8. Functions of Random Variables

- **Method of Distribution Functions**
- **One-to-One Transformations.**
- **Two-to Two Transformations. (Joint distribution of Functions of Random Variables)**
- **Method of Moment-Generating Functions.**

Q1) If $X\sim Uniform(0,1)$, find the pdf of Y=-2lnX. Name the distribution and its parameter values.

Q2) If $X\sim Uniform(a,b)$, find the constants c and d such that Y=c+dX $Uniform(0,1)$.

Q3) If $X\sim Normal(\mu,\sigma^2)$, find the pdf of Y= e^X.

Q4) If $X\sim Exponential(1)$, find the pdf of Y=-lnX.

Q5) If $X\sim Uniform(0,1)$, find the pdf of $Y=\sqrt{X}$.

Q6) The pdf of X is given by $f_X(x)=\frac{1}{2}x\,;0<x<2$.

a. Find the pdf of $Y=X^3$.

b. Find $P\left(\frac{1}{2}<X<1\right)$ and $P\left(\frac{1}{8}<Y<1\right)$. Are they the same or different? Why?

Q7) If $X\sim\chi_4^2$, find P(X > 5).

Q8) If $X\sim Uniform(0,1)$ independent of $Y\sim Exponential(1)$, find the distribution of Z=X+Y:

a. Using the pdf formula derived in class.

b. By first finding the cdf and then differentiating.

Q9) If $X \sim Gamma(2,3)$ independent of $Y \sim Uniform(0,2)$, and $Z \sim Gamma(5,3)$, what is the distribution of X+Y+Z if X, Y and Z are independent?

Q10) If $X \sim Normal(2,3)$ independent of $Y \sim Normal(5,1)$, and $Z \sim Normal(20,21)$, with X, Y and Z independent, find P(X+Y+Z < 25).

Q11) Let X and Y have joint pdf $f(x,y) = 1\ ;\ -y < x < y, 0 < y < 1$.

a. Find the conditional pdf of X|Y=y.

b. Find $P(X < 0|Y = y)$.

c. Find $P(X > \frac{1}{4}|Y = \frac{1}{3})$.

d. Find $P(0 < X < \frac{1}{4}|Y = \frac{1}{2})$.

Q12) Let X and Y have joint pdf $f(x,y) = \frac{2}{5}(x + 4y)\ ;\ 0 < x < 1, 0 < y < 1$.

a. Find the conditional pdf of Y|X=x.

b. Find $P(Y < \frac{1}{3}|X = \frac{1}{2})$.

Q13) If $X \sim Uniform(0,1)$ independent of $Y \sim Exponential(1)$, find

a. The joint density function of Z=X+Y and U=X/Y.

b. The density function of Z.

c. The density function of U.

Q14) Let (X,Y) have joint pdf $f(x,y) = \frac{1}{x^2y^2}; x \geq 1, y \geq 1$.

a. Find the joint density of U=XY and V=X/Y.

b. What are the marginal density of U and V?

Q15) Let X_1 and X_2 be independent $Exp(\lambda)$ r.v. Find the joint density of

$Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$.

Q16) Let $X_1 \sim Exp(\lambda_1)$ independent of $X_2 \sim Exp(\lambda_2)$ r.v.. Find:

a. The cumulative distribution function of $Z = \frac{X_1}{X_2}$.

b. $P(X_1 < X_2)$.

Q17) The joint pdf of (X,Y) is given by $f(x,y) = \frac{e^{-y}}{y}; 0 < x < y, 0 < y < \infty$. Find E(X), E(Y), V(X), V(Y) and Cov(X,Y).

Q18) Let X and Y be distributed as independent Uniform(0,1) r.v.

a. Find the joint density function of Z_1= X+Y and Z_1= Y.

b. Find the marginal pdf of Z_1 from the joint density.

Q19) Let X and Y be distributed as independent Exp(1) r.v., find:

a. The joint density function of $Z = X + Y$ and $U = \frac{X}{X+Y}$.

b. Find the marginal pdf of U.

Q20) Let (X,Y) have joint density given by $f(x,y) = 24xy; 0 < x < 1, 0 < y < 1$, $x + y < 1$, find the pdf of $Z = XY^2$.

Q21) Let X and Y have independent $Gamma(\alpha, \lambda)$ distributions.

a. Find the joint pdf of $U = \frac{X}{X+Y}$ and $V = X + Y$.

b. Show that the marginal density of U is a Beta distribution.

Q22) Let (X,Y) have joint density given by $f(x,y) = 24xy; 0 < x < 1, 0 < y < 1$, $x + y < 1$, find:

a. The marginal pdf's.

b. The following expectations:

i. E(X) and E(X^2).

ii. E(Y) and E(Y^2).

iii. E(XY) and E(X^2 Y^3).

iv. V(X), V(Y), Cov(X,Y). Do X and Y have a positive or negative relationship?

Q23) Let joint pdf of (X,Y) given by $f(x,y) = \frac{1}{y} e^{-y} e^{-x/y}$; $x > 0, y > 0$, find:

a. E(X) and E(X^2).

b. E(Y) and E(Y^2).

c. Show that Cov(X,Y)=1.

d. $\rho(X,Y)$.

Q24) If X, Y, Z ~ independent Exp(1), derive the joint distribution of U=X+Y, V=X+Z, and Z=Y+Z.

- If X_i $indpt.$ $Exp(\lambda), then\ the\ sum\ \sum_{i=1}^{i=n} X_i \sim Gamma\ (n, \lambda)$
- If X_i $indpt.$ $Gamma(\alpha_i, \beta), then\ the\ sum\ \sum_{i=1}^{i=n} X_i \sim Gamma\ (\sum_{i=1}^{i=n} \alpha_i, \beta)$
- If X_i $indpt.$ $Normal(\mu_i, \sigma_i^2), then\ the\ sum\ \sum_{i=1}^{i=n} X_i \sim Normal\ (\sum_{i=1}^{i=n} \mu_i, \sum_{i=1}^{i=n} \sigma_i^2)$
- If X_i $indpt.$ $Normal(\mu_0, \sigma_0^2), then\ the\ sum\ \sum_{i=1}^{i=n} X_i \sim Normal\ (n\mu_0, n\sigma_0^2)$
- If Z $Normal(0,1), then\ the\ Z^2 \sim \chi_{\mathbf{1}}^{\mathbf{2}}$
- If $X \sim \chi_{\mathbf{n}}^{\mathbf{2}},\ ind, of\ Y \sim \chi_{\mathbf{m}}^{\mathbf{2}},\ then\ the\ X + Y \sim \chi_{\mathbf{n+m}}^{\mathbf{2}}$
- If Z_1 $Normal(0,1), ind.of\ Z_2$ $Normal(0,1)\ then\ the\ Z_1 + Z_2 \sim \chi_{\mathbf{2}}^{\mathbf{2}}$