1. Find the Fourier integral for the following functions:

a) $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$.

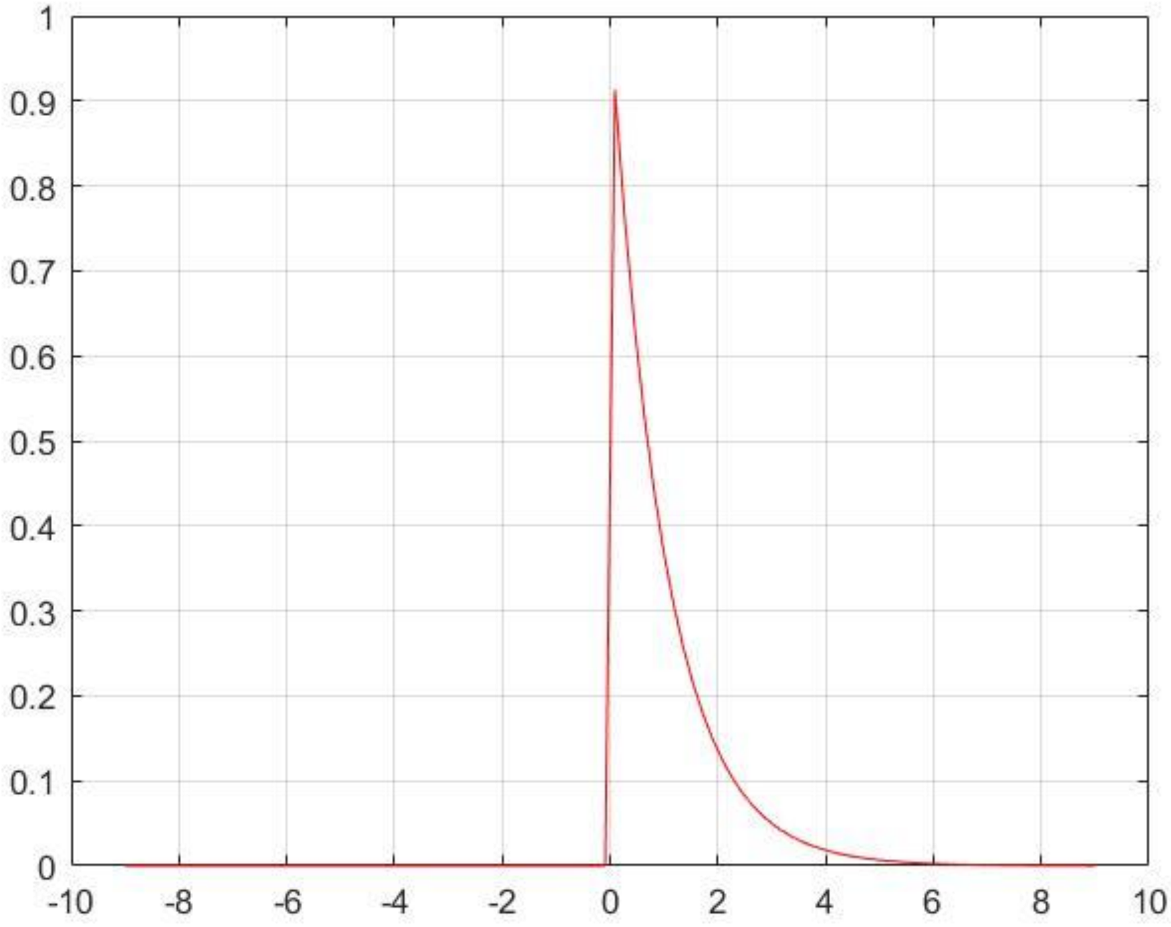
b) $g(x) = \begin{cases} 0, & -\infty < x < -2 \\ -2, & -2 < x < 0 \\ 2, & 0 < x < 2 \\ 0, & x > 2. \end{cases}$

c) $h(x) = e^{-|x|}\cos x,$

d) $k(x) = e^{-|x|}\sin x$

e) $M(x) = \begin{cases} 0, & -\infty < x < -1 \\ 2x, & -1 < x < 1 \\ 0, & 1 < x < \infty \end{cases}$

f) $N(x) = \begin{cases} 0, & x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$



DEFINITION 14.3.1 Fourier Integral

The **Fourier integral** of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha, \tag{4}$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx \tag{5}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx. \tag{6}$$

CONVERGENCE OF A FOURIER INTEGRAL Sufficient conditions under which a Fourier integral converges to $f(x)$ are similar to, but slightly more restrictive than, the conditions for a Fourier series.

THEOREM 14.3.1 Conditions for Convergence

Let f and f' be piecewise continuous on every finite interval and let f be absolutely integrable on $(-\infty, \infty)$.* Then the Fourier integral of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

1(a):

5. From formula (5) in the text,

$$A(\alpha) = \int_0^\infty e^{-x} \cos \alpha x \, dx.$$

Recall $\mathscr{L}\{\cos kt\} = s/(s^2 + k^2)$. If we set $s = 1$ and $k = \alpha$ we obtain

$$A(\alpha) = \frac{1}{1 + \alpha^2}.$$

Now

$$B(\alpha) = \int_0^\infty e^{-x} \sin \alpha x \, dx.$$

Recall $\mathscr{L}\{\sin kt\} = k/(s^2 + k^2)$. If we set $s = 1$ and $k = \alpha$ we obtain

$$B(\alpha) = \frac{\alpha}{1 + \alpha^2}.$$

Hence

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} \, d\alpha.$$

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b) Find the Fourier cosine of the functions:

i) $f(x) = e^{-x} \cos x$, $x > 0$, ii) $g(x) = xe^{-2x}$,

ii) $h(x) = \begin{cases} x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$.

The Laplace Transform

The Laplace Transform of a function, f(t), is defined as;

$$L[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

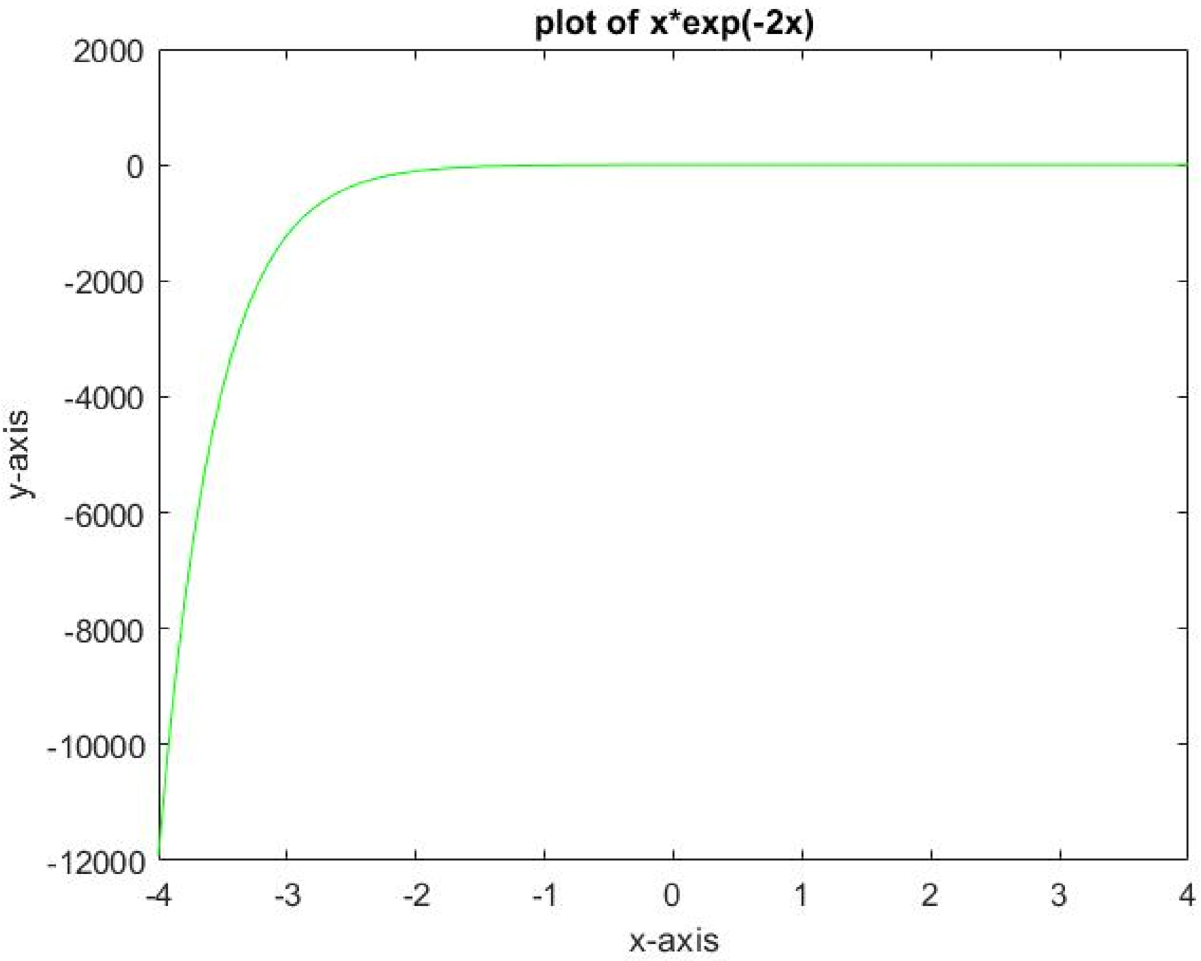
Eq A

The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{ts} ds$$

Eq B

*notes



DEFINITION 14.3.2 Fourier Cosine and Sine Integrals

(*i*) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the **cosine integral**

$$f(x) = \frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x\,d\alpha, \tag{8}$$

where

$$A(\alpha) = \int_0^\infty f(x)\cos\alpha x\,dx. \tag{9}$$

(*ii*) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the **sine integral**

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\sin\alpha x\,d\alpha, \tag{10}$$

where

$$B(\alpha) = \int_0^\infty f(x)\sin\alpha x\,dx. \tag{11}$$

b(ii):

5. For the cosine integral,

$$A(\alpha) = \int_0^\infty xe^{-2x} \cos \alpha x \, dx.$$

But we know

$$\mathscr{L}\{t \cos kt\} = -\frac{d}{ds} \frac{s}{(s^2 + k^2)} = \frac{(s^2 - k^2)}{(s^2 + k^2)^2}.$$

If we set $s = 2$ and $k = \alpha$ we obtain

$$A(\alpha) = \frac{4 - \alpha^2}{(4 + \alpha^2)^2}.$$

Hence

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{(4 - \alpha^2) \cos \alpha x}{(4 + \alpha^2)^2} \, d\alpha.$$

For the sine integral,

$$B(\alpha) = \int_0^\infty xe^{-2x} \sin \alpha x \, dx.$$

From Problem 12, we know

$$\mathscr{L}\{t \sin kt\} = \frac{2ks}{(s^2 + k^2)^2}.$$

If we set $s = 2$ and $k = \alpha$ we obtain

$$B(\alpha) = \frac{4\alpha}{(4 + \alpha^2)^2}.$$

Hence

$$f(x) = \frac{8}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{(4 + \alpha^2)^2} \, d\alpha.$$

4. Use Fourier integral to show that

$$\int_0^\infty \frac{\sin \pi\lambda}{1-\lambda^2}\sin x\lambda \, d\lambda = \begin{cases} \frac{\pi}{2}\sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases} \qquad \text{(a)}$$

$$\int_0^\infty \frac{\cos \lambda x}{1+\lambda^2} d\lambda = \frac{\pi}{2}e^{-x}, \quad x \ge 0. \qquad \text{(b)}$$

$$\int_0^\infty \frac{\lambda \sin \lambda x}{\beta^2+\lambda^2} d\lambda = \frac{\pi}{2}e^{-\beta x}, \quad x > 0,\ \beta > 0. \qquad \text{(c)}$$

هذا الهامش

A4 $\int_0^\infty \frac{\lambda \sin(\lambda x)}{\beta^2+\lambda^2} d\lambda = \frac{\pi}{2} e^{-\beta x}, \quad x>0, \ \beta>0$

Define $f(x) = \begin{cases} e^{-\beta x}, & x>0 \\ -e^{-\beta x}, & x<0 \end{cases}$

f is an odd $\Rightarrow A(\lambda) = \int_{-\infty}^{\infty} \underbrace{f(x)\cos(\lambda x)}_{\text{odd function}} dx = 0$

$B(\lambda) = 2\int_0^\infty e^{-\beta x} \sin(\lambda x)\, dx = \frac{2\lambda}{\beta^2+\lambda^2}$

$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)\, d\lambda$

$e^{-x} = \frac{1}{\pi}\int_0^\infty \frac{2\lambda}{\beta^2+\lambda^2}\sin(\lambda x)\, d\lambda$

$\frac{\pi}{2} e^{-\beta x} = \int_0^\infty \frac{\lambda \sin(\lambda x)}{\beta^2+\lambda^2}\, d\lambda$

3. a) Using the Fourier sine integral to show that

$$\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin x\lambda \, d\lambda = \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

A3

ممكن نضيف فرع آخر للسؤال

$$= \begin{cases} \frac{\pi}{2}, & 0 < x < \pi \\ \frac{\pi}{4}, & x = \pi \\ 0, & x > \pi \end{cases}$$

نمدد الدالة $f(x) = \begin{cases} 1, & 0<x<\pi \\ 0; & x>\pi \end{cases}$ كدالة فردية على R

وبالتالي $A(\lambda) = 0$ ونوجد $B(\lambda)$:

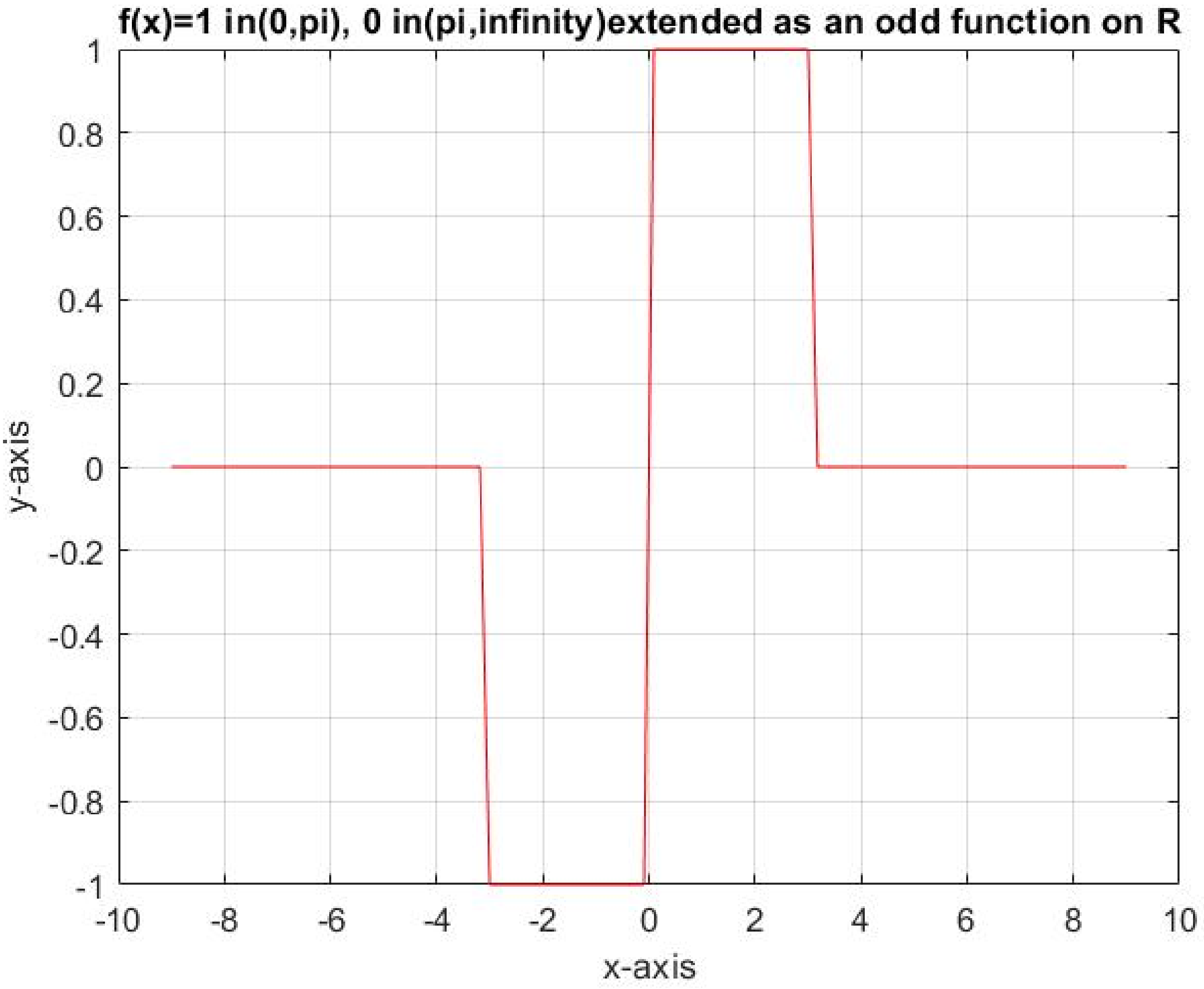
$$B(\lambda) = 2\int_0^\infty f(x)\sin(\lambda x)\,dx = 2\int_0^\infty 1 \cdot \sin(\lambda x)\,dx = -\frac{1}{\lambda}\cos(\lambda x)\Big]_0^\pi$$

$$= \frac{2}{\lambda}\left[1 - \cos(\lambda\pi)\right]$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)\,d\lambda$$

$$\begin{cases} 1; & 0<x<\pi \\ 0; & x>\pi \end{cases} = \frac{1}{\pi}\int_0^\infty \frac{2}{\lambda}\left[1-\cos(\lambda\pi)\right]\sin(\lambda x)\,d\lambda$$

$$\begin{cases} \frac{\pi}{2}; & 0<x<\pi \\ 0; & x>\pi \end{cases} = \int_0^\infty \frac{1-\cos(\lambda\pi)}{\lambda}\sin(\lambda x)\,d\lambda$$


f(x)=1 in(0,pi), 0 in(pi,infinity)extended as an odd function on R
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
y-axis
-10
-8
-6
-4
-2
0
2
4
6
8
10
x-axis