_{70}$	${}_{t\|}q_{70}$	Spot rate	$v^{(t+1)}$	EPV
70	0.03318	0	1	0.03318	0.016	0.984252	0.03266
71	0.03626	1	0.96682	0.03506	0.026	0.94996	0.03331
72		2	0.93176				
						Sum =	0.06597

The EPV of a two-year deferred insurance is
$0.93176(0.94996)A_{72} = 0.88513(0.54560) = 0.48293$,
where A_{72} is from the ILT at 6% (6% is correct since all forward rates are 6% after two years).

Then the answer is $1000(0.06597 + 0.48293) = 548.90$

Question # 12
Answer: B

Because it is impossible to return to state 0, ${}_1p_0^{\overline{00}}$ and ${}_1p_0^{00}$ are the same. Then,

$${}_tp_0^{00} = {}_tp_0^{\overline{00}} = e^{\left\{-\int_0^t \sum_{j=1}^{2} \mu_{0+s}^{0j}\, ds\right\}}$$

$$\mu_t^{01} + \mu_t^{02} = \left[0.01 + 0.02(2^t)\right] + 0.5\left[0.01 + 0.02(2^t)\right]$$
$$= 0.015 + 0.03(2^t)$$

which is Makeham's law with $A = 0.015, \quad B = 0.03, \quad c = 2$

$${}_1p_0^{00} = {}_1p_0^{\overline{00}} = e^{-0.015(1)}\, e^{\frac{-0.03(2^1-1)}{\ln(2)}} = 0.943$$

It is not necessary to recognize that this is Makeham's law. The value can be calculated directly as

$${}_1p_0^{00} = {}_1p_0^{\overline{00}} = \exp\left[-\int_0^1 0.015 + 0.03(2^t)dt\right]$$
$$= \exp\left[-\left(0.015t + \frac{0.03(2^t)}{\ln 2}\right)\Bigg|_0^1\right]$$
$$= \exp[-(0.015 + 0.08656 - 0 - 0.04328)]$$
$$= \exp(-0.05828) = 0.943$$

Question # 13
Answer: D

$$p_{50}^{00} = \frac{l_{51}^{(\tau)}}{l_{50}^{(\tau)}} = \frac{90,365}{100,000} = 0.90365$$

$$q'^{(3)}_{51} = q'^{(3)}_{50} = 1 - p'^{(3)}_{50} = 1 - p_{50}^{00\,\frac{p_{50}^{03}}{p_{50}^{0\bullet}}} = 1 - p_{50}^{00\left[\frac{d_{50}^{(3)}}{l_{50}^{(\tau)}}\right]\Big/\left(1-p_{50}^{00}\right)} = 1 - 0.90365^{\frac{\frac{1100}{100,000}}{1-0.90365}} = 0.0115$$

$$d_{51}^{30} = l_{51}^{(\tau)}\, p_{51}^{03} = l_{51}^{(\tau)} \frac{\ln\, p'^{(3)}_{51}}{\ln\, p_{51}^{00}}\, p_{51}^{0\bullet} = l_{51}^{(\tau)} \frac{\ln(1-0.0115)}{\ln\left(\frac{l_{52}^{(\tau)}}{l_{51}^{(\tau)}}\right)}\left(1 - \frac{l_{52}^{(\tau)}}{l_{51}^{(\tau)}}\right)$$

$$= 90,365 \frac{\ln(1-0.0115)}{\ln(80,000/90,365)}(1-80,000/90,365) = 984$$

$$d_{51}^{(1)} = l_{51}^{(\tau)} - l_{52}^{(\tau)} - d_{51}^{(2)} - d_{51}^{(3)} = 90,365 - 80,000 - 8200 - 984 = 1181$$

$$q'^{(1)}_{51} = 1 - p'^{(1)}_{51} = 1 - p_{51}^{00\,\frac{p_{51}^{01}}{p_{51}^{0\bullet}}} = 1 - p_{51}^{00\left[\frac{d_{51}^{(1)}}{l_{51}^{(\tau)}}\right]\Big/\left(1-p_{51}^{00}\right)} = 1 - (80,000/90,365)^{\frac{\frac{1181}{90,365}}{1-\frac{80,000}{90,365}}} = 0.0138$$

$$10,000 q'^{(1)}_{51} = 10,000(0.0138) = 138$$

Note: This solution uses multi-state notation for dependent probabilities. There is alternative notation for these when the context is strictly multiple decrement, as it is here.

Question # 14
Answer: A

$$Z = \begin{cases} 2v^{K+1}, & K < 20 \\ v^{K+1}, & K \ge 20 \end{cases}$$

$$E[Z] = 2A_{40} - {}_{20}E_{40}A_{60} = 2(0.36987) - 0.51276(0.62567) = 0.41892$$

$$Var(Z) = E[Z^2] - (E[Z])^2 = 0.24954 - 0.41892^2 = 0.07405$$

$$SD(Z) = \sqrt{0.07405} = 0.27212$$

An alternative way to obtain the mean is $E[Z] = 2A^{1}_{40:\overline{20|}} + {}_{20|}A_{40}$. Had the problem asked for the evaluation of the second moment, a formula is

$$E[Z^2] = (2^2)\left({}^2A^{1}_{40:\overline{20|}}\right) + (v^2)^{20}\left({}_{20}p_{40}\right)\left({}^2A_{60}\right).$$

Question # 15
Answer: D

Half-year	Benefit	PV of Benefit	$PV > 277,000$
1	300,000	275,229	if and only if (x) dies in the 2nd or 3rd half years.
2	330,000	277,754	
3	360,000	277,986	
4	390,000	276,286	

Under CF assumption, ${}_{0.5}p_x = {}_{0.5}p_{x+0.5} = (0.84)^{0.5} = 0.9165$ and ${}_{0.5}p_{x+1} = {}_{0.5}p_{x+1.5} = (0.77)^{0.5} = 0.8775$. Then the probability of dying in the 2nd or 3rd half-years is

$$\left({}_{0.5}p_x\right)\left(1-{}_{0.5}p_{x+0.5}\right)+\left(p_x\right)\left(1-{}_{0.5}p_{x+1}\right) = (0.9165)(0.0835)+(0.84)(0.1225) = 0.1794$$

Question # 16
Answer: D

For $t = 0$ and $h = 0.5$,

$$\begin{aligned}{}_{0.5}p^{10} &= {}_0p^{10} - 0.5\left[{}_0p^{10}\left(\mu^{01}+\mu^{02}\right) - {}_0p^{11}\mu^{10} - {}_0p^{12}\mu^{20}\right] \\ &= 0 - 0.5\left(0 - 1\mu^{10} - 0\right) = 0.5\mu^{10} = 0.03.\end{aligned}$$

Similarly ${}_{0.5}p^{12} = 0.5\mu^{12} = 0.05$.
Then, ${}_{0.5}p^{11} = 1 - 0.03 - 0.05 = 0.92$.
For $t = 0.5$ and $h = 0.5$,

$$\begin{aligned}{}_1p^{10} &= {}_{0.5}p^{10} - 0.5\left({}_{0.5}p^{10}\left(\mu^{01}+\mu^{02}\right) - {}_{0.5}p^{11}\mu^{10} - {}_{0.5}p^{12}\mu^{20}\right) \\ &= 0.03 - 0.5[0.03(0.02) - 0.92(0.06) - 0] = 0.0573.\end{aligned}$$

Question # 17
Answer: E

$i^{(4)} = 0.08$ means an interest rate of $j = 0.02$ per quarter. This problem can be done with two quarterly recursions or as a single calculation.
Using two recursions:

$$_{10.75}V = \frac{\left[_{10.5}V + 60(1-0.1)\right]1.02 - \frac{800-706}{800}1000}{\frac{706}{800}}$$

$$753.72 = \frac{\left[_{10.5}V + 54\right]1.02 - 117.5}{0.8825}$$

$$_{10.5}V = 713.31$$

$$_{10.5}V = \frac{\left[_{10.25}V\right]1.02 - \frac{898-800}{898}1000}{\frac{800}{898}}$$

$$713.31 = \frac{\left[_{10.25}V\right]1.02 - 109.13}{0.8909}$$

$$_{10.25}V = 730.02.$$

Using a single step:
$_{10.25}V$ is the *EPV* of cash flows through time 10.75 plus $_{0.5}E_{80.25}$ times the *EPV* of cash flows thereafter (that is, $_{10.75}V$).

$$_{10.25}V = 1000\left[\frac{898-800}{898(1.02)} + \frac{800-706}{898(1.02)^2}\right] - 60(1-0.1)\frac{800}{898(1.02)} + \frac{706}{898(1.02)^2}753.72$$

$$= 730.$$

Question # 18
Answer: B

EPV of benefits at issue $= 1000A_{40} + 4\ {}_{11}E_{40}(1000A_{51})$

$= 161.32 + (4)(0.50330)(259.61) = 683.97$

EPV of expenses at issue $= 100 + 10(\ddot{a}_{40} - 1) = 100 + 10(13.8166) = 238.17$

$\pi = (683.97 + 238.17) / \ddot{a}_{40} = 922.14 / 14.8166 = 62.24$

$G = 1.02\pi = 63.48$

EPV of benefits at time $1 = 1000A_{41} + 4\ {}_{10}E_{41}(1000A_{51})$

$= 168.69 + (4)(0.53499)(259.61) = 724.25$

EPV of expenses at time $1 = 10(\ddot{a}_{41}) = 10(14.6864) = 146.86$

Gross Prem Reserve $= 724.25 + 146.86 - G\ddot{a}_{41} = 871.11 - 63.48(14.6864) = -61.18$

Question # 19
Answer: E

Let X_i be the present value of a life annuity of 1/12 per month on life i for $i = 1, 2, \ldots, 200$.

Let $S = \sum_{i=1}^{200} 180 X_i$ be the present value of all the annuity payments.

$$E[X_i] = \ddot{a}_{62}^{(12)} = \frac{1 - A_{62}^{(12)}}{d^{(12)}} = \frac{1 - 0.4075}{0.05813} = 10.19267$$

$$Var(X_i) = \frac{{}^2A_{62}^{(12)} - \left(A_{62}^{(12)}\right)^2}{\left(d^{(12)}\right)^2} = \frac{0.2105 - 0.4705^2}{0.05813^2} = 13.15255$$

$$E[S] = (200)(180)(10.19267) = 366,936.12$$

$$Var(S) = 200(180)^2(13.15255) = 85,228,524$$

With the normal approximation, for $\Pr\{S \le M\} = 0.90$,

$$M = E[S] + 1.282\sqrt{Var(S)} = 366,936.12 + 1.282\sqrt{85,228,524} = 378,771.45 .$$

So $\pi = \dfrac{378,771.45}{200} = 1893.86$

Question # 20
Answer: C

Let C be the annual contribution, then $C = \dfrac{{}_{20}E_{45}\,\ddot{a}_{65}}{\ddot{a}_{45:\overline{20}|}}$

Let K_{65} be the future curtate lifetime of (65). The required probability is

$$\begin{aligned}\Pr\left(\frac{C\ddot{a}_{45:\overline{20}|}}{{}_{20}E_{45}} > \ddot{a}_{\overline{K_{65}+1}|}\right) &= \Pr\left(\frac{{}_{20}E_{45}\ddot{a}_{65}}{\ddot{a}_{45:\overline{20}|}}\frac{\ddot{a}_{45:\overline{20}|}}{{}_{20}E_{45}} > \ddot{a}_{\overline{K_{65}+1}|}\right)\\ &= \Pr\left(\ddot{a}_{65} > \ddot{a}_{\overline{K_{65}+1}|}\right)\\ &= \Pr\left(9.8969 > \ddot{a}_{\overline{K_{65}+1}|}\right)\end{aligned}$$

Thus, since $\ddot{a}_{\overline{14}|} = 9.8527$ and $\ddot{a}_{\overline{15}|} = 10.2950$ we have

$$\Pr\left(\ddot{a}_{\overline{K_{65}+1}|} < 9.8969\right) = \Pr\left(K_{65}+1 \le 14\right) = 1 - {}_{14}p_{65}$$

$$= 1 - \frac{l_{79}}{l_{65}} = 1 - \frac{4{,}225{,}163}{7{,}533{,}964} = 0.439$$

Note that it is not necessary to know the mortality or interest rates before age 65 because the values of ${}_{20}E_{45}$ and $\ddot{a}_{45:\overline{20}|}$ cancel out in determining the actuarial accumulated value of the contributions.

Question # 21
Answer: A

Let $\pi =$ benefit premium, then $\pi\,\bar{a}_{xy} = 100\,\bar{A}_{\overline{xy}}$

$$\bar{a}_{xy} = \int_0^\infty e^{-0.05t}\left[\frac{1}{4}e^{-0.01t} + \frac{3}{4}e^{-0.03t}\right]dt = \frac{1}{4}\ \frac{1}{0.06} + \frac{3}{4}\ \frac{1}{0.08} = 13.54167$$

$$\bar{A}_{xy} = 1 - \delta\bar{a}_{xy} = 0.3229167$$

$$\bar{A}_x = \frac{0.01}{0.06} = \frac{1}{6}$$

$$\bar{A}_y = \frac{0.02}{0.07} = \frac{2}{7}$$

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.1294643$$

$$\pi = \frac{100(0.1294643)}{13.54167} = 0.956044$$

Question # 22
Answer: D

The equation of value is given in words by:
Premiums for at most two years = 1000 for dying within two years + return of premium without interest (pay *P* if die in first year, 2*P* if die in second year).
In symbols, the equation is:

$$P\ddot{a}_{80:\overline{2}|} = 1000A^{1}_{80:\overline{2}|} + P(IA)^{1}_{80:\overline{2}|}$$

Solving for *P* we obtain $P = \dfrac{1000A^{1}_{80:\overline{2}|}}{\ddot{a}_{80:\overline{2}|} - (IA)^{1}_{80:\overline{2}|}}$.

Then,

$$\ddot{a}_{80:\overline{2}|} = 1 + vp_{80} = 1 + \frac{0.91970}{1.0175} = 1.90388$$

$$1000A^{1}_{80:\overline{2}|} = 1000\left(vq_{80} + v^2 p_{80} q_{81}\right) = 1000\left[\frac{0.08030}{1.0175} + \frac{0.91970(0.08764)}{1.0175^2}\right] = 156.77$$

$$(IA)^{1}_{80:\overline{2}|} = vq_{80} + 2v^2 p_{80} q_{81} = \frac{0.08030}{1.0175} + 2\frac{0.91970(0.08764)}{1.0175^2} = 0.23463$$

Then, $P = \dfrac{156.77}{1.90388 - 0.23463} = 93.92$

Question # 23
Answer: A

Net amount at risk $= 1000 - {}_3V = 987.82$
Expected deaths $= (10,000 - 30)q_{47} = 9970(0.00466) = 46.46$
Actual deaths = 18
Mortality Gain/Loss = (Expected deaths – Actual deaths)(Net amount at risk)
$= (46.46 - 18)(987.82) = 28,113$

Question # 24
Answer: D

possible transitions	probability	discounted benefits	APV
$H \to Z$	0.05	$250v$	11.904762
$H \to L$	0.04	$250v$	9.523810
$H \to Z \to D$	$0.05(0.7) = 0.035$	$1000v^2$	31.746032
$H \to L \to D$	$0.04(0.6) = 0.024$	$1000v^2$	21.768707
$H \to H \to Z$	$0.90(0.05) = 0.045$	$250v^2$	10.204082
$H \to H \to L$	$0.90(0.04) = 0.036$	$250v^2$	8.163265
		Sum of these gives total APV	=93.31066

Question # 25
Answer: E

$$G\ddot{a}_{x:\overline{10}|} = 100,000A^{1}_{x:\overline{10}|} + G(IA)^{1}_{x:\overline{10}|} + 0.45G + 0.05G\ddot{a}_{x:\overline{10}|} + 200\ddot{a}_{x:\overline{10}|}$$

$$G = \frac{100,000(0.17094) + 200(6.8865)}{(1-0.05)(6.8865) - 0.96728 - 0.45} = \frac{18,471.3}{5.124895} = 3604.23$$