Math 316

Second Midterm Exam 1435, 1st semester

Name: ID:

Q1 Prove or disprove each of the following statements:

(a) $\|H_2\|^2_{e^{-x^2}} = 4\sqrt{\pi}$, where H_n refers to Hermite polynomial.

(b) $L_{10}(x) = 1 - 10x - \frac{21}{10}x^5 + \frac{1}{896}x^8$, where L_n refers to Laguerre polynomials.

Q2 Consider the piecewise smooth function f defined by

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 2x, & 0 \le x < \pi \end{cases},$$

and

$$f(x + 2\pi) = f(x), \; x \in \mathbb{R}.$$

(a) Sketch the function f on the interval $[-3\pi, 3\pi]$. What is the period of f?

(b) Find the Fourier series representation for f.

(c) Find the sum of the Fourier series at $x = 0$.

(d) Does the Fourier series converge to f at $x = \pi$?Explain.

(e) Redefine the function f so that the Fourier series converge to f at every $x \in \mathbb{R}$.

Q3 Consider the function

$$f(t, x) = \frac{1}{\sqrt{1 - 2xt + t^2}}, \qquad |t| < 1, \; |x| \le 1,$$

(a) Show that

$$\left(1 - 2xt + t^2\right) \frac{\partial f}{\partial t} = (x - t) f$$

(b) If the Taylor series expansion of the analytic function f is written in the form

$$f(t, x) = \sum_{n=0}^{\infty} a_n(x) t^n, \qquad |t| < 1,$$

Prove that

$$(n + 1) a_{n+1}(x) + n a_{n-1}(x) = (2n + 1) x a_n(x), \qquad n \in \mathbb{N}_0$$

(c) Show that

$$a_0(x) = P_0(x), \quad a_1(x) = P_1(x)$$

where P_n is the Legendre polynomial. (Hint: $a_n(x) = \left.\frac{\partial^n f}{\partial t^n}\right|_{t=0}$).

(d) Deduce that $f(t,x)$ is the generating function of the Legendre polynomials, i.e.

$$f(t,x) = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Q4 The solution of the wave equation

$$u_{tt} = u_{xx}, \qquad 0 < x < l, \ t > 0,$$

subject to the boundary conditions

$$u(0,t) = u(l,t) = 0, \qquad t > 0,$$

is given by

$$u(x,t) = \sum_{n=0}^{\infty} \left(a_n \cos \frac{n\pi}{l} t + b_n \sin \frac{n\pi}{l} t\right) \sin \frac{n\pi}{l} x$$

Find a_n and b_n for all $n \in \mathbb{N}_0$ for the particular initial conditions

$$u(x,0) = x, \quad u_t(x,0) = 0, \quad 0 < x < l.$$

Good Luck
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