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Eur. Phys. J. Plus (2017) **132**: 181 DOI 10.1140/epjp/i2017-11501-2

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Springer

Eur. Phys. J. Plus (2017) **132**: 181
DOI 10.1140/epjp/i2017-11501-2

THE EUROPEAN
PHYSICAL JOURNAL PLUS

Regular Article

Thermodynamics of rotating Kaluza-Klein black holes in gravity's rainbow

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Received: 24 February 2017
Published online: 25 April 2017 –

Abstract. In this paper, a four-dimensional rotating Kaluza Klien (K-K) black hole was deformed using rainbow functions derived from loop quantum gravity and non-commutative geometry. We studied the thermodynamic properties and critical phenomena of this deformed black hole. The deformed temperature and entropy showed the existence of a Planckian remnant. The calculation of Gibbs free energy G for the ordinary and deformed black holes showed that both share a similar critical behaviour.

1 Introduction

The quest for a consistent theory of gravity is ongoing since the early 20's of the past century. Nevertheless, such a theory is still unavailable. Many programmes for quantum gravity however exists, like string theory, loop quantum gravity (LQG), causal dynamical triangulation (CDT), and many others. Most of these programmes predict that the spacetime admits a minimal length scale. Therefore, there is a maximal energy E_P that can be put into a system. This basic, yet important and universal prediction of quantum gravity programmes leads to phenomenological investigation of quantum gravity. The Hořava-Lifshitz gravity is based on such investigation, by imposing a deformation to the energy-momentum dispersion relations for energies close to E_p [1,2]. Another deformation is made by gravity's rainbow [3], where different wavelengths of light (having different energies) experience gravity differently. More generally, gravity is an energy-dependent phenomenon.

The deformation of energy-momentum dispersion relations can be derived from different quantum gravity programmes, in the UV limit. For instance in spacetime foam [4], spin-network in loop quantum gravity (LQG) [5], discrete spacetime [6], models based on string field theory [7] and non-commutative geometry [8]. This formalism has been heavily studied within string theory as well, the different Lifshitz scaling of space and time has been used to deform type-IIA string theory [9], type-IIB string theory [10], AdS/CFT correspondence [11–14], dilaton black branes [15,16], and dilaton black holes [17,18].

It has been shown that Hořava-Lifshitz gravity and gravity's rainbow produce similar physical results [19], as they are based on the same physical assumption. The Lifshitz deformation of geometries has produced interesting results, and rainbow deformation has the same motivation, in this paper we will study the rainbow deformation of rotating Kaluza-Klien black holes. In gravity's rainbow, the geometry depends on the energy of the probe, and thus probes of of different energy see the geometry differently. Thus, a single metric is replaced by a family of energy-dependent metrics forming a rainbow of metrics. Now the UV modification of the energy-momentum dispersion relation can be expressed as

$$E^2 f^2(E/E_P) - p^2 g^2(E/E_P) = m^2, \tag{1}$$

where E_P is the Planck energy, E is the energy at which the geometry is probed, and $f(E/E_P)$ and $g(E/E_P)$ are the rainbow functions. As the general relativity should be recovered in the IR limit, we have

$$\lim_{E/E_P \to 0} f(E/E_P) = 1, \qquad \lim_{E/E_P \to 0} g(E/E_P) = 1. \tag{2}$$

Now the metric in gravity's rainbow [20]

$$h(E) = \eta^{ab} e_a(E) \otimes e_b(E). \tag{3}$$

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So, the energy-dependent frame fields are

$$e_0(E) = \frac{1}{f(E/E_P)}\tilde{e}_0, \qquad e_i(E) = \frac{1}{g(E/E_P)}\tilde{e}_i. \tag{4}$$

Here $\tilde{e}_0$ and $\tilde{e}_i$ are the original energy-independent frame fields. The deformation of geometry by the rainbow functions has been studied extensively, such as the study of black rings [21], black branes [22], higher-dimensional microscopic black holes and the consequences of gravity's rainbow on their detection at the TeV scale at the LHC [23]. Gravity's rainbow has also been used to address the black hole information paradox [24], and in alternative theories of gravity [25–30].

In this paper, we shall study deformed rotating Kaluza-Kleinblack hole by the rainbow functions, and investigate its thermodynamic properties. We start by a review of rotating Kaluza-Klein black holes, and their thermodynamics, then we deform the metric via the rainbow functions and discuss the implications of this deformation on the thermodynamics of these black holes.

2 Thermodynamics of rotating Kaluza-Klein black holes

Kaluza-Klein black holes are a 5D uplifted solution of rotating black holes with electric Q and magnetic P charges [31–33]. This is a general solution to the dyonic solution (where $Q = P$). This solution is considered from the 4D Einstein-Maxwell-dilaton theory [34], or as a rotating D0-D6 bound state in string theory [35]. The rotating KK black hole contain a 4D dyonic Reissner-Nordstrøm black hole and Myers-Perry black hole [36]. The KK solution in 5D pure Einstein gravity has the following metric:

$$\mathrm{d}s^2_{(5)} = \frac{H_2}{H_1}\left(R\,\mathrm{d}\hat{y} + A\right)^2 - \frac{H_3}{H_2}\left(\mathrm{d}\hat{t} + B\right)^2 + H_1\left(\frac{\mathrm{d}\hat{r}^2}{\Xi} + \mathrm{d}\theta^2 + \frac{\Xi}{H_3}\sin^2\theta\,\mathrm{d}\phi\right), \tag{5}$$

where

$$H_1 = \hat{r}^2 + \mu^2 j^2\cos^2\theta + \hat{r}(p = 2\mu) + \frac{1}{2}\frac{p}{p+q}\,(p-2\mu)\,(q-2\mu) + \frac{1}{2}\frac{p}{p+q}\sqrt{(p^2-4\mu^2)(q^2-4\mu^2)}\,j\cos\theta. \tag{6a}$$

$$H_2 = \hat{r}^2 + \mu^2 j^2\cos^2\theta + \hat{r}(q - 2\mu) + \frac{1}{2}\frac{q}{p+q}(p-2\mu)(q-2\mu) - \frac{1}{2}\frac{p}{p+q}\sqrt{(p^2-4\mu^2)(q^2-4\mu^2)}\,j\cos\theta. \tag{6b}$$

$$H_3 = \hat{r}^2 + \mu^2 j^2\cos^2\theta - 2\mu\hat{r}. \tag{6c}$$

$$\Xi = \hat{r}^2 + \mu^2 j^2 - 2\mu\hat{r}, \tag{6d}$$

and

$$A = \left[\sqrt{\frac{q(q^2-4\mu^2)}{p+q}}\left(\hat{r} + \frac{p-2\mu}{2}\right) - \frac{1}{2}\sqrt{\frac{q^3(p^2-4\mu^2)}{p+q}}\,j\cos\theta\right]H_2^{-1}\mathrm{d}\hat{t} + \left[-\sqrt{\frac{q(q^2-4\mu^2)}{p+q}}\,(H_2 + \mu^2 j^2\sin^2\theta)\cos\theta + \frac{1}{2}\sqrt{\frac{q(q^2-4\mu^2)}{p+q}}\left\{p\hat{r} - \mu(p-2\mu) + \frac{q(q^2-4\mu^2)}{p+q}\right\}j\sin^2\theta\right]H_2^{-1}\mathrm{d}\phi \tag{7}$$

$$B = \frac{1}{2}\sqrt{pq}\,\frac{pq + 4\mu^2\hat{r} - \mu(p-2\mu)(q-2\mu)}{p+q}\,H_3^{-1}j\sin^2\theta\,\mathrm{d}\phi, \tag{8}$$

with R being the radius of the compactified fifth K-K dimension $\hat{y}$ with the condition $\hat{y} = \hat{y} + 2\pi$. We can obtain the 4D metric after the K-K reduction of $\hat{y}$:

$$\mathrm{d}s^2(4) = -\frac{H_3}{\sqrt{H_1 H_2}}\left(\mathrm{d}\hat{t} + B\right)^2 + \sqrt{H_1 H_2}\left(\frac{d\hat{r}^2}{\Xi} + d\theta^2 + \frac{\Xi}{H_3}\sin^2\theta\,\mathrm{d}\phi\right). \tag{9}$$

There are four physical parameters that characterises the rotating K-K black hole, the mass M, electric and magnetic charges Q, P and the angular momentum J. They are given in terms of the parameters μ, q, p and j:

$$M = \frac{p+q}{4}, \tag{10a}$$

$$Q = \frac{1}{2}\left(\frac{q(q^2-4\mu^2)}{p+q}\right)^{1/2}, \tag{10b}$$

$$P = \frac{1}{2}\left(\frac{p(p^2-4\mu^2)}{p+q}\right)^{1/2}, \tag{10c}$$

$$J = \frac{\sqrt{pq}(pq+4\mu^2)}{4(p+q)}\, j. \tag{10d}$$

The K-K black hole has an event horizon at $\Xi = 0$,

$$r_\pm = \mu\left(1 \pm \sqrt{1-j^2}\right). \tag{11}$$

The Hawking temperature is then

$$T_0 = \frac{\mu\hbar}{\pi\sqrt{pq}\left(\frac{2\mu}{\sqrt{1-j^2}} + \frac{4\mu^2+pq}{p+q}\right)}. \tag{12}$$

Using the relation $\mathrm{d}S = \mathrm{d}M/T$ we can obtain the entropy:

$$S_0 = \frac{2\pi\frac{p+q}{4}\left(\frac{3(p+q)}{4\sqrt{1-j^2}} + 12\mu + \frac{pq}{\mu}\right)}{3\hbar}. \tag{13}$$

We observe, from figs. 1 and 2, that the K-K black holes have a very similar thermodynamic behaviour as a Kerr-Neumann black holes with an effective $U(1)$ charge $\mathcal{Q} = Q + P$. The first law of thermodynamics is then written as [33]

$$\mathrm{d}M = T\,\mathrm{d}S - \Omega_d J + \Phi_E \mathrm{d}Q + \Phi_M \mathrm{d}P, \tag{14}$$

in which

$$\Omega = \frac{p+q}{\sqrt{pq}}\,\frac{2\mu j}{2\mu(p+q) + (pq+4\mu^2)\sqrt{1-j^2}}, \tag{15}$$

$$\Phi_E = \frac{\left(\frac{2\mu}{\sqrt{1-j}} + p\right)\sqrt{\frac{p(q^2-\mu^2)}{p+q}}}{2\sqrt{\mathrm{pq}}\left(\frac{2\mu}{\sqrt{1-j^2}} + \frac{4\mu^2+\mathrm{pq}}{p+q}\right)}, \tag{16}$$

$$\Phi_M = \frac{\left(\frac{2\mu}{\sqrt{1-j}} + q\right)\sqrt{\frac{q(p^2-\mu^2)}{p+q}}}{2\sqrt{\mathrm{pq}}\left(\frac{2\mu}{\sqrt{1-j^2}} + \frac{4\mu^2+\mathrm{pq}}{p+q}\right)}. \tag{17}$$

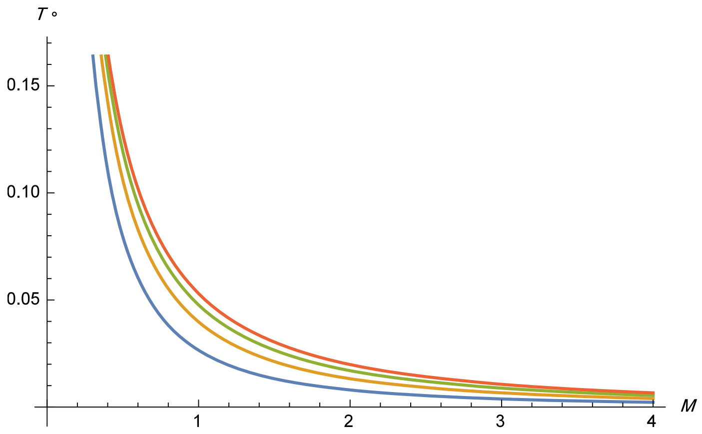

Fig. 1. Hawking temperature of different rotating K-K black holes (fixed Q, P and J) as a function of their mass M.

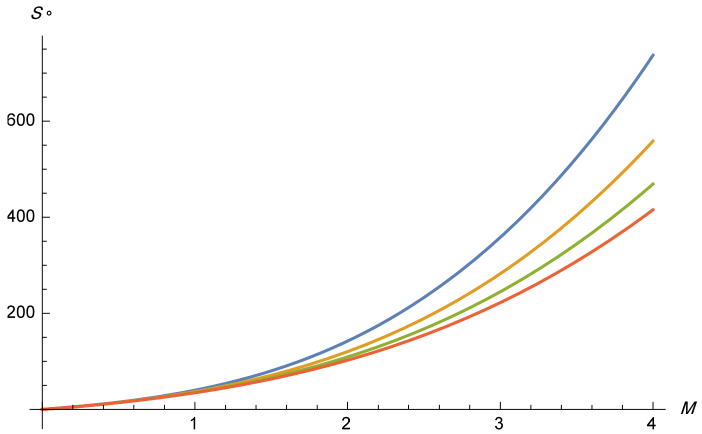

Fig. 2. The entropy of different rotating K-K black holes (fixed Q, P and J) as a function of their mass M.

The heat capacities can be calculated from the general relation $C_X := T(\partial S/\partial T)_X$ [37]

$$C_J=\frac{\pi}{2\mu\hbar}\left(-\frac{\mu\hbar(\sqrt{1-j^2}p+q+4\mu)}{\pi(\sqrt{1-j^2}\mu^2+\sqrt{1-j^2}\frac{1}{4}pq+\frac{1}{2}\mu(p+q))}-\left(\frac{2\mu}{\sqrt{1-j^2}}+q\right)\sqrt{\frac{q(p^2-\mu^2)}{p+q}}+\left(\frac{2\mu}{\sqrt{1-j^2}}+p\right)\left(-\sqrt{\frac{p(q^2-\mu^2)}{p+q}}\right)\right). \tag{18}$$

$$C_Q=\frac{\pi}{2\mu^2\hbar^2}\left(-\frac{4\mu^2\hbar^2(\sqrt{1-j^2}M+\mu)}{\pi(\sqrt{1-j^2}\mu^2+\sqrt{1-j^2}M^2+2\mu M)}\right.$$
$$\left.-\frac{2\pi^2(\sqrt{1-j^2}\mu^2+\sqrt{1-j^2}M^2+2\mu M)^3(\sqrt{1-j^2}q+2\mu)\sqrt{\frac{q(p^2-\mu^2)}{p+q}}}{(1-j^2)^{3/2}\mu\hbar(\sqrt{1-j^2}M+\mu)}-\mu\hbar\left(\frac{2\mu}{\sqrt{1-j^2}}+q\right)\sqrt{\frac{q(p^2-\mu^2)}{p+q}}\right). \tag{19}$$

$$C_P=\frac{\pi}{2\mu^2\hbar^2}\left(-\frac{4\mu^2\hbar^2(\sqrt{1-j^2}M+\mu)}{\pi(\sqrt{1-j^2}\mu^2+\sqrt{1-j^2}M^2+2\mu M)}\right.$$
$$\left.-\frac{2\pi^2(\sqrt{1-j^2}\mu^2+\sqrt{1-j^2}M^2+2\mu M)^3(\sqrt{1-j^2}q+2\mu)\sqrt{\frac{q(p^2-\mu^2)}{p+q}}}{(1-j^2)^{3/2}\mu\hbar(\sqrt{1-j^2}M+\mu)}-\mu\hbar\left(\frac{2\mu}{\sqrt{1-j^2}}+p\right)\sqrt{\frac{p(q^2-\mu^2)}{p+q}}\right). \tag{20}$$

Hereby, we finish the review on rotating K-K black holes. In the next section, they shall be deformed by gravity's rainbow and their thermodynamic properties will be discussed.

3 K-K black holes in gravity's rainbow

The rotating K-K black hole is deformed by the rainbow functions discussed earlier in (2), where E is the energy of a "quantum" particle near the outer horizon $\hat{r} \sim r_+$. Since the K-K black hole is 4-dimensional, since the fifth dimension is compactified and the motion on it resembles the $U(1)$ charge, the particle could - for instance - be emitted from the Hawking radiation, and this has been studied for other black holes [38]. In order to estimate E, we may use the uncertainty relation for position and momentum, and write $\Delta p \geq 1/\Delta x$. Thus, we can obtain a bound on energy of a black hole, $E \geq 1/\Delta x$ [39]. It should be noted that this uncertainty relation holds for the rotating K-K black hole like any other 4D black hole, in gravity's rainbow [38]. Thus we write

$$E \geq 1/\Delta x \approx 1/r_+. \tag{21}$$

The general relation for temperature of a black hole in gravity rainbow was found to be [39]

$$T = T_0 \frac{g(E)}{f(E)}, \tag{22}$$

where $f(E)$ and $g(E)$ are the rainbow function defined in (2). Observe that these deformations depend on the radial coordinates $\hat{r}$.

The deformation relation (22) is explained thoroughly in refs. [24,38–40] and many others. It is natural therefore to conjecture that this deformation holds for the rotating K-K black holes, as well. One may define the rainbow functions $f(E)$ and $g(E)$ in many ways, However, in this study these functions are chosen such that they are compatible with loop quantum gravity and non-commutative geometry [41,42]:

$$f(E) := 1, \qquad g(E) := \sqrt{1 - \eta(E/E_p)^\nu}. \tag{23}$$

Here, η and ν are free parameters. Now, we use (22), (12), and (23) to obtain the formula for the modified temperature:

$$T = \frac{\mu\hbar\sqrt{1-\eta(1/r_+E_p)^\nu}}{\pi\sqrt{pq}\left(\frac{2\mu}{\sqrt{1-j^2}} + \frac{4\mu^2+pq}{p+q}\right)}. \tag{24}$$

Since the area of the 4D black hole is spherically symmetric [37], we have $A = 4\pi r_+^2$ we may rewrite (22) in terms of A instead of r_+:

$$T(M) = \frac{\mu\hbar\sqrt{1-\eta\left(1/\sqrt{\frac{4\pi}{A}}E_p\right)^\nu}}{2\pi M\left(\frac{2\mu}{\sqrt{1-j^2}} + \frac{4\mu^2+4M^2}{M}\right)}. \tag{25}$$

Similarly, the deformed entropy is calculated from the integral $S = \int \frac{\mathrm{d}M}{T}$, it is found to be given by the hypergeometric functions ${}_2F_1(a,b;c;d)$,

$$S(M) = \frac{2\pi}{\mu\hbar}\left(\mu M\left(\frac{M\,{}_2F_1\left(\frac{1}{2}, -\frac{2}{\nu}; \frac{\nu-2}{\nu}; (\frac{1}{ME_p})^\nu\eta\right)}{\sqrt{1-j^2}} + \mu\,{}_2F_1\left(\frac{1}{2}, -\frac{1}{\nu}; \frac{\nu-1}{\nu}; \left(\frac{1}{ME_p}\right)^\nu\eta\right)\right) + \frac{1}{3}M^3\,{}_2F_1\left(\frac{1}{2}, -\frac{3}{\nu}; \frac{\nu-3}{\nu}; \left(\frac{1}{ME_p}\right)^\nu\eta\right)\right). \tag{26}$$

We observe from figs. 3 and 4 the existence of a remnant, like the other studied types of deformed black holes in gravity's rainbow [38]. The heat capacity at constant J is deformed in the following way:

$$C_J' = \frac{1}{\sqrt{1-\eta(E/E_p)^\nu}} C_J. \tag{27}$$

The same goes for other heat capacities. It is interesting to look at the criticality of rotating K-K black holes and their rainbow deformation, this can be done by studying the Gibbs free energy of this black hole. The Gibbs free energy is generally given by

$$G(M,J,Q,P) = M - TS. \tag{28}$$

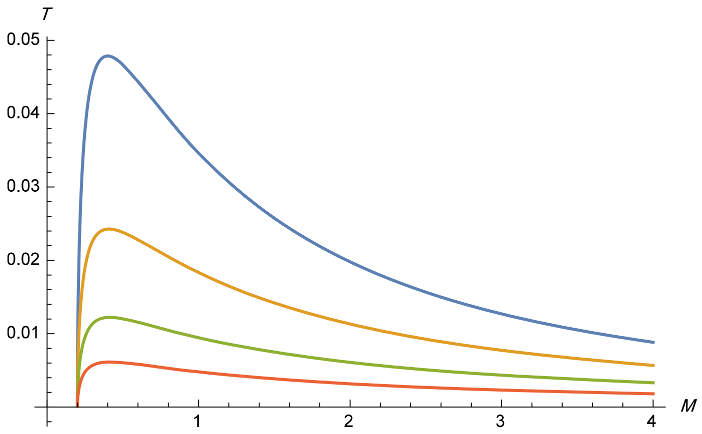

Fig. 3. Deformed Hawking temperature of different rotating K-K black holes (fixed Q, P and J) as a function of their mass M. We set $E_p = 5$, $\eta = 1$ and $\nu = 2$. The remnant can be observed at the same point for all black holes.

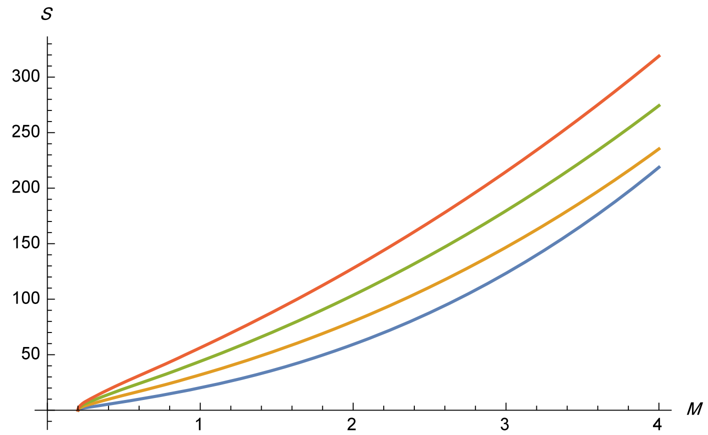

Fig. 4. The deformed entropy of different rotating K-K black holes (fixed Q, P and J) as a function of their mass M. We set $E_p = 5$, $\eta = 1$ and $\nu = 2$. The remnant can be observed at the same point for all black holes.

For the ordinary rotating K-K black hole it is found to be

$$G_0(M) = \frac{M^2(2\sqrt{1-j^2}M + 3\mu)}{3(\sqrt{1-j^2}\mu^2 + \sqrt{1-j^2}M^2 + 2\mu M)}. \tag{29}$$

We can plot (29) keeping Q, P fixed and vary M and J to obtain fig. 5, that shows a critical phenomena for the rotating K-K black holes. The deformed Gibbs free energy is calculated from (4) and (3) (see fig. 6),

$$G = M - \frac{\sqrt{1-\eta(\frac{M}{E_p})^\nu}}{\frac{2\mu M}{\sqrt{1-j^2}} + \mu^2 + M^2}\left(\frac{\mu M^2\, {}_2F_1(\frac{1}{2}, \frac{2}{\nu}; \frac{\nu+2}{\nu}; (\frac{M}{E_p})^\nu \eta)}{\sqrt{1-j^2}} \right.$$
$$\left. + \frac{1}{3}M^3\, {}_2F_1\left(\frac{1}{2}, \frac{3}{\nu}; \frac{\nu+3}{\nu}; \left(\frac{M}{E_p}\right)^\nu \eta\right) + \mu^2 M\, {}_2F_1\left(\frac{1}{2}, \frac{1}{\nu}; 1+\frac{1}{\nu}; \left(\frac{M}{E_p}\right)^\nu \eta\right)\right). \tag{30}$$

Both ordinary and deformed rotating K-K black holes show critical behaviour as the study of Gibbs free energy, if $G > 0$ the black hole is said to be "critical" and when $G < 0$ it is said that the black hole is uncritical.

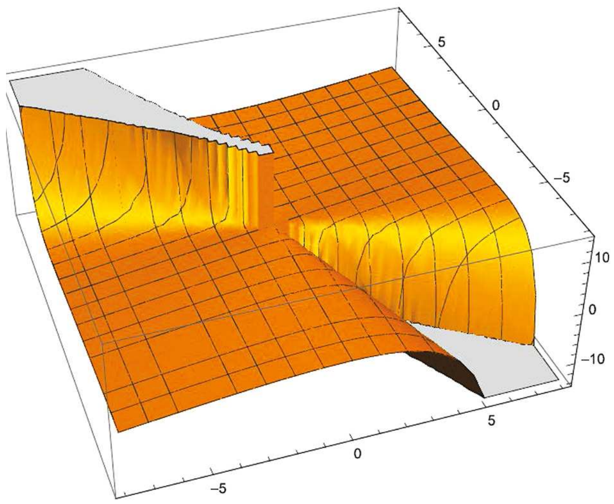

Fig. 5. A plot of $G_0(T, J, Q, P)$ of a critical rotating K-K black hole fixed Q, P, showing the critical phenomena.

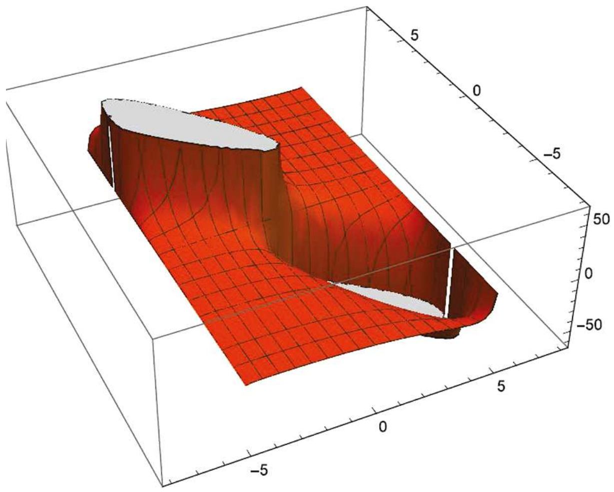

Fig. 6. A plot of the deformed Gibbs free energy $G(T, J, Q, P)$ of a deformed rotating K-K black hole fixed Q, P, showing the same critical phenomena, as the ordinary K-K black hole. We have set $\eta = 1$,$E_p = 5$ and $\nu = 2$.

4 Conclusion

In this paper, the geometry of 5D rotating Kaluza Klein black holes with electric and magnetic charges was deformed by the rainbow functions F, G motivated by loop quantum gravity and non-commutative geometry. Resulting a deformation on the thermodynamics of the 4D rotating K-K black hole. The deformed temperature and entropy indicate the existence of a remnant after the decay of the black hole to a "Plankckian" scale. This is independent of the compactification, or the K-K reduction of the 5D geometry. Moreover, the critical behaviour of this black hole was studied via calculating its Gibbs free energy, the ordinary and the deformed black holes appear to show the same critical behaviour.

Warm regards to Dr Mir Faizal for his generous help improving this work. This research project was supported by a grant from the "Research Center of the Female Scientific and Medical Colleges", Deanship of Scientific Research, King Saud University.

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