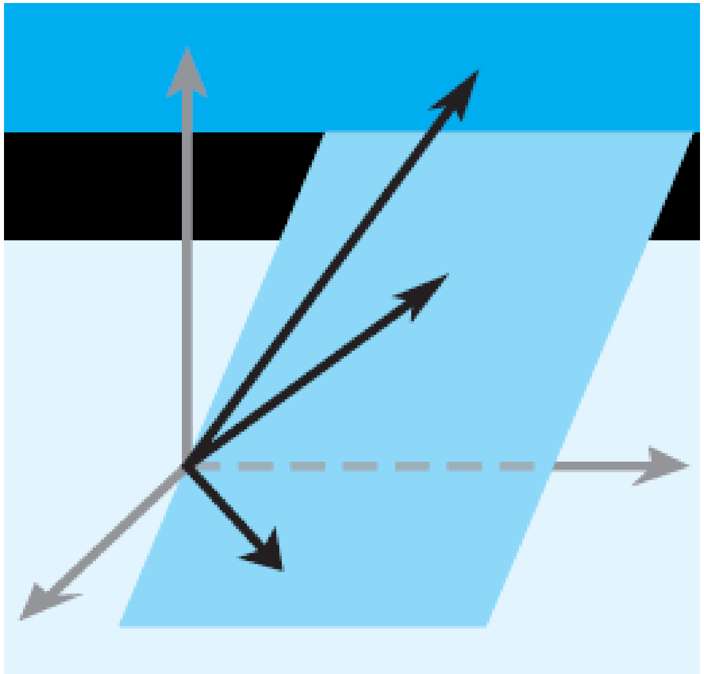
Faculty of Engineering
Mechanical Engineering Department

Linear Algebra and Vector Analysis

MATH 1120

Lecture 22

Elementary Linear Algebra



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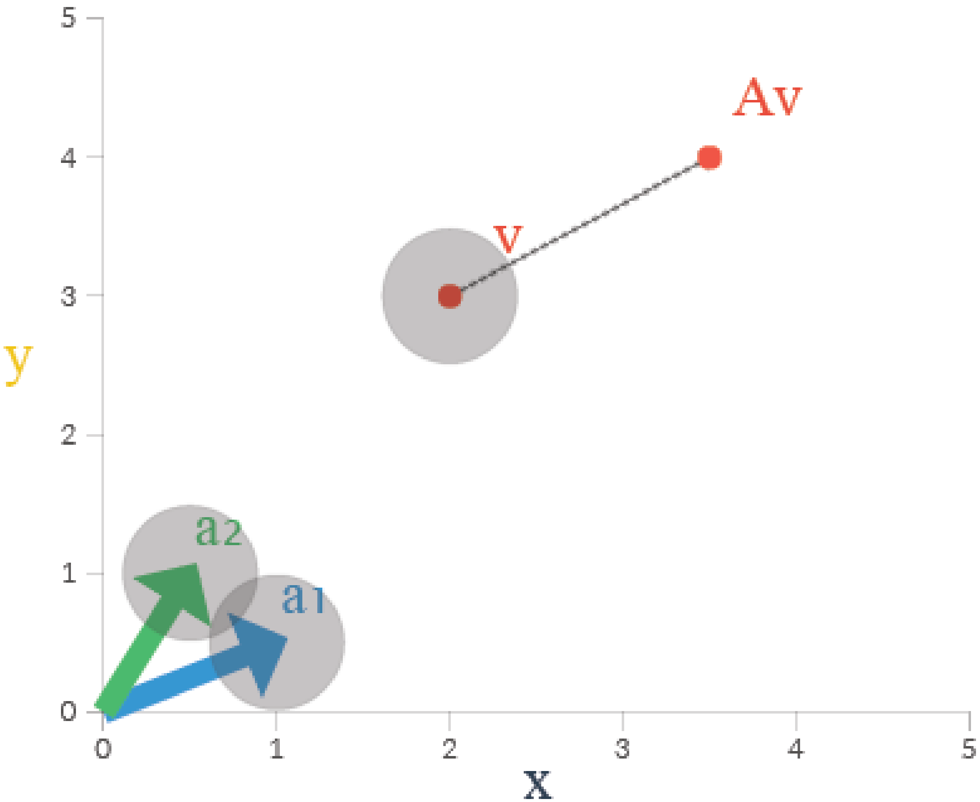
Chapter 5
Eigenvalues and Eigenvectors

- 5.1 Eigenvalues and Eigenvectors
- 5.2 Diagonalization
- 5.3 Complex Vector Spaces
- 5.4 Differential Equations

Eigenvalues and Eigenvectors

- Eigenvalues/vectors are instrumental to understanding electrical circuits, mechanical systems, ecology and even Google's PageRankalgorithm.

- To begin, let v be a vector (shown as a point) and AA be a matrix with columns a1a1 and a2a2 (shown as arrows). If we multiply vv by AA, then AA sends vv to a new vector AvAv.



$$A = \begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \end{bmatrix} = \begin{bmatrix} 1.00 & 0.50 \\ 0.50 & 1.00 \end{bmatrix}$$

$$v = \begin{bmatrix} 2.00 \\ 3.00 \end{bmatrix}$$

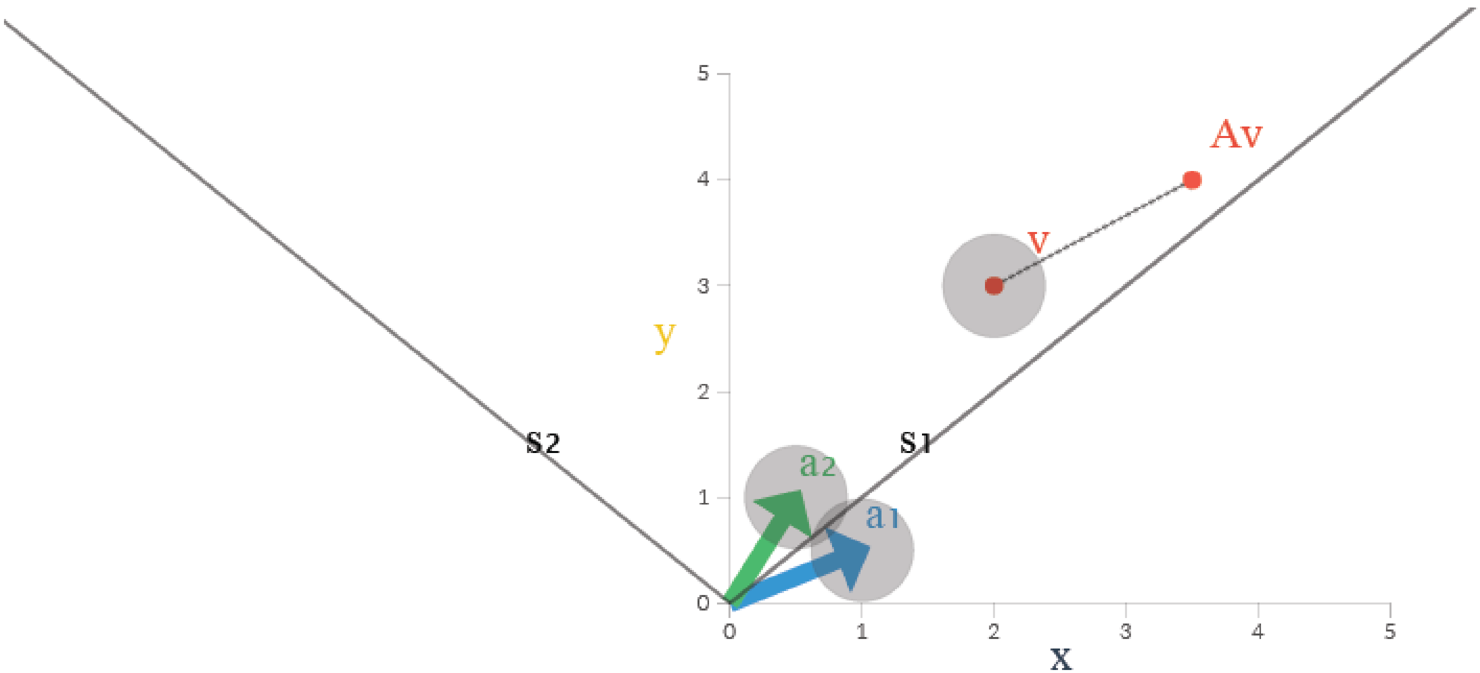
$$Av = \begin{bmatrix} 3.50 \\ 4.00 \end{bmatrix}$$

- If you can draw a line through the three points (0,0) v, and Av then Av, is just v multiplied by a number λ; that is, Av=λv. In this case,
- call λ an **eigenvalue** and v an **eigenvect or**. For example, here (1,2)is an eigenvector and 5 an eigenvalue.

$$Av = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda v.$$

- Below, change the columns of AA and drag vv to be an eigenvector. Note three facts: First, every point on the same line as an eigenvector is an eigenvector. Those lines are **eigenspaces**, and each has an associated eigenvalue. Second, if you place vv on an eigenspace (either s1s1 or s2s2) with associated eigenvalue λ<1λ<1, then AvAv is closer to (0,0)(0,0) than vv; but when λ>1λ>1, it's farther. Third, both eigenspaces depend on both columns of AA: it is not as though a1a1 only affects s1s1.

Below, change the columns of A and drag v to be an eigenvector. Note three facts: First, every point on the sam
eigenvector is an eigenvector. Those lines are **eigenspaces**, and each has an associated eigenvalue. Second, if y
eigenspace (either s_1 or s_2) with associated eigenvalue $\lambda < 1$, then Av is closer to $(0, 0)$ than v; but when λ >
both eigenspaces depend on both columns of A: it is not as though a_1 only affects s_1.



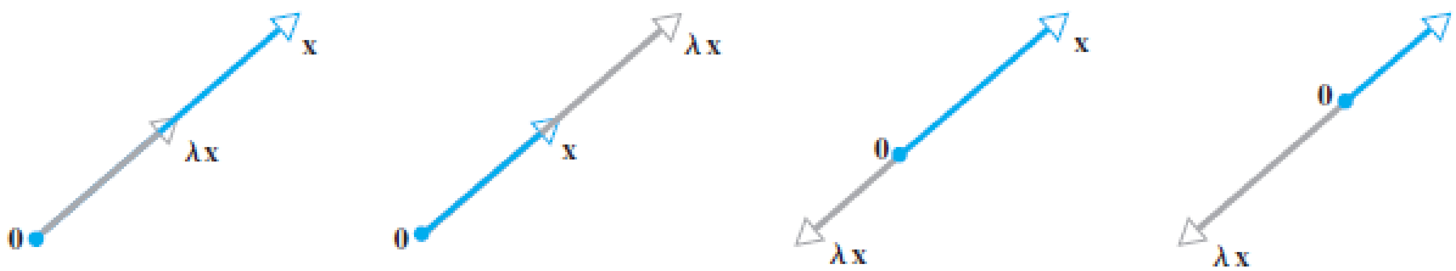
$\lambda_1 = 1.5$
$\lambda_2 = 0.5$

Section 5.1
Eigenvalues and Eigenvectors

DEFINITION 1 If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in R^n is called an *eigenvector* of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an *eigenvalue* of A (or of T_A), and $\mathbf{x}$ is said to be an *eigenvector corresponding to* λ.



(a) $0 \le \lambda \le 1$ (b) $\lambda \le 1$ (c) $-1 \le \lambda \le 0$ (d) $\lambda \le -1$

EXAMPLE 1 Eigenvector of a 2 × 2 Matrix

The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 3$, since

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{x}$$

Geometrically, multiplication by A has stretched the vector $\mathbf{x}$ by a factor of 3 (Figure 5.1.2).

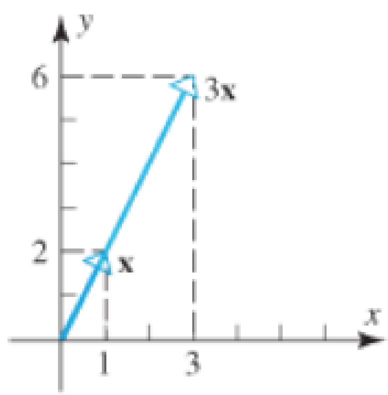


Figure 5.1.2

Computing Eigenvalues and Eigenvectors

The Characteristic Equation

THEOREM 5.1.1 *If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation*

$$\det(\lambda I - A) = 0 \tag{1}$$

This is called the characteristic equation of A.

THEOREM 5.1.3 *If A is an $n \times n$ matrix, the following statements are equivalent.*

(*a*) *λ is an eigenvalue of A.*

(*b*) *The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.*

(*c*) *There is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.*

(*d*) *λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.*

EXAMPLE 2 Finding Eigenvalues

In Example 1 we observed that $\lambda = 3$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

but we did not explain how we found it. Use the characteristic equation to find all eigenvalues of this matrix.

Solution It follows from Formula 1 that the eigenvalues of A are the solutions of the equation $\det(\lambda I - A) = 0$, which we can write as

$$\begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0$$

from which we obtain

$$(\lambda - 3)(\lambda + 1) = 0 \tag{2}$$

This shows that the eigenvalues of A are $\lambda = 3$ and $\lambda = -1$. Thus, in addition to the eigenvalue $\lambda = 3$ noted in Example 1, we have discovered a second eigenvalue $\lambda = -1$.

EXAMPLE 3 Eigenvalues of a 3 × 3 Matrix

Find the eigenvalues of

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\det(\lambda I-A)=\det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{bmatrix}=\lambda^3-8\lambda^2+17\lambda-4$$

The eigenvalues of A must therefore satisfy the cubic equation

$$\lambda^3-8\lambda^2+17\lambda-4=0$$

To solve this equation, we will begin by searching for integer solutions. This task can be simplified by exploiting the fact that all integer solutions (if there are any) of a polynomial equation with *integer coefficients*

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_n = 0$$

In applications involving large matrices it is often not feasible to compute the characteristic equation directly so other methods must be used to find eigenvalues. We will consider such methods in Chapter 9.

must be divisors of the constant term, c_n. Thus, the only possible integer solutions of 4 are the divisors of -4, that is, ± 1, ± 2, ± 4. Successively substituting these values in 4 shows that $\lambda = 4$ is an integer solution. As a consequence, $\lambda - 4$ must be a factor of the left side of 4. Dividing $\lambda - 4$ into $\lambda^3 - 8\lambda^2 + 17\lambda - 4$ shows that 4 can be rewritten as

$$(\lambda - 4)\left(\lambda^2 - 4\lambda + 1\right) = 0$$

Thus, the remaining solutions of 4 satisfy the quadratic equation

$$\lambda^2 - 4\lambda + 1 = 0$$

which can be solved by the quadratic formula. Thus the eigenvalues of A are

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \text{and} \quad \lambda = 2 - \sqrt{3}$$

Finding Eigenvectors and Bases for Eigenspaces

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

EXAMPLE 6 Bases for Eigenspaces

Find bases for the eigenspaces of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Solution In Example 1 we found the characteristic equation of A to be

$$(\lambda - 3)(\lambda + 1) = 0$$

from which we obtained the eigenvalues $\lambda = 3$ and $\lambda = -1$. Thus, there are two eigenspaces of A, one corresponding to each of these eigenvalues.

By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an eigenvector of A corresponding to an eigenvalue λ if and only if $\mathbf{x}$ is a nontrivial solution of $(\lambda I - A)\mathbf{x} = \mathbf{0}$, that is, of

$$\begin{bmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\lambda = 3$, then this equation becomes

$$\begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

whose general solution is

$$x_1 = \frac{1}{2}t, \quad x_2 = t$$

(verify) or in matrix form,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 3$. We leave it as an exercise for you to follow the pattern of these computations and show that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = -1$.

EXAMPLE 7 Eigenvectors and Bases for Eigenspaces

Find bases for the eigenspaces of

$$A=\begin{bmatrix}0 & 0 & -2\\ 1 & 2 & 1\\ 1 & 0 & 3\end{bmatrix}$$

Solution The characteristic equation of A is $\lambda^3-5\lambda^2+8\lambda-4=0$, or in factored form, $(\lambda-1)(\lambda-2)^2=0$ (verify). Thus, the distinct eigenvalues of A are $\lambda=1$ and $\lambda=2$, so there are two eigenspaces of A.

By definition,

$$\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

is an eigenvector of A corresponding to λ if and only if $\mathbf{x}$ is a nontrivial solution of $(\lambda I-A)\mathbf{x}=\mathbf{0}$, or in matrix form,

$$\begin{bmatrix}\lambda & 0 & 2\\ -1 & \lambda-2 & -1\\ -1 & 0 & \lambda-3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \tag{5}$$

In the case where $\lambda = 2$, Formula 5 becomes

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system using Gaussian elimination yields (verify)

$$x_1 = -s, \ \ x_2 = t, \ \ x_3 = s$$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$\mathbf{x} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent (why?), these vectors form a basis for the eigenspace corresponding to $\lambda = 2$.

If $\lambda = 1$, then 5 becomes

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system yields (verify)

$$x_1 = -2s, \;\; x_2 = s, \;\; x_3 = s$$

Thus, the eigenvectors corresponding to $\lambda = 1$ are the nonzero vectors of the form

$$\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ so that } \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 1$.