# Chapter(9)
# Estimation and Confidence Intervals
# (Examples)

# Types of estimation:

## i. Point estimation:

Example (1)
Consider the sample observations
17,3,25,1,18,26,16,10

$$\hat{\mu} = \bar{X} = \frac{\sum_{i=1}^{8} X_i}{8} = \frac{17+3+25+1+18+26+16+10}{8} = \frac{116}{8} = 14.5$$

14.5 is a point estimate for $\mu$ using the estimator $\bar{X}$ and the given sample observations.

## ii. Interval estimation:

**Constructing confidence interval**

The general form of an interval estimate of a population parameter:

| |
|---|
| Point Estimate ± Criticalvalue *Standard error |

This formula generates two values called the confidence limits;

- Lower confidence limit (LCL).
- Upper confidence limit (UCL).

Another way to find the confidence interval we used the **confidence**

# Confidence Interval for a Population Mean

**Case1: Confidence Interval for Population Mean with known variance (normal case):**

The confidence limits are:

$$\bar{X} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

**Steps for calculating:**

1. **Obtain** $Z_{\alpha/2}$ **,from the table of the area under the normal curve.**
2. **Calculate** $Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ **.**
3. **L=** $\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

   **U=** $\bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

$\bar{X}$ :The mean estimator

σ : The standard deviation of the population .

$\frac{\sigma}{\sqrt{n}}$ : The standard error of the mean $(\sigma_{\bar{x}})$.

$\pm Z_{\frac{\alpha}{2}}$ : **Critical value.**

## Example (2)

A sample of 49 observations is taken from a normal population with a standard deviation of 10.the sample mean is 55,determine the 99 percent confidence interval for the population mean

Solution:

$X \sim N(\mu, \sigma^2)$ $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ σ = 10 , n = 49, $\bar{X} = 55$

,Confidence level = 0.99,

$\therefore\ Z_{\frac{0.99}{2}} = Z_{0.495} = 2.58$ , The confidence limits are:

$$\bar{X} \pm Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} = 55 \pm 2.58\left(\frac{10}{\sqrt{49}}\right) = 55 \pm 3.6857$$

$$51.3143 \le \hat{\mu} \le 58.6857$$

$$(51.3143\ ,\ \ 58.6857)$$

**Case (2):**
**Confidence Interval for a Population Mean with unknown variance (Large Sample n≥30, normal case):**
The confidence limits are:

$$\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

S : Sample standard deviation (Estimate of $\sigma$)

## Example (3)

A scientist interested in monitoring chemical contaminants in food, and thereby the accumulation of contaminants in human diets, selected a random sample of n=50 male adults. it was found that the average daily intake of dairy products was $\bar{X} = 756$ grams per day with a standard deviation of35 grams per day .use this sample information to construct a 95% confidence interval for the mean daily intake of dairy products for men.

**Solution:**

**n =50  S=35**  $\bar{X} = 756$, confidence coefficient = 0.95

∴ Z =1.96

Since n≥30 (large), the distribution of the sample mean $\bar{X}$ is approximately normally distributed:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

The confidence limits are:

$$\bar{X} \pm Z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} = 756 \pm 1.96\left(\frac{35}{\sqrt{50}}\right) = 756 \pm 9.70$$

$$746.3 \le \hat{\mu} \le 765.7$$

$$(746.3\ ,\ 765.7)$$

Hence, the 95% confidence interval for μ is from 746.3 to 756.7 grams per day.

- IF you have (746.3, 765.7). Based on this information, you know that the best point estimate of the population mean $(\hat{\mu})$ is:

$$\hat{\mu} = \frac{upper + lower}{2} = \frac{765.57 + 746.3}{2} = \frac{1512}{2} = 756$$

## Example (4)

IF you have (746.3, 765.7). Based on this information, you know that the best point estimate of the population mean $(\hat{\mu})$ is:

$$\hat{\mu} = \frac{upper + lower}{2} = \frac{765.7 + 746.3}{2} = \frac{1512}{2} = 756$$

**Case3**:

**Confidence Interval for a Population Mean with unknown variance (**

## Example (5)

The owner of Britten's Egg Farm wants to estimate the mean number of eggs laid per chicken. A sample of 20 chickens shows they laid an average of 20 eggs per month with a standard deviation of 2 eggs per month (a sample is taken from a normal population).

i. What is the value of the population mean? What is the best estimate of this value?
ii. Explain why we need to use the t distribution. What assumption do you need to make?
iii. For a 95 percent confidence interval, what is the value of t?
iv. Develop the 95 percent confidence interval for the populationmean.
v. Would it be reasonable to conclude that the population mean is 25 eggs? What about 5 eggs?

## Solution:

i. the population mean is unknown, but the best estimate is 20,the sample mean
ii. Use the t distribution as the standard deviation is unknown and n<30.However, assume the population is normally distributed.

**iii.** $t_{n-1;\frac{\alpha}{2}} = t_{20-1,\frac{0.05}{2}} = t_{19,0.025} = 2.093024$

**iv.** $\bar{X} \pm t_{n-1;\frac{\alpha}{2}} \dfrac{S}{\sqrt{n}} = 20 \pm 2.093\left(\dfrac{2}{\sqrt{20}}\right) = 20 \pm 0.936$

$19.064 \le \hat{\mu} \le 20.936$

$(19.064\ ,\ 20.936)$

v. Neither value is reasonable, because they are not inside the interval.

## Example (6)

Find a 90% confidence interval for a population mean $\mu$ for these values:

**a.** $n = 50 \quad , \bar{x} = 21.9 \ \ , s^2 = 3.44$

**b.** $n = 14 \quad , \bar{x} = 1258 \ \ , s^2 = (214)^2$**,** $X \sim N(\mu, \sigma^2)$

## Solution:

**a. n ≥30**

$\bar{X} \pm Z_{\frac{\alpha}{2}} \dfrac{S}{\sqrt{n}}$

$= 21.9 \pm 1.65\left(\dfrac{1.855}{\sqrt{50}}\right)$

$= 21.9 \pm 0.4329$

$21.4671 \le \hat{\mu} \le 22.333$

$$\left(21.4671\ ,\ \ 22.333\right)$$

**b.** $\overline{X} \pm t_{n-1;\frac{\alpha}{2}} \dfrac{S}{\sqrt{n}}$

$$1258 \pm 1.77\left(\frac{214}{\sqrt{14}}\right)$$

$$= 1258 \pm 101.2332$$

$$1156.76 \le \widehat{\mu} \le 1359.23$$

$$\left(1156.76\ ,\ \ 1359.23\right)$$

# Choosing an appropriate sample size for the population mean

$$E = \pm Z \frac{\sigma}{\sqrt{n}} \textbf{Or} \quad \boldsymbol{E = \frac{UCL - LCL}{2}}$$

The length of confidence interval= UCL –LCL

The length of C.I=

$$\left(\bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) - \left(\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

$$= Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 2Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$E = Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

The sample size for estimating the population mean:

$$n = \left(\frac{Z_{\alpha/2}\,\sigma}{E}\right)^2$$

## Example (7)

A student in public administration wants to determine the mean amount members of city councils in large cities earn per month as remuneration for being a council member. The error in estimating the mean is to be less than \$100 with a 95 percent level of confidence. The student found a report by the Department of Labor that estimated the standard deviation to be \$1,000. What is the required sample size?

### Solution:

Given in the problem:

- E, the maximum allowable error, is \$100
- The value of z for a 95 percent level of confidence is 1.96,
- The estimate of the standard deviation is \$1,000.

$$n = \left(\frac{\left(Z_{\frac{\alpha}{2}}\right)\sigma}{E}\right)^2 = \left(\frac{(1.96)(1000)}{100}\right)^2 = 384.16 \approx 385$$

## Example (8)

A population is estimated to have a standard deviation of 10.if a 95 percent confidence interval is used and an interval of $\pm 2$ is desired .How large a sample is required?

**Solution:** Given in the problem:

- E, the maximum allowable error, is 2The value of z for a 95 percent level of confidence is 1.96,
- The estimate of the standard deviation is10.

$$n = \left( \frac{\left( Z_{\frac{\alpha}{2}} \right) \sigma}{E} \right)^2 = \left( \frac{(1.96)10}{2} \right)^2 = 96.04 \approx 97$$

## Example (9)

If a simple random sample of 326 people was used to make a 95% confidence interval of (0.57,0.67), what is the margin of error $(E)$?

**Solution:**

$$E = \frac{upper - lower}{2} = \frac{0.67 - 0.57}{2} = \frac{0.1}{2} = 0.05$$

## Example (10)

If n=34, the standard deviation 4.2 $(\sigma), 1-\alpha = 95\%$

What is the maximum allowable error $(E)$ ?

**Solution:**

$$n = \left( \frac{Z_{\frac{\alpha}{2}} \sigma}{E} \right)^2 = \frac{\left( Z_{\frac{\alpha}{2}} \sigma \right)^2}{E^2}$$

$$E^2 = \frac{\left( Z_{\frac{\alpha}{2}} \sigma \right)^2}{n}$$

$$E^2 = \frac{(1.96 \times 4.2)^2}{34} = \frac{(8.232)^2}{34} = \frac{67.77}{34} = 1.99$$

$$E = \pm\sqrt{1.99} = \pm 1.41$$

The maximum allowable error (E) = **1.41**

# Confidence Interval for a Population Proportion (Large Sample)

When the sample size is large $n \geq 100,\ \ 0.05 \leq \pi \leq 0.95\ \ , n\pi \geq 5, n(1-\pi) \geq 5$, the sample proportion,

$$P = \frac{X}{n} = \frac{Total\ number\ of\ successes}{Total\ number\ of\ trials}$$

$$P \sim N\left(\pi, \frac{\pi(1-\pi)}{n}\right)$$

The confidence interval for a population proportion:

$$\pi = P \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{P(1-P)}{n}}$$

$\sqrt{\frac{P(1-P)}{n}}$ , The standard error of the proportion

## Example (11)

The owner of the West End credit Kwick Fill Gas Station wishes to determine the proportion of customers who use a credit card or debit card to pay at the pump. He surveys 100 customers and finds that 80 paid at the pump.

a. Estimate the value of the population proportion.
b. Develop a 95 percent confidence interval for the population proportion.
c. Interpret your findings.

**Solution:**

**a.**

$$\pi = P = \frac{X}{n} = \frac{80}{100} = 0.8$$

**b.**

$$P \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{P(1-P)}{n}} = 0.8 \pm 1.96 \sqrt{\frac{(0.8)(0.2)}{100}} = 0.8 \pm 1.96\sqrt{0.0016} = 0.8 \pm 1.96(0.04) = 0.8 \pm 0.0784$$

$$0.72 \leq \hat{\pi} \leq 0.88$$

$$(0.72\ \ , 0.88)$$

c .We are reasonably sure the population proportion is between 0.72 and o.88 percent .

## Example (12)

The Fox TV network is considering replacing one of its prime-time crime investigation shows with a new family-oriented comedy show. Before a final decision is made, network executives commission a sample of 400 viewers. After viewing the comedy, 0.625 percent indicated they would watch the new show and suggested it replace the crime investigation show.

d. Estimate the value of the population proportion.
e. Develop a 99 percent confidence interval for the population proportion.
f. Interpret your findings.

## Solution:

**a.**

$\pi = P = 0.625$

**b.**

$$P \pm Z_{\frac{\alpha}{2}}\sqrt{\frac{P(1-P)}{n}} = 0.625 \pm 2.575\sqrt{\frac{(0.625)(0.375)}{400}} = 0.625 \pm 2.575\sqrt{0.000586}$$

$$= 0.625 \pm 2.575(0.0242) = 0.625 \pm 0.0623$$

$$0.56 \leq \hat{\pi} \leq 0.69$$

$$(0.56 \ , \ 0.69)$$

c .We are reasonably sure the population proportion is between 0.56 and o.69 percent .

**Note:**

If the value of estimated proportion(p) not mentioned we substitute it by o.5( as studies and reachears recommended)

# Choosing an appropriate sample size for the population proportion

The margin error for the confidence interval for a population proportion:

$$E = \pm Z_{\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}}$$

Solving "E" equation for "n" yields the following result:

$$n = \left(\frac{Z_{\frac{\alpha}{2}}\sqrt{p(1-p)}}{E}\right)^2$$

Or

$$n = p(1-p)\left(\frac{Z_{\frac{\alpha}{2}}}{E}\right)^2$$

**Example (13)**

The estimate of the population proportion is to be within plus or minus o.o5, with a 95 percent level of confidence. The best estimation of the population proportion is o.15.How large a sample is required?

**Solution:**

$$n = \left(\frac{\left(Z_{\frac{\alpha}{2}}\right)\sqrt{P(1-P)}}{E}\right)^2 = \left(\frac{1.96\sqrt{0.15\times 0.85}}{0.05}\right)^2 = \left(\frac{1.96\times 0.36}{0.05}\right)^2$$

$$= \left(\frac{0.69986}{0.05}\right)^2 = (13.9971)^2 = 195.92 \approx 196$$

## Example (14)

The estimate of the population proportion is to be within plus or minus o.10, with a 99 percent level of confidence. How large a sample is required?

## Solution:

$$n = \left( \frac{\left( Z_{\frac{\alpha}{2}} \right) \sqrt{P(1-P)}}{E} \right)^2 = \left( \frac{2.575\sqrt{0.5 \times 0.5}}{0.10} \right)^2 = \left( \frac{2.575 \times 0.5}{0.10} \right)^2$$

$$= \left( \frac{1.2875}{0.10} \right)^2 = (12.875)^2 = 165.7656 \approx 166$$

# Confidence interval for the Population Variance (Normal Case)

If we find the confidence interval fora **population variance**we must to begin with the statistic:

$$\chi^2 \sim \frac{(n-1)S^2}{\sigma^2}$$

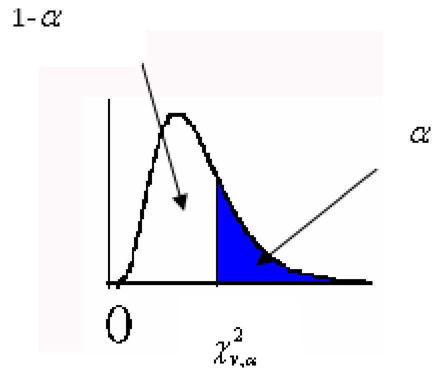
Is called a **chi-square variable** and has a sampling distribution called the **chi-square probability distribution**, with (n-1) degree of freedom.
It is not important to know the complex equation of the density function for $\chi^2$; only to use the well-tabulated critical values of $\chi^2$ given in $\chi^2$ table.
Under assumption the sample is randomly selected from a normal population:

$$X \sim N(\mu, \sigma^2)$$

The confidence interval for a **population variance ($\sigma^2$)** is :

$$\frac{(n-1)S^2}{\chi^2_{n-1;\frac{\alpha}{2}}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{n-1;1-\frac{\alpha}{2}}}$$

Where $\chi^2_{\alpha/2}$&$\chi^2_{(1-\alpha/2)}$are the upper and lower $\chi^2$value, which locate one – half of$\alpha$ in each tail of the chi-square distribution.



**The properties of the chi square distribution**.

- $\chi_v^2$ Is continuous distribution.
- $\chi_v^2$ Positive skewed curve (skewed to the right curve).
- $\chi_v^2$ It is not symmetric curve.

## Example (15)

The standard deviation of the lifetimes of 10 electric light bulbs manufactured by a company is 120 hours. Find 95% confidence limits for the standard deviation of all bulbs manufactured by the company.

## Solution:

$$\frac{(n-1)S^2}{\chi^2_{n-1;\frac{\alpha}{2}}} \le \hat{\sigma}^2 \le \frac{(n-1)S^2}{\chi^2_{n-1;1-\frac{\alpha}{2}}}$$

$$\frac{(9)120^2}{19.023} \le \hat{\sigma}^2 \le \frac{(9)120^2}{2.7004}$$

$$6812.81 \le \hat{\sigma}^2 \le 47992.89$$

$$82.54 \le \hat{\sigma} \le 219.07$$

## Example (16)

Suppose that $X \sim N(20, 4)$ and a random sample of size $n = 17$ is selected, prove that $4S^2 \sim \chi^2_{16}$...

## Solution:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_v^2$$

$$\frac{(17-1)S^2}{4} \sim \chi^2_{17-1}$$

$$\frac{16S^2}{4} \sim \chi^2_{16}$$