Final Examination | Math 244 | Semester I | 1439-1440 | Time: 3H

Calculators are not allowed **2 pages**

Question 1 : [7pts]

1. Let A, B, C and D be matrices of order 3 such that $AB + AC - D = 0$, $|D| = 6$, $B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & -1 & 3 \end{pmatrix}$.
   Find $|A|$.

2. Let R and S be matrices of order 3 such that $RS + R - 2I = 0$.
   Find R^{-1} if $S = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 0 & 2 & 5 \end{pmatrix}$.

3. Find a basis of the vector subspace $W = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} ;\ a - b - 2c - 3d = 0\}$.

Question 2 : [5pts]
Find the values of m for which the following linear system

$$\begin{cases} x + my + 2z & = & 3 \\ 4x + (6+m)y - mz & = & 13 - m \\ x + 2(m-1)y + (m+4)z & = & m + 2 \end{cases}$$

a) has a unique solution.
b) has infinite solutions.
c) has no solution.

Question 3 : [8pts]

1. Let V be a vector space and $B = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ a basis of V.
   Explain why $u_1 - 2u_2 + 3u_6 \neq 5u_3 + 7u_4 - 6u_5$.

2. Define $T\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ by:

$$T(x, y, z, t) = (x - 2y + z + 3t, 2x - 3y + 2t, -x + 3z + 5t).$$

   (a) Find the matrix of the linear transformation T with respect to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^3$.

(b) Find a basis for kernel T.

(c) Find a basis for Image T.

Question 4 : [7pts]

Let B and C be bases of a vector space V of dimension 3 such that ${}_CP_B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

(${}_CP_B$ is the transition matrix from the basis B to the basis C). Let $T\colon V \longrightarrow V$ be a linear transformation with $[T]_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$.

1. If $[v]_C = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, find $[v]_B$.

2. If $[w]_B = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$, find $[T(w)]_B$.

3. If $B = \{u_1, u_2, u_3\}$. Find the values of a, b, c such that $T(u_1) = au_1 - \frac{b}{5}u_2 + cu_3$.

Question 5 : [5pts]

1. Let F be the subspace of the Euclidean inner product space $\mathbb{R}^3$ spanned by $\{v_1 = (1,1,0), v_2 = (1,1,1)\}$.
   Use Gram-Schmidt process to get an orthonormal basis of F.

2. Let $\mathbb{R}^3$ be the Euclidean inner product space and $u = (1,-1,1)$, $v = (2,0,-2)$ in $\mathbb{R}^3$.

   (a) Find $\|u+v\|^2$.

   (b) Find $\cos\theta$, if θ is the angle between the vectors u and v.

Question 6 : [8pts]

1. Compute B^{10} if $B = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & -2 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A.

   (b) Find the eigenvalues and its corresponding eigenvectors of A.

   (c) Explain why A is not diagonalizable?