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**Definition:** A function $F$ is called an anti-derivative of $f$ on an interval $I$ if

$$F^{'}(x) = f(x)$$

for every $x \in I$

**Example:** Let $F(x) = x^2$ and $f(x) = 2x$,
then $F^{'}(x) = 2x = f(x)$.
This means $F^{'}(x)$ is an anti-derivative of $f(x) = 2x$.

**Note:** There are many anti-derivatives of $f(x)$. From the previous example, $f(x) = 2x$, the functions
$F(x) = x^2 + 2$
$F(x) = x^2 - \frac{1}{2}$
$F(x) = x^2 - \sqrt{2}$
...
....
$F(x) = x^2 + c$
where $c$ is constant.

Relationship between two different anti-derivatives of a function:

Let $F$ and $G$ be two anti-derivatives of $f$ on an interval $I$, then
$F(x) = G(x) + c$
$G(x) = F(x) + c$

**Example:** Let $F(x) = sin(x)$ and $G(x) = sin(x) + 2$ and let $f(x) = cos(x)$. Clearly, $F$ and $G$ are two anti-derivatives of $f$ and $F(x) = G(x) - 2$.

**Indefinite Integrals:**
The form of the indefinite integral is $\int f(x)\ dx = F(x) + c$
where
$\int f(x)\ dx$ is indefinite integral of $f(x)$,
$f(x)$ is the integrand,
$x$ is the variable of the integration and
$c$ is constant of the integral.

| Derivative | Indefinite Integrals |
|---|---|
| $\frac{d}{dx}(x) = 1$ | $\int 1\ dx = x + c$ |
| $\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = 1,\ n \neq 1$ | $\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$ |
| $\frac{d}{dx}(\sin\ x) = \cos\ x$ | $\int \cos\ dx = \sin\ x + c$ |
| $\frac{d}{dx}(-\cos\ x) = \sin\ x$ | $\int sin\ x\ dx = -cos\ x + c$ |
| $\frac{d}{dx}(\tan\ x) = \sec^2\ x$ | $\int \sec^2\ x\ dx = \tan\ x + c$ |
| $\frac{d}{dx}(-\cot\ x) = \csc^2\ x$ | $\int \csc^2\ x\ dx = -\cot\ x + c$ |
| $\frac{d}{dx}(\sec\ x) = \sec\ x\ \tan\ x$ | $\int \sec\ x \tan\ x\ dx = \sec\ x + c$ |
| $\frac{d}{dx}(-\csc\ x) = \csc\ x\ \cot\ x$ | $\int \csc\ x\ \cot\ x\ dx = -\csc\ x + c$ |

**Some Important Formulas:**
1) $\int \frac{d}{dx}(f(x))\ dx = f(x) + c$
2) $\frac{d}{dx}\int f(x)\ dx = f(x)$
3) $\int cf(x)\ dx = c\int f(x)\ dx$
4) $\int[f(x) \pm g(x)]\ dx = \int f(x) \pm \int g(x)\ dx$

**Exercise:** Evaluate the following integrals:
1) $\int 4x + 3\ dx$
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2) $\int \frac{4}{x^5} + \frac{2}{x^2} + x\ dx$
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3) $\int \frac{4x^2-x+3}{\sqrt{x}}\ dx$
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4) $\int \frac{\sec\ x}{\cos\ x}\ dx$
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**Exercise 2:** Evaluate the following integrals

(1) $\int(\sqrt{x}+\frac{1}{2\sqrt{x}})\ dx$

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(2) $\int(3\sin x+2\cos x)\ dx$

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(3) $\int(x+1)^5\ dx$

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(4) $\int x\sin x^2\ dx\ dx$

(5) $\int(3x+1)^4\ dx$

(6) $\int(2x^3+1)^7(6x^2)\ dx$

**Exercise:** Solve the differential equation $f^{'}(x)=6x^2+x-5$ subject to the initial condition $f(0)=2$.

**Solution:**

$\int f^{'}(x)\ dx=\int(6x^2+x-5)\ dx$

$f(x)=2x^3+\frac{1}{2}x^2-5x+c$

Let $x=0$ and use the condition $f(0)=2$. $f(0)=0+0-0+c\Rightarrow c=2$.

The solution of the differential equation is $f(x)=2x^3+\frac{1}{2}x^2-5x+2$ .

**Exercise:** Solve the differential equation $f^{''}(x)=5\cos x+2\sin x$ subject to the initial condition $f(0)=3$ and $f^{'}(0)=4$.

**Solution:**

$\int f^{''}(x)\ dx=\int(5\cos x+2\sin x)\ dx$

$f^{'}(x)=5\sin x-2\cos x+c$

Let $x=0$ and use the condition $f^{'}(0)=3$.

$f^{'}(0)=5\sin 0-2\cos 0+c\Rightarrow c=6$. Hence

$f^{'}(x)=5\sin x-2\cos x+6$

We integrate a second time:

$\int f^{'}(x)\ dx=\int(5\sin x-2\cos x+6)\ dx$

$f(x)=-5\cos x-2\sin x+6x+c$

Let $x=0$ and use the condition $f(0)=4$.

$f^{'}(0)=-5\cos 0-2\sin 0+6(0)+c\Rightarrow c=8$. Hence, the solution is

$f(x)=-5\cos x-2\sin x+6x+8$ .

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**Exercise:** Evaluate the following integrals:

1) $\int (x+1)\ dx =$

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2) $\int (x+1)^2\ dx =$

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3) $\int (3x+1)^4\ dx =$

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Here, we can't do the integration without the derivative of $3x$, so we use the substitution method.

The substitution method can be summarized in the following formula:

$\boxed{\int [f]^n f'\ dx = \frac{[f]^{n+1}}{n+1} + c}$, $n \neq -1$

To find the previous integration, we let $u = 3x+1 \Rightarrow du = 3dx \Rightarrow \frac{1}{3}du = dx$.

By substituting that into the integral, we have

$\frac{1}{3}\int u^4\ du = \frac{1}{3}\frac{u^5}{5} + c = \frac{u^5}{15} + c$

Then, $\int (3x+1)^4\ dx = \frac{1}{3}\frac{(3x+1)^5}{5} + c = \frac{(3x+1)^5}{15} + c$

Alternatively, we multiple and divide the integral by 3

$\frac{1}{3}\int 3\ (3x+1)^4\ dx = \frac{(3x+1)^5}{15} + c.$

4) $\int (4x+1)^n\ dx$

To do the integral, we the derivative of $4x$.

Let $u = 4x+1 \Rightarrow \frac{1}{4}\ du = dx$, by substituting

$\frac{1}{4}\int u^n\ dx = \frac{u^{n+1}}{n+1} + c$

But, $u = 4x+1$, the value of the integral is $\frac{(4x+1)^{n+1}}{n+1} + c$

Remember

$\int x^n\ dx =$

$\frac{x^{n+1}}{n+1} + c$

**Exercise:** Evaluate the following integrals:

1) $\int x^2\ (2x^3+1)^7\ dx$

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2) $\int 3x^2\ (7x^3+1)^{10}\ dx$

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3) $\int \frac{(\sqrt{x}+3)^4}{\sqrt{x}}\ dx$

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4) $\int \sqrt{x}\ \cos\ \sqrt{x^3}\ dx$

5) $\int 3\ \cos\ 6x\ dx$

6) $\int \frac{4}{\cos^2\ 4x}\ dx$

7) $\int (x^2+2)\ \sqrt{x^3+6x}\ dx$

8) $\int \frac{x\ cos\ (x^2+2)}{\sqrt{sin\ (x^2+2)}}\ dx$

9) $\int \frac{cos\ x}{\sqrt{4+sin\ x}}\ dx$

(a) $\frac{1}{2}\sqrt{\sin\ x+4}+c$ (b) $\sqrt{\sin\ x+4}+c$

(c) $2\sqrt{\sin\ x+4}+c$ (d) $-2\sqrt{\sin\ x+4}+c$

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Let $\{a_1, a_2, ..., a_n\}$ be a set of numbers, the symbol $\sum_{k=1}^{n} a_k$ represents their sum:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$$

**Example:** Evaluate the following

(1) $\sum_{k=1}^{3}(k+1)k^2$

$\sum_{k=1}^{3}(k+1)k^2 = (2)(1)^2 + (3)(2)^2 + (4)(3)^2 = 2 + 12 + 36 = 50$

(2) $\sum_{j=1}^{4}(j^2+1)$

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**Theorem:**

(1) $\sum_{k=1}^{n} c = c + c + ... + c = nc.$

(2) $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k.$

(3) $\sum_{k=1}^{n} c\ a_k = c\ \sum_{k=1}^{n} a_k$ for any $c \in \mathbb{R}$.

(4) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

(5) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(6) $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

(7) $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

**Exercise 1:** Evaluate the following

(1) $\sum_{k=1}^{10} 3$

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(2) $\sum_{k=1}^{20} k^2$

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(3) $\sum_{k=1}^{10} k^3$

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**Exercise 2:** Express the following sum in terms of $n$:
$\sum_{k=1}^{n}(k^3 + k^2 + 3k + 5)$

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**Exercise 3:** Choose the correct answer

1) If $\sum_{k=1}^{4}(k+a) = 14$, then the value of $a$ is equal to:

(a) 1 (b) 4 (c) -4 (d) -1

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2) If $\sum_{k=1}^{n}(k+\alpha) = \frac{n^2}{2}$ $(n \geq 1)$, then the value of $\alpha$ is equal to:

(a) $-\frac{n}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

3) The sum $\sum_{k=1}^{n^2}(k-1)$, is equal to:

(a) $\frac{n^2(n-1)}{2}$ (b) $\frac{n(n-1)}{2}$ (c) $\frac{n^2(n^2+1)}{2}$ (d) $\frac{n^2(n^2-1)}{2}$

4) $\sum_{k=1}^{n}(k^2+3k+2\alpha) = 130$, then $\alpha$ is equal to

(a) 2 (b) $-2$ (c) 1 (d) 3

5) $\sum_{k=1}^{5}(\alpha k^2+k-1) = 20$, then $\alpha$ is equal to

(a) $\frac{2}{11}$ (b) $\frac{-2}{11}$ (c) $\frac{1}{11}$ (d) $\frac{-1}{11}$

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**Definition:**
(1) A partition $P$ of a closed interval $[a, b]$ is decomposition of the interval into subintervals of form
$[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n]$
for any a positive integer $n$ such that $a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n$

| Sub-intervals | Length |
|---|---|
| First interval $= [x_0, x_1]$ | $\Delta x_1 = x_1 - x_0$ |
| Second interval $= [x_1, x_2]$ | $\Delta x_2 = x_2 - x_1$ |
| Third interval $= [x_2, x_3]$ | $\Delta x_3 = x_3 - x_2$ |
| ... | ... |
| n-th interval $= [x_{n-1}, x_n]$ | $\Delta x_n = x_n - x_{n-1}$ |

(2) Norm of partition is ($\| P \|$) is the largest number of $\Delta x_1, \Delta x_2, ..., \Delta x_n$.

**Example:** Let $P = \{0, 1.1, 2.6, 3.7, 4.1, 5\}$ be a partition of the interval $[0, 5]$.
(1) Find the length of each sub-intervals.
(2) Find the norm $\| P \|$.

**Solution:**

| Sub-intervals | Length |
|---|---|
| First interval $= [0, 1.1]$ | $\Delta x_1 = 1.1 - 0 = 1.1$ |
| Second interval = | |
| Third interval = | |
| Fourth interval = | |
| Fifth interval = | |

(2) The norm $\| P \|= 1.5$ .

**Riemann Sum**
Let $f$ be a defined function on the closed interval $[a, b]$ and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$.
Let $w_k \in [x_{k-1}, x_k]$, $k = 1, 2, 3, ..., n$.
Then a Riemann sum of $f$ for $P$ is

$$R_p = \sum_{k=1}^{n} f(w_k)\Delta x_k.$$

**Example:** Find the Riemann sum $R_p$ for the function $f(x) = 3 - 4x$ on the partition $P = \{-1, 0, 2, 4, 6\}$ of the interval $[-1, 6]$ by choosing
(i) the left-hand end point.
(ii) the right-hand end point.
(iii) the mid point.

**Solution:**

| Sub-intervals | Length |
|---|---|
| First interval $= [-10, 0]$ | $\Delta x_1 = 0 - (-1) = 1$ |
| Second interval = | |
| Third interval = | |
| Fourth interval = | |

(i) The left-hand end point.
$w_1 = -1,\ w_2 = 0, w_3 = -2,\ w_4 = 4$
$R_p = \sum_{k=1}^{n} f(w_k)\Delta x_k = f(-1)(1) + f(0)(2) + f(2)(2) + f(4)(2)$
$= 7 + (3)(2) + (-5)(2) + (-13)(2) = -23$

(ii) The right-hand end point.
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(iii) The mid point. (**Exercise**)
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**Exercise 1:** Let $A$ be the area under the graph of $f(x) = x^2 + 1$ from $x = 1$ to $x = 3$. Find the area $A$ by taking limit of Riemann sum.

**Exercise 2:** Choose the correct answer

The limit $\lim_{n\to\infty} \sum_{k=1}^{n} (\frac{k}{n^2})$

(a) 0 (b) $\infty$ (c) 2 (d) $\frac{1}{2}$

**Definite Integral:**

Let $f$ be a function defined on closed interval $[a,b]$. The definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(x)\ dx = \lim_{||P||\to 0} \sum_{k=1}^{n} f(w_k)\Delta x_k$$

The numbers $a$ and $b$ are called the limits of integration.

**Example:** Evaluate the following integrals:

(1) $\int_1^4 x^2+1\ dx$

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(2) $\int_0^2 \sqrt{4x+1}\ dx$

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(3) $\int_0^3 x^3(x^4-1)\ dx$

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(4) $\int_{-1}^4 |\ x\ |\ \ dx$

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(5) $\int_0^2 |\ x-1\ |\ \ dx$

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(6) $\int_0^{\frac{\pi}{2}} \sin\ x + \cos\ x\ dx$

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(7) $\int_0^{\frac{\pi}{4}} \sec^2\ x - 1\ dx$

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In the following, we give some important properties of definite integrals.

(1) $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

(2) If $f(a)$ exists, $\int_a^a f(x)\ dx = 0$

(3) $\int_a^b c\ dx = c(b-a)$

(4) If $f$ is integrable on $[a,b]$ and $c \in \mathbb{R}$, then

$$\int_a^b c\ f(x)\ dx = c\int_a^b f(x)\ dx$$

(5) $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$

(6) If $c \in [a,b]$ and $f$ is integrable on $[a,c]$ and $[c,b]$, then $f$ is integrable on $[a,b$ and

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

(7) If $f$ is integrable on $[a,b]$ and $f(x) \geq 0$, $x \in [a,b]$, then $\int_a^b f(x) \geq 0$.

(8) If $f$ and $g$ are integrable on $[a,b]$ and $f(x) \geq g(x)$ for every $x \in [a,b]$, then

$$\int_a^b f(x) \geq \int_a^b g(x)$$

**Example:** Evaluate the following integrals

(1) $\int_0^2 3\ dx$

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(2) $\int_2^2 x^2 + 4\ dx$

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(3) If $\int_0^2 f(x)\ dx = 4$ and $\int_0^2 g(x)\ dx = 2$, then find $\int_0^2 3f(x) - \frac{g(x)}{2}\ dx$ .

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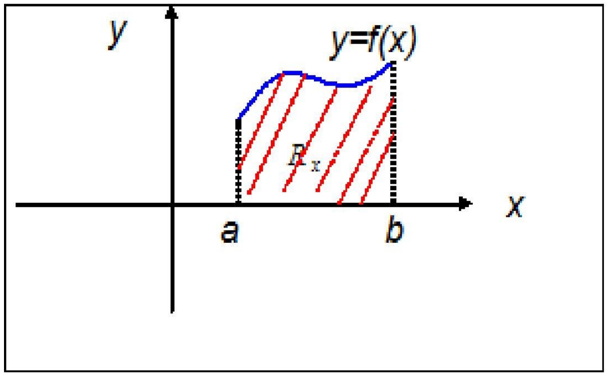
**An application of the definite integrals:**
**Theorem:** If $f$ is integrable and $f(x) \geq 0\ \forall x \in [a,b]$, the area $A$ of the region under the graph of $f$ from $a$ to $b$ is

$$A = \int_a^b f(x)\ dx$$




**Example:** Sketch the region bounded by $y = 2x - 1$ and $x > 0,\ y > 0$. Then, find the area.

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### Mean Value Theorem for Definite Integrals:

**Definition:** Let $f$ be continuous on $[a, b]$, then the average value $f_{av}$ of $f$ on $[a, b]$ is

$$f_{av} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

**Theorem:** If $f$ is continuous on $[a, b]$, there is exists a number $z \in (a, b)$ such that

$$\int_a^b f(x)\, dx = (b-a) f(z)$$

**Exercise:**
(1) If $f(x) = \sqrt{x+2}$, then
(i) Find the average value of $f$ on $[-2, 0]$.
(ii) Find a number $z$ that satisfies the Mean value theorem.

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(2) The number $z$ that satisfies the conclusion of the Mean Value Theorem for $f(x) = x^2 + 1$ on $[-2, 1]$ is:
(a) $\frac{1}{\sqrt{2}}$ (b) 2 (c) -1 (d) 0

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(3) The number $z$ that satisfies the conclusion of the Mean Value Theorem for $f(x) = x$ on $[\alpha, \beta]$ is:
(a) $\alpha$ (b) $\beta + 1$ (c) $\frac{\alpha+\beta}{2}$ (d) $\beta$

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(4) The average value of $f(x) = \mid x - 1 \mid$ on $[0, 2]$ is equal to:
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) 2

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**Theorem:** (Fundamental Theorem of Calculus )
Suppose $f$ is continuous on $[a, b]$.

1 If $F(x) = \int_a^x f(t)\, dt$ for every $x \in [a, b]$, then $F(x)$ is an anti-derivative of $f$ on $[a, b]$.

2 If $F(x)$ is an anti-derivative of $f$ on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

.

**Theorem:** If $g$ and $h$ are differentiable and $f$ is continuous, then

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\, dt = f(g(x))g^{'}(x) - f(h(x))h^{'}(x)$$

**Corollary:** Let $f$ be continuous on $[a, b]$. If $F(x) = \int_c^x f(t)\, dt$ where $c \in [a, b]$, then

$$F^{'}(x) = \frac{d}{dx}\Big[\int_c^x f(t)\, dt\Big] = f(x)$$

**Exercises:**

(1) Find $\frac{d}{dx}\int_0^{x^2} \sqrt{t^4+1}\, dt$

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(2) Find $\frac{d}{dx}\int_x^{-3} \sin^2\ t\, dt$

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(3) If $\frac{d}{dx}\int_0^{x^2} f(\sqrt{t})\, dt = x$ for $x > 0$, then $f(x)$ is equal to

(a) 1 (b) $\frac{1}{2x}$ (c) $\frac{1}{x^2}$ (d) $\frac{1}{2}$

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(4) If $F(x) = \int_1^{2x} f^{'}(t)\, dt$, then $F^{'}(x)$ is equal to

(a) $2f(2x) - f(1)$ (b) $2f(2x)$ (c) $2f^{'}(2x)$ (d) $f^{'}(2x)$

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Until this stage of this course, we can not evaluate some integrals such as $\int \frac{1}{x}\ dx$ and $\int \sqrt{x^2+3}\ dx$. In this section, we are going to study two techniques of the numerical integration: Trapezoidal rule and Simpson's rule. These techniques are used to approximate definite integrals.

**Trapezoidal Rule:**

We use Trapezoidal rule to approximate definite integrals of form $\int_a^b f(x)\ dx$.

Method:

(1) We want to divide the interval $[a,b]$ into sub-intervals, so find width of sub-intervals:

$$\Delta x = \frac{b-a}{n}$$

(2) Find the partition $P = \{x_0, x_1, x_2, ..., x_n\}$ where $x_k = x_0 + k(\Delta x) = x_0 + k\frac{(b-a)}{n}$.

(3) Approximate the integral:
$\int_a^b f(x)\ dx \approx \frac{(b-a)}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + ... + 2f(x_{n-1}) + f(x_n)\right]$

**Example 1:** By using trapezoidal rule, approximate the integral $\int_1^2 \frac{1}{x}\ dx$ with $n = 4$.

**Solution:**
(1) $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$.

(2) Partition: $x_0 = 1$, $x_1 = 1 + \frac{1}{4} = 1\frac{1}{4}$, $x_2 = 1 + 2(\frac{1}{4}) = 1\frac{1}{2}$, $x_3 = 1 + 3(\frac{1}{4}) = 1\frac{3}{4}$ and $x_4 = 1 + 4(\frac{1}{4}) = 2$

The partition is $P = \{1, 1.25, 1.5, 1.75, 2\}$.

| $n$ | $x_n$ | $f(x_n)$ | $m$ | $mf(x_n)$ |
|---|---|---|---|---|
| 0 | | | 1 | |
| 1 | | | 2 | |
| 2 | | | 2 | |
| 3 | | | 2 | |
| 4 | | | 1 | |
| Sum = $\sum_{k=1}^{4} mf(x_n)$ | | | | |

$\int_1^2 \frac{1}{x}\ dx \approx \frac{1}{8}[\qquad]$

**Error estimation:**
**Theorem:** Suppose $f''$ is continuous on $[a,b]$ and $M$ is the maximum value for $f''$ on $[a,b]$. If $E_T$ is the error in calculating $\int_a^b f(x)\ dx$ under trapezoidal rule, then

$$|E_T| < \frac{M(b-a)^3}{12n^2}$$

**Example:** Estimate the error in the previous example.
$f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3} \Rightarrow f'''(x) = \frac{-4}{x^4}$ .
Since $f''(x)$ is a decreasing function on the interval $[1,2]$, then $f''(x)$ is maximized at $x = 1$. This means $M = f''(1)| = 2$ and

$$|E_T| < \frac{2(2-1)^3}{(12)(4^2)} = \frac{2}{192} = \frac{1}{96}$$

**Example 2:** For the following integral $\int_0^2 \frac{1}{20}x^3 + 1\ dx$
(i) approximate the integral by using trapezoidal rule with $n = 4$.
(ii) estimate the error.

**Solution:**
(i) Homework

(ii) $f(x) = \frac{1}{20}x^2 + 1 \Rightarrow f'(x) = \frac{3}{20}x^2 \Rightarrow f''(x) = \frac{3}{10}x \Rightarrow f'''(x) = \frac{3}{10}$ .
Since $f''(x)$ is an increasing function on the interval $[0,2]$, then $f''(x)$ is maximized at $x = 2$. This means $M = |f''(2)| = \frac{(3)(2)}{10} = \frac{3}{5}$ and

$$|E_T| < \frac{\frac{3}{5}(2-0)^3}{(12)(4^2)} = \frac{3}{(30)(16)} = \frac{1}{160} = 0.0063$$

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## Simpson's Rule:

We can use Simpson's rule to approximate definite integrals of form $\int_a^b f(x)\,dx$.

Method:

(1) We want to divide the interval $[a,b]$ into sub-intervals, so find width of sub-intervals: $\Delta x = \frac{b-a}{n}$

(2) Find the partition $P = \{x_0, x_1, x_2, ..., x_n\}$ where $x_k = x_0 + k(\Delta x) = x_0 + k\frac{(b-a)}{n}$.

(3) Approximate the integral:
$\int_a^b f(x)\,dx \approx$
$\frac{(b-a)}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 2f(x_3) + ... + 4f(x_{n-1}) + f(x_n)\right]$

**Example:** By using Simpson's rule, approximate the integral $\int_1^3 \frac{1}{x+1}\,dx$ with $n = 4$.

**Solution:**
(1) $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$.

(2) Partition: $x_0 = 1$,
$x_1 = 1 + \frac{1}{2} = 1\frac{1}{2}$,
$x_2 = 1 + 2(\frac{1}{2}) = 2$,
$x_3 = 1 + 3(\frac{1}{2}) = 2\frac{1}{2}$ and
$x_4 = 1 + 4(\frac{1}{2}) = 3$

The partition is $P = \{1, 1.5, 2, 2.5, 3\}$.

| $n$ | $x_n$ | $f(x_n)$ | $m$ | $mf(x_n)$ |
|---|---|---|---|---|
| 0 | | | 1 | |
| 1 | | | 4 | |
| 2 | | | 2 | |
| 3 | | | 4 | |
| 4 | | | 1 | |
| Sum $= \sum_{k=1}^{4} mf(x_n)$ | | | | |

$\int_1^3 \frac{1}{x+1}\,dx \approx \frac{1}{6}[\qquad]$

**Error estimation:**

**Theorem:** Suppose $f^{(4)}$ is continuous on $[a,b]$ and $M$ is the maximum value for $f^{(4)}$ on $[a,b]$. If $E_s$ is the error in calculating $\int_a^b f(x)\,dx$ under Simpson's rule, then

$$|E_s| < \frac{M(b-a)^5}{180n^4}$$

**Example:** Estimate the error in the previous example.
$f(x) = \frac{1}{x+1} \Rightarrow f'(x) = \frac{-1}{(x+1)^2} \Rightarrow f''(x) = \frac{2}{(x+1)^3} \Rightarrow f'''(x) = \frac{-6}{(x+1)^4}$
$\Rightarrow f^{(4)}(x) = \frac{24}{(x+1)^5} \Rightarrow f^{(5)}(x) = \frac{-120}{(x+1)^6}$ .
Since $f^{(5)}(x)$ is a decreasing function on the interval $[1,3]$, then $f^{(4)}(x)$ is maximized at $x = 1$. This means $M = |f^{(4)}(1)| = 0.75$ and

$$|E_s| < \frac{(0.75)(3-1)^3}{(180)(4^4)} = 0.00013$$

**Exercise 1:** By using trapezoidal rule, approximate the following integrals and then estimate the error

(1) $\int_0^\pi \sin\ x\,dx$ with $n = 4$.

(2) $\int_{-2}^3 e^{-x}\,dx$ with $n = 4$.

(3) $\int_1^3 \frac{1}{x^2}\,dx$ with $n = 4$.

**Exercise 2:** By using Simpson's rule, approximate the following integrals and then estimate the error

(1) $\int_0^4 \sqrt{1+x^3}\,dx$ with $n = 4$.

(2) $\int_0^1 \frac{4}{1+x^2}\,dx$ with $n = 4$.

(3) $\int_0^{\frac{\pi}{2}} \cos\ x\,dx$ with $n = 4$.

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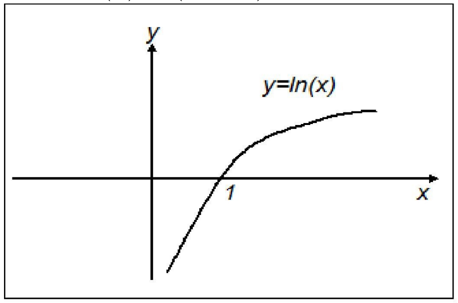
We have seen that $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$ where $n \neq -1$.
This formula can not be used when $n = -1$ where then the denominator becomes zero. This means we do not know the value of the integral $\int \frac{1}{x} \, dx$. Alternatively, we are looking for a function $F(x)$ such that $F^{'}(x) = \frac{1}{x}$.

**Definition:** The natural logarithm function is defined as follows:
$ln : (0, \infty) \to \mathbb{R}, \; ln(x) = \int_1^x \frac{1}{t} \, dt$

Note that the function $f(t) = \frac{1}{t}$ is continuous on any interval that does not contain 0.

**Remark:**
1) Domain of the function $\ln(x)$ is $(0, +\infty)$.




2) Range of the function $\ln(x)$ is $\mathbb{R}$.

3) Values of $\ln(x)$:

(i) $\ln(x) > 0$ if $x > 1$
(ii) $\ln(x) = 0$ if $x = 1$
(iii) $\ln(x) < 0$ if $0 < x < 1$
(iv) $\ln(e) = 1$ where $e \approx 2.718$
(4) The function $ln(x)$ is differentiable and continuous on the domain $(0, \infty)$. Also,

$$\frac{d}{dx}(\ln(x)) = \frac{d}{dx}\int_1^x \frac{1}{t} \, dt = \frac{1}{x}$$

(5) $ln(x)$ is increasing function and it is concave on the domain $(0, \infty)$.

(6) $\lim_{x \to 0^+} ln(x) = -\infty$ and $\lim_{x \to \infty} ln(x) = +\infty$ .

**Theorem:** For every $a, b > 0$ and $n \in \mathbb{Q}$ [a] , then

(1) $\ln(ab) = \ln(a) + \ln(b)$
(2) $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$
(3) $\ln(a^n) = n \ln(a)$

**Theorem:** If $u = g(x)$ is differentiable and $u \neq 0$ for every $x$ in an interval $I$ , then

$$\frac{d}{dx} \ln(|\, u \,|) = \frac{1}{u}\frac{du}{dx}, \text{ for every } x \in I.$$

**Exercise 1:** Find $f^{'}(x)$

(1) $f(x) = \ln(x^2 + 4)$
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(2) $f(x) = cos(\ln(x^3))$
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(3) If $f(x) = \ln(\ln(x))$, then $f^{'}(e)$ is
(a) $e$ (b) $-e$
(c) $\frac{1}{e}$ (d) -$\frac{1}{e}$
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**Exercise 2:** If $y = \ln(\sqrt{\frac{x^2-1}{x^2+1}})$, find $y^{'}$

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**Exercise 3:** If $y = (1 + x^2)^{\sin\ x}$, find $y^{'}$

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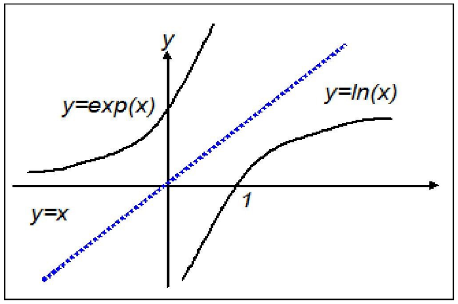
[a] $Q$ is a set of rational numbers.

The natural exponential function ($exp$ or $e$) is the inverse function of the natural logarithm function. Therefore,

$$exp : \mathbb{R} \to (0, \infty)$$




**Theorem:** For any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}^+$ such that

$$x = \ln(y) \Leftrightarrow y = exp(x)$$

**Remark:**

1) From the above discussion, the domain of $exp$ is $\mathbb{R}$ and the range is $(0, +\infty)$.

2) $\lim_{x\to+\infty} exp(x) = \infty$ and $\lim_{x\to-\infty} exp(x) = 0$

3) $\ln(e^x) = x\ln(e) = x(1) = x$ for $x \in \mathbb{R}$.

4) $e^{\ln(x)} = x$ for $x \in \mathbb{R}^+$.

**Theorem:** For every $a, b > 0$ and $n \in \mathbb{Q}$ [a] , then

(1) $e^a e^b = e^{a+b}$

(2) $\frac{e^a}{e^b} = e^{a-b}$

(3) $(e^a)^b = e^{ab}$

[a] $Q$ is a set of rational numbers.

**Theorem:** If $u = f(x)$ is differentiable, then

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

**Exercise 1:** Find value of $x$:

(1) $\ln(x) = 2$

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(2) $\ln(\ln(x)) = 0$

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**Exercise 2:** Find $f'$

(1) $f(x) = e^{3\cos\ x - 4x^2}$

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(2) $f(x) = \sin\ (e^{x^2})$

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**Exercise 3:** If $y = x^{e^x}$, find $y'$.

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**Remember:**

$\frac{d}{dx}\ln(x) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\ dx = \ln \mid x \mid +c$

and

$\frac{d}{dx}\ln(u) = \frac{1}{u}\ \frac{du}{dx} \Rightarrow \int \frac{1}{u}u'\ dx = \ln \mid u \mid +c$

Also,

$\frac{d}{dx}e^x = e^x \Rightarrow \int e^x\ dx = e^x + c$

and

$\frac{d}{dx}e^u = e^u\ \frac{du}{dx} \Rightarrow \int e^u\ u'\ dx = e^u + c$

**Exercise:** Evaluate the following integrals:

(1) $\int \frac{2}{2x+7}\ dx$

(2) $\int \frac{x^2+1}{x^3+3x+1}\ dx$

(3) $\int \frac{\sin\ x - \cos\ x}{\sin\ x + \cos\ x}\ dx$

(4) $\int x\ e^{-x^2}\ dx$

(5) $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\ dx$

(6) $\int_0^{\ln(5)} e^x(3-4e^x)\ dx$

(7) $\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}\ dx$

## (1) General Exponential Function

In the previous lecture, we defined the natural exponential function ($e^x$). In this lecture, we generalize that function for any base other than $e$.

**Definition:** For any $x \in \mathbb{R}, a^x = e^{x\ln(a)}$.

The definition is derived from the natural logarithm function. We know from the properties that

$$\ln(a^x) = x\ln(a)$$

By taking the *exp* for both sides,

$$e^{\ln(a^x)} = e^{x\ln(a)} \Rightarrow a^x = e^{x\ln(a)}$$

The function $f(x) = a^x$ is an exponential function with base $a$ where $x$ is called exponent.

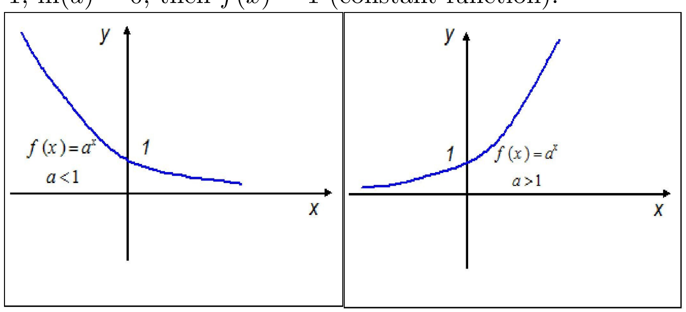
**Remark:**

1) The domain of $f(x) = a^x$ is $\mathbb{R}$ the range is $(0, +\infty)$.

2) For values of the base $a$, we have,
(i)If $a > 1$, $\ln(a) > 0$, then $x\ln(a)$ increases as $x$ increases. Hence, $f(x) = a^x$ is increasing function.

(ii) If $0 < a < 1$, $\ln(a) < 0$, then $x\ln(a)$ decreases as $x$ increases. Hence, $f(x) = a^x$ is decreasing function.

(iii) If $a = 1$, $\ln(a) = 0$, then $f(x) = 1$ (constant function).




3) The function $f(x) = \frac{1}{a^x}$ can be written as $f(x) = a^{-x}$.

**Theorem:** For every $a, b > 0$ and $x, y \in \mathbb{R}$, then

(1) $a^x b^y = a^{x+y}$ (3) $(a^x)^y = a^{xy}$
(2) $\frac{a^x}{a^y} = a^{x-y}$ (4) $(ab)^x = a^x b^y$

**Differentiation and Integration of $f(x) = a^x$ :**

(1) If $u = f(x)$ is differentiable, then

$$\frac{d}{dx}(a^x) = a^x \ \ln(a)$$

$$\frac{d}{dx}(a^u) = a^u \ \ln(a) \ u'$$

(2) Integration of the general exponential function:

$$\int a^x \ dx = \frac{1}{\ln(a)} \ a^x + c$$

$$\int a^u u' \ dx = \frac{1}{\ln(a)} \ a^u + c$$

**Exercise 1:** Find $f'(x)$

(1) $f(x) = 2^{\sqrt{x}\sin \ x}$

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(2) $f(x) = \cos(3^{x^2})$

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**Remember:**

If $u = f(x)$ is differentiable, then

$$\frac{d}{dx}(a^x) = a^x \ \ln(a) \Rightarrow \int a^x \ dx = \frac{1}{\ln(a)} \ a^x + c$$

$$\frac{d}{dx}(a^u) = a^u \ \ln(a) \ u^{'} \Rightarrow \int a^u u^{'} \ dx = \frac{1}{\ln(a)} \ a^u + c$$

**Exercise 1:** If $f(x) = 4^{x \tan(x)}$, find $f^{'}$

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**Exercise 2:** If $y = (x^2 + 1)^x$, find $y^{'}$.

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**Exercise 3:** Evaluate the following integrals:

(1) $\int 5^{3x} \ dx$

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(2) $\int_0^2 x 3^{-x^2} \ dx$

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(3) $\int \frac{3^{\sqrt{x}}}{\sqrt{x}} \ dx$

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(4) $\int_0^1 4^x \ dx$

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(5) $\int \frac{2^x}{2^x+1} \ dx$

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(6) $\int 3^x(3 + \sin 3^x) \ dx$

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### (2) General logarithmic Function

The inverse function of $y = a^x$ is the general logarithm function $x = \log_a y$. Since $a^x : \mathbb{R} \to (0, \infty)$, then

$$\log_a : (0, \infty) \to \mathbb{R}$$

$$y = a^x \Leftrightarrow x = \log_a y$$

The function $\log_a$ is called the logarithm function with the base $a$.

**Remarks:**

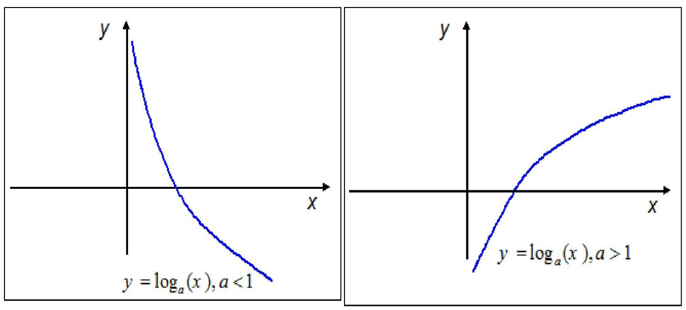
1) The natural logarithm function $\ln = \log_e$.

2) Usually, the logarithm function $\log_{10} = \log$.

3) $\log_a x = \frac{\ln x}{\ln a} \Rightarrow \log_a a = 1$.

Let $y = \log_a x \Rightarrow x = a^y \Rightarrow \ln x = \ln a^y \Rightarrow \ln x = y \ln a \Rightarrow y = \frac{\ln x}{\ln a}$.

4)The graph of $\log_a x$ depends on $\ln a$ meaning that if $a > 1$ or $0 < a < 1$.




**Theorem:** For every $x, y \in \mathbb{R}^+$ and for every $n \in \mathbb{R}$,

(1) $\log_a(xy) = \log_a(x) + \log_a(y)$

(2) $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$

(3) $\log_a(x^n) = n \log_a(x)$

**Exercise 1:** If $\log_2(\frac{x}{x-1}) = 1$, then $x$ is equal to:

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) $-1$

**Differentiation and integration of the general logarithmic function:**

Let $u = f(x)$ be differentiable, then

$$\frac{d}{dx} log_a x = \frac{1}{\ln a}\,\frac{1}{x} \Rightarrow \int \frac{1}{x \ln a} = \log_a x + c$$

$$\frac{d}{dx} log_a u = \frac{1}{\ln a}\,\frac{1}{u}\,u' \Rightarrow \int \frac{u'}{u \ln a} = \log_a u + c$$

**Exercise 1:** Find $y'$

1) $y = \log_7 \sqrt{x^2 + \sin^2 x}$

2) $y = \log(\ln x)$

**Exercise 2:** Evaluate the following integrals

1) $\int \frac{1}{x \log x}\, dx$

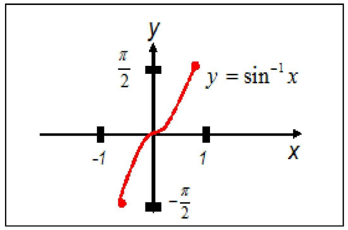
2) $\int_3^9 \frac{\log_3(x^2)}{x}\, dx$

In this lecture, we are going to define the inverse trigonometric functions.

(1) $\sin^{-1}$:

$x = \sin\ y \Leftrightarrow y = \sin^{-1}\ x$

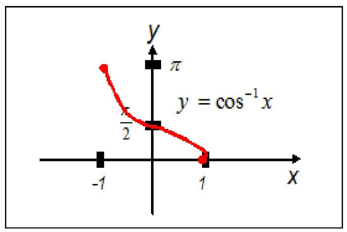
$-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$




2) $\cos^{-1}$:

$x = \cos\ y \Leftrightarrow y = \cos^{-1}\ x$

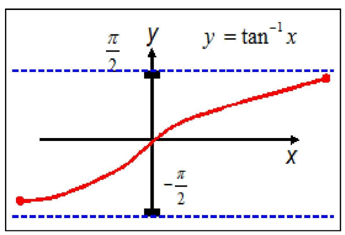
$-1 \leq x \leq 1$ and $0 \leq y \leq \pi$




3) $\tan^{-1}$:

$x = \tan\ y \Leftrightarrow y = \tan^{-1}\ x$

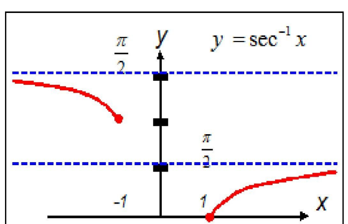
$x \in \mathbb{R}$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$




4) $\sec^{-1}$:

$x = \sec\ y \Leftrightarrow y = \sec^{-1}\ x$

$x \leq -1$ or $x \geq 1$ and
$0 \leq y < \frac{\pi}{2}$ or $\pi \leq y < \frac{3\pi}{2}$




**Differentiation of inverse trigonometric functions :**
Let $u = f(x)$ be differentiable, then

| $y$ | $y'$ | $y$ | $\int y\ dx$ |
|---|---|---|---|
| $\sin^{-1} u$ | $\frac{1}{\sqrt{1-u^2}}u'$ | $\int \frac{1}{\sqrt{1-x^2}}\ dx$ | $\sin^{-1}(x) + c$ |
| $\cos^{-1} u$ | $\frac{-1}{\sqrt{1-u^2}}u'$ | $\int \frac{-1}{\sqrt{1-x^2}}\ dx$ | $\cos^{-1}(x) + c$ |
| $\tan^{-1} u$ | $\frac{1}{1+u^2}u'$ | $\int \frac{1}{1+x^2}\ dx$ | $\tan^{-1}(x) + c$ |
| $\sec^{-1} u$ | $\frac{1}{u\sqrt{u^2-1}}u'$ | $\int \frac{1}{x\sqrt{x^2-1}}\ dx$ | $\sec^{-1}(x) + c$ |

**Exercise 1:** Choose the correct answer:

1) The derivative of $\sec^{-1}(e^x)$ is equal to

(a) $\frac{1}{e^x\sqrt{e^x-1}}$ (b) $\frac{1}{\sqrt{e^x-1}}$ (c) $\frac{1}{\sqrt{e^{2x}-1}}$ (d) $\frac{1}{\sqrt{e^{2x}+1}}$

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2) The value of the integral $\int \frac{\sin x}{\sqrt{4-cos^2 x}}\ dx$ is equal to

(a) $\sin^{-1}(\cos x) + c$ (b) $\cos^{-1}(\frac{\cos x}{2}) + c$
(c) $-\cos^{-1}(\frac{\cos x}{2}) + c$ (d) $\sin^{-1}(\frac{\sin x}{2}) + c$

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**Exercise 2:** Evaluate the following integrals:

1) $\int \frac{e^x}{1+e^{2x}}\ dx$

2) $\int \frac{1}{x\sqrt{x^6-4}}\ dx$

3) $\int_0^1 \frac{e^x}{1+e^{2x}}\ dx$

4) $\int_0^1 \frac{1}{e^x+e^{-x}}\ dx$

5) $\int \frac{1}{\sqrt{1-9x^2}}\ dx$

6) $\int \frac{1}{\sqrt{x}(1+x)}\ dx$

7) $\int \frac{1}{\sqrt{e^{2x}-1}}\ dx$

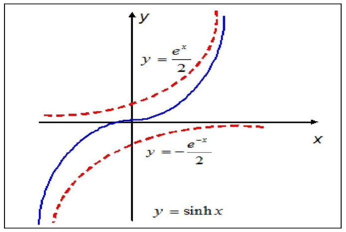
8) $\int \frac{x+1}{\sqrt{4-x^2}}\ dx$

In this section, we study hyperbolic functions that depend on functions $e^x$ and $e^{-x}$.

**(1) Hyperbolic Functions :**

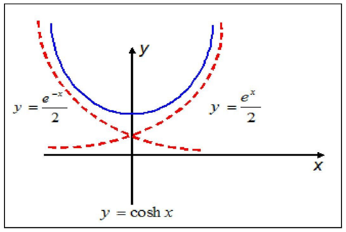
**Hyperbolic Sine:**

$$\sinh x = \frac{e^x - e^{-x}}{2}, \forall x \in \mathbb{R}$$




**Hyperbolic Cosine:**

$$\cosh x = \frac{e^x + e^{-x}}{2}, \forall x \in \mathbb{R}$$




**Hyperbolic Tangent:**

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \forall x \in \mathbb{R}$$

**Hyperbolic Cotangent:**

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \forall x \neq 0$$

**Hyperbolic Secant:**

$$\sec hx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \forall x \in \mathbb{R}$$

**Hyperbolic Cosecant:**

$$\csc hx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \forall x \neq 0$$

**Remarks:**

1) For $x \in \mathbb{R}$, $\cosh^2 x - \sinh^2 x = 1$

2) $1 - \tanh^2 x = \sec h^2 x$

3) $\coth^2 x - 1 = \csc h^2 x$

**Differentiation Hyperbolic Functions:**

| $y$ | $y'$ | $y$ | $y'$ |
|---|---|---|---|
| $\sinh u$ | $\cosh u \ u'$ | $\coth u$ | $-\csc h^2 u \ u'$ |
| $\cosh u$ | $\sinh u \ u'$ | $\sec hu$ | $-\sec hu \tanh u \ u'$ |
| $\tanh u$ | $\sec h^2 u \ u'$ | $\csc hu$ | $-\csc hu \coth u \ u'$ |

**Exercise 1:** Choose the correct answer:
1) The derivative of the function $f(x) = \tan^{-1}(\sinh x)$ is equal to
(a) $\sec hx$ (b) $\csc hx$ (c) $\tanh x$ (d) $-x$

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2) The value of the integral $\int_{-1}^{1} \sinh(x)$ is equal to
(a) 0 (b) $2e$ (c) $2e^{-1}$ (d) $\frac{1}{2}e$

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**Exercise 2:** If $f(x) = \cosh(\sqrt{4x^2 + 3})$ find $f'(x)$

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## (2) Inverse Hyperbolic Functions :

1)
$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$$
$$\sinh y = x \Leftrightarrow y = \sinh^{-1} x$$

2)
$$\cosh^{-1} : [1, \infty) \to [0, \infty)$$
$$\cosh y = x \Leftrightarrow y = \cosh^{-1} x$$

3)
$$\tanh^{-1} : (-1, 1) \to \mathbb{R}$$
$$\tanh y = x \Leftrightarrow y = \tanh^{-1} x$$

4)
$$\sec h^{-1} : (0, 1] \to [0, \infty)$$
$$\sec hy = x \Leftrightarrow y = \sec h^{-1} x$$

## Differentiation of Inverse Hyperbolic Functions :

| $y$ | $y'$ | $y$ | $y'$ |
|---|---|---|---|
| $\sinh^{-1} u$ | $\frac{1}{\sqrt{u^2+1}} u'$ | $\tanh^{-1} u$ | $\frac{1}{1-u^2} u'$, $\|u\| < 1$ |
| $\cosh^{-1} u$ | $\frac{1}{\sqrt{u^2-1}} u'$, $u > 1$ | $\sec h^{-1} u$ | $\frac{-1}{u\sqrt{1-u^2}} u'$, $0 < u < 1$ |

**Exercise 3:** Evaluate the following integrals

1) $\int \frac{1}{x\sqrt{1-x^4}}\, dx$

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2) $\int \frac{1}{\sqrt{25x^2-9}}\, dx$

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3) $\int \frac{1}{\sqrt{1-e^{2x}}}\, dx$

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4) $\int \frac{x+1}{x\sqrt{25-x^2}}\, dx$

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**Exercise 1:** Find the following limits

1) $\lim_{x\to 5} x - 5$

2) $\lim_{x\to 5} \frac{x^2-25}{x-5}$

3) $\lim_{x\to 5} \frac{x-5}{x^2-25}$

4) $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25}$

5) $\lim_{x\to 0} \frac{\sin\ x}{x}$

**Indeterminate Forms:**

| Form | Indeterminate Forms |
|---|---|
| Quotient | $\frac{0}{0}$ and $\frac{\infty}{\infty}$ |
| Product | $0.\infty$ and $0.(-\infty)$ |
| Sum & Difference | $(-\infty)+\infty$ and $\infty-\infty$ |
| Exponential | $0^0$, $1^\infty$, $1^{-\infty}$ and $\infty^0$ |

**L'Hopital Rule:**

Suppose $f(x)$ and $g(x)$ are differentiable on an interval $I$ and $c \in I$ where $f$ and $g$ may not be differentiable at $c$. If $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, if $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists or equals to $\infty$ or $-\infty$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

**Exercise:** Find the following limits:

1) $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25}$

2) $\lim_{x\to 0} \frac{\sin x}{x}$

3) $\lim_{x\to 0} \frac{\cos x+3x-1}{4x}$

5) $\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$

6) $\lim_{x\to 0} \frac{e^{ax}}{x^k}$

7) $\lim_{x\to 0^+} x^2 \ln x$

8) $\lim_{x\to \frac{\pi}{4}} (1-\tan x)\sec(2x)$

9) $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

10) $\lim_{x\to 0} \frac{x-tan^{-1}x}{x\sin x}$

11) $\lim_{x\to 0} (1+\frac{1}{x})^x$

12) $\lim_{x\to \infty} (1+\frac{1}{x})^{5x}$

**Theorem:** Let $u = f(x)$ and $v = g(x)$. If $f^{'}$ and $g^{'}$ are continuous, then

$$\int u\ dv = uv - \int v\ du$$

Explanation:

We know that

$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g^{'}(x) + f^{'}(x)g(x)$

This implies

$f(x)g^{'}(x) = \frac{d}{dx}\big(f(x)g(x)\big) - f^{'}(x)g(x)$

By integrating both sides

$\int f(x)g^{'}(x)\ dx = \int \frac{d}{dx}\big(f(x)g(x)\big)\ dx - \int f^{'}(x)g(x)\ dx$
$\int f(x)g^{'}(x)\ dx = f(x)g(x) - \int f^{'}(x)g(x)\ dx$

Since $du = f^{'}(x)dx$ and $dv = g^{'}(x)dx$, then

$$\int u\ dv = uv - \int v\ du$$

**Question:** Why do we use the integration by parts?
We use this technique to simplify the original integral by dividing the integrand into two parts $u$ and $dv$.

**Example:** Evaluate the following integrals $\int x\cos x\ dx$ .

**Solution:**

Let $I = \int x\cos x\ dx$ and
$u = x \Rightarrow du = dx$
$dv = \cos x \Rightarrow v = \sin x$

$I = x\sin x - \int \sin x\ dx$

$\Rightarrow I = x\sin x + \cos x$ .

**Exercise:** Evaluate the following integrals

1) $\int x\sec^2 x\ dx$

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2) $\int e^x \sin x\ dx$

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3) $\int_1^2 x^3 \ln x \; dx$

4) $\int \tan^{-1} x \; dx$

5) $\int x e^x \; dx$

6) $\int x^3 \sqrt{1+x^2} \; dx$

In this section, we evaluate integrals of forms:

1) $\int \sin^n x \cos^m x \; dx$

2) $\int \tan^n x \sec^m x \; dx$

3) $\int \sin nx \sin mx \; dx$, $\int \cos nx \cos mx \; dx$, $\int \sin nx \cos mx \; dx$

Before we start considering the previous forms, we present some important formulas that are used in this section.

| | |
|---|---|
| $\cos^2 x + \sin^2 x = 1$ | $1 + \tan^2 x = \sec^2 x$ |
| $\cot^2 x + 1 = \csc^2 x$ | $2\cos^2 x - 1 = \cos 2x$ |
| $\sin^2 x = \frac{1-\cos 2x}{2}$ | $\cos^2 x - \sin^2 x = \cos 2x$ |
| $\cos^2 x = \frac{1+\cos 2x}{2}$ | $1 - 2\sin^2 x = \cos 2x$ |
| $\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$ | |
| $\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$ | |
| $\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$ | |

**Form** 1

$$\int \sin^n x \;\cos^m x \; dx$$

Method:

① If $n$ is odd, we write $\sin^n x \;\cos^m x = \sin^{n-1} x \;\cos^m x \sin x$.

Then, we use $\cos^2 x + \sin^2 x = 1$ and the substitution $u = \cos x$.

② If $m$ is odd, we write $\sin^n x \;\cos^m x = \sin^n x \;\cos^{m-1} x \cos x$.

Then, we use $\cos^2 x + \sin^2 x = 1$ and the substitution $u = \sin x$.

③ If $n$ and $m$ are even, we use the formula $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.

**Exercise:** Evaluate the following integrals

(1) $\int \sin^3 x \; dx$

Since $n = 3$ is odd, we use the substitution method. Write

$$\sin^3 x = \sin^2 x \;\sin x$$

$$\sin^3 x = (1 - \cos^2 x)\;\sin x$$

Let $u = \cos x \Rightarrow du = -\sin x$.
By substituting, the integral becomes

$$-\int (1-u^2)\; du = -(u - \frac{u^3}{3}) + c$$

but $u = \cos x$, this implies

$$\int \sin^3 x \; dx = -\cos x + \frac{\cos^3 x}{3} + c$$

(2) $\int \sin^4 x \; dx$

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(3) $\int \sin^5 x \ \cos^4 x \ dx$

(4) $\int \sin^2 x \ \cos^4 x \ dx$

(5) $\int_0^{\frac{\pi}{2}} \sin^3 x \ \cos^3 x \ dx$

(6) $\int \cos^5 x \ dx$

**Remember:**

| | |
|---|---|
| $\cos^2 x + \sin^2 x = 1$ | $1 + \tan^2 x = \sec^2 x$ |
| $\cot^2 x + 1 = \csc^2 x$ | $2\cos^2 x - 1 = \cos 2x$ |
| $\sin^2 x = \frac{1-\cos 2x}{2}$ | $\cos^2 x - \sin^2 x = \cos 2x$ |
| $\cos^2 x = \frac{1+\cos 2x}{2}$ | $1 - 2\sin^2 x = \cos 2x$ |

**Remark:**

(1) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x| + c.$

(2) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \ln|\sin x| + c.$

(3) $\int \sec x \, dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x} \, dx = \int \frac{\sec^2 x+\sec x \tan x}{\sec x+\tan x} = \ln|\sec x + \tan x| + c.$

**Form** 2

$$\int \tan^n x \ \sec^m x \ dx$$

Method:

① If $n = 0$ and

(i) $m$ is odd, then write $\sec^m x = \sec^{m-2} x \ \sec^2 x$. Use the integration by parts and let $u = \sec^{m-2} x$ and $dv = \sec^2 x$.

(ii) $m$ is even, then write $\sec^m x = \sec^{m-2} x \ \sec^2 x \, dx$ and use $\sec^2 x = 1 + \tan^2 x$ with the substitution $u = \tan x$.

② If $m = 0$, write $\tan^n x = \tan^{n-2} x \tan^2 x$ and use $\tan^2 x = \sec^2 x - 1$ with the substitution $u = \tan x$.

③ If $n$ is even and $m$ is odd, use $\tan^2 x = \sec^2 x - 1$ to change the integral to becomes $\int \sec^r x \, dx$, then use ①.

④ $n$ is odd and $m$ is even OR if $n$ and $m$ are even, then write $\tan^n x = \sec^m x = \tan^n x \sec^{m-2} x \ \sec^2 x \ dx$ and use $\sec^2 x = 1 + \tan^2 x$ with the substitution $u = tanx$.

**Exercise:** Evaluate the following integrals

(1) $\int \tan^3 x \, dx$

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(2) $\int \tan^4 x \, dx$

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(3) $\int \sec^4 x \, dx$

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(4) $\int \sec^5 x \; dx$

(5) $\int \tan^2 x \; \sec^4 x \; dx$

(6) $\int \tan^5 x \; \sec^6 x \; dx$

(7) $\int \tan^2 x \; \sec^3 x \; dx$

**Remember:**

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]$$
$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]$$
$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]$$

**Exercise:** Evaluate the following integrals

(1) $\int \sin 5x \ \sin 3x \ dx$

(2) $\int \cos 3x \ \cos 2x \ dx$

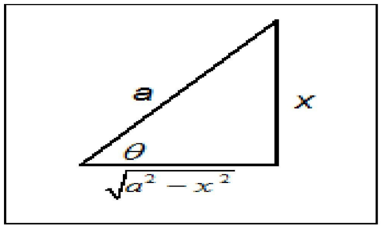
(3) $\int \sin 5x \ \cos 3x \ dx$

(4) $\int_0^{\frac{\pi}{2}} \sin 3x \ \cos 2x \ dx$

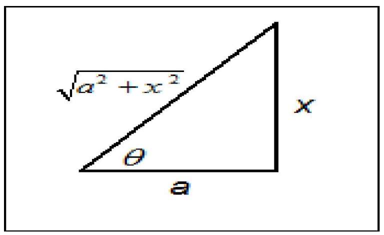
We are going to study integrals contains of $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$.

1 $\sqrt{a^2-x^2} = a\cos\theta$ if $x = a\sin\theta$.

If $x = a\sin\theta$, then $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$

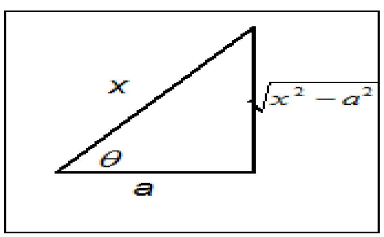



2 $\sqrt{a^2+x^2} = a\sec\theta$ if $x = a\tan\theta$.




If $x = a\tan\theta$, then $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$.

3 $\sqrt{x^2-a^2} = a\tan\theta$ if $x = a\sec\theta$.




If $x = a\sec\theta$, then $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta-a^2} = \sqrt{a^2(\sec^2\theta-1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$.

**Exercise:** Evaluate the following integrals

(1) $\int x^3\sqrt{4-x^2}\,dx$

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(2) $\int \frac{dx}{\sqrt{x^2+9}}$

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(3) $\int \frac{\sqrt{x^2-25}}{x^4}$

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Let assume that the rational function takes the form: $q(x) = \frac{f(x)}{g(x)}$. So, $f(x)$ is the numerator and $g(x)$ is the denominator. We will take 4 cases for integrating the rational functions.

Before we start, you should know about how to factor an expression $g(x)$ into irreducible factors.

(1) $a^2 - b^2 = (a-b)(a+b)$
$(x^2 - 64) = (x-8)(x+8)$

(2) $ax^2 \pm bx \pm c$
Let $ax^2 \pm bx \pm c = 0$, then $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.
$x^2 + 3x + 2 \Rightarrow x^2 + 3x + 2 = 0$, then $x = \frac{-3 \pm \sqrt{9-4(1)(2)}}{2(1)}$
$\Rightarrow x = -1$ or $x = -2 \Rightarrow (x+1)(x-2) = 0$.

(3) $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
$x^3 \pm 27 = (x \pm 3)(x^2 \mp 3x + 9)$

**Theorem:** If $q(x) = \frac{f(x)}{g(x)}$ is a rational function such that degree of $f(x)$ is less than the degree of $g(x)$, then there exists partial fractions $F_1, F_2, ..., F_n$ where $q(x) = F_1(x) + f_2(x) + ... + F_n(x)$. Each of $F_i(x)$ takes the form $\frac{A}{(ax+b)^m}$ $m \in \mathbb{N}$ or $\frac{Ax+B}{(ax^2+bx+c)^m}$ $b^2 - 4ac < 0$.

**Case 1:** If degree of $f(x)$ is greater than or equal to degree of $g(x)$.
For this case, we have three steps:

Step 1: Long division: $\frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)}$ where $h(x)$ is and $r(x)$ is the remaining.

Step 2: Factor $g(x)$ into irreducible factors.

Step 3: Write the fraction $\frac{r(x)}{g(x)}$ as partial factors.

**Example:** Evaluate the following integral: $\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8} dx$

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**Case 2:** If degree of $f(x)$ is less than degree of $g(x)$.
We ignore step 1 in **Case 1**, so we have two steps:

Step 1: Factor $g(x)$ into irreducible factors.

Step 2: Write the fraction $\frac{f(x)}{g(x)}$ as partial factors.

Example: Evaluate the following integral: $\int \frac{x^2+12x+12}{x^3-4x}dx$

**Exercise:** Evaluate the following integrals:

- $\int \frac{4x^4+2x^2-1}{x^2-1}dx$
- $\int \frac{x^2-4x+2}{x^3-4x^2-5x}dx$
- $\int \frac{x-1}{x^2+3x+2}dx$
- $\int \frac{dx}{x^2+3x-4}$
- $\int \frac{dx}{x^2+8x+7}$
- $\int \frac{5x-4}{x^2-4x}dx$
- $\int \frac{x^3+x}{x-1}dx$
- $\int \frac{e^x}{e^{2x}-4}dx$
- $\int_0^1 \frac{2x^5-x^3-1}{x^3-4x}dx$

**Case 3:** Repeated factors of the denominator $g(x)$.

Example: Evaluate the following integral: $\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}dx$

**Case 4:** Quadratic factors of the denominator $g(x)$.

Example: Evaluate the following integral: $\int \frac{8x^3+13x}{(x^2+2)^2}dx$

**Exercise:** Evaluate the following integrals:

- $\int \frac{2x^3-4x-8}{x^3-2x^2+x}\,dx$
- $\int \frac{2x^3-4x-8}{x(x-1)^2}\,dx$
- $\int \frac{12}{x^4-x^3-2x^2}\,dx$
- $\int \frac{4x+1}{(x-3)(x^2+6x+12)}\,dx$
- $\int \frac{x^3+x^2+x+2}{x^4+3x^2+2}\,dx$
- $\int \frac{4x^2-13x+6}{(x+2)(x-2)^2}\,dx$
- $\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx$
- $\int \frac{\cos x}{\sin^2 x+4\sin x-5}\,dx$
- $\int \frac{x+1}{(x^2+x+2)^2}\,dx$

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Sometimes quadratic expression $ax^2 + bx + c$ is irreducible. Because of that we use new technique that is completing square method: $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

Ex 1. The quadratic expression $x^2 - x - 12$ is reducible. $x^2 - x - 12 = (x+3)(x-4)$

Ex 2. The quadratic expression $x^2 - 6x + 13$ is irreducible.

By completing square i.e., we need to rewrite the previous expression as $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

Put $2b = 6 \Rightarrow b = 3 \Rightarrow b^2 = 9$. So, we add and subtract 9
$\Rightarrow (x^2 - 6x + 9) + 13 - 9$.
$\Rightarrow (x - 3)^2 + 4$

Ex 3. The quadratic expression $x^2 + 8x + 25$ is irreducible.

By completing square, $2b = 8 \Rightarrow b = 4 \Rightarrow b^2 = 16$. So, we add and subtract 16, then the previous expression becomes
$(x^2 + 8x + 16) + 25 - 16 = (x + 4)^2 + 9$.

Now, we use the idea to solve the following integrals.

**Exercise:** Evaluate the following integrals

① $\int \frac{1}{x^2-2x+2} dx$.

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② $\int_2^3 \frac{x^2-4x+6}{x^2-4x+5} dx$.

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③ $\int \frac{1}{\sqrt{7+6x-x^2}} dx$.

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We use the substitution $u = tan(\frac{x}{2})$ to reduce the integration of rational functions that contain on $sin(x)$ and $cos(x)$ in their denominators.

Let $u = tan(\frac{x}{2}) \Rightarrow sec^2(\frac{x}{2}) = u^2 + 1$.
Also, this implies $du = \frac{u^2+1}{2} dx$.

Now, we write $sin(x)$ and $cos(x)$ using the previous substitution.

$$\begin{aligned} sin(x) &= sin2(\tfrac{x}{2}) = 2sin(\tfrac{x}{2})cos(\tfrac{x}{2}) \\ &= 2\frac{sin(\frac{x}{2})}{cos(\frac{x}{2})}cos(\tfrac{x}{2})cos(\tfrac{x}{2}) \\ &= 2tan(\tfrac{x}{2})cos^2(\tfrac{x}{2}) = 2\frac{tan(\frac{x}{2})}{sec^2(\frac{x}{2})} = \frac{2u}{u^2+1}. \end{aligned}$$

This means if $u = tan(\frac{x}{2})$, then
$du = \frac{u^2+1}{2} dx$ and $sin(x) = \frac{2u}{u^2+1}$.

Similarly, we can find that $cos(x) = \frac{1-u^2}{u^2+1}$.

Hint: use the formula $sin(u)sin(v) = \frac{1}{2}[cos(u-v) - cos(u+v)]$, put $u = v = x$.

**Exercise:** Evaluate the following integrals

① $\int \frac{1}{3sin(x)+4cos(x)} dx$.

② $\int \frac{1}{1-sin(x)} dx$ (Homework)

③ $\int \frac{1}{1-cos(x)} dx$ (Homework)

④ $\int \frac{1}{1+cos(x)} dx$ (Homework)

⑤ $\int \frac{1}{5+3cos(x)} dx$ (Homework)

==========================================================================

We call $\int_a^b f(x)\ dx$ a proper integral if
1) the interval $[a, b]$ is finite and
2) $f(x)$ is continuous on $[a, b]$.
This means if condition 1 or 2 is not satisfied, the integral is **improper**.

We going to study two cases of improper integrals:

**Case 1:** Interval of the integral is infinite
$\int_a^{\infty} f(x)\ dx,\ \int_{-\infty}^{a} f(x)\ dx,\ \int_{\infty}^{-\infty} f(x)\ dx$ .

**Definition:** (1) if $f$ is continuous on $[a, +\infty)$, then

$$\int_a^{\infty} f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$$

(2) if $f$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$$

(3) if $f$ is continuous on $R$ and $a \in R$, then

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{t\to-\infty} \int_t^a f(x)\ dx + \lim_{t\to\infty} \int_a^t f(x)\ dx$$

Note: An improper integral is convergent if the limit exists as a finite number.

**Exercise:** Determine whether the integral converges or diverges:

1 $\int_0^{\infty} \frac{1}{4+x^2}\ dx$ .

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2 $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx$ .

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3 $\int_0^{\infty} \frac{x}{1+x^2}\ dx$ .

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**Case 2:** Integrals with infinite discontinuities.

**Definition:** (1) if $f$ is continuous on $[a, b)$ and has an infinite discontinuity at $b$, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)\ dx$$

(2) if $f$ is continuous on $(a, b]$ and has an infinite discontinuity at $a$, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^a f(x)\ dx$$

The integral is convergent if the limit exists as a finite number.

**Exercise:** Determine whether the integral converges or diverges:

1. $\int_{-3}^{1} \frac{1}{x^2}\ dx$ .

2. $\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\ dx$ .

3. $\int_{-\infty}^{0} \frac{1}{(x-8)^{\frac{2}{3}}}\ dx$ .

4. $\int_0^{\infty} \sin x\ dx$ .

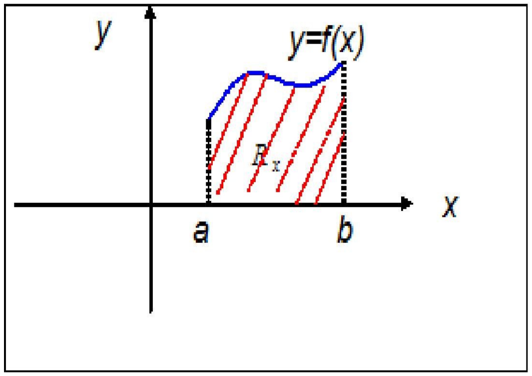
5. $\int_{-1}^{1} x^{\frac{-4}{3}}\ dx$ .

==================================================================

**Remember:** If $f(x) \geq 0$ and continuous on $[a, b]$, then the area of the region under the graph **See Figure 1** is given by

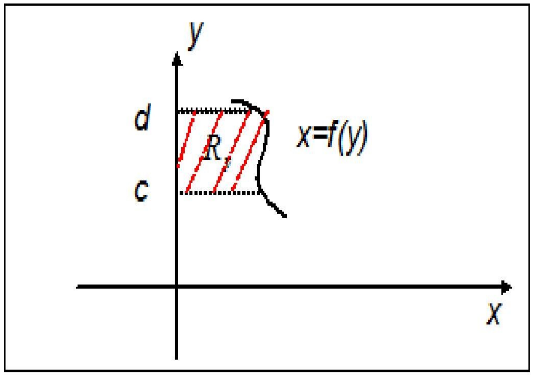
$$A = \int_a^b f(x)dx$$

(Figure 1)




**Remark:** (1) If we have an equation of the form $x = f(y)$ instead of $y = f(x)$ where $f$ is continuous on $[c, d]$. We let $y$ be the variable of the integral. Then, the area is

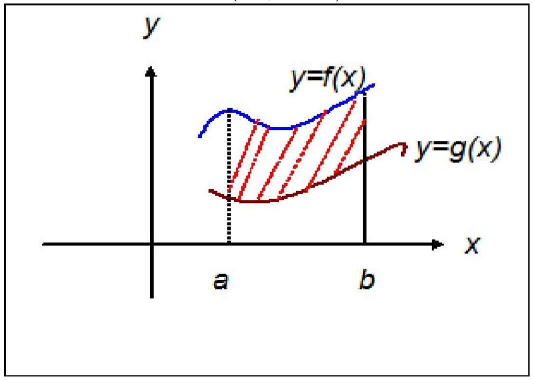
$$A = \int_c^d f(y)\ dy$$




**Theorem:** If $f(x)$ and $g(x)$ continuous and $f(x) \geq g(x)$ for every $x \in [a, b]$, then the area A of the region bounded by the graphs of $f$ and $g$ is

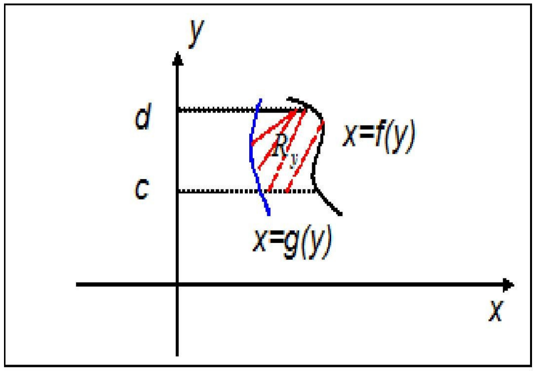
$$A = \int_a^b \big(f(x) - g(x)\big)dx$$

(Figure 2)



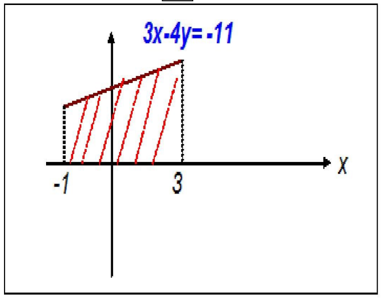

(2) If $f(y)$ and $g(y)$ are two continuous functions such that $f(y) \geq g(y)$ for every $y \in [c, d]$, then the area $A$ of the region bounded by the graphs of $f$ and $g$ is

$$A = \int_c^d \big(f(y) - g(y)\big)dy$$



**Exercise 1:** Express the area of the shaded region as a definite integral then find the area.

1

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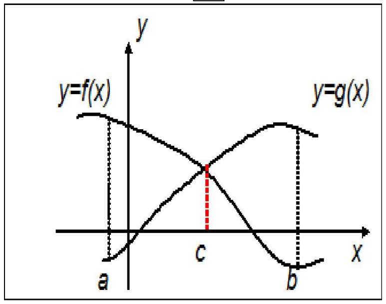


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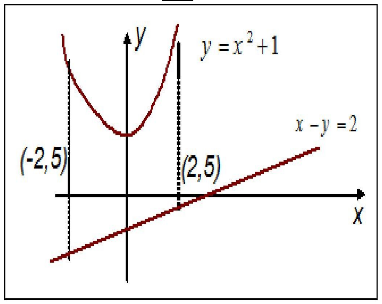


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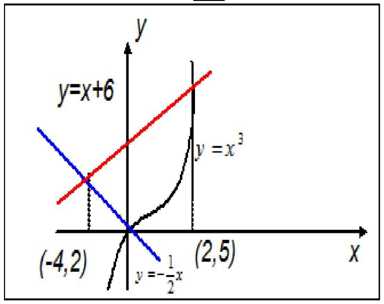


**Exercise 2:** Sketch the region by the graphs of $y = x^3$ and $y = x$, then find its area.

**Exercise 3:** Sketch the region by the graphs of $x = 3 - y^2$ and $x = y + 1$, then find its area.

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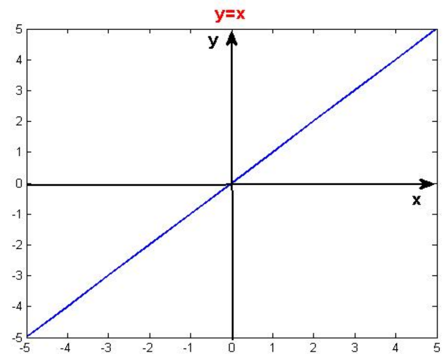


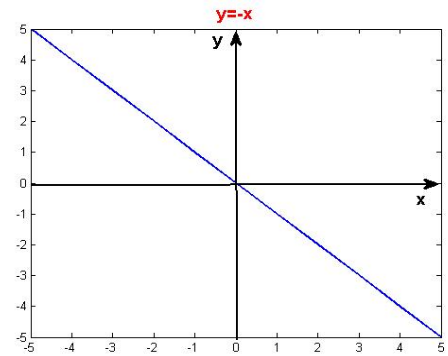


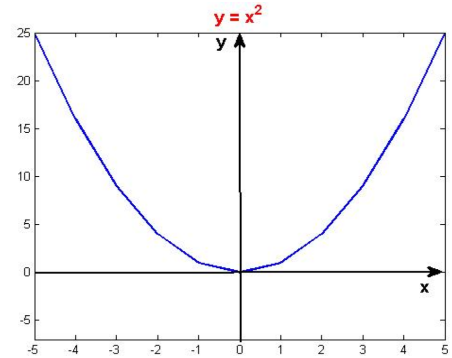


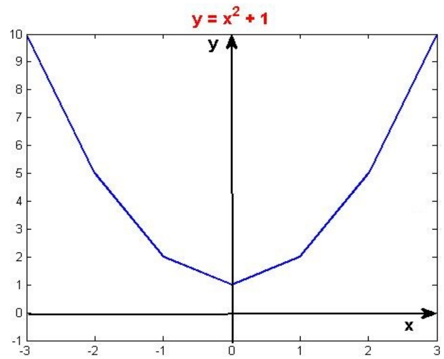


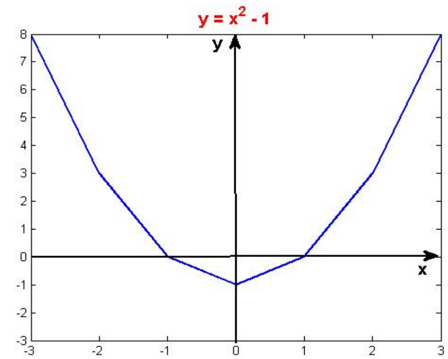


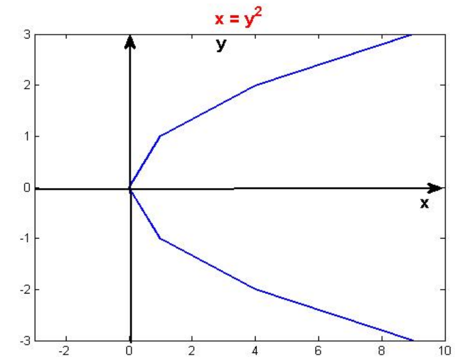


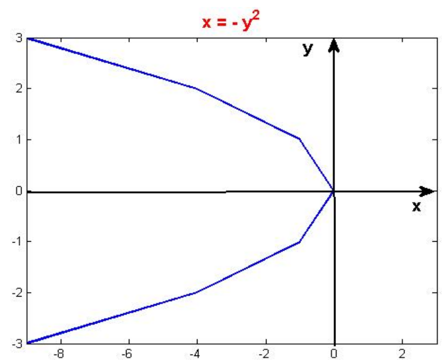


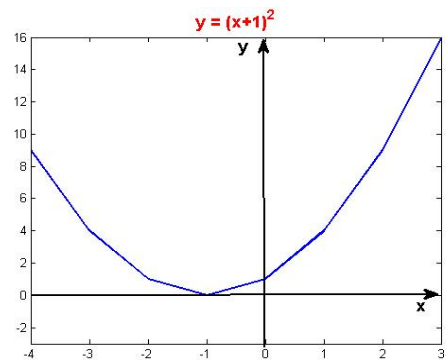


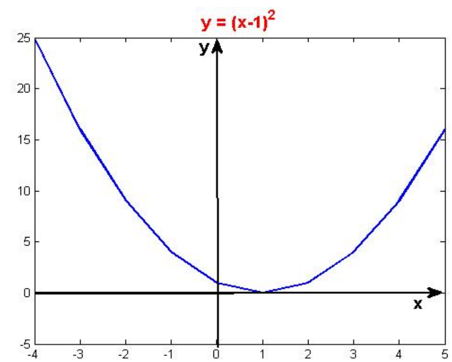

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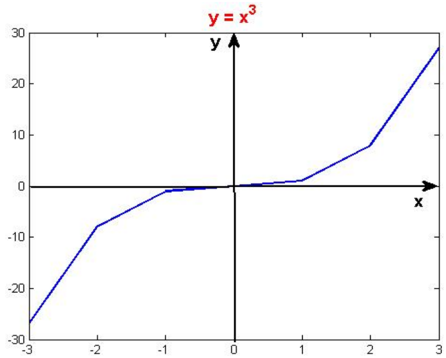
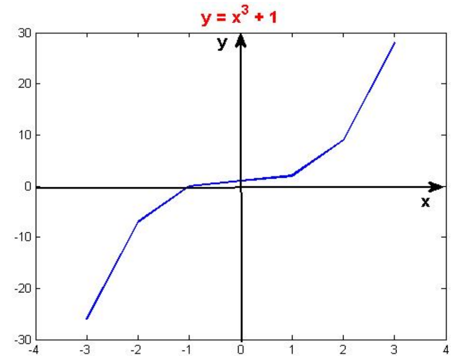
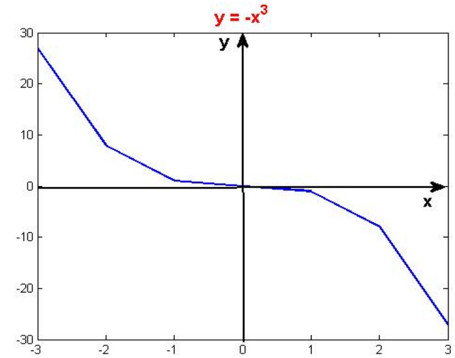
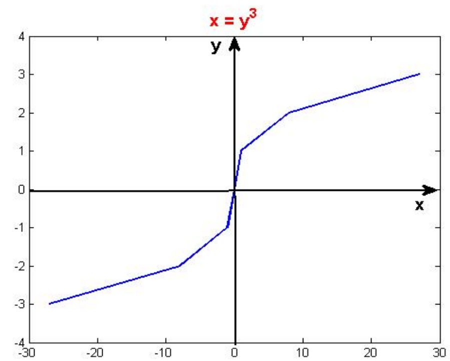
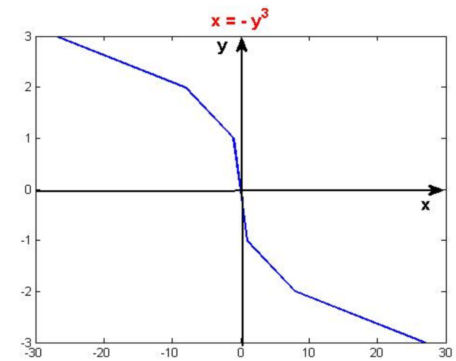
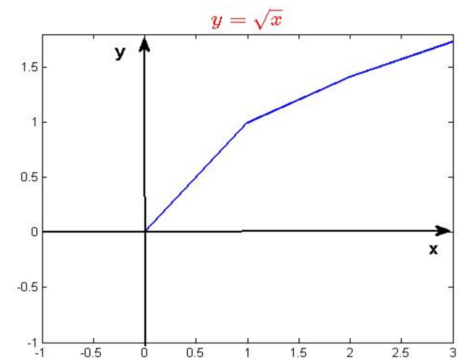

y = x3

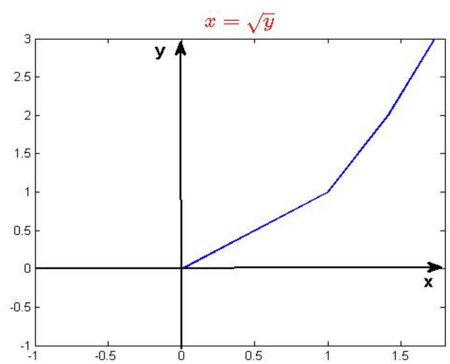

y = x3 + 1

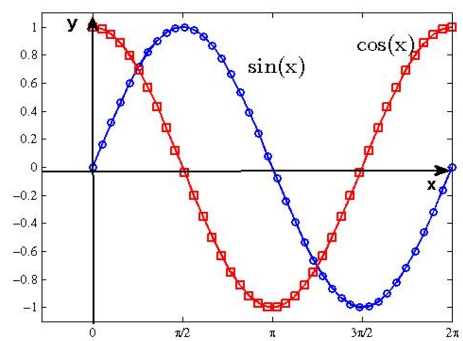

y = -x3


x = y3


x = - y3


$y = \sqrt{x}$

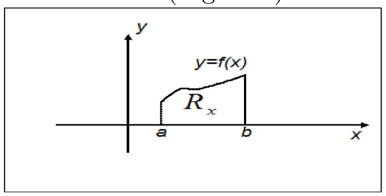

$x = \sqrt{y}$


sin(x)
cos(x)

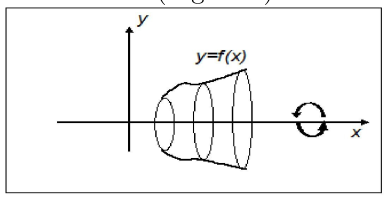
Let $R_x$ be a region of $f(x)$ where $f$ is continuous and $f \geq 0$. Let $R_x$ be bounded by the graph and x-axis and $x = a, x = b$. Revolution of the region about a line (x-axis or y-axis) generates a solid called solid of revolution.

**Example 1:**
Consider the graph of $f(x)$ and the region $R_x$ in **Figure 1**. Revolution of $R_x$ about x-axis generates a solid of revolution given in **Figure 2**.

(Figure 1)

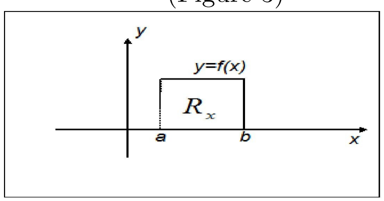



(Figure 2)

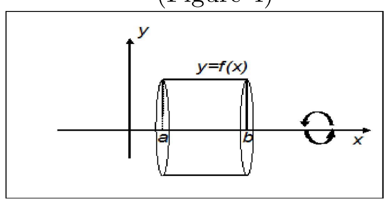



**Example 2:**
Let $f(x)$ be a constant function, for example $f(x) = 3$ as in **Figure 3**. The region $R_x$ is a rectangular and revolution of $R_x$ about x-axis generates a circular cylinder given in **Figure 4**.

(Figure 3)

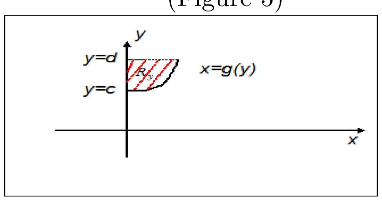



(Figure 4)

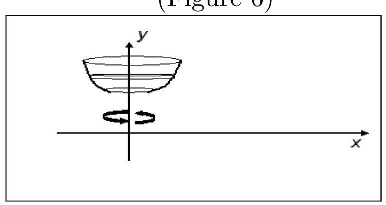



**Example 3:**
Consider the graph of $f(y)$ and the region $R_y$ in **Figure 5**. Revolution of $R_x$ about x-axis generates a solid of revolution given in **Figure 6**.

(Figure 5)




(Figure 6)




**Volume of the Solid of Revolution:**

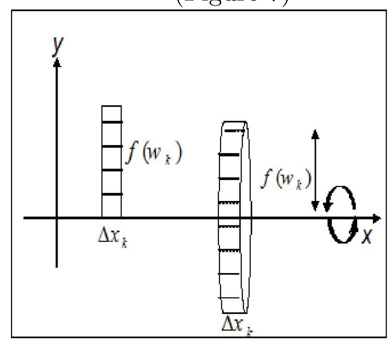
**(1) Disk Method**
Let $f$ be continuous on $[a, b]$ and let $R_x$ be a region bounded by the graphs, x-axis and the points $x = a,\ x = b$. Let $S$ be a solid generated by revolving $R_x$ about x-axis.

Let $P$ be a partition of $[a, b]$ and $w_k \in [x_{k-1}, x_k]$. For each $[x_{k-1}, x_k]$, we form a rectangular, its high is $f(w_k)$ and its width is $\Delta x_k$.

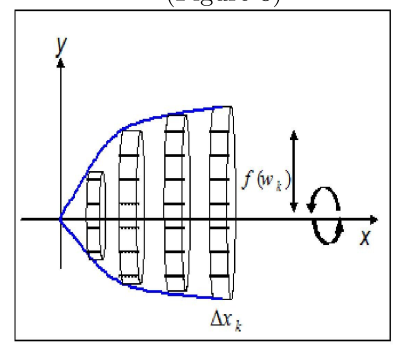
Revolution of the rectangular about x-axis generates a circular disk as shown in **Figure 7**. Its radius and high are

$$r = f(w_k)$$
$$h = \Delta x_k$$

(Figure 7)




(Figure 8)




From this, the volume of the circular disk is

$$V_k = \pi(f(w_k))^2 \Delta x_k$$

The sum of volumes of circular disks approximately gives the volume of the solid of revolution given in **Figure 8**:

$$V = \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} \pi(f(w_k))^2 \Delta x_k = \int_a^b \pi[f(x)]^2 \, dx$$

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From the previous discussion,
(1) The volume $V$ of the solid of the revolution of the region bounded by the graph of $y = f(x)$ and $x = a, x = b$ about x-axis is

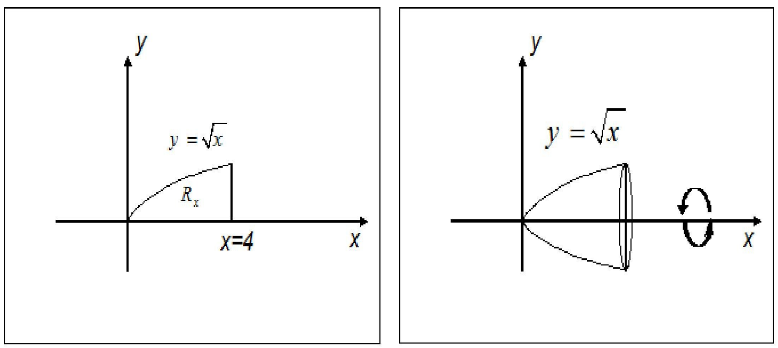
$$V = \int_a^b \pi[f(x)]^2 \, dx$$

(2) Let $f$ be continuous on $[c, d]$. The volume $V$ of the solid of the revolution of the region bounded by the graph of $x = f(y)$ and $y = c, y = d$ about y-axis is

$$V = \int_c^d \pi[f(y)]^2 \, dy$$

**Example:** Sketch the region $R$ bounded by the graphs of the equations $y = \sqrt{x}$, $x = 4$, $y = 0$.
Then, find the volume of the solid generated if $R$ is revolved about x-axis.

**Solution:**




The volume:

$V = \int_0^4 \pi[\sqrt{x}]^2 \, dx$

$= \pi \int_0^4 x \, dx = \pi \left[\frac{x^2}{2}\right]_0^4 = 8\pi.$

**Exercise:** Sketch the region $R$ bounded by the graphs of the equations $y = x^3 - 1$, $x = 0$, $y = 7$.
Then, find the volume of the solid generated if $R$ is revolved about y-axis.

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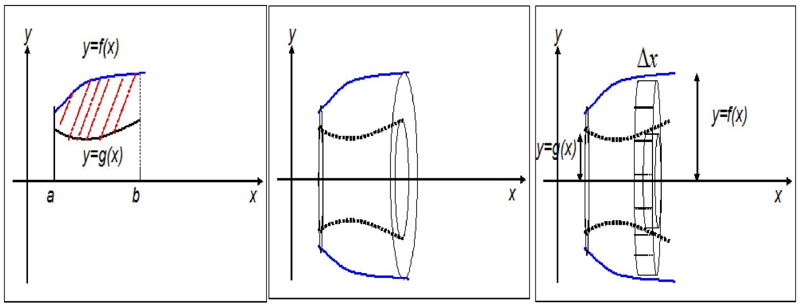
## Washer Method

The washer method is generalization of the disk method for a region between two functions $f(x)$ and $g(x)$ as shown in the following **Figure 1**. Let $R_x$ be a region bounded by the graphs of $f(x)$ and $g(x)$ such that $f(x) > g(x)$ and by $x = a,\ x = b$.




Note: Revolution of a rectangular generates a solid likes a washer where there are two radius: outer radius and inner radius.

Volume of the washer $= \pi\ [r_1 - r_2]$ (thickness)

where $r_1$ is the outer radius and $r_2$ is the inner radius. For a partition $P = \{x_1, x_2, ..., x_n\}$ and $w_k \in [x_{k-1}, x_k]$, the volume of the washer is

$$V = \pi\big([f(w_k)]^2 - [g(w_k)]^2\big)\Delta x_k$$

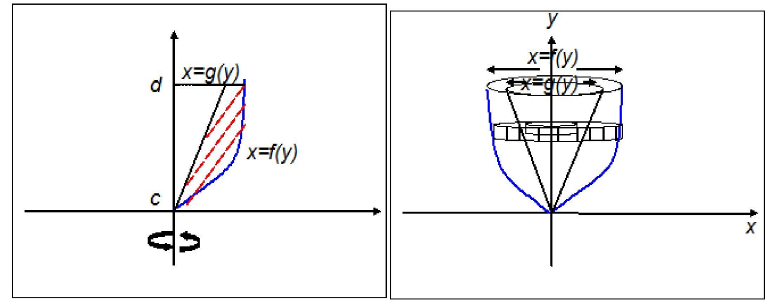
Since the whole solid $S$ is formed by a set of washers, then the volume of $S$ can be obtained by summing the volume of washers.

**Summary:**

(1) If $R_x$ is revolved about x-axis, we have a solid $S$ with a hole through that solid. The volume of $S$ is the difference between the volumes of two solids generated by $f$ and $g$:

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right)\, dx$$

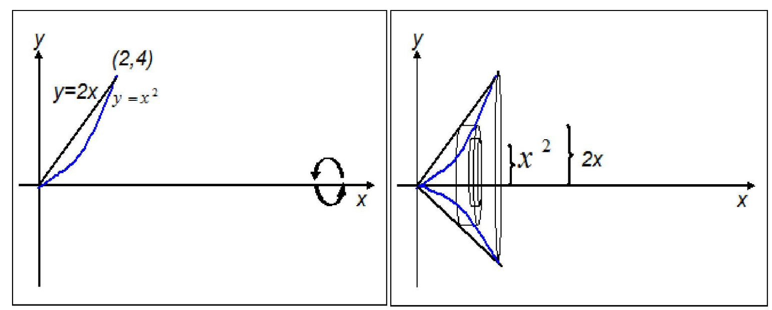
(2) If $R_y$ is revolved about y-axis, we have a solid $S$ with a hole through that solid. The volume of $S$ is the difference between the volumes of two solids generated by $f$ and $g$:




$$V = \pi \int_c^d \left([f(y)]^2 - [g(y)]^2\right)\, dy$$

**Example:** Evaluate the volume of the solid generated by revolution of the bounded region by graphs of the following two functions $y = x^2$ and $y = 2x$ about x-axis.

**Solution:**




The volume of the solid $S$ is
$V = \pi \int_0^2 [2x]^2 - [x^2]^2\ dx = \pi \int_0^2 [4x^2] - [x^4]\ dx = \pi \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{64}{15}\pi$

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**Exercise 1:** Evaluate the volume of the solid generated by revolution of the bounded region by graphs of the following two functions $x = \sqrt{y}$ and $x = \frac{y}{2}$ about y-axis.

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**Exercise 2:** Evaluate the volume of the solid generated by revolving the bounded region by graphs $y = \sin$ and $y = \cos\ x$, $x = 0$, $x = \frac{\pi}{4}$ about x-axis.

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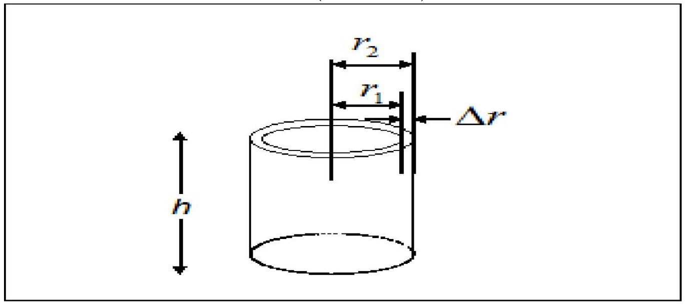
(Figure 1)




As shown in **Figure 1**, let
$r_1$ be outer radius of the shell
$r_2$ be inner radius of the shell
$h$ be high of the shell
$\Delta r = r_2 - r_1$ be thickness of the shell
$r = \frac{r_1+r_2}{2}$ be average radius of the shell

The volume of the cylindrical shell

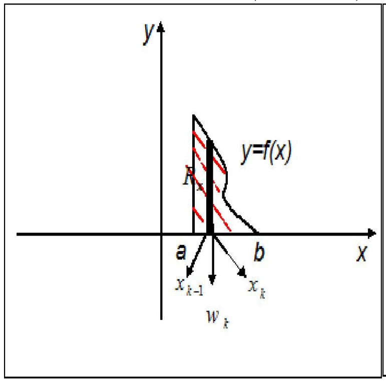
$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi(\tfrac{r_2+r_1}{2})h(r_2 - r_1) \\ &= 2\pi r h \Delta r \end{aligned}$$

Now, consider the graph given in **Figure 2**. Revolution of the region $R_x$ about y-axis generates a solid given in **Figure 3**.

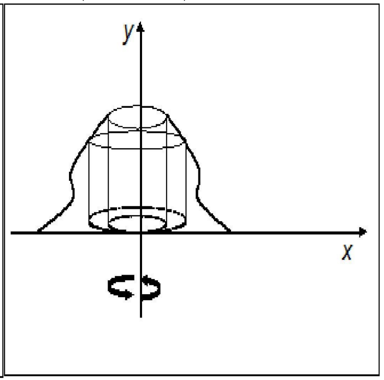
Let $P$ be a partition of the interval $[a, b]$ and let $w_k$ be the mid point of $[x_{k-1}, x_k]$ (see **Figure 2**).

Revolution of the rectangular given in **Figure 2** about y-axis generates a cylindrical shell where
average radius $= w_k$
high $= f(w_k)$
thickness $= \Delta x_k$

(Figure 2)




(Figure 3)




Hence, the volume of the cylindrical shell

$$V_k = 2\pi w_k f(w_k)\Delta x_k$$

To evaluate the volume of the whole solid, we sum the volume of all cylindrical shells. This means

$$V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} w_k f(w_k)\Delta x_k$$

From Riemann Sum $\sum_{k=1}^{n} w_k f(w_k)\Delta x_k = \int_a^b x f(x)\, dx$, we have

$$V = 2\pi \int_a^b x f(x)\, dx$$

Similarly, if the revolution of the region about x-axis, the volume of the solid of revolution is

$$V = 2\pi \int_c^d y f(y)\, dy$$

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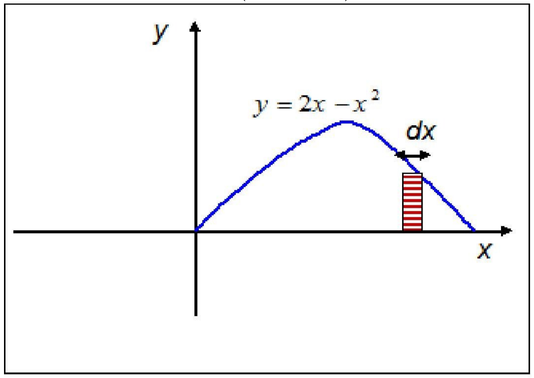
**Example:** Sketch the region $R$ bounded by the graphs of the equations $y = 2x - x^2$, $x = 0$.
Then, by method of cylindrical shells, find the volume of the solid generated if $R$ is revolved about y-axis.

(Figure 4)




Since the revolution of the region $R_x$ about y-axis, then
$V = 2\pi \int_0^2 x(2x - x^2)\ dx$

$= 2\pi \int_0^2 2x^2 - x^3\ dx$

$= 2\pi \left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2$

$= 2\pi \left[\left(2\frac{8}{3} - \frac{16}{4}\right) - (0)\right]$

$= 2\pi \left[\frac{16}{3} - 4\right]$

$= 2\pi \left[\frac{16-12}{3}\right]$

$= 2\pi \left[\frac{4}{3}\right]$

$= \frac{8}{3}\pi$

**Exercise:** Find the volume of the region bounded by $y = \sin\ x$, $x = 0$, $y = 0$, $x = \pi$ and revolved about y-axis.

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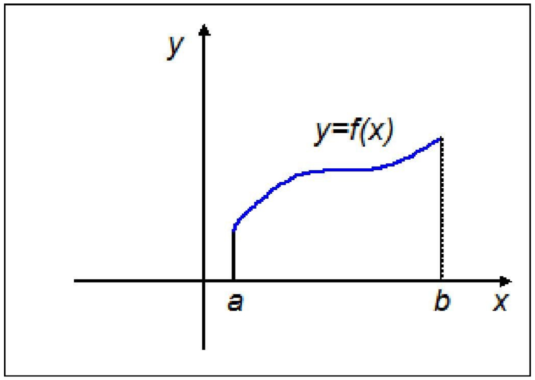
## (1) Arc Length:

**Definition:**

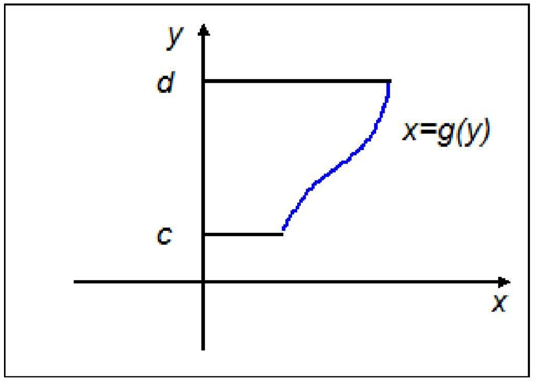
(1) Let $y = f(x)$ is a smooth function (it has derivatives of all orders everywhere in its domain) on $[a, b]$. The length of the arc of $f$ from $x = a$ to $x = b$ is

$$\boxed{L(f) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx},\ f'(x) = \frac{dy}{dx}$$




(2) Let $x = g(y)$ is a smooth function on $[c, d]$. The length of the arc of $g$ from $y = c$ to $y = d$ is

$$\boxed{L(g) = \int_c^d \sqrt{1 + [g'(x)]^2}\, dy},\ g'(x) = \frac{dx}{dy}$$




**Exercise 1:** Choose the correct answer

(1) The arc length of the graph of the curve $y = coshx$, $0 \le x \le 4$ is equal to

(a) $\sinh(4) - 1$ (b) $\cosh(4) - 1$ (c) $\sinh(4)$ (d) $\cosh(4)$

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(2) The arc length of the graph of the curve $y = 4x$, from $A(0, 0)$ to $B(1, 4)$ is equal to

(a) $\sqrt{17}$ (b) $\sqrt{5}$ (c) $4\sqrt{17}$ (d) $4\sqrt{5}$

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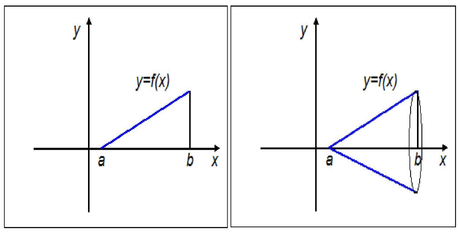
**(2) Surfaces of Revolution:**

Let $y = f(x)$ be a smooth function on $[a, b]$. Revolution of the curve about x-axis or y-axis generates surface called **Surface of Revolution**.

**Definition:**

(1) Let $y = f(x)$ be a smooth function on $[a, b]$. The surface area $S.A$ generated by revolving the curve of $f$ about **x-axis** from $x = a$ to $x = b$ is

$$S.A = 2\pi \int_a^b \mid f(x) \mid \sqrt{1 + [f'(x)]^2}\, dx$$

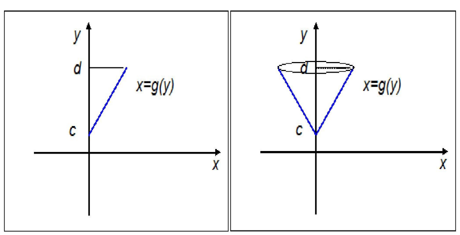



**NOTE:** If the revolution of the curve of $f$ is about **y-axis**, then

$$S.A = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\, dx$$

(2) Let $x = g(y)$ be a smooth function on $[c, d]$. The surface area $S.A$ generated by revolving the curve of $g$ about **y-axis** from $y = c$ to $y = d$ is

$$S.A = 2\pi \int_c^d \mid g(y) \mid \sqrt{1 + [g'(x)]^2}\, dy$$




**NOTE:** If the revolution of the curve of $g$ is about **x-axis**, then

$$S.A = 2\pi \int_c^d y\sqrt{1 + [g'(x)]^2}\, dy$$

**Exercise 2:** Choose the correct answer

(1) The surface area generated by revolving the curve of the function $\sqrt{4 - x^2}$, $-2 \leq x \leq 2$ around x-axis is equal to

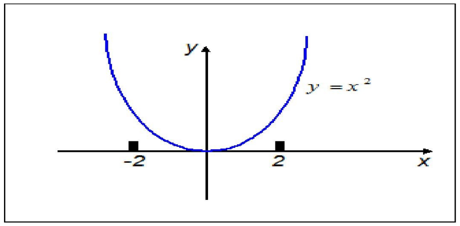
(a) $16\pi$ (b) $4\pi$ (c) $8\pi$ (d) $6\pi$

(2) The surface area resulting by revolving the graph of the equation $y = x$, $0 \leq y \leq 4$ around y-axis is equal to

(a) $16\sqrt{2}\pi$ (b) $\sqrt{2}\pi$ (c) $16\sqrt{2}$ (d) $8\sqrt{2}\pi$

If $f$ is continuous, the graph of $y = f(x)$ is called a plane curve.

**Example:** Let $y = x^2$ for $-2 \leq x \leq 2$. The equation is continuous and its graph given in the following figure.




Now, let $x = t$ and $y = t^2$ for $-2 \leq t \leq t$. then, we have the same graph. The last equations are called parametric equations for the curve $C$.

**Note:**

(1) Parametric equations give the same graph of $y = f(x)$

(2)Parametric equations give the orientation of $C$.

(3) To find the parametric equations, we introduce a third variable $t$ called a parameter. Rewrite $x$ and $y$ as functions of $t$, then we have the parametric equations
$x = f(t)$ parametric equation for $x$
$y = g(t)$ parametric equation for $y$

**Exercise 1:** For the following curves,
(a) find an equation in $x$ and $y$ whose graph contains the points on the curve.
(b) sketch the graph of $C$.
(c) indicate the orientation.

(1) $x = t - 2$, $y = 2t + 3$, $0 \leq t \leq 5$

(2) $x = 2\sin t$, $y = 3\cos t$, $0 \leq t \leq 2\pi$

(3) $x = t^2$, $y = 2\ln t$, $t > 0$

(4) $x = \sin t$, $y = \cos t$, $0 \leq t \leq \pi$

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Here, we are going to find **the slope of the tangent**, **the second derivative**, **the length of the arc** and **the area of the surface of revolution**.

**(1) Slope of the tangent line at a point:**
If a smooth curve $C$ given by $x = f(t)$ and $y = g(t)$, then the slope of tangent line to $C$ at point $P(x, y)$ is

$$y^{'} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ if } \frac{dx}{dt} \neq 0$$

**(2) Second derivative in a parametric form:**

$$y^{''} = \frac{d^2y}{dx^2} = \frac{d(y^{'})}{dx} = \frac{\frac{dy^{'}}{dt}}{\frac{dx}{dt}}$$

**(3) Length of the arc of the curve:**
The length of the curve $x = f(t)$, $y = g(t)$ where $a \leq t \leq b$ is given by

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

**(4) Area of the surface of revolution of a curve:**
(i) Let the curve $C$ is given by $x = f(t)$, $x = g(t)$ where $a \leq t \leq b$. If $y \geq 0$ on $[a, b]$, then the area $S.A$ of the surface generated by revolving $C$ about **x-axis** is

$$S.A = 2\pi \int_a^b y\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

(ii) If the curve $C$ is revolved about **y-axis** where $x = f(t) \geq 0$ on $[a, b]$, then the area $S.A$ of the surface

$$S.A = 2\pi \int_a^b x\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

**Exercise 2:** Choose the correct answer
1) The slope of the tangent line at the point corresponding to $t = \frac{\pi}{4}$ on the curve given parametrically by the equations $x = \sin t$, $y = \cos t$; $0 \leq t \leq 2\pi$ is
(a) $-1$ (b) 1 (c) 0 (d) $\frac{1}{3}$

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2) The length of the curve $C$: $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq 1$ is equal to
(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4

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3) The surface area resulting by revolving the graph of the parametric equation $x = 3t$, $y = 3t$, $0 \leq t \leq 1$ around the x-axis is equal to
(a) $9\sqrt{2}\pi$ (b) $18\sqrt{2}\pi$ (c) $24\sqrt{2}\pi$ (d) $\frac{9}{2}\sqrt{2}\pi$

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**Exercise 3:** Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, then evaluate each at the indicated value of the parameter.

$x = 2\cos t$, $y = 2\sin t$ at $t = \frac{\pi}{4}$.

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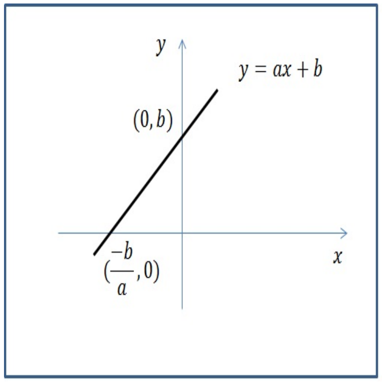
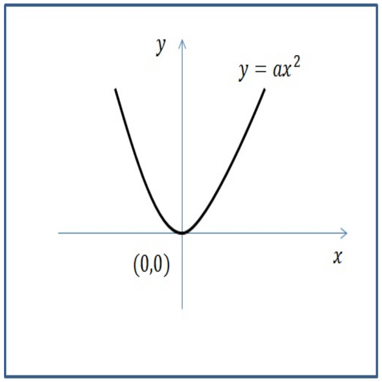
(1) Straight line

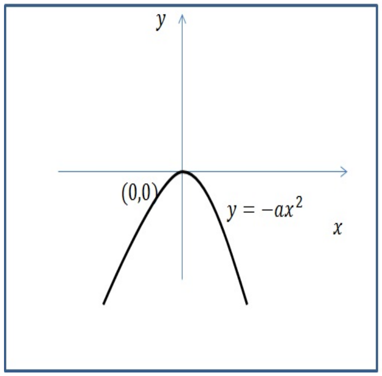



(2) Parabola

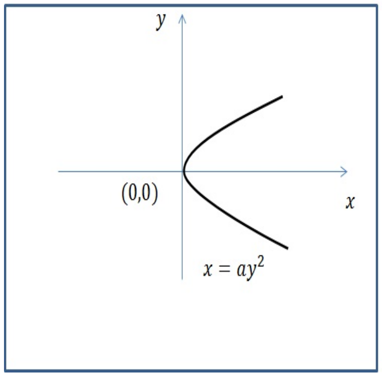
(A)



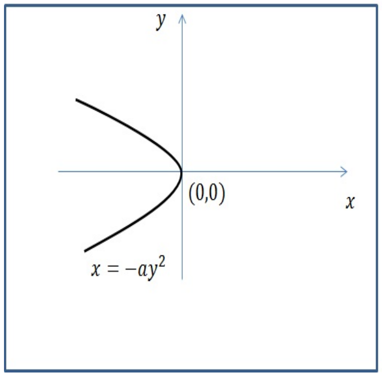

(B)



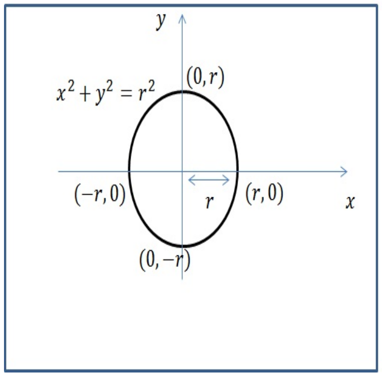

(C)




(D)

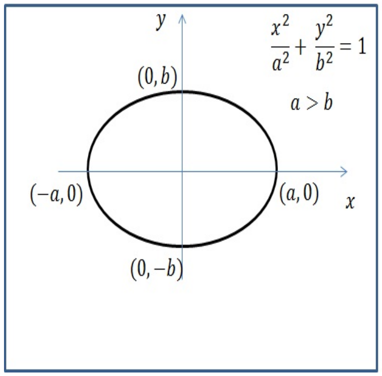



(3) Circle

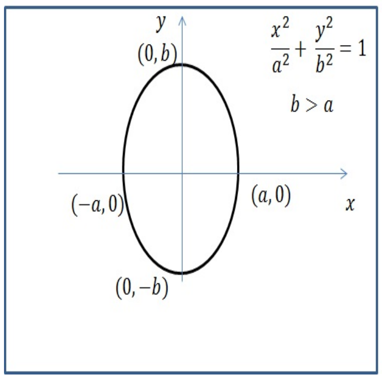


(4) Ellipse

(A)

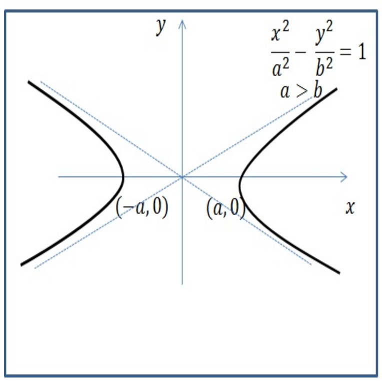



(B)

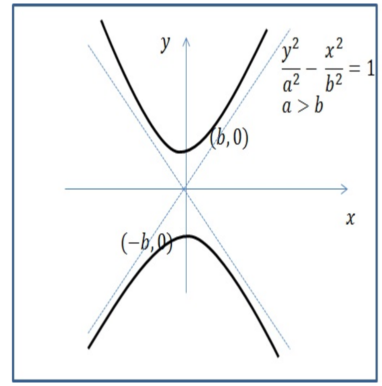



(5) Hyperbola

(A)




(B)



Previously, we used Cartesian coordinate to determine points $(x, y)$ as shown in **Figure 1**. We are going to study a new coordinate system called **Polar Coordinates**.

The polar coordinate is a two-dimensional coordinate system. It contains a fixed point $O$ (Pole) and each point on a plane is determined by a distance $(r)$ from the pole and an angle $(\theta)$ from a fixed direction as shown in **Figure 2**.

**Figure 1**

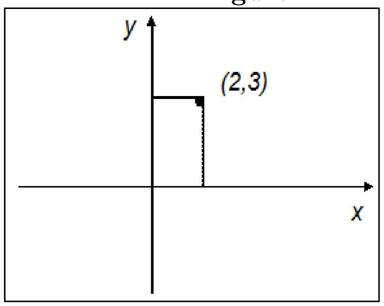
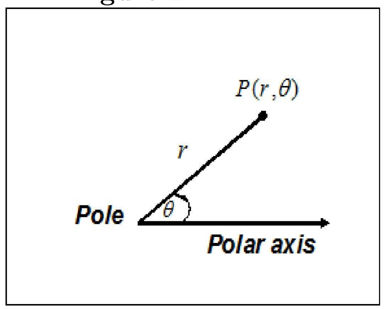
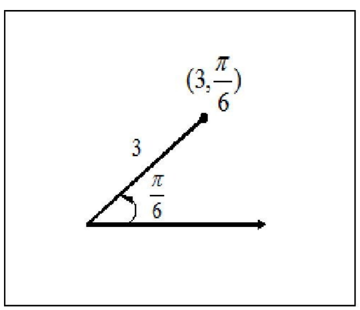
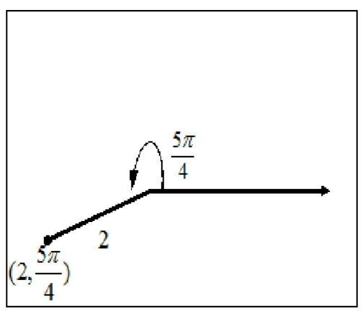



**Figure 2**



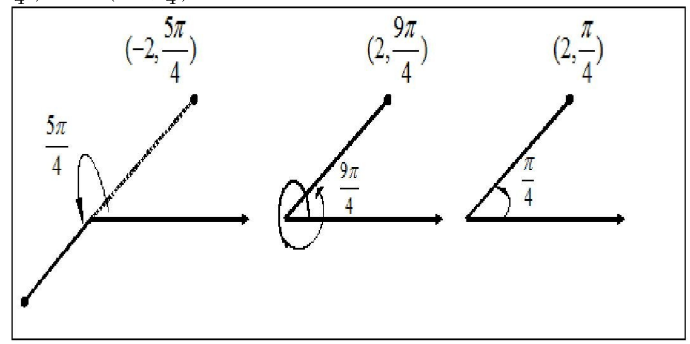

**Example 1:**




**Note:** In the Cartesian coordinate system, each point in the plane corresponds to a unique ordered pair $(x, y)$ of numbers. However, this is not true in the polar coordinate where each point has infinite number of polar coordinate pairs.

**Example 2:** Represent the following polar coordinates $(2, \frac{\pi}{4})$, $(2, \frac{9\pi}{4})$ and $(-2, \frac{\pi}{4})$.

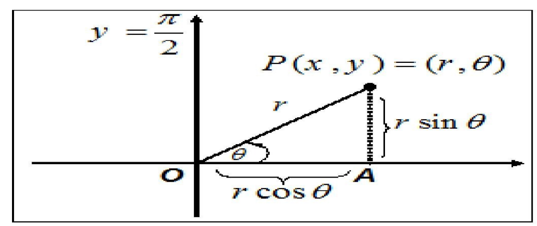



Conclusion: the polar coordinates represent the same point. Generally, we can write

$$(r, \theta + 2n\pi) = (r, \theta) = (-r, \theta + (2n+1)\pi) \quad n \in \mathbb{Z}$$

**(1) Relationship between Polar and Rectangular Coordinates**

Let $(x, y)$ be rectangular a coordinate and $(r, \theta)$ be a polar coordinate. Let the pole on the origin point and polar axis on x-axis, and the line $\theta = \frac{\pi}{2}$ on y-axis as shown in the following figure.




From the triangle O A P

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

From this,
$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 \Rightarrow x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$
Then, $x^2 + y^2 = r^2$ .

**Exercise 1:** Choose the correct answer

(1) If a point has $(r,\theta)$-coordinates where $(r,\theta)=(1,\frac{\pi}{6})$, then its $(x,y)$-coordinate is
(a) $(\frac{\sqrt{3}}{2},\frac{1}{2})$ (b) $(\frac{\sqrt{1}}{2},\frac{\sqrt{3}}{2})$ (c) $(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$ (d) $(1,0)$

(2) If a point has xy-coordinates where $(x,y)=(1,1)$, then its $(r,\theta)$-coordinate is
(a) $(1,\frac{\pi}{4})$ (b) $(2,\frac{\pi}{4})$ (c) $(\sqrt{2},\frac{-\pi}{4})$ (d) $(-\sqrt{2},\frac{5\pi}{4})$

**(2) Slope of a tangent line**

The slope of a tangent line to the graph of $r=f(\theta)$ is given by

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

**Note:**

(1) If $\frac{dy}{d\theta}=0$ such that $\frac{dx}{d\theta}\neq 0$, the curve has a horizontal tangent line.

(2) If $\frac{dx}{d\theta}=0$ such that $\frac{dy}{d\theta}\neq 0$, the curve has a vertical tangent line.

**Exercise:** Choose the correct answer

(1) The slope of the tangent line to the curve: $r=\cos\theta$ at $\theta=\frac{\pi}{4}$ is
(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{\pi}{4}$ (d) 1

(2) The slope of the tangent line to the curve: $r=2$ at $\theta=\frac{\pi}{4}$ is
(a) 1 (b) $-1$ (c) 0 (d) $\infty$

## (3) Graphs in Polar Coordinates

**Test of Symmetry in Polar System:**

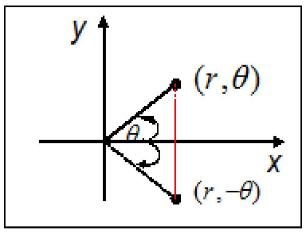
(a) Symmetry about the polar axis (x-axis)

The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if substitution of

$$-\theta \text{ for } \theta$$

does not change the equation $r = f(\theta)$.

**Example 1:** Consider the graph of $r = 4\cos\theta$.
Since $\cos(-\theta) = \cos\theta$, then the graph is symmetric about the polar axis.

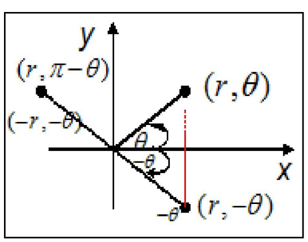



(b) Symmetry about the vertical line $\theta = \frac{\pi}{2}$ (y-axis)

The graph of $r = f(\theta)$ is symmetric with respect to the vertical line if substitution of
(i) $\pi - \theta$ for $\theta$ OR

(ii) $-r$ for $r$ and $-\theta$ for $\theta$
does not change the equation $r = f(\theta)$.

**Example 2:** Consider the graph of $r = 4\sin\theta$.
Since $\sin(\pi-\theta) = \sin\theta$ and also, $-r\sin(-\theta) = r\sin\theta$, then the graph is symmetric about the vertical line $\theta = \frac{\pi}{2}$.

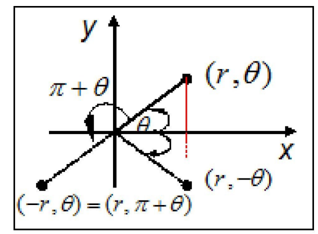



(c) Symmetry about the pole $\theta = 0$ (origin in xy-plane)

The graph of $r = f(\theta)$ is symmetric with respect to the pole if substitution of
(i) $-r$ for $r$ OR

(ii) $\pi + \theta$ for $\theta$
does not change the equation $r = f(\theta)$.

**Example 3:** Consider the graph of $r^2 = a^2\sin 2\theta$.
Note: $(-r)^2 = a^2\sin 2\theta = r^2 = a^2\sin 2\theta$. Also,
$r^2 = a^2\sin[2(\pi+\theta)] = a^2\sin(2\pi + 2\theta) = a^2\sin 2\theta$.
This means the graph is symmetric about the pole.




### 1 Lines in Polar Coordinates

(i) General equation of a straight line $ax + bx = c$, its polar equation is

$$r = \frac{c}{a\cos\theta + b\sin\theta}$$

(ii) Equation of a vertical line $x = k$, its polar equation is

$$r = k\sec\theta$$

**HOW?** $r = k\sec\theta \Rightarrow r = \frac{k}{\cos\theta} \Rightarrow r\cos\theta = k \Rightarrow x = k$.
(iii) Equation of a horizontal line $y = k$, its polar equation is

$$r = k\csc\theta$$

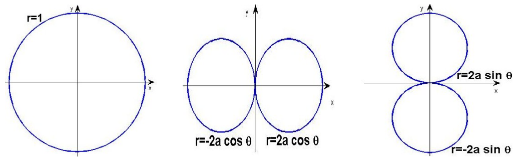
**HOW?** $r = k\csc\theta \Rightarrow r = \frac{k}{\sin\theta} \Rightarrow r\sin\theta = k \Rightarrow y = k$.
(iv) Equation of a line that passes the origin point and makes an angle $\theta_0$:

$$\theta = \theta_0$$

**Example:** Sketch the graph of the polar equation $\theta = \frac{\pi}{2}$.

## 2 Circles in Polar Coordinates

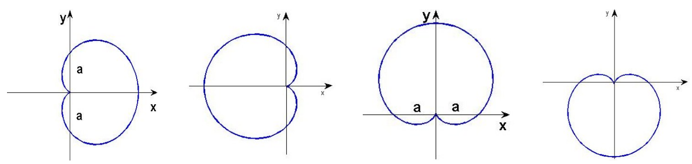
(i) A circle its center at $O$ and radius $a$: $r = a$
(ii) A circle its center at $(a, 0)$ and radius $|a|$: $r = 2a\cos\theta$
(iii) A circle its center at $(0, a)$ and radius $|a|$: $r = 2a\sin\theta$




## 3 Cardioid

$r = a(1 \pm \cos\theta)$ OR $r = a(1 \pm \sin\theta)$

$r = a(1+\cos\theta)$ $r = a(1-\cos\theta)$ $r = a(1+\sin\theta)$ $r = a(1-\sin\theta)$



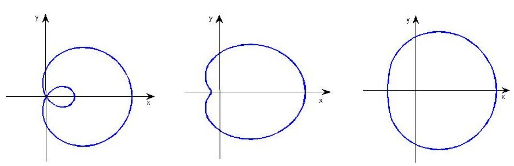
## 4 Limacons

$r = a \pm b\cos\theta$ OR $r = a \pm b\sin\theta$

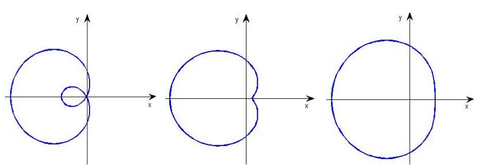
(1) $r = a \pm b\cos\theta$

(i) $r = a + b\cos\theta$

$\frac{a}{b} < 1$ $1 < \frac{a}{b} < 2$ $\frac{a}{b} \geq 2$

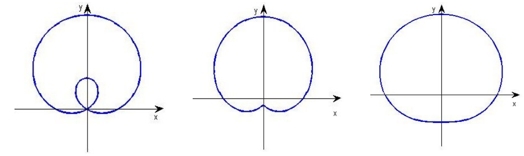


(ii) $r = a - b\cos\theta$

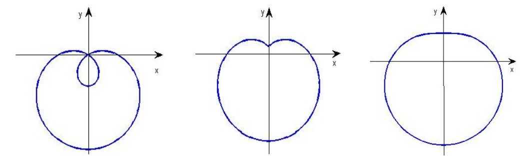


(2) $r = a \pm b\sin\theta$
(i) $r = a + b\sin\theta$

$\frac{a}{b} < 1$ $1 < \frac{a}{b} < 2$ $\frac{a}{b} \geq 2$



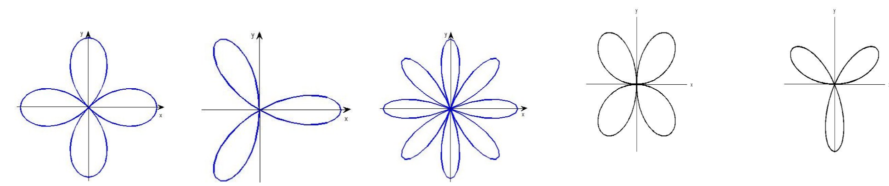
(ii) $r = a - b\sin\theta$



## 5 Roses

$r = a\cos n\theta$ OR $r = a\sin n\theta$ where $n \in \mathbb{N}$.

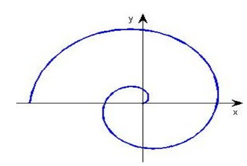
(i) $r = a\cos n\theta$: $n = 2$, $n = 3$, $n = 4$ (ii) $r = a\sin n\theta$: $n = 2$, $n = 3$



**Note:** If $n$ is odd, there are $n$ petals. If $n$ is even, there are $2n$ petals.

## 6 Spiral of Archimedes

$r = a\theta$ where $a > 0$.



**Exercise:** Sketch the following:

- $r = 3$.
- $r = 2\cos\theta$.
- $r = 6\sin\theta$.
- $r = 6 - 6\sin\theta$.

**Theorem:** (1) Let $f$ is continuous and non-negative on $[\alpha, \beta]$, where $0 \leq \alpha < \beta \leq 2\pi$. The area $A$ of the region bounded by the graphs of $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

(2) Let $f$ and $g$ are continuous and non-negative on $[\alpha, \beta]$, the area of the region $R = \{(r, \theta) : \alpha \leq \theta \leq \beta, g(\theta) \leq r \leq f(\theta)\}$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right] \, d\theta$$

**Exercise 1:** Find the area of the region bounded by the graph of the polar equation

(i) $r = 2\cos\theta$.

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(ii) $r = 6\sin\theta$.

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(iii) $r = 6 - 6\sin\theta$.

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| $\theta$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|---|---|---|
| $\sin\theta$ | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $0$ | $-1$ | $0$ |
| $\cos\theta$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ | $-1$ | $0$ | $1$ |
| $\tan\theta$ | $0$ | $\frac{1}{\sqrt{3}}$ | $1$ | $\sqrt{3}$ | $\infty$ | $0$ | $-\infty$ | $0$ |

**Exercise 2:** Find the area of the region that is inside the graphs of both the equations $r = \sin\theta$ , $r = \sqrt{3}\cos\theta$.

**Exercise 3:** Find the area of the region that is outside the graph of $r = 3$ and inside the graph of $r = 2 + 2\cos\theta$.

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**Exercise:** Find the area of the region that is inside both of graph of equations $r = 3 + 3\cos\theta$ and $r = 3 - 3\cos\theta$.

**(1) Arc Length of a Curve $C$ in Polar Coordinates:**
To find arc length of a curve $C$ in polar coordinates, we use

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$$

**Exercise 1:** Find the length of the curve
(i) $r = 1 + \cos\theta$

(iii) $r = 2 - 2\cos\theta$

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**(2) Surface of revolution in Polar Coordinates:**
Surface of revolution generated by revolving $C$ about
**(A) Polar axis:**

$$S = \int_{\alpha}^{\beta} 2\pi |y| \sqrt{(r)^2 + (\frac{dr}{d\theta})^2}\, d\theta$$

**(B) The line $\theta = \frac{\pi}{2}$:**

$$S = \int_{\alpha}^{\beta} 2\pi |x| \sqrt{(r)^2 + (\frac{dr}{d\theta})^2}\, d\theta$$

**Remember:**

$$x = r\cos\theta$$

$$y = r\sin\theta$$

**Exercise 2:** Find the area of the surface generated by revolving the curve $C$: $r = 2 + 2\cos\theta$ about the polar axis.

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**Exercise 3:** Find the area of the surface generated by revolving the curve $C$: $r = \sin^2\theta$ about the line $\theta = \frac{\pi}{2}$.

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**Exercise 4:** Find the area of the surface generated by revolving the curve $C$: $r = 2a\sin\theta$ about
(i) the line polar axis.
(ii) the line $\theta = \frac{\pi}{2}$.

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