Shaft Design for Stress : Stress Analysis

- Assuming a solid shaft with round cross section, appropriate geometry terms can be introduced for *c*, *I,* and *J* resulting in the fluctuating stresses due to bending and torsion as

$$\sigma_a = K_f \frac{32M_a}{\pi d^3} \qquad \sigma_m = K_f \frac{32M_m}{\pi d^3} \qquad \tau_a = K_{fs} \frac{16T_a}{\pi d^3} \qquad \tau_m = K_{fs} \frac{16T_m}{\pi d^3}$$

- Combining these stresses in accordance with the distortion energy failure theory, the von Mises stresses for rotating round, solid shafts, neglecting axial loads, are given by

$$\sigma'_a = (\sigma_a^2 + 3\tau_a^2)^{1/2} = \left[\left(\frac{32K_f M_a}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs}T_a}{\pi d^3}\right)^2\right]^{1/2} \qquad \sigma'_m = (\sigma_m^2 + 3\tau_m^2)^{1/2} = \left[\left(\frac{32K_f M_m}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs}T_m}{\pi d^3}\right)^2\right]^{1/2}$$

- These equivalent alternating and midrange stresses can be evaluated using an appropriate failure curve on the modified Goodman diagram as

$$\frac{1}{n} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}}$$

$$\frac{1}{n} = \frac{16}{\pi d^3}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs}T_a)^2\right]^{1/2} + \frac{1}{S_{ut}}\left[4(K_f M_m)^2 + 3(K_{fs}T_m)^2\right]^{1/2}\right\}$$

- A von Mises maximum stress for static failure is calculated

$$\sigma'_{\max} = \left[(\sigma_m + \sigma_a)^2 + 3(\tau_m + \tau_a)^2\right]^{1/2}$$

$$= \left[\left(\frac{32K_f(M_m + M_a)}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs}(T_m + T_a)}{\pi d^3}\right)^2\right]^{1/2}$$

$$d = \left(\frac{16n}{\pi} \left\{ \frac{1}{S_e} \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2 \right]^{1/2} + \frac{1}{S_{ut}} \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2 \right]^{1/2} \right\} \right)^{1/3}$$

DE-Goodman

$$\frac{1}{n} = \frac{16}{\pi d^3} \left\{ \frac{1}{S_e} \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2 \right]^{1/2} + \frac{1}{S_{ut}} \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2 \right]^{1/2} \right\} \tag{7–7}$$

$$d = \left(\frac{16n}{\pi} \left\{ \frac{1}{S_e} \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2 \right]^{1/2} + \frac{1}{S_{ut}} \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2 \right]^{1/2} \right\} \right)^{1/3} \tag{7–8}$$

DE-Gerber

$$\frac{1}{n} = \frac{8A}{\pi d^3 S_e} \left\{ 1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}} \right)^2 \right]^{1/2} \right\} \tag{7–9}$$

$$d = \left(\frac{8nA}{\pi S_e} \left\{ 1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}} \right)^2 \right]^{1/2} \right\} \right)^{1/3} \tag{7–10}$$

where

$$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$$

$$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$$

DE-ASME Elliptic

$$\frac{1}{n} = \frac{16}{\pi d^3} \left[4 \left(\frac{K_f M_a}{S_e} \right)^2 + 3 \left(\frac{K_{fs} T_a}{S_e} \right)^2 + 4 \left(\frac{K_f M_m}{S_y} \right)^2 + 3 \left(\frac{K_{fs} T_m}{S_y} \right)^2 \right]^{1/2} \tag{7–11}$$

$$d = \left\{ \frac{16n}{\pi} \left[4 \left(\frac{K_f M_a}{S_e} \right)^2 + 3 \left(\frac{K_{fs} T_a}{S_e} \right)^2 + 4 \left(\frac{K_f M_m}{S_y} \right)^2 + 3 \left(\frac{K_{fs} T_m}{S_y} \right)^2 \right]^{1/2} \right\}^{1/3} \tag{7–12}$$

DE-Soderberg

$$\frac{1}{n} = \frac{16}{\pi d^3} \left\{ \frac{1}{S_e} \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{yt}} \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2} \right\} \tag{7–13}$$

$$d = \left(\frac{16n}{\pi} \left\{ \frac{1}{S_e} \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{yt}} \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2} \right\} \right)^{1/3} \tag{7–14}$$

Fig. 6–27. The Gerber and modified Goodman criteria do not guard against yielding, requiring a separate check for yielding. A von Mises maximum stress is calculated for this purpose.

$$\sigma'_{\max} = \left[(\sigma_m + \sigma_a)^2 + 3(\tau_m + \tau_a)^2\right]^{1/2}$$

$$= \left[\left(\frac{32K_f(M_m + M_a)}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs}(T_m + T_a)}{\pi d^3}\right)^2\right]^{1/2} \qquad (7\text{–}15)$$

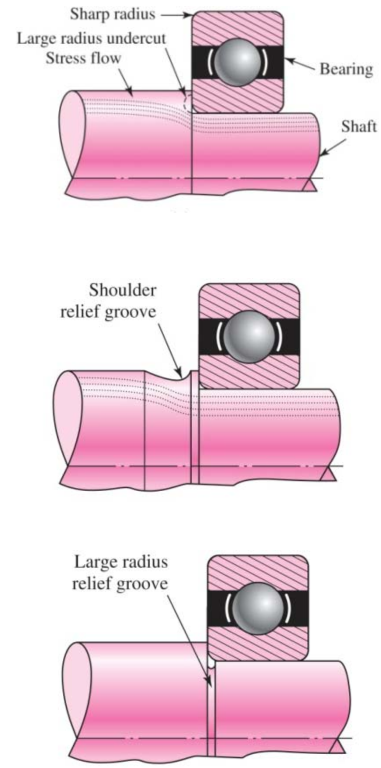
To check for yielding, this von Mises maximum stress is compared to the yield strength, as usual.

$$n_y = \frac{S_y}{\sigma'_{\max}} \qquad (7\text{–}16)$$

For a quick, conservative check, an estimate for $\sigma'_{\max}$ can be obtained by simply adding σ'_a and σ'_m. $(\sigma'_a + \sigma'_m)$ will always be greater than or equal to $\sigma'_{\max}$, and will therefore be conservative.

Shaft Design for Stress : Stress Concentration

- Stress concentrations for shoulders and keyways are dependent on size specifications that are not known the first time through the process.
- These stress concentrations will be fine-tuned in successive iterations, once the details are known.
- In cases where the shoulder at the bearing is found to be critical, the designer should plan to select a bearing with generous fillet radius, or consider providing for a larger fillet radius on the shaft by relieving it into the base of the shoulder.
- A keyway will produce a stress concentration near a critical point where the load-transmitting component is located.
- Some typical stress concentration facto iteration in the design of a shaft.



	Bending	Torsional	Axial
Shoulder fillet—sharp (r/d = 0.02)	2.7	2.2	3.0
Shoulder fillet—well rounded (r/d = 0.1)	1.7	1.5	1.9
End-mill keyseat (r/d = 0.02)	2.14	3.0	—
Sled runner keyseat	1.7	—	—
Retaining ring groove	5.0	3.0	5.0

Missing values in the table are not readily available.

EXAMPLE 7–1 At a machined shaft shoulder the small diameter d is 28 mm, the large diameter D is 42 mm, and the fillet radius is 2.8 mm. The bending moment is 142.4 N·m and the steady torsion moment is 124.3 N·m. The heat-treated steel shaft has an ultimate strength of $S_{ut} = 735$ MPa and a yield strength of $S_y = 574$ MPa. The reliability goal is 0.99.

(a) Determine the fatigue factor of safety of the design using each of the fatigue failure criteria described in this section.

(b) Determine the yielding factor of safety.

Solution (*a*) $D/d = 42/28 = 1.50$, $r/d = 2.8/28 = 0.10$, $K_t = 1.68$ (Fig. A–15–9), $K_{ts} = 1.42$ (Fig. A–15–8), $q = 0.85$ (Fig. 6–20), $q_{\text{shear}} = 0.92$ (Fig. 6–21).

From Eq. (6–32),

$$K_f = 1 + 0.85(1.68 - 1) = 1.58$$

$$K_{fs} = 1 + 0.92(1.42 - 1) = 1.39$$

Eq. (6–8): $$S'_e = 0.5(735) = 367.5 \text{ MPa}$$

Eq. (6–19): $$k_a = 4.51(735)^{-0.265} = 0.787$$

Eq. (6–20): $$k_b = \left(\frac{28}{7.62}\right)^{-0.107} = 0.870$$

$$k_c = k_d = k_f = 1$$

Table 6–6: $$k_e = 0.814$$

$$S_e = 0.787(0.870)0.814(367.5) = 205 \text{ MPa}$$

For a rotating shaft, the constant bending moment will create a completely reversed bending stress.

$$M_a = 142.4 \text{ N}\cdot\text{m} \qquad T_m = 124.3 \text{ N}\cdot\text{m} \qquad M_m = T_a = 0$$

Applying Eq. (7–7) for the DE-Goodman criteria gives

$$\frac{1}{n} = \frac{16}{\pi(0.028)^3}\left\{\frac{\left[4(1.58 \cdot 142.4)^2\right]^{1/2}}{205 \times 10^6} + \frac{\left[3(1.39 \cdot 124.3)^2\right]^{1/2}}{735 \times 10^6}\right\} = 0.615$$

Answer $n = 1.62$ DE-Goodman

Similarly, applying Eqs. (7–9), (7–11), and (7–13) for the other failure criteria,

Answer $n = 1.87$ DE-Gerber

Answer $n = 1.88$ DE-ASME Elliptic

Answer $n = 1.56$ DE-Soderberg

For comparison, consider an equivalent approach of calculating the stresses and applying the fatigue failure criteria directly. From Eqs. (7–5) and (7–6),

$$\sigma'_a = \left[\left(\frac{32 \cdot 1.58 \cdot 142.4}{\pi(0.028)^3}\right)^2\right]^{1/2} = 104.4 \text{ MPa}$$

$$\sigma'_m = \left[3\left(\frac{16 \cdot 1.39 \cdot 124.3}{\pi(0.028)^3}\right)^2\right]^{1/2} = 69.4 \text{ MPa}$$

Taking, for example, the Goodman failure critera, application of Eq. (6–46) gives

Taking, for example, the Goodman failure critera, application of Eq. (6–46) gives

$$\frac{1}{n} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{104.4}{205} + \frac{69.4}{735} = 0.604$$

$$n = 1.62$$

which is identical with the previous result. The same process could be used for the other failure criteria.

(*b*) For the yielding factor of safety, determine an equivalent von Mises maximum stress using Eq. (7–15).

$$\sigma'_{\max} = \left[\left(\frac{32(1.58)(142.4)}{\pi(0.028)^3}\right)^2 + 3\left(\frac{16(1.39)(124.3)}{\pi(0.028)^3}\right)^2\right]^{1/2} = 125.4\ \text{MPa}$$

Answer

$$n_y = \frac{S_y}{\sigma'_{\max}} = \frac{574}{125.4} = 4.58$$

For comparison, a quick and very conservative check on yielding can be obtained by replacing $\sigma'_{\max}$ with $\sigma'_a + \sigma'_m$. This just saves the extra time of calculating $\sigma'_{\max}$ if σ'_a and σ'_m have already been determined. For this example,

$$n_y = \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{574}{104.4 + 69.4} = 3.3$$

which is quite conservative compared with $n_y = 4.58$.

EXAMPLE 7–2

This example problem is part of a larger case study. See Chap. 18 for the full context.

A double reduction gearbox design has developed to the point that the general layout and axial dimensions of the countershaft carrying two spur gears has been proposed, as shown in Fig. 7–10. The gears and bearings are located and supported by shoulders, and held in place by retaining rings. The gears transmit torque through keys. Gears have been specified as shown, allowing the tangential and radial forces transmitted through the gears to the shaft to be determined as follows.

$$W_{23}^{t} = 2400 \text{ N} \qquad W_{54}^{t} = -10\,800 \text{ N}$$

$$W_{23}^{r} = -870 \text{ N} \qquad W_{54}^{r} = -3900 \text{ N}$$

where the superscripts *t* and *r* represent tangential and radial directions, respectively; and, the subscripts 23 and 54 represent the forces exerted *by* gears 2 and 5 (not shown) *on* gears 3 and 4, respectively.

Proceed with the next phase of the design, in which a suitable material is selected, and appropriate diameters for each section of the shaft are estimated, based on providing sufficient fatigue and static stress capacity for infinite life of the shaft, with minimum safety factors of 1.5.

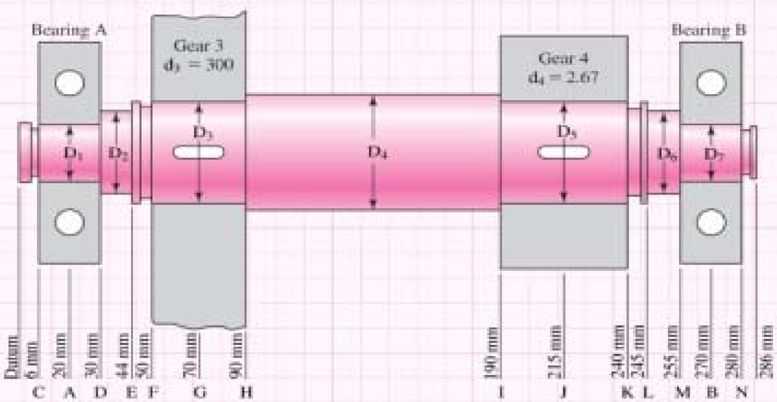


Figure 7–10

Shaft layout for Ex. 7–2. Dimensions in millimeters.

Solution

Perform free body diagram analysis to get reaction forces at the bearings.

$R_{Az} = 422$ N

$R_{Ay} = 1439$ N

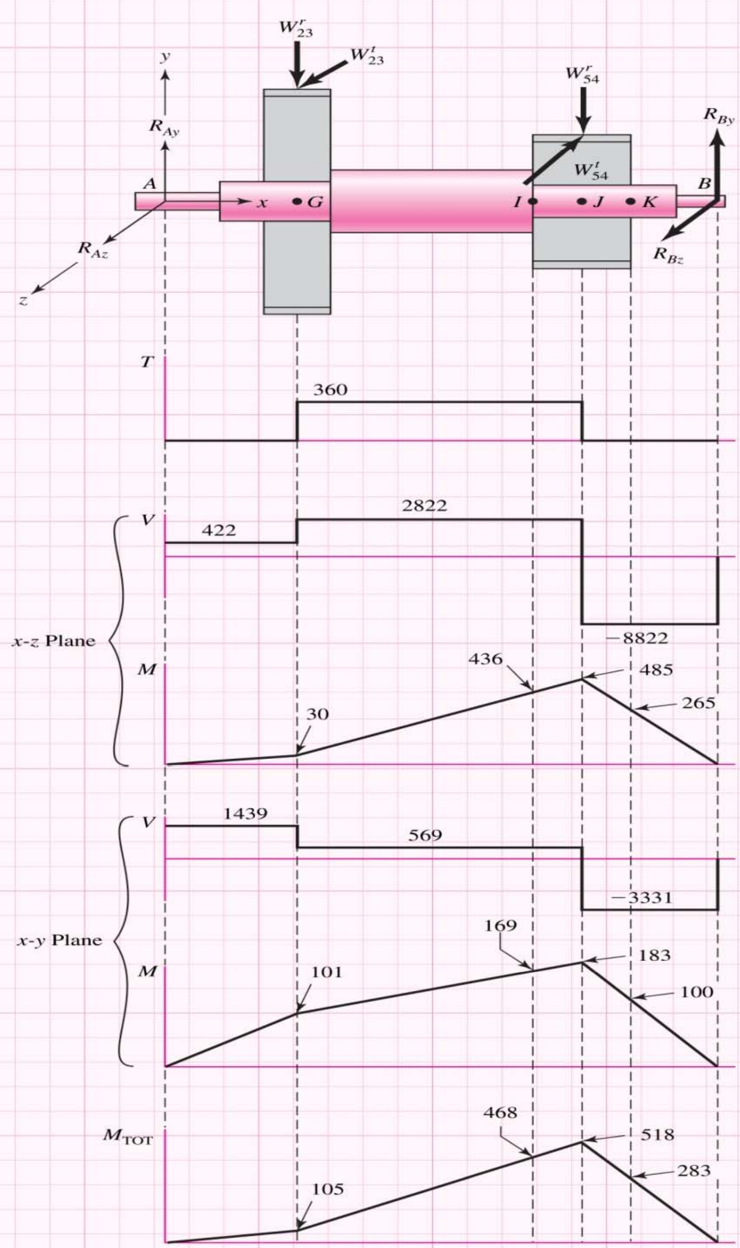
$R_{Bz} = 8822$ N

$R_{By} = 3331$ N

From ΣM_x, find the torque in the shaft between the gears, $T = W_{23}^t(d_3/2) = 2400\ (0.3/2) = 360$ N·m

Generate shear-moment diagrams for two planes.

Combine orthogonal planes as vectors to get total moments, e.g. at J, $\sqrt{485^2 + 183^2} = 518$ N·m.



Start with Point I, where the bending moment is high, there is a stress concentration at the shoulder, and the torque is present.

At I, $M_a = 468$ N·m, $T_m = 360$ N·m, $M_m = T_a = 0$

Assume generous fillet radius for gear at I.

From Table 7–1, estimate $K_t = 1.7$, $K_{ts} = 1.5$. For quick, conservative first pass, assume $K_f = K_t$, $K_{fs} = K_{ts}$.

Choose inexpensive steel, 1020 CD, with $S_{ut} = 469$ MPa. For S_e,

Eq. (6–19) $$k_a = aS_{ut}^b = 4.51\,(469)^{-0.265} = 0.883$$

Guess $k_b = 0.9$. Check later when d is known.

$$k_c = k_d = k_e = 1$$

Eq. (6–18) $$S_e = (0.883)(0.9)(0.5)(469) = 186 \text{ MPa.}$$

For first estimate of the small diameter at the shoulder at point I, use the DE-Goodman criterion of Eq. (7–8). This criterion is good for the initial design, since it is simple and conservative. With $M_m = T_a = 0$, Eq. (7–8) reduces to

$$d = \left\{ \frac{16n}{\pi} \left(\frac{2\,(K_f M_a)}{S_e} + \frac{\left[3\,(K_{fs} T_m)^2\right]^{1/2}}{S_{ut}} \right) \right\}^{1/3}$$

$$d = \left\{ \frac{16(1.5)}{\pi} \left(\frac{2\,(1.7)\,(468)}{186 \times 10^6} + \frac{\{3\,[(1.5)\,(360)]^2\}^{1/2}}{469 \times 10^6} \right) \right\}^{1/3}$$

$$d = 0.0432 \text{ m} = 43.2 \text{ mm}$$

All estimates have probably been conservative, so select the next standard size below 43.2 m. and check, $d = 42$ mm.

A typical D/d ratio for support at a shoulder is $D/d = 1.2$, thus, $D = 1.2 \times 42 = 50.4$ mm. Use D = 50 mm. A nominal 50-mm cold-drawn shaft diameter can be used. Check if estimates were acceptable.

$$D/d = 50/42 = 1.19$$

Assume fillet radius $r = d/10 \cong 4$ mm $\qquad r/d = 0.1$

$$K_t = 1.6 \text{ (Fig. A–15–9)}, \; q = 0.82 \text{ (Fig. 6–20)}$$

Eq. (6–32) $$K_f = 1 + 0.82(1.6 - 1) = 1.49$$

$$K_{ts} = 1.35 \quad \text{(Fig. A–15–8)}, \; q_s = 0.95 \text{ (Fig. 6–21)}$$

$$K_{fs} = 1 + 0.95(1.35 - 1) = 1.33$$

$$k_a = 0.883 \text{ (no change)}$$

Eq. (6–20) $$k_b = \left(\frac{42}{7.62}\right)^{-0.107} = 0.833$$

$$S_e = (0.883)(0.833)(0.5)(469) = 172 \text{ MPa}$$

Eq. (7–5) $$\sigma_a' = \frac{32 K_f M_a}{\pi d^3} = \frac{32(1.49)(468)}{\pi (0.042)^3} = 96 \text{ MPa}$$

Eq. (7–6) $$\tau_m' = \left[3\left(\frac{16 K_{fs} T_m}{\pi d^3}\right)^2\right]^{1/2} = \frac{\sqrt{3}(16)(1.33)(360)}{\pi (0.042)^3} = 57 \text{ MPa}$$

Using Goodman criterion

$$\frac{1}{n_f} = \frac{\sigma_a'}{S_e} + \frac{\sigma_m'}{S_{ut}} = \frac{96}{172} + \frac{57}{469} = 0.68$$

$$n_f = 1.55$$

Note that we could have used Eq. (7–7) directly.
Check yielding.

$$n_y = \frac{S_y}{\sigma'_{\max}} > \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{393}{96 + 57} = 2.57$$

Also check this diameter at the end of the keyway, just to the right of point I, and at the groove at point K. From moment diagram, estimate M at end of keyway to be $M = 443\ \mathrm{N \cdot m}$.

Assume the radius at the bottom of the keyway will be the standard $r/d = 0.02$, $r = 0.02\ d = 0.02\ (42) = 0.84\ \mathrm{mm}$

$$K_t = 2.14 \text{ (Fig. A–15–18)}, \ q = 0.65 \text{ (Fig. 6–20)}$$

$$K_f = 1 + 0.65(2.14 - 1) = 1.74$$

$$K_{ts} = 3.0 \text{ (Fig. A–15–19)}, \ q_s = 0.9 \text{ (Fig. 6–21)}$$

$$K_{fs} = 1 + 0.9(3 - 1) = 2.8$$

$$\sigma'_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(1.74)(443)}{\pi(0.042)^3} = 106\ \mathrm{MPa}$$

$$\sigma'_m = \sqrt{3}(16)\frac{K_{fs}T_m}{\pi d^3} = \frac{\sqrt{3}(16)(2.8)(443)}{\pi(0.042)^3} = 148\ \mathrm{MPa}$$

$$\frac{1}{n_f} = \frac{\sigma'_a}{S_e} + \frac{\sigma'_m}{S_{ut}} = \frac{106}{172} + \frac{148}{469} = 0.93$$

$$n_f = 1.08$$

The keyway turns out to be more critical than the shoulder. We can either increase the diameter, or use a higher strength material. Unless the deflection analysis shows a need for larger diameters, let us choose to increase the strength. We started with a very low strength, and can afford to increase it some to avoid larger sizes. Try 1050 CD, with $S_{ut} = 690\ \mathrm{MPa}$.

Recalculate factors affected by S_{ut}, i.e. $k_a \rightarrow S_e$; $q \rightarrow K_f \rightarrow \sigma'_a$

$$k_a = 4.51(690)^{-0.265} = 0.797, \quad S_e = 0.797(0.833)(0.5)(690) = 229\ \mathrm{MPa}$$

$$q = 0.72, \ K_f = 1 + 0.72(2.14 - 1) = 1.82$$

$$\sigma'_a = \frac{32(1.82)(443)}{\pi(0.042)^3} = 110.8\ \mathrm{MPa}$$

$$\frac{1}{n_f} = \frac{110.8}{229} + \frac{148}{690} = 0.7$$

$$n_f = 1.43$$

Since the Goodman criterion is conservative, we will accept this as close enough to the requested 1.5.

Check at the groove at K, since K_t for flat-bottomed grooves are often very high. From the torque diagram, note that no torque is present at the groove. From the moment diagram, $M_a = 283\ \mathrm{N \cdot m}$, $M_m = T_a = T_m = 0$. To quickly check if this location is potentially critical just use $K_f = K_t = 5.0$ as an estimate, from Table 7–1.

$$\sigma_a = \frac{32K_f M_a}{\pi d^3} = \frac{32(5)(283)}{\pi(0.042)^3} = 194.5\ \mathrm{MPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{229}{194.5} = 1.18$$

This is low. We will look up data for a specific retaining ring to obtain K_f more accurately. With a quick on-line search of a retaining ring specification using the website www.globalspec.com, appropriate groove specifications for a retaining ring for a shaft diameter of 42 mm are obtained as follows: width, $a = 1.73$ mm; depth, $t = 1.22$ mm; and corner radius at bottom of groove, $r = 0.25$ mm.
From Fig. A–15–16, with $r/t = 0.25/1.22 = 0.205$, and $a/t = 1.73/1.22 = 1.42$

$$K_t = 4.3,\ q = 0.65\ \text{(Fig. 6–20)}$$

$$K_f = 1 + 0.65(4.3 - 1) = 3.15$$

$$\sigma_a = \frac{32 K_f M_a}{\pi d^3} = \frac{32(3.15)(283)}{\pi(0.042)^3} = 122.6\ \text{MPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{229}{122.6} = 1.87$$

Quickly check if point M might be critical. Only bending is present, and the moment is small, but the diameter is small and the stress concentration is high for a sharp fillet required for a bearing. From the moment diagram,
$M_a = 113\ \text{N}\cdot\text{m}$, and $M_m = T_m = T_a = 0$.

Estimate $K_t = 2.7$ from Table 7–1, $d = 25$ mm, and fillet radius r to fit a typical bearing.

$$r/d = 0.02,\ r = 0.02(25) = 0.5$$

$$q = 0.7\ \text{(Fig. 6–20)}$$

$$K_f = 1 + (0.7)(2.7 - 1) = 2.19$$

$$\sigma_a = \frac{32 K_f M_a}{\pi d^3} = \frac{32(2.19)(113)}{\pi(0.025)^3} = 161\ \text{MPa}$$

$$n_f = \frac{S_e}{\sigma_a} = \frac{229}{161} = 1.42$$

Should be OK. Close enough to recheck after bearing is selected.

With the diameters specified for the critical locations, fill in trial values for the rest of the diameters, taking into account typical shoulder heights for bearing and gear support.

$$D_1 = D_7 = 25\ \text{mm}$$

$$D_2 = D_6 = 35\ \text{mm}$$

$$D_3 = D_5 = 42\ \text{mm}$$

$$D_4 = 50\ \text{mm}$$

The bending moments are much less on the left end of shaft, so D_1, D_2, and D_3 could be smaller. However, unless weight is an issue, there is little advantage to requiring more material removal. Also, the extra rigidity may be needed to keep deflections small.