Eigenvalues and Eigenvectors

Linear Algebra and Vector Analysis

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Definition: A nonzero vector **x** is an ***eigenvector*** (or *characteristic vector*) of a square matrix **A** if there exists a scalar **λ** such that **Ax = λx**. Then **λ** is an ***eigenvalue*** (or *characteristic value*) of **A**.

Note: The zero vector can not be an eigenvector even though **A0 = λ0**. But **λ = 0** can be an eigenvalue.

Example:

$$Show\ \ x=\begin{bmatrix}2\\1\end{bmatrix}\ is\,an\,eigenvector\ \ for\ \ A=\begin{bmatrix}2 & -4\\3 & -6\end{bmatrix}$$

$$Solution:Ax=\begin{bmatrix}2 & -4\\3 & -6\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$But\quad for\ \ \lambda=0,\ \ \lambda x=0\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$Thus, x\,is\,an\,eigenvector\ of\ A, and\ \lambda=0\ is\,an\,eigenvalue.$$

Geometric interpretation of Eigenvalues and Eigenvectors

An n×n matrix **A** multiplied by n×1 vector **x** results in another n×1 vector **y=Ax**. Thus **A** can be considered as a transformation matrix.

In general, a matrix acts on a vector by changing both its magnitude and its direction. However, a matrix may act on certain vectors by changing only their magnitude, and leaving their direction unchanged (or possibly reversing it). These vectors are the **eigenvectors** of the matrix.

A matrix acts on an eigenvector by multiplying its magnitude by a factor, which is positive if its direction is unchanged and negative if its direction is reversed. This factor is the **eigenvalue** associated with that eigenvector.

Eigenvalues: examples

Example 1: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 \Rightarrow (\lambda + 1)(\lambda + 2)$$

two eigenvalues: –1, – 2

Note: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

$\lambda = 2$ is an eigenvector of multiplicity 3.

Eigenvectors

To each distinct eigenvalue of a matrix **A** there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations. If λ_i is an eigenvalue then the corresponding eigenvector $\mathbf{x_i}$ is the solution of ***$(A - \lambda_i I)x_i = 0$***

Example 1 (cont.):

$$\lambda = -1 : (-1)I - A = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 = 0 \Rightarrow x_1 = 4t,\ x_2 = t$$

$$\mathbf{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda = -2 : (-2)I - A = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \end{bmatrix}, s \neq 0$$

Eigenvectors

Example 2 (cont.): Find the eigenvectors of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

- Recall that λ = 2 is an eigenvector of multiplicity 3.
- Solve the homogeneous linear system represented by

$$(2I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = s,\ x_3 = t$$

- The eigenvectors of λ = 2 are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

s and t not both zero.

Properties of Eigenvalues and Eigenvectors

Definition: The trace of a matrix A, designated by tr(A), is the sum of the elements on the main diagonal.

Property 1: The sum of the eigenvalues of a matrix equals the trace of the matrix.

Property 2: A matrix is singular if and only if it has a zero eigenvalue.

Property 3: The eigenvalues of an upper (or lower) triangular matrix are the elements on the main diagonal.

Property 4: If λ is an eigenvalue of **A** and **A** is invertible, then $1/\lambda$ is an eigenvalue of matrix $\mathbf{A}^{-1}$.

Property 5: If λ is an eigenvalue of **A** then **k**λ is an eigenvalue of **kA** where **k** is any arbitrary scalar.

Property 6: If λ is an eigenvalue of **A** then λ^k is an eigenvalue of A^k for any positive integer **k**.

Property 8: If λ is an eigenvalue of **A** then λ is an eigenvalue of A^T.

Property 9: The product of the eigenvalues (counting multiplicity) of a matrix equals the determinant of the matrix.