Call - Put parity condition:

Ex:

$$S + P = C + k e^{-rT}$$

$$\Rightarrow P = C - S + k e^{-rT}$$

Ex: $S_0 = 50\$$.

$K = 45\$$

$T = 6 \text{ months} = 0.5$

$RFR = 5\%$ (annual)

$C = 3.5\$/\text{Call}$.

Calculate the put price:

$$P = 3.5 - 50 + 45 e^{-0.05 \times 0.5}$$

The BOPM

Main Idea:

We use the Delta Hedging and risk-neutral valuation in Option pricing.

We first present an example using an option that only has one period to maturity and where we assume we know the possible outcomes for the Stock price (no dividends).

The Basic idea is to construct a synthetic portfolio, which contains a call option

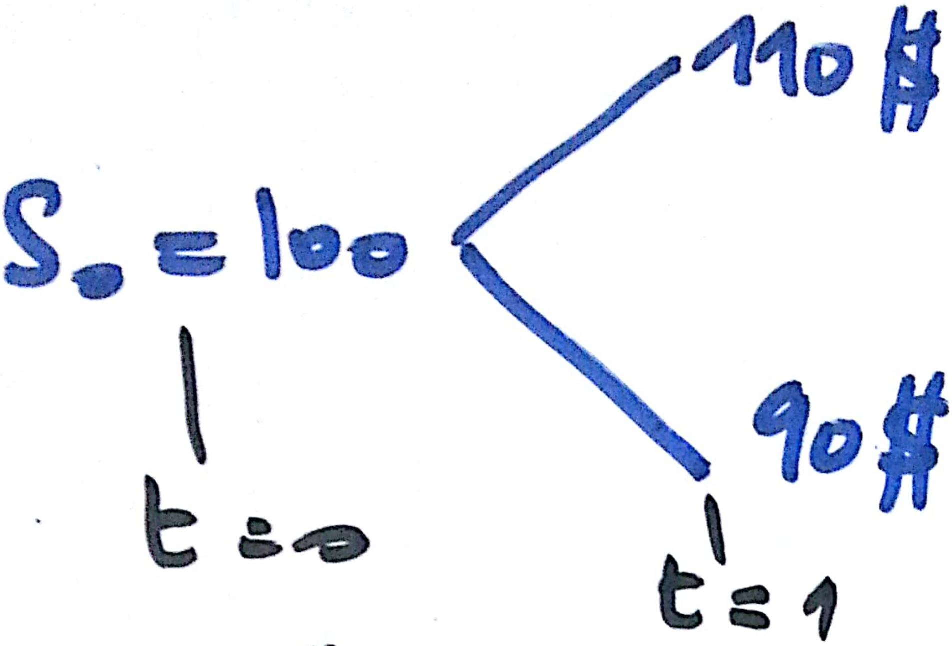
and a Stock in such a way that this portfolio is risk-free.

We can then equate the return from this synthetic portfolio to the known risk-free interest rate and solve for the unknown price of call option.

We assume that the Stock pay no divident.

One period BOPM

Consider a one-period problem where there are only two possible outcomes for the stock price



$S_t = 100$ \$ = Stock price today.

$K = 100$ \$ = Strike price (at-the-money call)

C = the unknown call premium.

$r = 5\% = 0.05$ = free interest rate

We Assume that the "real World" probability of the Stock price moving up or down is $p = \frac{1}{2}$.

S_T	Prob
110	0.5
90	0.5

$$E(S_T) = \sum p_i S_i = 110 \times 0.5 + 90 \times 0.5 = 100.$$

$$E(\text{Return}) = \frac{100 - 100}{100} = 0\%.$$

the payoffs to holding either one stock or long call will be shown as follows:

A Long 1 Stock

$$
\begin{array}{lcl}
 & \overset{0.5}{\nearrow} & S_u = uS = 110 \\
S_0 = 100 & & \\
 & \underset{0.5}{\searrow} & S_d = dS = 90 \\
t = 0 & & t = 1\ (T)
\end{array}
$$

Long one Call: ($k=100$, $S_0=100$)

$C_u = \text{Max}\left[0, S_u - k\right]$

C

?

$C_d = \text{Max}\left[0, S_d - k\right]$

$C_u = \text{Max}\left[0, 110-100\right]$

C

$C_d = \text{Max}\left[0, 90-100\right]$

$C_u = 10\$$

$\Rightarrow$ C

?

$C_d = 0\$$

unknown ?
Call price

the possible payoffs on the long call are

$$Max[0, S_u - k] = 10 = C_u$$

$$Max[0, S_d - k] = 0 = C_d$$

- the difference in the 2 payoffs for holding a long call is +10.
- the difference in the payoffs of stock is 20 (110 - 90).

Our portfolio is risk - free

$\Rightarrow$ we could write $\Delta S = 20$

$\Delta C = 10$

$$\Rightarrow h = \frac{\Delta C}{\Delta S} = \frac{1}{2}$$

h is the Hedging Ratio

Comment:

the difference in the payoff on $\frac{1}{2}$ Stock which equals 10 \$ $(\frac{1}{2} \times 20)$, will just offset the difference in the payoff of 10 \$ on the call.

What we are going to do now with (h). we construct a portfolio (Stock + Call) that has a known payoff at t=1 no matter what the value of the Stock price and the Call premium at t=1. The hedge ratio (h) will involve a cash outlay at t=0

but because the payoff at $t=1$ is known, our (h) portfolio must have a return equal to the RFR, or profitable arbitrage opportunities are possible.

So, Consider a portfolio A
{ Long on $\frac{1}{2}$ Stock
Short on 1 call.

<u>Payoff of the Portfolio</u>:

- Payoff for price rise $= \frac{1}{2}(110) - 10 = \underline{\underline{45}}$
$= \frac{1}{2} SU - C_u = 55 - 10$

- Payoff for price fall $= \frac{1}{2} Sd - Cd$
$= \frac{1}{2} \times 90 - 0 = \underline{\underline{45}}$

The payoff to portfolio A at $t=1$ is known with certainty, no matter what the outcome for the Stock price (110 or 90).

$\Rightarrow$ We have created a risk-free portfolio using a hedge ratio $h = -\frac{1}{2}$.

$\Rightarrow$ the cost of constructing A at $t=0$ is the cost of buying the stocks less the receipts from the sale of the <u>Call</u> that is:

$$\frac{1}{2} S_0 - C_0 = 45$$

Hence, the payoff of 45\$,

discounted back to today at the risk free rate, must equal to the cost of portfolio A.

Therefore,

$$\frac{45}{1.05} = \text{Cost of the portfolio at } t=0$$

$$= \frac{1}{2} S_0 - C_0$$

$\Rightarrow$ =

$$42.8572 = \frac{1}{2} 100 - C_0$$

$$C_0 = \frac{1}{2} \times 100 - 42.8572$$

$$\boxed{C_0 = 7.1428571}$$

Equivalently, the payoff from portfolio A is the risk-free and therefore must earn RFR.

$$1 + RFR = \frac{\text{Certain Payoff } t=1}{\text{Initial Invest } t=0}$$

$$1 + RFR = \frac{45}{\frac{1}{2} S_0 - C}$$

$\Rightarrow$ From this equality, we find that

$$\underline{C_0 = 7.142857\overline{1}}$$

Fair Pricing And Arbitrage Opportunity

the fair price of the Optim must be $C = 7.14$ \$. Otherwice risk free arbitrage profit can be made.

How?

Suppose that the quoted Optim premium is 10 \$. which exceeds the fair price. You are an optims trader and you spot this pricing anomaly.

You sell high and you buy low.

At $t=0$, you sell the call at 10 \$. To garantee you have a payout at $t=1$ (T)

You Also purchase $\frac{1}{2}$ Stock. the guaranteed payoff to this hedged portfolio A is 45\$.

With a premium of 10\$, the Net Cost of portfolio A at t=0 is 40\$ $= \frac{1}{2}(100) - 10 = 40$.

Subsequently, the return of the hedged portfolio is:

$$\frac{45}{40} - 1 = 12.5\% >> RFR = 5\%.$$

therefore, You could borrow 40\$ from the Bank.

At $t=1$, you will pay $40(1.05) = 42$\$. But the portfolio worth 45\$ -86- $\Rightarrow$

The risk Free Arbitrage profit will be 3 $.

Pricing of a put option

We can price a put option by constructing a similar risk-free portfolio of Stocks and the PUT.

Portfolio B:

- Long on Some Stocks ($h\,S$)
- Long on One put.

So, if S falls, the Loss on the Stock will be offset by the Gain on the put.

making the portfolio (Stock + Put) risk - free. (K = 100)

the payoff of the put.

$$S_0 = 100 \begin{cases} uS = 110\$ \\ dS = 90\$ \end{cases}$$

$t = 0$ ... $t = 1\ (T)$

$$P_0 = ? \begin{cases} Max[0, K - uS] = 0 \\ Max[0, K - dS] = 10 \end{cases}$$

$t = 0$ ... $t = 1\ (T)$

Let $h = \frac{1}{2}$, the number of Stocks to purchase, and we also hold one Long put, then:

Payoff to the portfolio B ($\frac{1}{2}$ Stock + 1 Long put)

Payoff for Price rise : $\frac{1}{2}(110)+0=55$

Payoff for Price fall : $\frac{1}{2}(90)+10=55$

Equating the return on the risk-free hedge portfolio B to the risk-free rate gives :

$$\frac{\text{Certain Payout } t=1}{\text{Cost of invest at } t=0}=(1+RFR)$$

$$\frac{55}{\frac{1}{2}(100)+P}=1.05$$

From this Eq. With

$P = 2.38\ \$$

Formal Derivation: BOPM

Let's now derive the price of the Call using Algebra.
The 2 outcomes of the Stock are:

$$\begin{tikzcd}[row sep=small] & uS = 110 \\ S_0 \arrow[ur] \arrow[dr] & \\ & dS = 90 \end{tikzcd}$$

Similarly,
C_u = payoff of the Long call if the Stock price is $\underline{\underline{S_u}}$

C_d = payoff of the Long call if the Stock price is $\underline{\underline{S_u}}$

$$\begin{tikzcd}[row sep=small] & C_u = Max\boxed{0, S_u - K} \\ C_0 = ? \arrow[ur] \arrow[dr] & \\ & C_d = Max\boxed{0, S_d - K} \end{tikzcd}$$

-go-

Portfolio A (Long H Stock, Short 1 call).

- Payoff for Price rise $t = 1$ =
$$hS_u - C_u.$$
- Payoff for Price fall $t = 1$
$$hS_d - C_d.$$

$$\begin{tikzcd}[column sep=large]
 & hS_u - C_u \\
hS_0 - C_0 \arrow[ur, no head] \arrow[dr, no head] & \\
 & hS_d - C_d.
\end{tikzcd}$$

$t=0$ $\qquad\qquad\qquad\qquad$ $t=1$

The minus sign indicates that portfolio A consists of one short call. For 2 payoffs to be equal at $t = 1$
$$hS_u - C_u = hS_d - C_d$$

$$h = \frac{C_u - C_d}{S_u - S_d} = \frac{C_u - C_d}{S(u-d)}$$

$$= \frac{10-0}{110-90} = \frac{10}{20} = \frac{1}{2}.$$

$$\boxed{h = \frac{1}{2}}$$

the hedge ratio is also known as the option <u>Delta</u> (Δ)

and This approach is called <u>Delta Hedging</u>

PV of payoff of PF A = Cost of the portfolio A At = t = 0

$$\frac{hSu - C_u}{1+r} = hS - C$$

Substituting for h and rearranging, the call premium is :

$$\boxed{C = \frac{1}{R}\left[qC_u + (1-q)C_d\right]}$$

Where, $R = 1 + RFR$

$$q = \frac{R-D}{U-D} = \underline{0,75}$$

$R = 1.05$

$D = \frac{90}{100} = 0,90$ (downward coef)

$U = \frac{110}{100} = 1.1$ (upward coef)

$$q = \frac{1.05 - 0,9}{1.1 - 0.9} = \frac{0.15}{0.2} = 0.75$$

Notes :

1- The Formula of the call premium does not depend on the real world prob. of an up or down move of the Stock price.

2- In the formula for the CALL premium the weights applied to the 2 Options payoffs Cu and Cd are (q) and (1-q) (Sum to 1)

3- The Call premium C is a weighted average of the payoffs of the call (Cu and Cd) discounted using the risk free rate.

4 - the Weight(q) is known as the risk-neutral probability of a rise in the Stock price. but this must not be Confused with the actual probability in the Stock price, which does not affect the Option premium.

5. the Risk-neutral Prob. is simply a number in $[0, 1]$ is derived under the assumption that the PF is Risk-Free earn the risk-free Rate. ✓.