M 106

Integral Calculus and Applications

Contents

Chapter 1

The Indefinite Integrals

1.1 Anti-derivatives and Definition of Indefinite Integrals

1.1.1 Anti-derivatives

Definition 1.1.1 *A function F is called an anti-derivative of f on an interval I if*

$$F^{'}(x) = f(x) \text{ for every } x \in I\ .$$

Example 1.1.1 1. *Let $F(x) = x^2+3x+1$ and $f(x) = 2x+3$.*
Since $F^{'}(x) = f(x)$, the function $F(x)$ is an anti-derivative of $f(x)$.

2. *Let $G(x) = \sin(x)+x$ and $g(x) = \cos(x)+1$.*
We know that $G^{'}(x) = \cos(x)+1$ and this means the function $G(x)$ is an anti-derivative of $g(x)$.

Generally, if $F(x)$ is an anti-derivative of $f(x)$, then every function $F(x)+c$ is also anti-derivative of $f(x)$, where c is a constant.

Theorem 1.1.1 *If the functions $F(x)$ and $G(x)$ are anti-derivatives of a function $f(x)$ on the interval I, there exists a constant c such that $G(x) = F(x)+c$.*

The last theorem means that any anti-derivative $G(x)$, which is different from the function $F(x)$ can be expressed as $F(x)+c$ where c is an arbitrary constant. The following examples clarify this point.

Example 1.1.2 *Let $f(x) = 2x$. The functions*

$F(x) = x^2+2,$

$G(x) = x^2-\frac{1}{2},$

$H(x) = x^2-\sqrt[3]{2},$

and many other functions are anti-derivatives of a function $f(x)$. Generally, for the function $f(x) = 2x$, the function $F(x) = x^2+c$ is the anti-derivative where c is an arbitrary constant.

Example 1.1.3 *Find the general form of the anti-derivative of $f(x) = 6x^5$.*

Solution:

Let $F(x) = x^6$, then $F'(x) = 6x^5$. Thus, $F(x) = x^6+c$ is the general anti-derivative of $f(x)$.

1.1.2 Indefinite Integrals

Definition 1.1.2 *Let f be a continuous function on an interval I. The indefinite integral of $f(x)$ is the general anti-derivative of $f(x)$ on I and symbolized by $\int f(x)\,dx$.*

Remark 1.1.1 *If $F(x)$ is an anti-derivative of f, then*

$$\int f(x)\,dx = F(x) + c\ .$$

The function $f(x)$ is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

Now, by using the previous remark, the general anti-derivatives in Example 1.1.1 are

1. $\int 2x+3\,dx = x^2+3x+c$.
2. $\int \cos(x)+1\,dx = \sin(x)+x+c$.

The list of the basic indefinite integrals:

Derivative	Indefinite Integrals
$\frac{d}{dx}(x) = 1$	$\int 1\,dx = x+c$
$\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = 1,\, n \neq -1$	$\int x^n\,dx = \frac{x^{n+1}}{n+1}+c$
$\frac{d}{dx}(\sin\,x) = \cos\,x$	$\int \cos\,dx = \sin\,x+c$
$\frac{d}{dx}(-\cos\,x) = \sin\,x$	$\int \sin\,x\,dx = -\cos\,x+c$
$\frac{d}{dx}(\tan\,x) = \sec^2\,x$	$\int \sec^2\,x\,dx = \tan\,x+c$
$\frac{d}{dx}(-\cot\,x) = \csc^2\,x$	$\int \csc^2\,x\,dx = -\cot\,x+c$
$\frac{d}{dx}(\sec\,x) = \sec\,x\,\tan\,x$	$\int \sec\,x\,\tan\,x\,dx = \sec\,x+c$
$\frac{d}{dx}(-\csc\,x) = \csc\,x\,\cot\,x$	$\int \csc\,x\,\cot\,x\,dx = -\csc\,x+c$

Table 1.1: The list of the basic integration rule.

Example 1.1.4 *Evaluate the following integrals:*

1. $\int x^{-3}\,dx$

2. $\int \frac{1}{\cos^2 x}\,dx$

Solution:

1. $\int x^{-3}\,dx = \frac{x^{-2}}{-2}+c = -\frac{1}{2x^2}+c\ .$
2. $\int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx = \tan x+c\ .$

Remember: $sec\ x = \frac{1}{\cos x}$

Exercise 1:

1 - 8 ■ Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{x}}\,dx$
2. $\int \frac{1}{x^{\frac{5}{4}}}\,dx$
3. $\int \frac{1}{\sin^2 x}\,dx$
4. $\int -csc^2 x\,\tan^2 x\,dx$
5. $\int \frac{1}{\sqrt[5]{x}}\,dx$
6. $\int \frac{\tan x}{\cos x}\,dx$
7. $\int \frac{\sqrt{x}}{x^3}\,dx$
8. $\int \sqrt{\sin^4 x}\,cscx\,dx$

1.2 Properties of Indefinite Integrals

Theorem 1.2.1 *Let f and g be integrable functions, then*

1. $\frac{d}{dx}\int f(x)\,dx = f(x)$.
2. $\int \frac{d}{dx}(F(x))\,dx = F(x)+c$.
3. $\int \big(f(x)\pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.
4. $\int kf(x)\,dx = k\int f(x)\,dx$*, where k is a constant*

In the following example, we use the previous properties and the table of the basic integrals to evaluate some indefinite integrals.

Example 1.2.1 *Evaluate the following integrals:*

1. $\int (4x+3)\,dx$
2. $\int (2\sin\,x+3\cos\,x)\,dx$
3. $\int (\sqrt{x}+\sec^2\,x)\,dx$
4. $\int \frac{d}{dx}(\sin\,x)\,dx$
5. $\frac{d}{dx}\int \sqrt{x+1}\,dx$

Solution:

1. $\int (4x+3)\,dx = \frac{4x^2}{2}+3x+c = 2x^2+3x+c$.
2. $\int (2\sin\,x+3\cos\,x)\,dx = -2\cos\,x+3\sin\,x+c$.
3. $\int (\sqrt{x}+\sec^2\,x)\,dx = \frac{x^{\frac{3}{2}}}{3/2}+\tan\,x+c = \frac{2x^{\frac{3}{2}}}{3}+\tan\,x+c$.
4. $\int \frac{d}{dx}(\sin\,x)\,dx = \sin\,x+c$.
5. $\frac{d}{dx}\int \sqrt{x+1}\,dx = \sqrt{x+1}$.

Example 1.2.2 *If $\int f(x)\,dx = x^2+c$ and $\int g(x)\,dx = \tan x+c$, then find $\int \big(3f(x)-2g(x)\big)\,dx$.*

Solution:

From the third and fourth property, $\int \big(3f(x)-2g(x)\big)\,dx = 3\int f(x)\,dx - 2\int g(x)\,dx = 3x^2-2\tan x+c$.

Example 1.2.3 *Solve the differential equation $f'(x)=x^3$ subject to the initial condition $f(0)=1$.*

Solution:

$$\int f'(x)\,dx = \int x^3\,dx$$
$$f(x) = \frac{1}{4}x^4 + c\,.$$

If $x = 0$, $f(0) = \frac{1}{4}(0)^4 + c = 1$ and this implies $c = 1$. Thus, the solution of the differential equation is $f(x) = \frac{1}{4}x^4 + 1$.

Example 1.2.4 *Solve the differential equation $f'(x) = 6x^2 + x - 5$ subject to the initial condition $f(0) = 2$.*

Solution:

$$\int f'(x)\,dx = \int (6x^2 + x - 5)\,dx$$
$$f(x) = 2x^3 + \frac{1}{2}x^2 - 5x + c\,.$$

Use the condition $f(0) = 2$ i.e., substitute $x = 0$ into the function $f(x)$. We have $f(0) = 0 + 0 - 0 + c = 2 \Rightarrow c = 2$. Hence, the solution of the differential equation is $f(x) = 2x^3 + \frac{1}{2}x^2 - 5x + 2$.

Example 1.2.5 *Solve the differential equation $f''(x) = 5\cos\,x + 2\sin\,x$ subject to the initial condition $f(0) = 3$ and $f'(0) = 4$.*

Solution:

$$\int f''(x)\,dx = \int (5\cos\,x + 2\sin\,x)\,dx$$
$$f'(x) = 5\sin\,x - 2\cos\,x + c$$

The condition $f'(0) = 4$ yields $f'(0) = 0 - 2 + c = 4 \Rightarrow c = 6$. Thus, $f'(x) = 5\sin\,x - 2\cos\,x + 6$. Now, again

$$\int f'(x)\,dx = \int (5\sin\,x - 2\cos\,x + 6)\,dx$$
$$f(x) = -5\cos\,x - 2\sin\,x + 6x + c\,.$$

Use the condition $f(0) = 3$ by substituting $x = 0$ into $f(x)$. This yields $f(0) = -5 - 0 + 0 + c = 3 \Rightarrow c = 8$. Thus, the solution of the differential equation is $f(x) = -5\cos\,x - 2\sin\,x + 6x + 8$.

Note that, in the previous examples, we use x as the variable of the integration. However, for this role, we can use any variable y, z, t, ... That is, instead of $f(x)\,dx$, we can integrate $f(y)\,dy$, $f(t)\,dt$.

Exercise 2:

1 - 10 ■ Evaluate the following integrals:

1. $\int \sqrt{x^5}\,dx$
2. $\int (x^{\frac{3}{4}} + x^2 + 1)\,dx$
3. $\int x(x^3 + 2x + 1)\,dx$
4. $\int x^2 + \sec^2 x\,dx$
5. $\int \frac{3\sin^2 x + 4}{\sin^2 x}\,dx$
6. $\int \frac{x^2 - 1}{x^4}\,dx$
7. $\int 4x^{\frac{2}{5}} - 2x^{\frac{2}{3}} + x\,dx$
8. $\int \frac{3}{x^3} + \frac{2}{x^4} + 1\,dx$
9. $\int \csc^2 x - \sqrt{x}\,dx$
10. $\int \frac{x^2 + x + 1}{\sqrt[3]{x}}\,dx$

11 - 12 ■ Evaluate the following:

11. $\frac{d}{dx}(\int \sqrt{\cos^3 x + 1}\,dx)$
12. $\int \frac{d}{dx}(\sqrt{\cos^3 x + 1})\,dx$

13 - 17 ■ Solve the differential equation

13. $f'(x) = 4x^3 + 2x + 1$ subject to the initial condition $f(0) = 1$.
14. $f''(x) = \sin x + 2\cos x$ subject to the initial conditions $f(0) = 1$ and $f'(0) = 3$.
15. $f'(x) = \sqrt{x}$ subject to the initial condition $f(0) = 0$.
16. $f'(x) = \cos x$ subject to the initial condition $f(\pi) = 1$.
17. $f'(x) = \sec^2 x$ subject to the initial condition $f(\frac{\pi}{4}) = 0$.

1.3 Integration By Substitution

The integration by substitution (known as u-substitution) is one technique for solving some complex integrals. The goal of changing the variable of the integration is to obtain a simple indefinite integral. In a sense that the substitution method turns the integral into a simpler integral involving the variable u that can be solved by using either the table of the basic integrals or other techniques of integration. The following definition shows how the substitution technique works.

Theorem 1.3.1 *Let g be a differentiable function on the interval I where the derivative is continuous. Let f be a continuous on an interval I involves the range of the function g. If F is an anti-derivative of the function f on I, then*

$$\int f(g(x))g^{'}(x)\, dx = F(g(x)) + c, \ \ x \in I \ .$$

Steps of integration by substitution:

Step 1: Choose a new variable u.

Step 2: Determine the value of du.

Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral is in terms of u.

Step 4: Evaluate the new integral.

Step 5: Return the evaluation to the initial variable x.

Example 1.3.1 *Evaluate the integral* $\int 2x(x^2+1)^3\, dx$.

Solution:

One can use the previous theorem as follows:

let $f(x) = x^3$ and $g(x) = x^2+1$. Since $g^{'}(x) = 2x$, then from Theorem 1.3.1, we have

$$\int 2x(x^2+1)^3\, dx = \frac{(x^2+1)^4}{4} + c \ .$$

We can end with the same solution by using the five steps of the substitution method.

Let $u = x^2+1$, then $du = 2x\, dx$. By substituting that into the original integral, we have $\int u^3\, du = \frac{u^4}{4} + c$. Now, by returning the evaluation to the initial variable x, we have $\int 2x(x^2+1)^3\, dx = \frac{(x^2+1)^4}{4} + c$.

Example 1.3.2 *Evaluate the integral* $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\, dx$.

Solution:

Let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\, dx$. By substitution, we have $2\int \sec^2(u)\, du = 2\tan(u) + c = 2\tan(\sqrt{x}) + c$.

Example 1.3.3 *Evaluate the integral* $\int \frac{x^2-1}{(x^3-3x+1)^6}\, dx$.

Solution:

let $u = x^3-3x+1$, then $du = 3(x^2-1)\, dx$. By substitution, we have

$$\frac{1}{3}\int u^{-6}\, du = \frac{1}{3}\ \frac{1}{-5u^5} + c = \frac{-1}{15(x^3-3x+1)^5} + c \ .$$

Corollary 1.3.1 *If* $\int f(x)\,dx = F(x)+c$, *then for any* $a \neq 0$,

$$\int f(ax \pm b)\,dx = \frac{1}{a}F(ax \pm b) + c\ .$$

Example 1.3.4 *Evaluate the following integrals:*

1. $\int \sqrt{2x-5}\,dx$

2. $\int \cos(3x+4)\,dx$

Solution:

From Corollary 1.3.1, we have

1. $\int \sqrt{2x-5}\,dx = \frac{1}{2}\frac{(2x-5)^{3/2}}{3/2} + c = \frac{(2x-5)^{3/2}}{3} + c\ .$
2. $\int \cos(3x+4)\,dx = \frac{1}{3}\sin(3x+4) + c\ .$

Exercise 3:

1 - 16 ■ Evaluate the following integrals:

1. $\int x\sqrt{1+x^2}\,dx$
2. $\int x\sqrt{x-1}\,dx$
3. $\int x^2\sqrt{x-1}\,dx$
4. $\int \frac{\tan x}{\cos^2 x}\,dx$
5. $\int \sin^5 x\,\cos x\,dx$
6. $\int \frac{x}{\sqrt{2x^2+1}}\,dx$
7. $\int \cos\,t\sqrt{1-\sin\,t}\,dt$
8. $\int \frac{\cos^3 x}{\csc x}\,dx$
9. $\int \cos(3x+4)\,dx$
10. $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx$
11. $\int \sec 4x\,\tan 4x\,dx$
12. $\int \frac{\sqrt{\cot x}}{\sin^2 x}\,dx$
13. $\int (1+\frac{1}{t})t^{-2}\,dt$
14. $\int \frac{x}{\sqrt{2x-1}}\,dx$
15. $\int x^2(4x^3-6)^7\,dx$
16. $\int \sin^2(3x)\cos(3x)\,dx$

Chapter 2

The Definite Integrals

2.1 Summation Notation

Summation is the addition of a sequence of numbers and the result is their sum or total.

Definition 2.1.1 *Let* $\{a_1, a_2, \ldots, a_n\}$ *be a set of numbers. The symbol* $\sum_{k=1}^{n} a_k$ *represents their sum:*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \ .$$

Example 2.1.1 *Evaluate the following sums:*

1. $\sum_{i=0}^{3}(i^3)$.

2. $\sum_{j=1}^{4}(j^2+1)$.

3. $\sum_{k=1}^{3}(k+1)k^2$.

Solution:

1. $\sum_{i=0}^{3}(i^3) = 0^3 + 1^3 + 2^3 + 3^3 = 0 + 1 + 8 + 27 = 36$.
2. $\sum_{j=1}^{4}(j^2+1) = (1^2+1) + (2^2+1) + (3^2+1) + (4^2+1) = 2 + 5 + 10 + 17 = 50$.
3. $\sum_{k=1}^{3}(k+1)k^2 = (2)(1)^2 + (3)(2)^2 + (4)(3)^2 = 2 + 12 + 36 = 50$.

Properties of Sum Notation:

1. $\sum_{k=1}^{n} c = \underbrace{c + c + \ldots + c}_{\text{n-times}} = nc$ for any $c \in \mathbb{R}$.
2. $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$.
3. $\sum_{k=1}^{n} c\ a_k = c\ \sum_{k=1}^{n} a_k$ for any $c \in \mathbb{R}$.

Example 2.1.2 *Evaluate the following sums:*

1. $\sum_{k=1}^{10} 15$.

2. $\sum_{k=1}^{4} k^2 + 2k$.

3. $\sum_{k=1}^{3} 3(k+1)$.

Solution:

1. $\sum_{k=1}^{10} 15 = (10)(15) = 150$.

2. $\sum_{k=1}^{4} k^2 + k = \sum_{k=1}^{4} k^2 + \sum_{k=1}^{4} k = (1^2+2^2+3^2+4^2) + (1+2+3+4) = 30+10 = 40$.
3. $\sum_{k=1}^{3} 3(k+1) = 3\sum_{k=1}^{3}(k+1) = 3(2+3+4) = 27$.

Theorem 2.1.1

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. *2.* $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. *3.* $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Example 2.1.3 *Evaluate the following sums:*

1. $\sum_{k=1}^{100} k$. *2.* $\sum_{k=1}^{10} k^2$. *3.* $\sum_{k=1}^{10} k^3$.

Solution:

1. $\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 5050$.
2. $\sum_{k=1}^{10} k^2 = \frac{10(11)(21)}{6} = 385$.
3. $\sum_{k=1}^{10} k^3 = \left[\frac{10(11)}{2}\right]^2 = 3025$.

Example 2.1.4 *Express the following sums in terms of n:*

1. $\sum_{k=1}^{n}(k+1)$. *2.* $\sum_{k=1}^{n}(k^2-k+1)$.

Solution:

1. $\sum_{k=1}^{n}(k+1) = \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$.
2. $\sum_{k=1}^{n}(k^2-k-1) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} - n = \frac{n(n^2-1)-3}{3}$.

Exercise 1:

1 - 6 ■ Evaluate the following sums:

1. $\sum_{i=1}^{3}(i+1)$
2. $\sum_{j=0}^{5} j^2$
3. $\sum_{k=1}^{4} \frac{k}{k+1}$
4. $\sum_{i=1}^{10} 5i$
5. $\sum_{k=1}^{30} 4$
6. $\sum_{j=1}^{3}(3-2j)^2$

7 - 9 ■ Express the following sums in terms of n:

7. $\sum_{k=1}^{n}(k-1)$
8. $\sum_{k=1}^{n}(k^2+1)$
9. $\sum_{k=1}^{n}(k^3+2k^2-k+1)$

2.2 Riemann Sum and Area

A Riemann sum is a mathematical form and one of its applications is approximating the area underneath a curve of a function. Before start-up in this issue, we would like to provide some basic definitions that we need in the Riemann sum.

Definition 2.2.1 *A set* $P = \{x_0, x_1, x_2, ..., x_n\}$ *is called a partition of a closed interval* $[a,b]$ *if for any positive integer n,*

$$a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b \ .$$

Note that,

1. the division of the interval $[a,b]$ by the partition P generates n sub-intervals: $[x_0,x_1],[x_1,x_2],[x_2,x_3],...,[x_{n-1},x_n]$.
2. The length of each sub-interval $[x_{k-1},x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
3. The union of sub-intervals gives the main interval $[a,b]$.

Definition 2.2.2 *The norm of the partition of P is the largest length among* $\Delta x_0, \Delta x_1, \Delta x_2, ..., \Delta x_n$ *i.e.,*

$$|| P || = max\{\Delta x_0, \Delta x_1, \Delta x_2, ..., \Delta x_n\} \ .$$

Example 2.2.1 *If $P = \{0, 1.2, 2.3, 3.6, 4\}$ is a partition of the interval $[0,4]$, find the norm of the partition P.*

Solution:

We need to find the sub-intervals and their lengths.

Sub-interval $[x_{k-1}, x_k]$	Length Δx_k
$[0, 1.2]$	$1.2 - 0 = 1.2$
$[1.2, 2.3]$	$2.3 - 1.2 = 1.1$
$[2.3, 3.6]$	$3.6 - 2.3 = 1.3$
$[3.6, 4]$	$4 - 3.6 = 0.4$

From the table, the norm is $\| P \| = 1.3$.

Remark 2.2.1

1. *The partition P of the interval $[a,b]$ is regular if $\Delta x_0 = \Delta x_1 = \Delta x_2 = \ldots = \Delta x_n = \Delta x$.*
2. *For any positive integer n, if the partition P is regular then*

$$\Delta x = \frac{b-a}{n} \text{ and } x_k = x_0 + k\,\Delta x .$$

To explain the previous result, let P be a regular partition for the interval $[a,b]$. We know that $x_0 = a$ and $x_n = b$. Then,

$$x_1 = x_0 + \Delta x,$$
$$x_2 = x_1 + \Delta x = (x_0 + \Delta x) + \Delta x = x_0 + 2\Delta x,$$
$$x_3 = x_2 + \Delta x = (x_0 + 2\Delta x) + \Delta x = x_0 + 3\Delta x .$$

By continuing doing so, we have $x_k = x_0 + k\,\Delta x$.

Example 2.2.2 *Define a regular partition P that divides the interval $[1,4]$ into 4 sub-intervals.*

Solution:

Since P is a regular partition of $[1,4]$ where $n = 4$, then $\Delta x = \frac{4-1}{4} = \frac{3}{4}$ and $x_k = 1 + k\,\frac{3}{4}$.

Thus,

$x_0 = 1$
$x_1 = 1 + \frac{3}{4} = \frac{7}{4}$
$x_2 = 1 + 2(\frac{3}{4}) = \frac{5}{2}$
$x_3 = 1 + 3(\frac{3}{4}) = \frac{13}{4}$
$x_4 = 1 + 4(\frac{3}{4}) = 4$

The regular partition is $P = \{1, \frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 4\}$.

Now, we are ready to define the Riemann sum that will be used to evaluate the definite integrals.

Definition 2.2.3 *Let f be a defined and bounded function on the closed bounded interval $[a,b]$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Let $\omega_k \in [x_{k-1}, x_k]$, $k = 1, 2, 3, \ldots, n$ where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a mark on the partition P. Then, the Riemann sum of f for P is*

$$R_p = \sum_{k=1}^{n} f(\omega_k)\Delta x_k .$$

Consider Figure 2.1, we want to explain the definition of the Riemann sum of a function f for the partition P. As shown in the figure, the amount $f(\omega_1)\Delta x_1$ is the area of the rectangle A_1, $f(\omega_2)\Delta x_2$ is the area of the rectangle A_2 and so on. The sum of these areas approximates the whole area under the graph of the function f. This indicates that, the area under f bounded by $x = a$ and $x = b$ can be estimated by the Riemann sum where as the number of the sub-intervals increases (i.e., $n \to \infty$), the estimation becomes better. Note that, when $n \to \infty$, the norm $\| P \| \to 0$. From this,

$$A = \lim_{n\to\infty} R_p = \lim_{n\to\infty} \sum_{k=1}^{n} f(\omega_k)\Delta x_k \,.$$

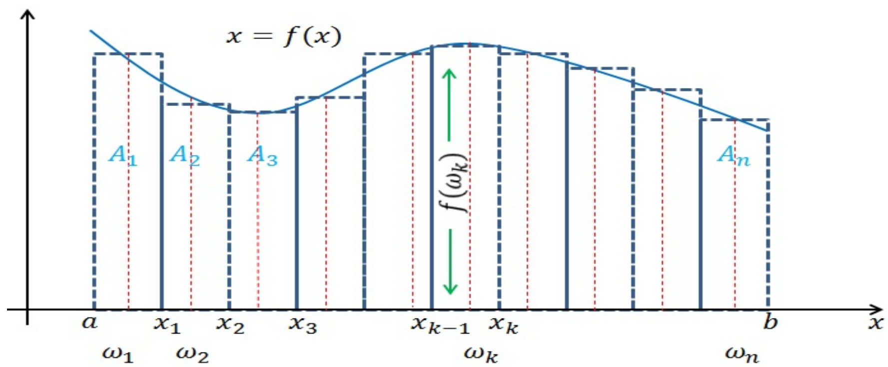


Figure 2.1: The Riemann sum of the function $f(x)$ for the partition P.

Example 2.2.3 *Find the Riemann sum R_p of the function $f(x) = 2x-1$ for the partition $P = \{-2,0,1,4,6\}$ of the interval $[a,b]$ by choosing the mark as follows:*

1. *the left-hand end point,*
2. *the right-hand end point,*
3. *the midpoint.*

Solution:

1. The left-hand end point.

Sub-intervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2,0]$	$0-(-2)=2$	-2	-5	-10
$[0,1]$	$1-0=1$	0	-1	-1
$[1,4]$	$4-1=3$	1	1	3
$[4,6]$	$6-4=2$	4	7	14
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				6

2. The right-hand end point.

Sub-intervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2,0]$	$0-(-2)=2$	0	-1	-2
$[0,1]$	$1-0=1$	1	1	1
$[1,4]$	$4-1=3$	4	7	21
$[4,6]$	$6-4=2$	6	11	22
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				42

3. The midpoint.

Sub-intervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\ \Delta x_k$
$[-2,0]$	$0-(-2)=2$	-1	-3	-6
$[0,1]$	$1-0=1$	0.5	0	0
$[1,4]$	$4-1=3$	2.5	4	12
$[4,6]$	$6-4=2$	5	9	18
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				24

Example 2.2.4 *Let A be the area under the graph of $f(x) = x+1$ from $x = 1$ to $x = 3$. Find the area A by taking the limit of the Riemann sum such that the partition P is regular and the mark ω is the right end point of each sub-interval.*

Solution:

For a regular partition P, we have

1. $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$, and
2. $x_k = x_0 + k\ \Delta x$ where $x_0 = 1$.

Since the mark ω is the right end point of the sub-interval $[x_{k-1}, x_k]$, then $\omega_k = x_k = 1 + \frac{2k}{n}$. Hence,

$$f(\omega_k) = (1 + \frac{2k}{n}) + 1 = \frac{2k}{n} + 2 = \frac{2}{n}(n+k)\ .$$

Now,

$$\begin{aligned} R_p = \sum_{k=1}^{n} f(w_k)\Delta x_k &= \frac{4}{n^2}\sum_{k=1}^{n}(n+k) \\ &= \frac{4}{n^2}\left[n^2 + \frac{n(n+1)}{2}\right] \\ &= 4 + \frac{2(n+1)}{n}\ . \end{aligned}$$

Remember:
(1) $\sum_{k=1}^{n}(n+k) = \sum_{k=1}^{n} n + \sum_{k=1}^{n} k$
(2) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Therefore, $\lim_{n\to\infty} R_p = 4 + 2 = 6$.

Exercise 2:

1 - 8 ■ If P is a partition of the interval $[a,b]$, find the norm of the partition P:

1. $P = \{-1, 0, 1.3, 4, 4.1, 5\}, \quad [-1, 5]$
2. $P = \{0, 0.5, 1, 2.5, 3.1, 4\}, \quad [0, 4]$
3. $P = \{-3, 0, 2.3, 4.6, 4.8, 5.5, 6\}, \quad [-3, 6]$
4. $P = \{-2, 0, 2.3, 3, 3.5, 4\}, \quad [-2, 4]$
5. $P = \{3, 3.5, 3.6, 4, 4.9, 7\}, \quad [3, 7]$
6. $P = \{-2, 0, 1.3, 2, 2.5, 3.4, 5.5\}, \quad [-2, 5.5]$
7. $P = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}, \quad [-1, 2]$
8. $P = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}, \quad [0, \pi]$

9 - 12 ■ Define a regular partition P that divides the interval $[a,b]$ into n sub-intervals:

9. $[a,b] = [0,3] \quad n = 5$
10. $[a,b] = [-1,4] \quad n = 6$
11. $[a,b] = [-4,4] \quad n = 8$
12. $[a,b] = [0,1] \quad n = 4$

13 - 15 ■ Find the Riemann sum R_p of the function $f(x) = x^2 + 1$ for the partition $P = \{0, 1, 3, 4\}$ of the interval $[a,b]$ by choosing the mark as follows:

13. the left-hand end point,
14. the right-hand end point,
15. the midpoint.

16 - 19 ■ Let A be the area under the graph of $f(x)$ from a to b. Find the area A by taking the limit of the Riemann sum such that the partition P is regular and the mark ω is the right end point of each sub-intervals:

16. $f(x) = x/3 \quad a = 1,\ b = 2$
17. $f(x) = x - 1 \quad a = 0,\ b = 3$
18. $f(x) = 5 - x^2 \quad a = -1,\ b = 1$
19. $f(x) = x^3 - 1 \quad a = 0,\ b = 4$

2.3 Definite Integrals

Definition 2.3.1 *Let f be a defined and bounded function on a closed bounded interval $[a,b]$ and let P be a partition of $[a,b]$. If f is integrable on that interval, the definite integral of f is*

$$\int_a^b f(x)\ dx = \lim_{\|P\|\to 0}\sum_k f(\omega_k)\Delta x_k = A\ .$$

if the limit exists. The numbers a and b are called the limits of the integration.

Example 2.3.1 *Evaluate the following integral* $\int_2^4 x+2\,dx$.

Solution:

We try to solve the example by using the previous definition.

The function $f(x) = x+2$ is integrable since it is continuous. Let $P = \{x_0, x_1, ..., x_n\}$ be a regular partition of $[2,4]$ and $\omega_k \in [x_{k-1}, x_k]$.

Since P is regular, $\Delta = \frac{4-2}{n} = \frac{2}{n}$. Let $\omega_k = x_k$, then $\omega_k = \frac{2}{n}(n+k)$. Hence,

$$R_p = \sum_k f(\omega_k)\Delta x_k = \sum_k (\frac{2(2n+k)}{n})\frac{2}{n} = \frac{4}{n^2}\sum_k (2n+k) = \frac{4}{n^2}(2n^2 + \frac{n(n+1)}{2}) = 8 + \frac{2(n+1)}{n} \ .$$

This implies $\lim_{n\to\infty} R_p = 8 + \lim_{n\to\infty} \frac{2n(n+1)}{n^2} = 8+2 = 10$.

The following remark simplifies the process of calculating the definite integrals. This remark will be stated later in Theorem 2.5.1.

Remark 2.3.1 *To find the value of a definite integral* $\int_a^b f(x)\,dx$*, we first find the value of the indefinite integral* $\int f(x)\,dx = F(x)+c$ *as shown in Chapter 1. Then, we substitute a and b into* $F(x)$ *as follows:*

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) \ .$$

Example 2.3.2 *Evaluate the following integrals:*

1. $\int_{-1}^2 2x+1\,dx$
2. $\int_0^3 x^2+1\,dx$
3. $\int_1^2 \frac{1}{\sqrt{x^3}}\,dx$
4. $\int_0^{\frac{\pi}{2}} \sin(x)+1\,dx$
5. $\int_{\frac{\pi}{4}}^{\pi} \sec^2(x)-4\,dx$
6. $\int_0^{\frac{\pi}{3}} \sec(x)\,tan(x)+x\,dx$

Solution:

1. $\int_{-1}^2 2x+1\,dx = \Big[x^2+x\Big]_{-1}^2 = (4+2) - ((-1)^2 + (-1)) = 6-0 = 6$.
2. $\int_0^3 x^2+1\,dx = \Big[\frac{x^3}{3}+x\Big]_0^3 = (\frac{27}{3}+3) - 0 = 12$.
3. $\int_1^2 \frac{1}{\sqrt{x^3}}\,dx = \Big[\frac{-2}{\sqrt{x}}\Big]_1^2 = \frac{-2}{\sqrt{2}} - (-2) = \frac{-2+\sqrt{2}}{\sqrt{2}}$.
4. $\int_0^{\frac{\pi}{2}} \sin(x)+1\,dx = \Big[-\cos(x)+x\Big]_0^{\frac{\pi}{2}} = (-\cos(\frac{\pi}{2}) + \frac{\pi}{2}) - (-\cos(0)+0) = \frac{\pi}{2}+1$.
5. $\int_{\frac{\pi}{4}}^{\pi} \sec^2(x)-4\,dx = \Big[\tan(x)-4x\Big]_{\frac{\pi}{4}}^{\pi} = (\tan(\pi)-4\pi) - (\tan(\frac{\pi}{4}) - 4.\frac{\pi}{4}) = -4\pi - (1-\pi) = -3\pi - 1$.
6. $\int_0^{\frac{\pi}{3}} \sec(x)\,tan(x)+x\,dx = \Big[\sec(x)+\frac{x^2}{2}\Big]_0^{\frac{\pi}{3}} = (\sec(\frac{\pi}{3}) + \frac{(\frac{\pi}{3})^2}{2}) - (\sec(0)+\frac{0}{2}) = 2 + \frac{\pi^2}{18} - 1 = 1 + \frac{\pi^2}{18}$.

One application of the definite integrals is to find the area under the graph of a non-negative function f on the interval $[a,b]$. This is clear from Definition 2.3.1,

$$A = \int_a^b f(x)\,dx \ .$$

The application of the definite integrals will be discussed in detail in Chapter 7.

Exercise 3:

1 - 8 ■ Evaluate the following integrals:

1. $\int_0^3 (2-x+x^2)\,dx$
2. $\int_{-1}^1 (x^2+3x+1)\,dx$
3. $\int_0^{10} (x^{\frac{3}{2}}+1)\,dx$
4. $\int_1^2 \frac{2}{\sqrt{x}}\,dx$
5. $\int_0^{\pi} \cos x\,dx$
6. $\int_0^{\frac{\pi}{4}} \sin x+\cos x\,dx$
7. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec x(\tan x+\sec x)\,dx$
8. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\,dx$

2.4 Properties of Definite Integrals

Theorem 2.4.1 *If f is integrable on $[a,b]$, then*

1. $\int_a^b c\,dx = c(b-a)$,
2. $\int_a^a f(x)\,dx = 0$,
3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
4. *If f and g are integrable on $[a,b]$, then $f+g$ and $f-g$ are integrable on $[a,b]$ and*
$$\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x) \pm \int_a^b g(x)\,dx\,.$$
5. *If f is integrable on $[a,b]$ and $k\in\mathbb{R}$, then $k\,f$ is integrable on $[a,b]$ and*
$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx\,.$$
6. *If f and g are integrable on $[a,b]$ and $f(x)\geq g(x)$ for all $x\in[a,b]$, then*
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx\,.$$
7. *If f is integrable on $[a,b]$ and $f(x)\geq 0$ for all $x\in[a,b]$, then*
$$\int_a^b f(x)\,dx \geq 0\,.$$
8. *If f is integrable on the intervals $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and*
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

Example 2.4.1 *Evaluate the following integrals:*

1. $\int_0^2 3\,dx$.
2. $\int_2^2 x^2+4\,dx$.

Solution:

1. $\int_0^2 3\,dx = 3(2-0) = 6$.
2. $\int_2^2 x^2+4\,dx = 0$.

Example 2.4.2 *If $\int_a^b f(x)\,dx = 4$ and $\int_a^b g(x)\,dx = 2$, then find $\int_a^b 3f(x)-\frac{g(x)}{2}\,dx$.*

Solution:
$$\int_a^b 3f(x)-\frac{g(x)}{2}\,dx = 3\int_a^b f(x)\,dx - \frac{1}{2}\int_a^b g(x)\,dx = 3(4)-\frac{1}{2}(2) = 11\,.$$

Example 2.4.3 *Prove that* $\int_0^2 (x^3+x^2+2)\ dx \geq \int_0^2 (x^2+1)\ dx$ *without evaluating the integrals.*

Solution:

Put $f(x) = x^3+x^2+2$ and $g(x) = x^2+1$. We find that $f(x)-g(x) = x^3+1 > 0$ for all $x \in [0,2]$. This implies $f(x) > g(x)$ and from Theorem **??**, we have

$$\int_0^2 (x^3+x^2+2)\ dx \geq \int_0^2 (x^2+1)\ dx\ .$$

Example 2.4.4 *If* $f(x) = \begin{cases} x^2 & : x < 0 \\ x^3 & : x \geq 0 \end{cases}$, *find* $\int_{-1}^2 f(x)\ dx$ . .

Solution:

Since $[-1,2] = [-1,0] \cup [0,2]$, then from Theorem **??**,

$$\begin{aligned} \int_{-1}^2 f(x)\ dx &= \int_{-1}^0 f(x)\ dx + \int_0^2 f(x)\ dx \\ &= \int_{-1}^0 x^2\ dx + \int_0^2 x^3\ dx \\ &= \left[\frac{x^3}{3}\right]_{-1}^0 + \left[\frac{x^4}{4}\right]_0^2 \\ &= \frac{-1}{3} + \frac{16}{4} = \frac{44}{12} = \frac{11}{3}\ . \end{aligned}$$

Example 2.4.5 *Evaluate the integral* $\int_0^2 |\ x-1\ |\ \ dx$.

Solution:

$$|\ x-1\ | = \begin{cases} -(x-1) & : x < 1 \\ x-1 & : x \geq 1 \end{cases}$$

Since $[0,2] = [0,1] \cup [1,2]$, then from Theorem **??**,

$$\begin{aligned} \int_0^2 |\ x-1\ |\ \ dx &= \int_0^1 -x+1\ dx + \int_1^2 x-1\ dx \\ &= \left[\frac{-x^2}{2} + x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2 \\ &= (\frac{-1}{2} - 0) + (2 - \frac{1}{2}) = 1\ . \end{aligned}$$

Mean Value Theorem for Integrals

Theorem 2.4.2 *If f is continuous on the interval $[a,b]$, then there is at least one number $z \in (a,b)$ such that*

$$\int_a^b f(x)\ dx = f(z)(b-a)\ .$$

Example 2.4.6 *Find the number z that satisfies the conclusion of the mean value theorem for the function f on the given interval $[a,b]$:*

1. $f(x) = 1+x^2, \qquad [0,2]\ .$
2. $f(x) = \sqrt[3]{x}, \qquad [0,1]\ .$

Solution:

1. From Theorem 2.4.2,

$$\begin{aligned}\int_0^2 1+x^2\,dx &= (2-0)f(z)\\ \left[x+\frac{x^3}{3}\right]_0^2 &= 2(1+z^2)\\ 3 &= 2(1+z^2)\\ \frac{3}{2} &= 1+z^2\end{aligned}$$

This implies $z^2=\frac{1}{2}\Rightarrow z=\pm\frac{1}{\sqrt{2}}$. However, $-\frac{1}{\sqrt{2}}\notin(0,2)$, so $z=\frac{1}{\sqrt{2}}\in(0,2)$.

2. From Theorem 2.4.2,

$$\begin{aligned}\int_0^1 \sqrt[3]{x}\,dx &= (1-0)f(z)\\ \frac{3}{4}\left[x^{\frac{4}{3}}\right]_0^1 &= \sqrt[3]{z}\end{aligned}$$

This implies $z=\frac{27}{64}\in(0,1)$.

From the previous theorem, we define the average value of the function f on the interval $[a,b]$.

Definition 2.4.1 *If f is continuous on the interval $[a,b]$, then the average value f_{av} of the function f on that interval is*

$$f_{av}=\frac{1}{b-a}\int_a^b f(x)\,dx\,.$$

Example 2.4.7 *Find the average value of the function f on the given interval $[a,b]$:*

1. $f(x)=x^3+x-1,\quad [0,2]\,.$

2. $f(x)=\sqrt{x},\quad [1,3]\,.$

Solution:

1. $f_{av}=\frac{1}{2-0}\int_0^2 x^3+x-1\,dx=\frac{1}{2}\left[\frac{x^4}{4}+\frac{x^2}{2}-x\right]_0^2=\frac{1}{2}\Big[(4+2-2)-(0)\Big]=2\,.$

2. $f_{av}=\frac{1}{3-1}\int_0^2 \sqrt{x}\,dx=\frac{1}{2}\cdot\frac{2}{3}\left[\,x^{\frac{3}{2}}\,\right]_1^3=\frac{3\sqrt{3}-1}{3}\,.$

Exercise 4:

1 - 4 ■ Evaluate the following integrals:

1. $\int_0^5 7\,dx$
2. $\int_1^1 (x^3-5x+1)\,dx$
3. $\int_0^2 |\,x-1\,|\,dx$
4. $\int_{-1}^1 |\,3x+1\,|\,dx$

5 - 8 ■ If $\int_a^b f(x)\,dx=2$ and $\int_a^b g(x)\,dx=3$, then find

5. $\int_a^b 6f(x)-\frac{g(x)}{3}\,dx.$
6. $\int_b^a f(x)+g(x)\,dx.$
7. $\int_a^a \sqrt{f(x).g(x)}\,dx.$
8. $\int_c^a f(x)\,dx+\int_b^c f(x)\,dx$ where $c\in(a,b)$.

9 - 14 ■ Verify that the function f satisfies the hypotheses of the Mean Value Theorem on the interval $[a,b]$. Then, find all numbers z that satisfy the conclusion of the Mean value Theorem.

9. $f(x) = (x+1)^3, \ [a,b] = [-1,1]$
10. $f(x) = 1 - x^3, \ [a,b] = [-2,0]$
11. $f(x) = \sqrt{x}, \ [a,b] = [1,4]$
12. $f(x) = \frac{2}{\sqrt{x}}, \ [a,b] = [1,4]$
13. $f(x) = \sin x, \ [a,b] = [0,\pi]$
14. $f(x) = \cos x, \ [a,b] = [0, \frac{\pi}{2}]$

15 - 18 ■ Find the average value of the function $f(x)$ on the given interval $[a,b]$:

15. $f(x) = x^3 + x^2 - 1, \ [a,b] = [0,2]$
16. $f(x) = \sqrt[3]{x}, \ [a,b] = [-1,3]$
17. $f(x) = \frac{1}{x^3}, \ [a,b] = [1,5]$
18. $f(x) = \sin x, \ [a,b] = [0, \frac{\pi}{6}]$

2.5 The Fundamental Theorem of Calculus

Theorem 2.5.1 *Suppose f is continuous on the closed interval $[a,b]$.*

1. *If $F(x) = \int_a^x f(t)\, dt$ for every $x \in [a,b]$, then $F(x)$ is an anti-derivative of f on $[a,b]$.*
2. *If $F(x)$ is any anti-derivative of f on $[a,b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$.*

From the previous theorem, if f is continuous on $[a,b]$ and $F(x) = \int_c^x f(t)\, dt$ where $c \in [a,b]$, then

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x) \ \ \forall x \in [a,b] \ .$$

This result can be generalized as follows:

Theorem 2.5.2 *Let f be continuous on $[a,b]$. If $g(x)$ and $h(x)$ are differentiable, then*

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\, dt\right] = f(h(x))h'(x) - f(g(x))g'(x) \ \ \forall x \in [a,b] \ .$$

The following corollary is stated without proof since the proof is straightforward from the previous theorem.

Corollary 2.5.1 *Let f be continuous on $[a,b]$. If $g(x)$ and $h(x)$ are differentiable, then*

1. $\frac{d}{dx}\left[\int_a^{h(x)} f(t)\, dt\right] = f(h(x))h'(x) \ \ \forall x \in [a,b]$,
2. $\frac{d}{dx}\left[\int_{g(x)}^{a} f(t)\, dt\right] = -f(g(x))g'(x) \ \ \forall x \in [a,b]$.

Example 2.5.1 *Find the following derivatives:*

1. $\frac{d}{dx}\int_1^x \sqrt{\cos t}\, dt$
2. $\frac{d}{dx}\int_1^{x^2} \frac{1}{t^3+1}\, dt$
3. $\frac{d}{dx}\int_x^{x^2} x(t^3-1)\, dt$
4. $\frac{d}{dx}\int_1^{\sin x} \frac{1}{t^2-1}\, dt$
5. $\frac{d}{dx}\int_{-x}^{x} \cos(t^2+1)\, dt$
6. $\frac{d}{dx}\int_{-x}^{x^2} \frac{1}{t^3+1}\, dt$
7. $\frac{d}{dx}\int_{x+1}^{3} \sqrt{t+1}\, dt$
8. $\frac{d}{dx}\int_{\cos x}^{\sin x} \sqrt{1+t^4}\, dt$

Solution:

1. $\frac{d}{dx}\int_1^x \sqrt{\cos t}\, dt = \sqrt{\cos x}\,(1) = \sqrt{\cos x}$.
2. $\frac{d}{dx}\int_1^{x^2} \frac{1}{t^3+1}\, dt = \frac{1}{(x^2)^3+1}(2x) = \frac{2x}{x^6+1}$.

3. $\frac{d}{dx}\int_x^{x^2} x(t^3-1)\,dt = \frac{d}{dx}\big(x\int_x^{x^2}(t^3-1)\,dt\big)$
$= \int_x^{x^2}(t^3-1)\,dt + x\big(2x(x^6-1)-(x^3-1)\big)$.

Let $f(x)=x$ and $g(x)=\int_x^{x^2}(t^3-1)\ dt$. Then, find $\frac{d}{dx}(f.g)$

4. $\frac{d}{dx}\int_{x+1}^{3}\sqrt{t+1}\,dt = 0-\sqrt{(x+1)+1} = -\sqrt{x+2}$.

5. $\frac{d}{dx}\int_{1}^{\sin x}\frac{1}{1-t^2}\,dt = \frac{1}{1-\sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x$.

6. $\frac{d}{dx}\int_{-x}^{x}\cos(t^2+1)\,dt = \cos(x^2+1)+\cos(x^2+1) = 2\cos(x^2+1)$.

7. $\frac{d}{dx}\int_{-x}^{x^2}\frac{1}{t^2+1}\,dt = \frac{2x}{x^4+1}+\frac{1}{x^2+1}$.

8. $\frac{d}{dx}\int_{\cos x}^{\sin x}\sqrt{1+t^4}\,dt = \sqrt{1+\sin^4 x}\ \cos x + \sqrt{1+\cos^4 x}\ \sin x$.

Example 2.5.2 *If* $F(x) = (x^2-2)\int_2^x \big(t+3F'(t)\big)\,dt$*, find* $F'(2)$.

Solution:

$$F'(x) = 2x\int_2^x \big(t+3F'(t)\big)\,dt + (x^2-2)(x+3F'(x))$$

Then,

$$F'(2) = 4\int_2^2 \big(t+3F'(t)\big)\,dt + (4-2)(2+3F'(2))$$

This implies $-5F'(2) = 4 \Rightarrow F'(2) = -\frac{4}{5}$.

Exercise 5:

1 - 8 ■ Find the following derivatives:

1. $\frac{d}{dx}\int_{\cos x}^{\sin x}\sqrt{t+1}\,dt$
2. $\frac{d}{dx}\int_{x}^{\sqrt{x}}\frac{1}{t^2+1}\,dt$
3. $\frac{d}{dx}\int_{1}^{x}(t-1)\,dt$
4. $\frac{d}{dx}\int_{1}^{x}(\sin x)\sqrt{t}\,dt$
5. $\frac{d}{dx}\int_{-2x}^{x}\sin(t+1)\,dt$
6. $\frac{d}{dx}\int_{x^3}^{\pi}\frac{1}{t^4+1}\,dt$
7. $\frac{d}{dx}\int_{x+1}^{3(x-1)}\frac{1}{t-1}\,dt$
8. $\frac{d}{dx}\int_{\tan x}^{\sec x}\sqrt{1+t^4}\,dt$

9 - 12 ■ Find the derivative for the given values:

9. $F(x) = \int_2^x \sqrt{3t^2+1}\,dt$, $F(2)$, $F'(2)$ and $F''(2)$.
10. $G(x) = \int_x^0 \frac{\sin t}{t+1}\,dt$, $G(0)$, $G'(0)$ and $G''(0)$.
11. $H(x) = \int_x^{x^2}\sqrt[5]{t+1}\,dt$, $H'(2)$.
12. $F(x) = \sin x\int_0^x \big(1+F'(t)\big)\,dt$, $F(0)$ and $F'(0)$.

2.6 Numerical Integration

Sometimes we face definite integrals that cannot be solved even if the integrands are integrable functions such as $\sqrt{1+x^3}$ and e^{x^2}. In our discussion in this book so far, we are not able to evaluate such integrals. We exploit this to show the reader a new technique depends on numerical methods. In this section, we will discuss two techniques of numerical integration to approximate definite integrals : Trapezoidal rule and Simpson's rule.

2.6.1 Trapezoidal Rule

As discussed in Section 2.2, a Riemann sum approximates the area underneath a curve of a function f from $x = a$ to $x = b$ as follows. First, we divide the interval $[a,b]$ by a regular partition P to generate n sub-intervals : $[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n]$. Then, we find the length of the sub-intervals: $\Delta x_k = \frac{b-a}{n}$. From the Riemann sum, we have

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(\omega_k)\Delta x_k = \frac{b-a}{n}\sum_{k=1}^{n} f(\omega_k)\ ,$$

where $\omega_k \in \omega$ and ω is a mark on the partition P.

As shown in Figure 2.2, we take the mark as follows:

1. the left-hand end point. We choose $\omega_k = x_{k-1}$ in each sub-interval. Then,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{k=1}^{n} f(\omega_{k-1})\ .$$

2. the right-hand end point. We choose $\omega_k = x_k$ in each sub-interval. Then,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\sum_{k=1}^{n} f(\omega_k)\ .$$

The average of the previous two cases is called the trapezoidal rule - see Figure 2.2 (C). Thus, by the trapezoidal rule, we have

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[\sum_{k=1}^{n} f(\omega_{k-1}) + \sum_{k=1}^{n} f(\omega_k)\right] = \frac{b-a}{2n}\Big[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\Big]\ .$$

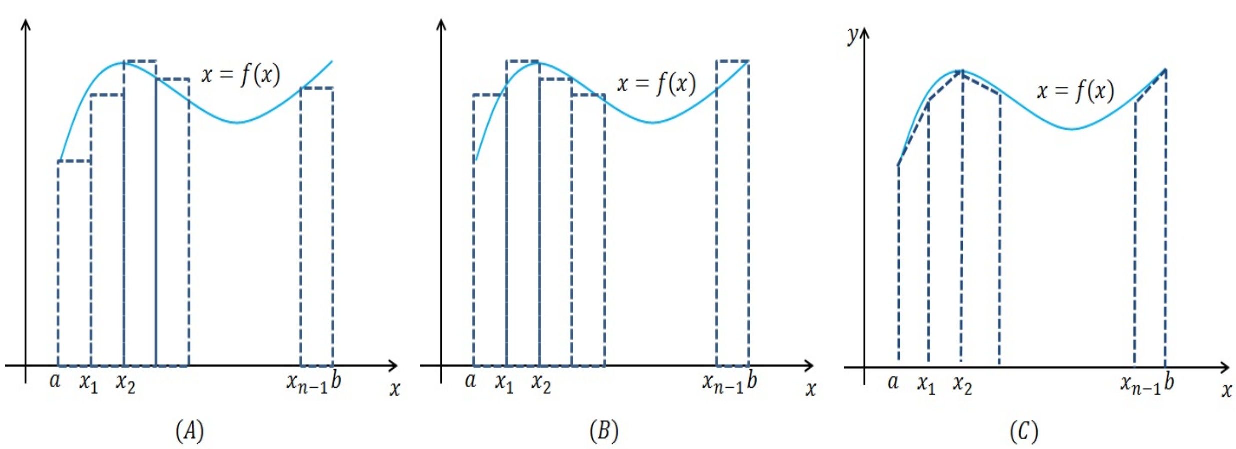


Figure 2.2: Approximation of the integral by the trapezoidal rule.

Error Estimation

Although the numerical methods give an approximated value of a definite integral, there is a possibility that an error occurs. The numerical method and number of the sub-intervals play a role in determining that possibility.

Theorem 2.6.1 *Suppose f'' is continuous on $[a,b]$ and M is the maximum value for f'' over $[a,b]$. If E_T is the error in calculating $\int_a^b f(x)\,dx$ under the trapezoidal rule, then*

$$|\,E_T\,| \le \frac{M(b-a)^3}{12\,n^2}\ .$$

Example 2.6.1 *By using the trapezoidal rule with $n=4$, approximate the integral $\int_1^2 \frac{1}{x}\,dx$. Then, estimate the error.*

Solution:

1. We approximate the integral $\int_1^2 \frac{1}{x}\,dx$ by the trapezoidal rule.
   (a) Divide the interval $[1,2]$ into sub-intervals. The length of each sub-intervals is $\Delta x = \frac{2-1}{4} = \frac{1}{4}$.
   (b) Find the partition $P = \{x_0, x_1, x_2, ..., x_n\}$ where $x_k = x_0 + k\Delta x = x_0 + k\frac{(b-a)}{n}$.
   The partition:
   $x_0 = 1,$
   $x_1 = 1 + \frac{1}{4} = 1\frac{1}{4},$
   $x_2 = 1 + 2(\frac{1}{4}) = 1\frac{1}{2},$
   $x_3 = 1 + 3(\frac{1}{4}) = 1\frac{3}{4}$, and
   $x_4 = 1 + 4(\frac{1}{4}) = 2$.
   Thus $P = \{1, 1.25, 1.5, 1.75, 2\}$.
   (c) Approximate the integral by using the following table:

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	1	1	1	1
1	1.25	0.8	2	1.6
2	1.5	0.6667	2	1.3334
3	1.75	0.5714	2	1.1428
4	2	0.5	1	0.5
Sum = $\sum_{k=1}^{4} m_k f(x_n)$				5.7762

   Thus, $\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{8}\left[5.5762\right] = 0.697$.

2. We estimate the error by using Theorem 2.6.1.

$$f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3}\ .$$

   Since $f''(x)$ is a decreasing function on the interval $[1,2]$, then $f''(x)$ is maximized at $x=1$. Hence, $M = |\,f''(1)\,| = 2$ and

$$|\,E_T\,| < \frac{2(2-1)^3}{12(4)^2} = \frac{1}{96} = 0.0104\ .$$

Remark 2.6.1 *By knowing the error amount, we can determine the number of the sub-intervals n before starting approximating.*

Example 2.6.2 *Find number of the sub-intervals to approximate the integral $\int_1^2 \frac{1}{x}\,dx$ such that the error is less than 10^{-3}.*

Solution:

From the previous example, we know that $M = 2$. Thus, $|E_T| < \frac{2(2-1)^3}{12n^2} < 10^{-3}$. This implies that

$$n^2 > \frac{2(2-1)^3}{12}10^3 = \frac{10^3}{6} \Rightarrow n > \sqrt{\frac{500}{3}} = 12.91$$

This means we consider $n = 13$.

2.6.2 Simpson's Rule

Simpson's rule is another numerical method to approximate the definite integrals. The question that can be raised here is that how the trapezoidal method differs from the Simpson's method? The trapezoidal method depends on building trapezoids from the sub-intervals, then taking the average of the left and right hands end point. Whereas, the Simpson's rule is built on approximating area of the graph in each sub-interval with parabola (Figure 4.2).

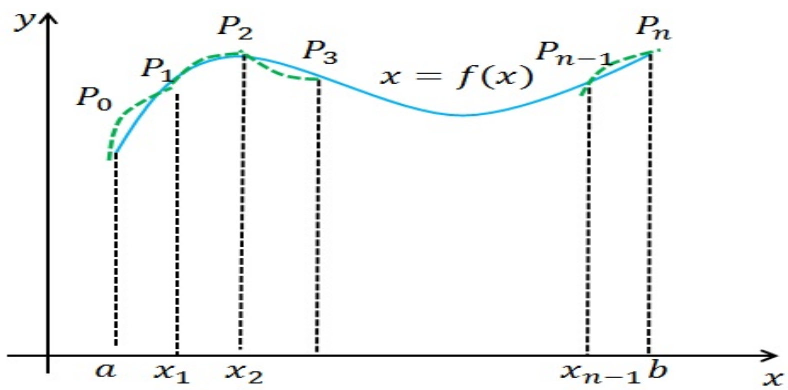
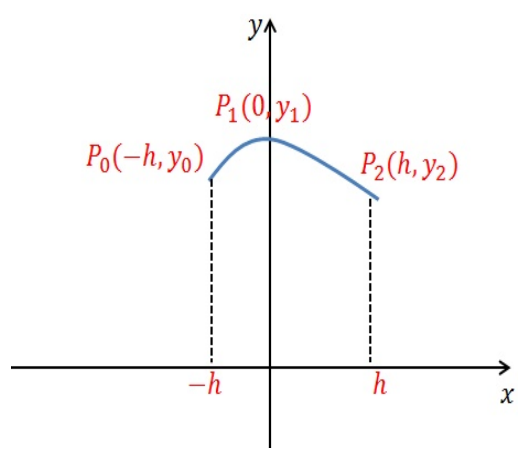


Figure 2.3: Approximation of the integral by Simpson's rule.

First, let P be a regular partition of the interval $[a,b]$ to generate n sub-intervals such that $|P| = \frac{(b-a)}{2n}$ and n is an even number. Consider the first sub-interval as shown in Figure 4.2.

Now, take three points lie on the parabola as shown in the next figure. Assume for simplicity that $x_0 = -h$, $x_1 = 0$ and $x_2 = h$. Since the parabola equation passes through three points is $ax^2 + bx + c$, then from the figure, the area under the graph bounded by $[-h,h]$ is

$$\int_{-h}^{h} ax^2 + bx + c \, dx = \frac{h}{3}(2ah^2 + 6c) \ .$$



Thus, since the points P_0, P_1 and P_2 pass through the parabola, then

$$y_0 = ah^2 - bh + c$$
$$y_1 = c$$
$$y_2 = ah^2 + bh + c \ .$$

We can find that $2ah^2+6c = y_0+4y_1+y_2$. Thus,

$$\int_{-h}^{h} ax^2+bx+c\,dx = \frac{h}{3}(y_0+4y_1+y_2) = \frac{h}{3}\big(f(x_0)+4f(x_1)+f(x_2)\big)\ .$$

Generally, for any three points P_{k-1}, P_k and P_{k+1}, we have

$$\frac{h}{3}(y_{k-1}+4y_k+y_{k+1}) = \frac{h}{3}\big(f(x_{k-1})+4f(x_k)+f(x_{k+1})\big)\ .$$

By summing the areas under the graphs, we have

$$\begin{aligned}\int_a^b f(x)\,dx = \frac{h}{3}\big(f(x_0)+4f(x_1)+f(x_2)\big)\\ &+\frac{h}{3}\big(f(x_2)+4f(x_3)+f(x_4)\big)\\ &\ldots\\ &+\frac{h}{3}\big(f(x_{n-2})+4f(x_{n-1})+f(x_n)\big)\\ &=\frac{b-a}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots\\ &+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]\end{aligned}$$

Hence, under the Simpson's rule, the integral $\int_a^b f(x)\,dx$ is approximated as follows:

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{3n}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]\ .$$

Error Estimation

The estimation of the error under the Simpson's method is calculated by the following theorem.

Theorem 2.6.2 *Suppose $f^{(4)}$ is continuous on $[a,b]$ and M is the maximum value for $f^{(4)}$ on $[a,b]$. If E_S is the error in calculating $\int_a^b f(x)\,dx$ under Simpson's rule, then*

$$|\,E_S\,| \le \frac{M(b-a)^5}{180\,n^4}\ .$$

Example 2.6.3 *By using the Simpson's rule with $n=4$, approximate the integral $\int_1^3 \sqrt{x^2+1}\,dx$. Then, estimate the error.*

Solution:

1. We approximate the integral $\int_1^3 x^2+1\,dx$ by the Simpson's rule.

   (a) Divide the interval $[1,3]$ into sub-intervals. The length of each sub-intervals is $\Delta x = \frac{3-1}{4} = \frac{1}{2}$.

   (b) Find the partition $P=\{x_0,x_1,x_2,\ldots,x_n\}$ where $x_k = x_0+k\Delta x = x_0+k\ \frac{(b-a)}{n}$.

   The partition:

   $x_0 = 1$,
   $x_1 = 1+\frac{1}{2} = 1\frac{1}{2}$,
   $x_2 = 1+2(\frac{1}{2}) = 2$,
   $x_3 = 1+3(\frac{1}{2}) = 2\frac{1}{2}$, and
   $x_4 = 1+4(\frac{1}{2}) = 3$.

Thus $P = \{1, 1.5, 2, 2.5, 3\}$.

(c) Approximate the integral by using the following table:

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	1	1.4142	1	2
1	1.5	1.8028	4	7.2112
2	2	2.2361	2	4.4722
3	2.5	2.6926	4	10.7704
4	3	3.1623	1	10
Sum = $\sum_{k=1}^{4} m_k f(x_k)$				27.0302

Thus, $\int_1^3 \sqrt{x^2+1}\, dx \approx \frac{2}{12}[27.0302] = 4.5050$.

2. We estimate the error by using Theorem 2.6.2.

Since $f^{(5)}(x) = -(15x(4x^2-3))/\sqrt{(x^2+1)^9}$, then $f^{(4)}(x)$ is a decreasing function on the interval $[1,3]$. Hence, $f^{(4)}(x)$ is maximized at $x = 1$. Then, $M = \mid f^{(4)}(1) \mid = 0.7955$ and

$$\mid E_s \mid < \frac{0.7955(3-1)^5}{180(4)^4} = 5.5243 \times 10^{-4} .$$

Example 2.6.4 *Find number of the sub-intervals to approximate the integral* $\int_1^3 \sqrt{x^2+1}\, dx$ *such that the error is less than* 10^{-2}.

Solution:

From the previous example, we know that $M = 0.7955$. Thus, $\mid E_S \mid < \frac{0.7955(3-1)^5}{180n^4} < 10^{-2}$. This implies that

$$n^4 > \frac{0.7955(32)}{180} 10^2 \Rightarrow n > 14.14$$

This means we consider take $n = 14$.

Exercise 6:

1 - 4 ■ By using trapezoidal rule, approximate the definite integral for the given n, then estimate the error:

1. $\int_{-1}^{1} \sqrt{x^2+1}\, dx,\qquad n = 4$
2. $\int_{2}^{4} \sqrt{x}\, dx,\qquad n = 5$

$\int_{0}^{4} \frac{x}{x+1}\, dx,\qquad n = 4$

4. $\int_{0}^{\pi} \sin x\, dx,\qquad n = 4$

5 - 8 ■ By using Simpson's rule, approximate the definite integral for the given n, then estimate the error:

5. $\ln(2) = \int_{1}^{2} \frac{1}{x}\, dx,\qquad n = 4$
6. $\int_{0}^{1} \frac{x}{\sqrt{x^4+1}}\, dx,\qquad n = 6$
7. $\int_{0}^{2} \sqrt{x^3+1}\, dx,\qquad n = 10$
8. $\int_{1}^{3} \sqrt{\ln x}\, dx,\qquad n = 4$

9 - 10 ■ Consider function $f(x)$, and the integral $I(f)$. What is the minimum number of points to be used to ensure an error $\leq 5 \times 10^{-2}$ in the following:

9. $f(x) = e^x$ and $I(f) = \int_0^2 e^x\, dx$ in the trapezoid rule.
10. $f(x) = \cos x^2$ and $I(f) = \int_0^2 \cos x\, dx$ in the Simpson's rule.

Chapter 3

Logarithmic and Exponential Functions

3.1 Natural Logarithmic Function

As mentioned in Chapter 1, the integral $\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$ if $r \neq -1$. This means, the previous formula cannot be used when $r = -1$ because the denominator will become zero. The task in this section is to find general anti-derivative of the function $\frac{1}{x}$ i. e., we are looking for a function $F(x)$ such that $F^{'}(x) = \frac{1}{x}$.

Consider the function $f(t) = \frac{1}{t}$. It is continuous on the interval $(0,+\infty)$ and this implies that the function is integrable on the interval $[1,x]$. The area under the graph of the function $f(t) = \frac{1}{t}$ bounded from $t = 1$ to $t = x$ as shown in the Figure **??** is

$$f(x) = \int_1^x \frac{1}{t}\,dx$$

Definition 3.1.1 *The natural logarithmic function is defined as follows:*

$$ln : (0,\infty) \to \mathbb{R}\ ,$$

$$ln(x) = \int_1^x \frac{1}{t}\,dt$$

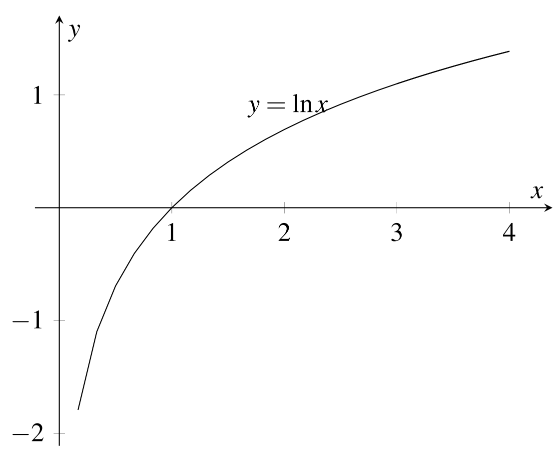


Figure 3.2: The graph of the function $y = \ln x$.

3.1.1 Properties of the Natural Logarithmic Function

1. The domain of the function $\ln(x)$ is $(0,\infty)$.

2. The range of the function $\ln(x)$ is $\mathbb{R}$ as follows:

$$\ln(x) = \begin{cases} y > 0 & : x > 1 \\ y = 0 & : x = 1 \\ y < 0 & : 0 < x < 1 \end{cases}$$

3. The function $\ln(x)$ is differentiable and continuous on the domain. From the fundamental theorem of calculus, we have

$$\frac{d}{dx}(\ln(x)) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}, \forall x > 0 \,.$$

From this, the function $\ln(x)$ is increasing on the interval $(0,\infty)$.

4. The second derivative $\frac{d^2}{dx^2}(\ln(x)) = \frac{-1}{x^2} < 0$ for all $x \in (0,\infty)$. Hence, the function $\ln(x)$ is concave downward on the interval $(0,\infty)$.

5. $\lim_{x\to 0^+} \ln(x) = -\infty$ and $\lim_{x\to\infty} \ln(x) = +\infty$.

Theorem 3.1.1 *For every $a,b > 0$ and $r \in \mathbb{Q}$, then*

1. $\ln(a\,b) = \ln(a) + \ln(b)$.
2. $\ln(\frac{a}{b}) = \ln(a) - \ln(b)$.
3. $\ln(a^r) = r\ln(a)$.

3.1.2 Differentiating and Integrating Natural Logarithm Function

From our discussion above, we know that

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Hence,

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x} \,.$$

From this, we have

$$\frac{d}{dx}\ln(|\,x\,|) = \frac{1}{x} \;\forall x \neq 0 \,.$$

Generally, if $u = g(x)$ is differentiable and $u \neq 0$ for every x in an interval I , then

$$\frac{d}{dx}\ln(|\,u\,|) = \frac{1}{u}\frac{du}{dx}, \forall x \in I$$

Example 3.1.1 *Find the derivative of the following functions:*

1. $f(x) = \ln(x+1)$
2. $g(x) = \ln(x^3 + 2x - 1)$
3. $h(x) = \ln(\sqrt{x^2+1})$
4. $y = \sqrt{\ln(x)}$
5. $f(x) = \ln(\cos x)$
6. $g(x) = \sqrt{x}\ln(x)$
7. $h(x) = \sin(\ln(x))$
8. $y(x) = \ln(x + \ln x)$

Solution:

1. $f'(x) = \frac{1}{x+1}$.
2. $g'(x) = \frac{3x^2+2}{x^3+2x-1}$.
3. $h'(x) = \frac{1}{\sqrt{x^2+1}} \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{x^2+1}$.
4. $y'(x) = \frac{1}{2\sqrt{\ln x}} . \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$.
5. $f'(x) = \frac{-\sin x}{\cos x} = -\tan x$.
6. $g'(x) = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x}\frac{1}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{lnx+2}{2\sqrt{x}}$.
7. $h'(x) = \cos(\ln x)(\frac{1}{x}) = \frac{\cos(\ln x)}{x}$.
8. $y'(x) = \frac{1}{x+\ln x}(1+\frac{1}{x}) = \frac{x+1}{x(x+\ln x)}$.

In the following, we present one of the simple applications of the natural logarithm function. We know that the derivative of composite functions takes efforts and time. This problem can be solved by using the derivative of the natural logarithm function. In a sense, Theorem 3.1.1 and the derivative of *ln* are used to simplify the differentiation of the composite functions.

Example 3.1.2 *Find the derivative of the function* $y = \sqrt[5]{\frac{x-1}{x+1}}$.

Solution:

We can solve this example using the derivative rules, but this will take time. Instead, we use the natural logarithm function as follows:

Take logarithm function (ln) for both sides. This implies $\ln y = \ln|\sqrt[5]{\frac{x-1}{x+1}}| = \frac{1}{5}\Big(\ln|x-1| - \ln|x+1|\Big)$. Differentiate both sides

$$\frac{y'}{y} = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big) \qquad \text{Remember: } \frac{d}{dx}lny = \frac{y'}{y}$$

This implies

$$y' = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big)y = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big)\sqrt[5]{\frac{x-1}{x+1}} .$$

Example 3.1.3 *Find the derivative of the function:* $y = \dfrac{\sqrt{x}\cos x}{(x+1)\sin x}$.

Solution:

For simplicity, we use the natural logarithm function (ln). Take ln for both sides, this implies

$$\ln|y| = \ln|\frac{\sqrt{x}\cos x}{(x+1)\sin x}| = \ln\sqrt{x} + \ln|\cos x| - \ln|x+1| - \ln|\sin x| .$$

By differentiating both sides, we have

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} - \frac{\sin x}{\cos x} - \frac{1}{x+1} - \frac{\cos x}{\sin x} .$$

This implies

$$y' = \Big(\frac{1}{2\sqrt{x}} - \tan x - \frac{1}{x+1} - \cot x\Big)\frac{\sqrt{x}\cos x}{(x+1)\sin x} .$$

Recall, $\frac{d}{dx}\ln|u| = \frac{u'}{u}$ where $u = g(x)$ is a differentiable function. By integrating both sides, we have

$$\begin{aligned}\int \frac{u'}{u}\,dx &= \int \frac{d}{dx}\ln|u|\;dx \\ &= \ln|u| + c .\end{aligned}$$

This can be stated as follows:

$$\int \frac{u'}{u}\,dx = \ln|u| + c \quad \Rightarrow \quad \int \frac{1}{x}\,dx = \ln|x| + c$$

Example 3.1.4 *Evaluate the following integrals*

1. $\int \frac{2x}{x^2+1}\,dx$
2. $\int \frac{6x^2+1}{4x^3+2x+1}\,dx$
3. $\int_2^e \frac{dx}{x\ln x}$
4. $\int_1^4 \frac{dx}{\sqrt{x}(1+\sqrt{x})}$
5. $\int \tan x\,dx$
6. $\int \cot x\,dx$
7. $\int \sec x\,dx$
8. $\int \csc x\,dx$

Solution:

1. $\int \frac{2x}{x^2+1}\,dx = \ln|x^2+1| + c.$

2. $\int \frac{6x^2+1}{4x^3+2x+1}\,dx = \frac{1}{2}\int \frac{12x^2+2}{4x^3+2x+1}\,dx = \frac{1}{2}\ln|4x^3+2x+1| + c.$

3. $\int_2^e \frac{dx}{x\ln x}$

   Put $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$. By substitution, we have $\int \frac{1}{u}\,du = \ln|u| = \left[\ln(\ln x)\right]_2^e = \ln(\ln e) - \ln(\ln 2) = \ln(1) - \ln(\ln 2) = -\ln(\ln 2)$.

4. $\int_1^4 \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

   Put $u = 1+\sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx$. By substitution, we have

$$2\int \frac{1}{u}\,du = 2\ln|u| = 2\left[\ln|1+\sqrt{x}|\right]_1^4 = 2(\ln 3 - \ln 2)\ .$$

5. $\int \tan x\,dx$

$$\begin{aligned}\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx &= -\int \frac{-\sin x}{\cos x}\,dx \\ &= -\ln|\cos x| + c \\ &= \ln|\sec x| + c\ .\end{aligned}$$

Remember: $\sec x = \frac{1}{\cos x}$

6. $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c.$

7. $\int \sec x\,dx = \int \frac{\sec x\,(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + c.$

8. $\int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{(\csc x - \cot x)}\,dx = \int \frac{\csc^2 x - \csc x\,\cot x}{\csc x - \cot x}\,dx = \ln|\csc x - \cot x| + c.$

Exercise 1:

1 - 18 ■ Find the derivative of the following functions:

1. $y = \ln(x+1)$
2. $y = \ln(x^3+2x-4)$
3. $y = \ln(\sqrt{x})$
4. $y = \ln(\sqrt[3]{x^2})$
5. $y = \ln(\frac{1}{x})$
6. $y = \ln(\sin x + x + 1)$
7. $y = \ln(\sec x + x^2)$
8. $y = \ln(\cos x)$
9. $y = \ln(\sin^2 x)$
10. $y = \ln(\sec x\ \tan x)$
11. $y = \csc x\ \ln(x)$
12. $y = \sqrt[3]{x^2}\ln(x^3+1)$
13. $y = \ln\left(\sqrt{\frac{x^2-1}{x+2}}\right)$
14. $y = \ln\left((x^2+1)(x-1)\right)$
15. $y = \ln(\sqrt{x}-\sqrt{x+1})$
16. $y = \frac{x}{\ln(x^2)}$
17. $y = \ln(x^3+1)$
18. $y = \ln(\ln(\sin x))$

19 - 24 ■ Find the derivative of the following functions:

19. $y = \sqrt[5]{\frac{2x+1}{3x-1}}$
20. $y = \frac{(x-1)(\sqrt{x^3+2x+1})}{x^3+2x^2+x-1}$
21. $y = \frac{x^2\sqrt{7x+3}}{(1+x^2)^3}$
22. $y = \sqrt[3]{\frac{\tan^2 x\ \sin x\ \cos x}{\sqrt{x^3}}}$
23. $y = (\frac{x\sec x^2}{\sqrt{x}(x+1)})^{\frac{7}{2}}$
24. $y = \frac{\sqrt[3]{x+1}\cos^2 x}{(x+1)^2\cos(3x)}$

25 - 36 ■ Evaluate the following integrals:

25. $\int \frac{3x}{x^2+1}\,dx$
26. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$
27. $\int \frac{1}{x\ln x^2}\,dx$
28. $\int \sec x\,dx$
29. $\int \frac{csc^2 x}{1+\cot x}\,dx$
30. $\int_{-1}^{4} \frac{x}{x^2+1}\,dx$
31. $\int \csc x\,dx$
32. $\int \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$
33. $\int \frac{\sqrt{\ln x^2}}{x}\,dx$
34. $\int_1^2 \frac{x+3}{x^2}\,dx$
35. $\int \frac{\cos(\ln x)}{x}\,dx$
36. $\int_2^3 \frac{1}{x(\ln x)^5}\,dx$

3.2 Natural Exponential Function

Since the natural logarithm function $\ln : (0,\infty) \longrightarrow \mathbb{R}$ is a strictly increasing function (see Figure 3.3), it is one-to-one. The function ln is also onto and this implies that the natural logarithm function has an inverse function. This function is called the natural exponential function.

Definition 3.2.1 *The natural exponential function is defined as follows:*

$$exp : \mathbb{R} \longrightarrow (0,\infty)\ ,$$

$$y = exp(x) \Leftrightarrow \ln y = x$$

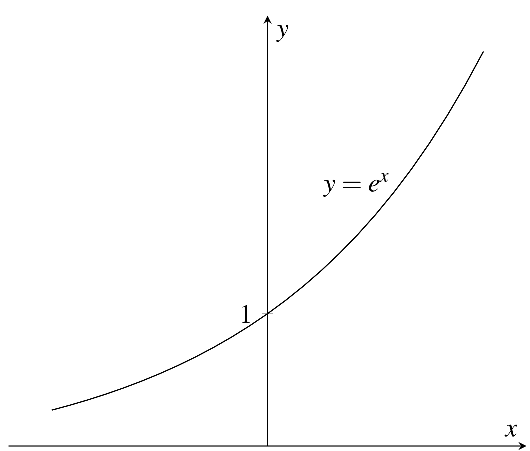


Figure 3.3: The graph of the function $y = e^x$.

3.2.1 Properties of the Natural Exponential Function

1. The domain of the function $exp(x)$ is $\mathbb{R}$.
2. The range of the function $exp(x)$ is $(0,\infty)$ as follows:

$$exp(x) = \begin{cases} y > 1 & : x > 0 \\ y = 1 & : x = 0 \\ y < 1 & : x < 0 \end{cases}$$

3. Usually, the symbol $exp(x)$ is written as e^x. Thus, $exp(1) = e$ and from Definition 3.2.1, we have $\ln(e) = 1$. Also, $\ln(e^r) = r\ln e = r \;\forall\; r \in \mathbb{Q}$.

4. The function e^x is differentiable and continuous on the domain
$$\frac{d}{dx}(e^x) = e^x, \forall x \in \mathbb{R} .$$
From this, the function e^x is increasing on $\mathbb{R}$.

5. The second derivative $\frac{d^2}{dx^2}(e^x) = e^x > 0$ for all $x \in \mathbb{R}$. Hence, the function e^x is concave upward on the domain $\mathbb{R}$.

6. $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

7. Since e^x and $\ln x$ are inverse functions, then
$$\ln(e^x) = x, \;\forall x \in \mathbb{R} ,$$
$$e^{\ln x} = x, \;\forall x \in (0,\infty) .$$

Theorem 3.2.1 *For every $a,b > 0$ and $r \in \mathbb{Q}$, then*
1. $e^a e^b = e^{a+b}$.
2. $\dfrac{e^a}{e^b} = e^{a-b}$.
3. $(e^a)^r = e^{ar}$.

Example 3.2.1 *Find value of x:*

1. $\ln x = 2$
2. $\ln(\ln x) = 0$
3. $(x-1)e^{-\ln\frac{1}{x}} = 2$
4. $xe^{2\ln x} = 8$

Solution:

1. $\ln x = 2 \Rightarrow e^{\ln x} = e^2 \Rightarrow x = e^2$. Take *exp* for both sides
2. $\ln(\ln x) = 0 \Rightarrow e^{\ln(\ln x)} = e^0 \Rightarrow \ln x = 1 \Rightarrow e^{\ln x} = e^1 \Rightarrow x = e$. Take *exp* twice
3. $(x-1)e^{-\ln\frac{1}{x}} = 2 \Rightarrow (x-1)e^{\ln(x^{-1})^{-1}} = 2 \Rightarrow (x-1)e^{\ln x} = 2$. This implies
$$x(x-1) = 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1 \text{ or } x = 2 .$$
4. $xe^{2\ln x} = 8 \Rightarrow xe^{\ln x^2} = 8 \Rightarrow x^3 = 8 \Rightarrow x = 2$.

Example 3.2.2 *Simplify the following:*

1. $\ln(e^{\sqrt{x}})$
2. $e^{\frac{1}{3}\ln x}$
3. $(x+1)\ln(e^{x-1})$
4. $e^{(\sqrt{x}+2\ln x)}$

Solution:

1. $\ln(e^{\sqrt{x}}) = \sqrt{x}$.
2. $e^{\frac{1}{3}\ln x} = e^{\ln\sqrt[3]{x}} = \sqrt[3]{x}$.
3. $(x+1)\ln(e^{x-1}) = (x+1)(x-1) = x^2 - 1$.
4. $e^{(\sqrt{x}+2\ln x)} = e^{\sqrt{x}}e^{\ln x^2} = x^2 e^{\sqrt{x}}$.

3.2.2 Differentiating and Integrating Natural Exponential Function

From the discussion above, if $u = g(x)$ is differentiable on the interval I, then

$$y = exp(x) \Rightarrow \ln y = x \ .$$

Also,

$$\frac{d}{dx}\ln y = \frac{y'}{y} = 1 \Rightarrow y' = y$$

This implies

$$\frac{d}{dx}e^u = e^u u',\ \forall x \in I \quad \Rightarrow \quad \frac{d}{dx}e^x \ = \ e^x$$

Example 3.2.3 *Find the derivative of the following functions:*

1. $y = e^{\sqrt[3]{x+1}}$
2. $y = e^{-5x^2}$
3. $y = e^{3\cos x - 4x^2}$
4. $y = e^{\frac{1}{x}} - \frac{1}{e^x}$
5. $y = e^{\ln \sin x}$
6. $y = \ln(e^{2x} + \sqrt{1-e^x})$

Solution:

1. $y' = e^{\sqrt[3]{x+1}}(\frac{1}{3\sqrt[3]{(x+1)^2}})$.
2. $y' = e^{-5x^2}(-10x)$.
3. $y' = e^{3\cos x - 4x^2}(-3\sin x - 8x)$.
4. $y' = e^{\frac{1}{x}}(\frac{-1}{x^2}) - (-e^{-x}) = \frac{1}{e^x} - \frac{e^{\frac{1}{x}}}{x^2}$.
5. $y' = e^{\ln \sin x}(\frac{\cos x}{\sin x}) = \cos x$.
6. $y' = \frac{1}{e^{2x}+\sqrt{1-e^x}}(2e^{2x} - \frac{e^x}{2\sqrt{1-e^x}})$.

Recall, $\frac{d}{dx}e^u = e^u u'$ where $u = g(x)$ is a differentiable function. By integrating both sides, we have

$$\begin{aligned}\int e^u u'\ dx &= \int \frac{d}{dx}e^u\ dx \\ &= e^u + c\ .\end{aligned}$$

This can be stated as follows

$$\int e^u u'\ dx = e^u + c \quad \Rightarrow \quad \int e^x\ dx = e^x + c$$

Example 3.2.4 *Evaluate the following integrals:*

1. $\int xe^{-x^2}\ dx$
2. $\int_0^{ln5} e^x(3-4e^x)\ dx$
3. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\ dx$
4. $\int \frac{e^{\tan x}}{\cos^2 x}\ dx$

Solution:

1. $\int xe^{-x^2}\ dx$

   Put $u = -x^2 \Rightarrow du = -2x\ dx$.

   By substitution, we have

   $\frac{-1}{2}\int e^u\ du = \frac{-1}{2}e^u + c = \frac{-1}{2}e^{-x^2} + c$.

2. $\int_0^{ln5} e^x(3-4e^x)\,dx$

Put $u = 3-4e^x \Rightarrow du = -4e^x\,dx$.

By substituting into the integral, we have

$\frac{-1}{4}\int u\,du = \frac{u^2}{-8}$

Thus

$\int_0^{ln5} e^x(3-4e^x)\,dx = \frac{-1}{8}\Big[(3-4e^x)^2\Big]_0^{\ln 5} = \frac{-1}{8}\big[(-17)^2-(-1)^2\big] = -36$.

3. $\int \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$.

Put $u = e^x - e^{-x} \Rightarrow du = e^x + e^{-x}\,dx$.

By substitution, we have

$\int \frac{1}{u}\,du = \ln|u| + c = \ln|e^x - e^{-x}| + c$.

4. $\int \frac{e^{\tan x}}{\cos^2 x}\,dx = \int e^{\tan x}\sec^2 x\,dx$

Put $u = \tan x \Rightarrow du = \sec^2 x\,dx$.

By substitution, we have

$\int e^u\,du = e^u + c = e^{\tan x} + c$.

Exercise 2:

1 - 4 ■ Simplify the following:

1. $\sin^2 x + e^{2\ln\cos x}$
2. $\ln e^{\sqrt[5]{x}}$
3. $(x+2)e^{\ln(x-2)}$
4. $\ln(e^{3+2\ln x})$

5 - 8 ■ Find value of x:

5. $\ln x^2 = 4$
6. $\ln(\ln x) = 1$
7. $xe^{\ln x} = 27$
8. $\ln e^x(x+2) = 3$

9 - 18 ■ Find the derivative of the following functions:

9. $y = e^{\sin x - 3x^2}$
10. $y = xe^{x\sqrt{x}}$
11. $y = e^x\cos(\ln x)$
12. $y = e^{\frac{1}{x}}\ln x$
13. $y = \ln(e^{-x} + \sqrt{x}e^{-x})$
14. $y = \sin(x)e^{\sqrt[3]{x}}$
15. $y = \ln(\tan e^x)$
16. $y = \sqrt{e^x}$
17. $y = (e^x+1)(\sqrt{e^{-x}+1})$
18. $y = \sec^2(e^{3x})$

19 - 28 ■ Evaluate the following integrals:

19. $\int_0^1 e^{2x+1}\,dx$
20. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$
21. $\int \frac{e^{\sin x}}{\sec x}\,dx$
22. $\int \frac{(1-2\sqrt{x}\sin x)e^{\sqrt{x}+\cos x}}{\sqrt{x}}\,dx$
23. $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$
24. $\int_0^{\pi/4} \frac{e^{\sec x}\sin x}{\cos^2 x}\,dx$
25. $\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}\,dx$
26. $\int \frac{e^x}{(1+e^x)^5}\,dx$
27. $\int e^{\ln\cos x}\,dx$
28. $\int_1^2 \frac{e^x}{e^x+1}\,dx$

3.3 General Exponential and Logarithmic Functions

3.3.1 General Exponential Function

Definition 3.3.1 *The general exponential function is defined as follows:*

$$a^x : \mathbb{R} \to (0,\infty) ,$$

$$a^x \ = \ e^{x \ln a} .$$

Since $\ln a^x = x \ln a \ \forall x \in \mathbb{Q}$, then by taking *exp* for both sides, we can write $a^x \ = \ e^{x \ln a}$. The function a^x is called the general exponential function for the base a.

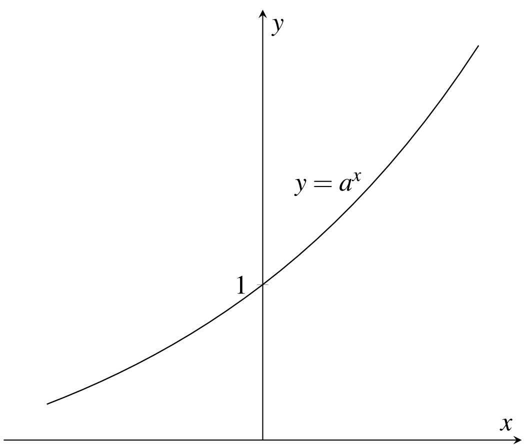


Figure 3.4: The function $y = a^x$ for $a > 1$.

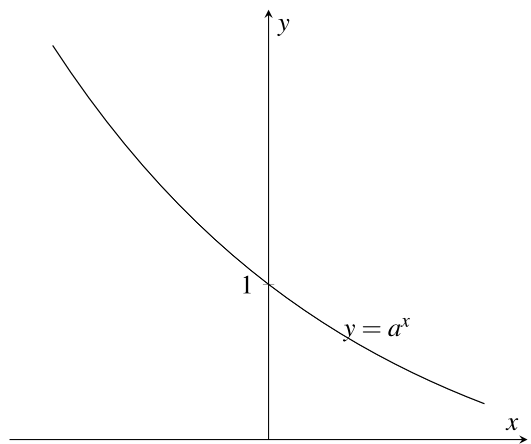


Figure 3.5: The function $y = a^x$ for $a < 1$.

In the following, we provide the main properties of the general exponential function.

Properties of the General Exponential Function

Let $f(x) = a^x \ \forall x \in \mathbb{R}$.

1. The domain of $f(x)$ is $\mathbb{R}$ and the range is $(0,\infty)$.

2. If $a > 1$, $\ln a > 0$ and this implies that $x \ln a$ is an increasing function with x. This indicates that $f(x)$ is an increasing function (see Figure 3.4 for $a > 1$).

3. If $a < 1$, $\ln a < 0$ and this implies that $x \ln a$ and $f(x)$ are decreasing functions (see Figure 3.5 for $a < 1$).

Theorem 3.3.1 *For every $x, y > 0$ and $a, b \in \mathbb{R}$,*

1. $x^a x^b = x^{a+b}$.

2. $\frac{x^a}{x^b} = x^{a-b}$.

3. $(x^a)^b = x^{a\,b}$.

4. $(xy)^a = x^a y^a$.

Differentiating and Integrating General Exponential Function

Since $a^x = e^{x \ln a}$, then

$$\begin{aligned} \frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} \\ &= e^{x \ln a} \ln a \\ &= a^x \ln a \, . \end{aligned}$$

This can be stated as follows:

$$\frac{d}{dx} a^x \; = a^x . \ln a \;\; \Rightarrow \;\; \int a^x \, dx = \frac{1}{\ln a} a^x + c$$

Generally, if $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx} a^u \; = a^u . u' . \ln a \;\; \Rightarrow \;\; \int a^u . u' \, dx = \frac{1}{\ln a} a^u + c$$

Example 3.3.1 *Find the derivative of the following functions:*

1. $y = 2^{\sqrt{x}}$
2. $y = 3^{x^2 \sin x}$
3. $y = \sin 3^x$
4. $y = x(7^{-3x})$
5. $y = \ln(\tan 5^x)$
6. $y = (10^x + 10^{-x})^{10}$

Solution:

1. $y' = 2^{\sqrt{x}} \; \ln 2 \; \frac{1}{2\sqrt{x}} = \frac{2^{\sqrt{x}} \ln 2}{2\sqrt{x}}$.
2. $y' = 3^{x^2 \sin x} \; \ln 3 \; (2x \sin x + x^2 \cos x)$.
3. $y' = (3^x \; \ln 3) \cos 3^x$.
4. $y' = 7^{-3x} + x\,(-3 \ln 7 \; 7^{-3x}) = 7^{-3x}(1 - 3\ln 7x)$.
5. $y' = \frac{(5^x \ln 5) \; \sec^2(5^x)}{\tan(5^x)}$.
6. $y' = 10\,(10^x + 10^{-x})^9 \; (10^x \; \ln 10 - 10^{-x} \; \ln 10) =$ $10 \; \ln 10 \; (10^x + 10^{-x})^9 \; (10^x - 10^{-x})$.

Example 3.3.2 *Find the derivative of the following function* $y = (\sin x)^x$.

Solution:

Take ln for both sides. This implies $\ln y = x \ln(\sin x)$. Now, find the derivative of both sides

$$\begin{aligned} \frac{y'}{y} &= \ln(\sin x) + \frac{x \cos x}{\sin x} \\ \Rightarrow y' &= \big(\ln(\sin x) + x \cot x\big)(\sin x)^x \, . \end{aligned}$$

Example 3.3.3 *Evaluate the following integrals:*

1. $\int x 3^{-x^2} \, dx$
2. $\int 5^x \sqrt{5^x + 1} \, dx$
3. $\int 3^x \sin 3^x \, dx$
4. $\int \frac{2^x}{2^x + 1} \, dx$

Solution:

1. $\int x 3^{-x^2} \, dx$. Put $u = -x^2 \Rightarrow du = -2x \, dx$. By substitution, we have

$$\frac{-1}{2} \int 3^u \, du = \frac{-1}{2 \ln 3} 3^u + c = \frac{-1}{2 \ln 3} 3^{-x^2} + c \, .$$

2. $\int 5^x\sqrt{5^x+1}\,dx$. Put $u = 5^x+1 \Rightarrow du = 5^x \ln 5\,dx$. The substitution implies

$$\frac{1}{\ln 5}\int u^{\frac{1}{2}}\,dx = \frac{1}{\ln 5}\frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2(5^x+1)^{\frac{3}{2}}}{3\ln 5} + c\ .$$

3. $\int 3^x \sin 3^x\,dx$. Put $u = 3^x \Rightarrow du = 3^x \ln 3\,dx$. By substitution, we have

$$\frac{1}{\ln 3}\int \sin u\,du = -\frac{1}{\ln 3}\cos u + c = -\frac{1}{\ln 3}\cos 3^x + c\ .$$

4. $\int \frac{2^x}{2^x+1}\,dx$. Put $u = 2^x+1 \Rightarrow du = 2^x \ln 2\,dx$. By substituting that into the integral, we have

$$\frac{1}{\ln 2}\int \frac{1}{u}\,du = \frac{1}{\ln 2}\ln|\,u\,| + c = \frac{1}{\ln 2}\ln|\,2^x+1\,| + c\ .$$

3.3.2 General Logarithmic Function

We know that if $a \neq 1$, the function a^x is strictly increasing or decreasing, depending on the value of a. Thus, the function a^x is one-to-one. The function is also onto and this implies that the function a^x has an inverse function. The inverse is the general logarithmic function $\log_a$ for the base a.

Definition 3.3.2 *The general logarithmic function is defined as follows:*

$$\log_a : (0,\infty) \to \mathbb{R}\ ,$$

$$x = a^y \Leftrightarrow y = \log_a(x)\ .$$

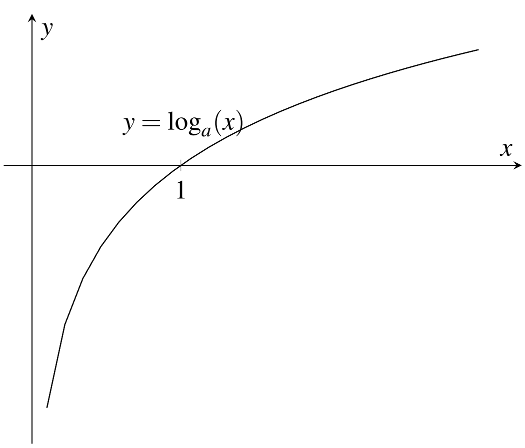


Figure 3.6: The function $y = \log_a(x)$ for $a > 1$.

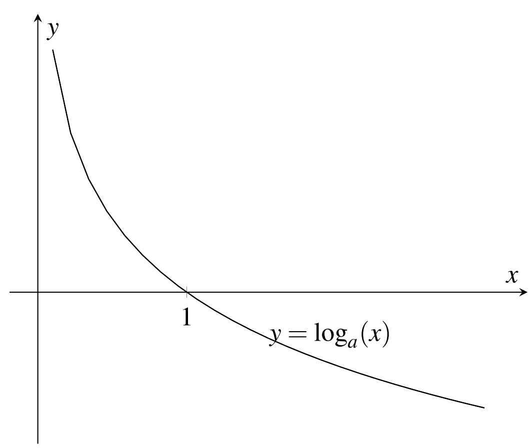


Figure 3.7: The function $y = \log_a(x)$ for $a < 1$.

Properties of the General Logarithm Function

1. The general logarithm function $\log_a x = \frac{\ln x}{\ln a}$.

   To see this, let $y = \log_a x \Rightarrow x = a^y$. Take ln for both sides,

$$\ln x = \ln a^y = y \ln a \Rightarrow y = \frac{\ln x}{\ln a}\ .$$

2. If $a > 1$, the function $\log_a(x)$ is increasing function, but if $0 < a < 1$, the function $\log_a(x)$ is decreasing function (see Figures 3.6 and 3.7).
3. The natural logarithm function $\ln x = \log_e x$.
4. The general logarithm function $\log_{10} x = \log x$.
5. The general logarithm function $\log_a(a) = 1$.

Theorem 3.3.2 *For every $x, y > 0$ and $r \in \mathbb{R}$, then*
1. $\log_a(xy) = \log_a(x) + \log_a(y)$.
2. $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$.
3. $\log_a(x^r) = r\log_a(x)$.

Differentiating and Integrating General Logarithmic Function

From the previous properties, we know that $\log_a(x) = \frac{\ln x}{\ln a}$. Thus,

$$\frac{d}{dx}\big(\log_a x\big) = \frac{d}{dx}\Big(\frac{\ln x}{\ln a}\Big) = \frac{1}{x\ln a} \ .$$

Hence, we have

$$\int \frac{1}{x\ln a}\ dx = \log_a(x) + c \ .$$

Generally, this can be stated as follows. If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\big(\log_a u\big) = \frac{1}{u\ln a}.u' \ \Rightarrow \ \int \frac{1}{u\ln a}.u'\ dx = \log_a(u) + c$$

Example 3.3.4 *Find the derivative of the following functions:*

1. $y = \log_3 \sin x$.
2. $y = \log\sqrt{x}$.

Solution:

1. $y' = \frac{1}{\ln 3}\frac{1}{\sin x}\cos x = \frac{\cos x}{\ln 3 \sin x} = \frac{\cot x}{\ln 3}$.
2. $y' = \frac{1}{2x\ln 10}$.

Example 3.3.5 *Evaluate the following integrals:*

1. $\displaystyle\int \frac{1}{x\log x}\ dx$.
2. $\displaystyle\int \frac{1}{\sqrt{x}\ \log_2\sqrt{x}}\ dx$.

Solution:

1. $\displaystyle\int \frac{1}{x\log x}\ dx$.

   Put $u = \log x \Rightarrow du = \frac{dx}{x\ln 10}$. By substitution, we have

   $\ln(10)\displaystyle\int \frac{1}{u}\ du = \ln(10)\ \ln|\,u\,| + c = \ln(10)\ \ln|\log x\,| + c.$

2. $\displaystyle\int \frac{1}{\sqrt{x}\ log_2\sqrt{x}}\ dx$.

   Put $u = \log_2\sqrt{x} \Rightarrow du = \frac{dx}{2\ln 2\sqrt{x}}$. By substitution, we have

$$2\ln(2)\int \frac{1}{u}\,du = 2\ln(2)\ \ln|\,u\,| + c = 2\ln(2)\ \ \ln|\,\log_2\sqrt{x}\,| + c.$$

Exercise 3:

1 - 10 ■ Find the derivative of the following functions:

1. $y = 3^x$
2. $y = 2^{\sin x\ \cos x}$
3. $y = \ln(2^x)$
4. $y = \log_2 \cos x$
5. $y = \log \sqrt[3]{x+1}$
6. $y = 5^{\sqrt{x}\ \tan x}$
7. $y = x\,4^{-2x}$
8. $y = \log(x+1)$
9. $y = \ln(\sec 5^{x+1})$
10. $y = \log_5 x^{\frac{3}{2}}$

11 - 14 ■ Find the derivative of the following functions:

11. $y = (\sin x)^x$
12. $y = (e^x)^x$
13. $y = x^{e^x}$
14. $y = (x^2 - x)^{\ln x}$

15 - 20 ■ Evaluate the following integrals:

15. $\int x^2 5^{x^3}\,dx$
16. $\int 2^x \cos(2^x+1)\,dx$
17. $\int \frac{1}{x\log x^2}\,dx$
18. $\int \frac{3^x}{\sqrt{3^x+1}}\,dx$
19. $\int 7^{3x}\sqrt{7^{3x}+1}\,dx$
20. $\int \frac{\log_2 \sin x}{\tan x}\,dx$

Chapter 4

Inverse Trigonometric and Hyperbolic Functions

4.1 Inverse Trigonometric Functions

The inverse trigonometric functions are the inverse functions of the trigonometric functions: the sine, cosine, tangent, cotangent, secant, and cosecant functions. The trigonometric functions give trigonometric ratios; meaning that they are used to obtain an angle from the angle trigonometric ratios. The most common notations to name the inverse trigonometric functions are $arcsin(x)$, $arccos(x)$, $arctan(x)$, etc. However, the notations $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$, etc., are often used as well. In this book, we use the latter notations.

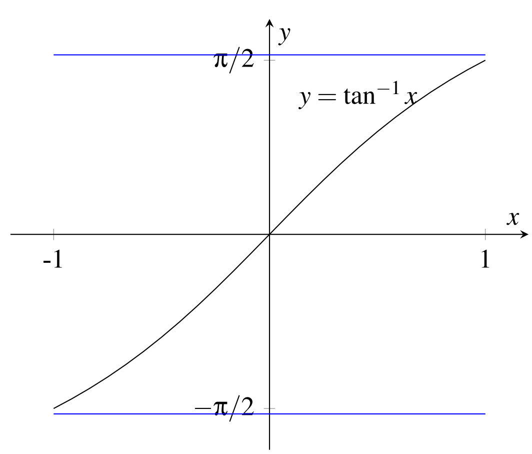
To find the inverse of any function, we need to show bijection of that function (i.e., is it one-to-one and onto?). From your knowledge, none of the six trigonometric functions are bijective. Therefore, in order to have inverse trigonometric functions, we consider subsets of the domain . In the following, we plot the inverse trigonometric functions and determine their domains and ranges.

Common mistake

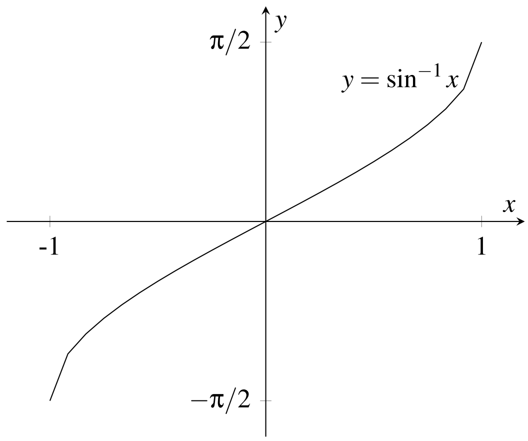
$$\sin^{-1}(x) = (\sin(x))^{-1} = \frac{1}{\sin(x)}$$

which is not true.

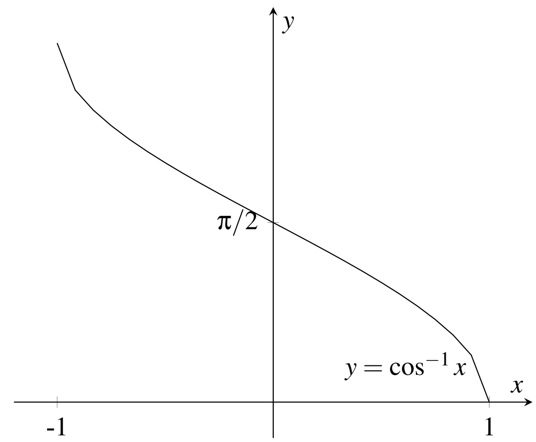
(3) The inverse tangent
$\tan y = x \Leftrightarrow y = \tan^{-1} x$
Domain: $\mathbb{R}$
Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$



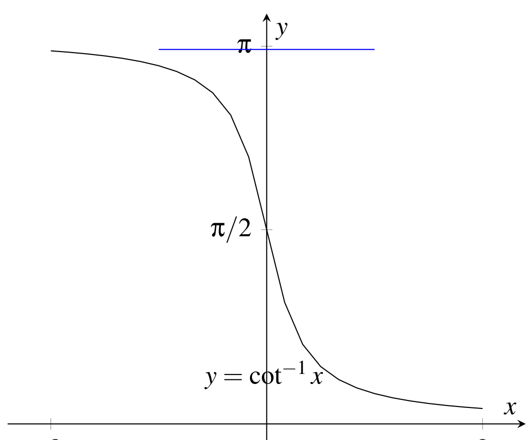
(1) The inverse sine
$\sin y = x \Leftrightarrow y = \sin^{-1} x$
Domain: $[-1,1]$ Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$



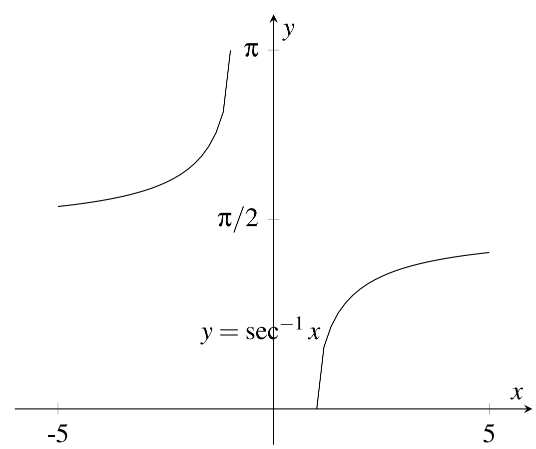
(2) The inverse cosine
$\cos y = x \Leftrightarrow y = \cos^{-1} x$
Domain: $[-1,1]$ Range: $[0,\pi]$



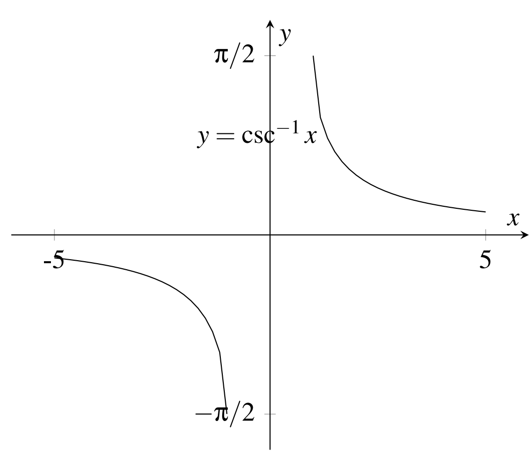
(4) The inverse cotangent
$\cot y = x \Leftrightarrow y = \cot^{-1} x$
Domain: $\mathbb{R}$ Range: $(0,\pi)$



(5) The inverse secant
$\sec y = x \Leftrightarrow y = \sec^{-1} x$
Domain: $\mathbb{R} \setminus (-1,1)$ Range: $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$



(6) The inverse cosecant
$csc\ y = x \Leftrightarrow y = csc^{-1}\ x$
Domain: $\mathbb{R} \setminus (-1,1)$
Range: $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$



Differentiating and Integrating Inverse Trigonometric Functions

In general, if $u = g(x)$ is differentiable function, then

1. $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}u'$
2. $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}}u'$
3. $\frac{d}{dx}\tan^{-1} u = \frac{1}{u^2+1}u'$
4. $\frac{d}{dx}\cot^{-1} u = \frac{-1}{u^2+1}u'$
5. $\frac{d}{dx}\sec^{-1} u = \frac{1}{u\sqrt{u^2-1}}u'$
6. $\frac{d}{dx}\csc^{-1} u = \frac{-1}{u\sqrt{u^2-1}}u'$

Example 4.1.1 *Find the derivatives of the following functions:*

1. $y = \sin^{-1}(5x)$
2. $y = \tan^{-1}(e^x)$
3. $y = \sec^{-1}(2x)$
4. $y = \sin^{-1}(x-1)$

Solution:

1. $y' = \frac{5}{\sqrt{1-25x^2}}$.
2. $y' = \frac{e^x}{(e^x)^2+1}$.
3. $y' = \frac{1}{x\sqrt{4x^2-1}}$.
4. $y' = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{2x-x^2}}$.

From the list of the derivatives of the inverse trigonometric functions, we have the following integral rules:

For $a > 0$,

1. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\tfrac{x}{a}) + c$.
2. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \tfrac{1}{a}\tan^{-1}(\tfrac{x}{a}) + c$.
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \tfrac{1}{a}\sec^{-1}(\tfrac{x}{a}) + c$.

Example 4.1.2 *Evaluate the following integrals:*

1. $\displaystyle\int \frac{1}{\sqrt{4-25x^2}}\,dx$.
2. $\displaystyle\int \frac{1}{x\sqrt{x^6-4}}\,dx$.
3. $\displaystyle\int \frac{1}{9x^2+5}\,dx$.
4. $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$.

Solution:

1. $\displaystyle\int \frac{1}{\sqrt{4-25x^2}}\,dx = \int \frac{1}{\sqrt{4-(5x)^2}}\,dx$.

   Put $u = 5x \Rightarrow du = 5dx \Rightarrow dx = \frac{du}{5}$. By substitution, we have

   $\displaystyle\tfrac{1}{5}\int \frac{1}{\sqrt{4-u^2}}\,du = \frac{1}{5}\sin^{-1}(\frac{u}{2}) + c = \frac{1}{5}\sin^{-1}(\frac{5x}{2}) + c$.

2. $\int \frac{1}{x\sqrt{x^6-4}}\,dx = \int \frac{1}{x\sqrt{(x^3)^2-4}}\,dx$.

   Put $u = x^3 \Rightarrow du = 3x^2dx$. Then , we have

   $\displaystyle\tfrac{1}{3}\int \frac{1}{u\sqrt{u^2-4}}\,du = \frac{1}{3}\frac{1}{2}\sec^{-1}(\frac{u}{2}) + c = \frac{1}{6}\sec^{-1}(\frac{x^3}{2}) + c$.

3. $\displaystyle\int \frac{1}{9x^2+5}\,dx = \int \frac{1}{(3x)^2+5}\,dx$.

   Put $u = 3x \Rightarrow du = 3dx$. By substitution, we have

$\frac{1}{3}\int \frac{1}{u^2+5}\, du = \frac{1}{3}\frac{1}{5}\tan^{-1}(\frac{u}{5})+c = \frac{1}{15}\tan^{-1}(\frac{3x}{5})+c$.

4. $\int \frac{1}{\sqrt{e^{2x}-1}}\, dx = \int \frac{1}{\sqrt{(e^x)^2-1}}\, dx$.

   Put $u = e^x \Rightarrow du = e^x dx$. After substitution , we have

   $\int \frac{1}{u\sqrt{u^2-1}}\, du = \sec^{-1}(u)+c = \sec^{-1}(e^x)+c$.

Exercise 1:

1 - 8 ■ Find the derivative of the following functions:

1. $y = \sin^{-1}(\ln x)$
2. $y = \cos^{-1}(4x^2)$
3. $y = \tan^{-1}(\sqrt{x})$
4. $y = csc^{-1}(\frac{5}{3}x)$
5. $y = \sin^{-1}(x^2+x-1)$
6. $y = \tan^{-1}(\frac{1}{x})$
7. $y = \cot^{-1}(e^{\frac{1}{x}})$
8. $y = \sec^{-1}(\ln \sqrt[3]{x})$

9 - 16 ■ Evaluate the following integrals:

9. $\int \frac{1}{\sqrt{9-x^2}}\, dx$
10. $\int \frac{1}{x^2+81}\, dx$
11. $\int \frac{1}{\sqrt{e^{2x}-4}}\, dx$
12. $\int \frac{1}{\sec x(\sin^2 x+1)}\, dx$
13. $\int \frac{1}{x\sqrt{x^8-9}}\, dx$
14. $\int \frac{e^x}{e^{2x}+1}\, dx$
15. $\int \frac{1}{x\sqrt{1-(\ln x)^2}}\, dx$
16. $\int \frac{\cot x}{\cos^2 x\sqrt{\tan^2 x-3}}\, dx$

4.2 Hyperbolic Functions

Definition 4.2.1 *The hyperbolic sine (*sinh*) and the hyperbolic cosine (*cosh*) are defined as follows:*

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \forall x \in \mathbb{R}\ ,$$

$$\cosh x = \frac{e^x + e^{-x}}{2},\ \forall x \in \mathbb{R}\ .$$

The remaining hyperbolic functions can be defined from the hyperbolic sine and the hyperbolic cosine as follows:

$$\tanh\ x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},\ \forall x \in \mathbb{R} \qquad sech\ x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},\ \forall x \in \mathbb{R}$$

$$\coth\ x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},\ \forall x \neq 0 \qquad csch x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}},\ \forall x \neq 0$$

4.2.1 Properties of Hyperbolic Functions

In this section, we provide the main characteristics of the hyperbolic functions.

1. The graph of the hyperbolic sine (sinh) and the hyperbolic cosine depends on the natural exponential functions e^x and e^{-x} (as shown in Figure 4.1).
2. From Figure 4.1, the range of sinh is $\mathbb{R}$ and the range of cosh is $[1,\infty)$.

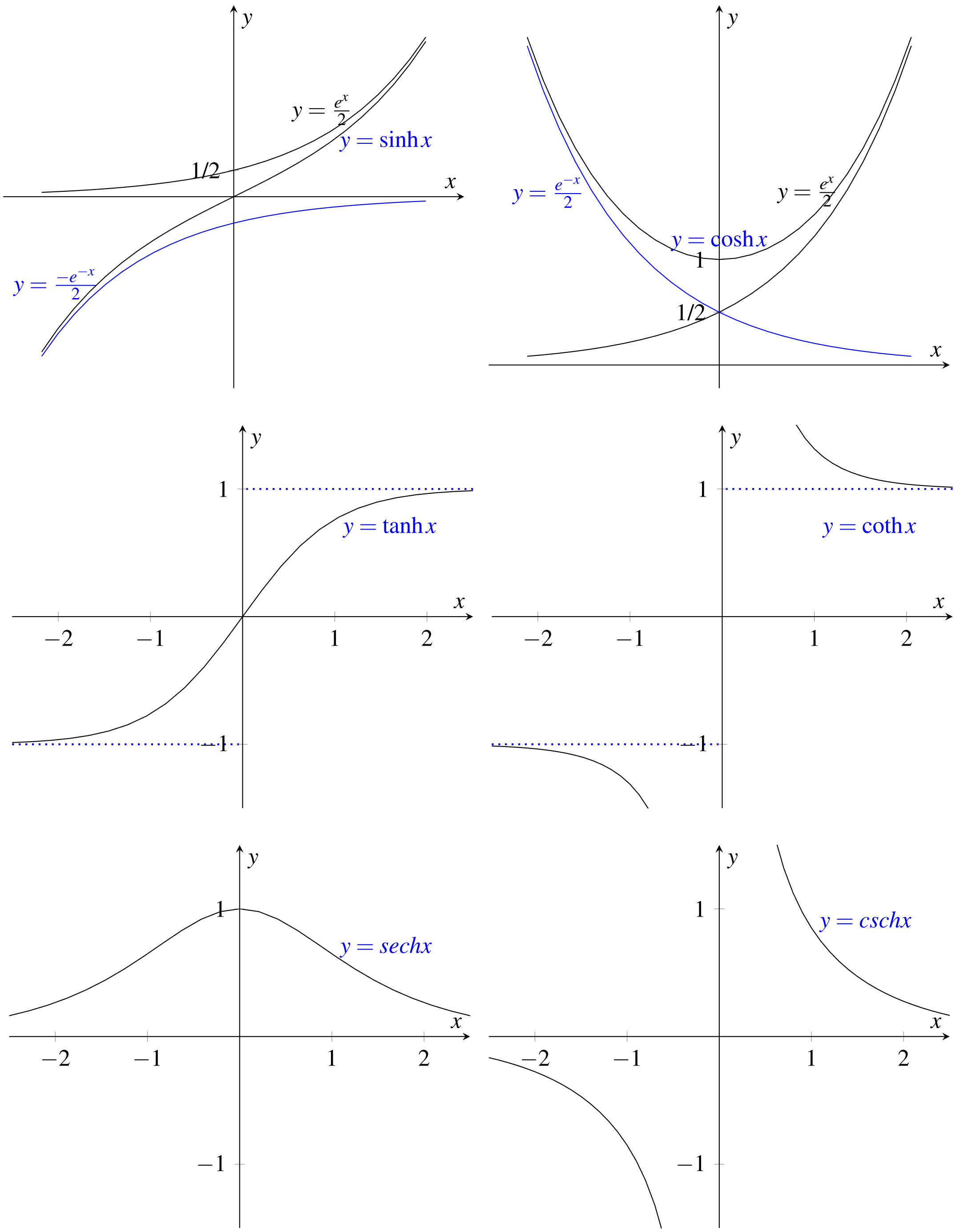


Figure 4.1: The hyperbolic functions.

3. The hyperbolic sine is an odd function (i.e., $\sinh(-x) = -\sinh x$); whereas the hyperbolic cosine is an even function (i.e., $\cosh(-x) = \cosh x$). Hence, the functions tanh, coth and are odd functions and the function is even. This means that the graphs of the functions sinh, tanh, coth and are symmetric around the original point; whereas the graph of the functions cosh and are symmetric around the y-axis.

4. $\cosh^2 x - \sinh^2 x = 1, \ \forall x \in \mathbb{R}$.

To see this, from Definition 4.2.1,

$$\cosh x - \sinh x = e^{-x} \text{ and } \cosh x + \sinh x = e^{x} .$$

Therefore,

$$(\cosh x - \sinh x)(\cosh x + \sinh x) = \cosh^2 x - \sinh^2 x = e^{-x}e^{x} = e^0 = 1 .$$

5. Since $\cos^2 t + \sin^2 t = 1$ for any $t \in \mathbb{R}$, then the point $P(\cos t, \sin t)$ located on the unit circle $x^2 + y^2 = 1$. However, for any $t \in \mathbb{R}$, the point $P(\cosh t, \sinh t)$ located on the hyperbola $x^2 - y^2 = 1$. Figure 4.2 illustrates this item.

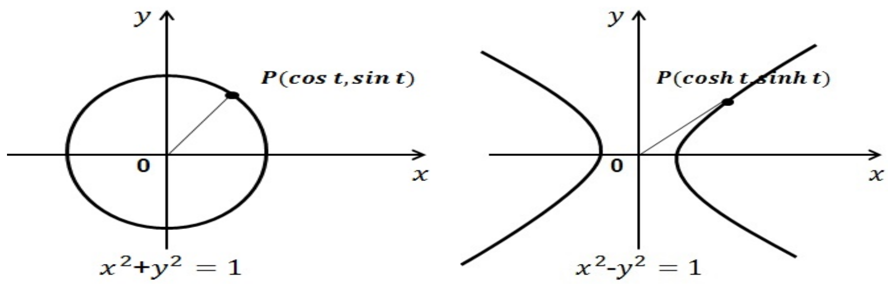


Figure 4.2: $\sin x$ and $\cos x$ versus $\sinh x$ and $\cosh x$.

As we know many identities interrelate the trigonometric functions. Similarly, the hyperbolic functions satisfies some identities given in Theorem 4.2.1.

Theorem 4.2.1

1. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
2. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
3. $\sinh(2x) = 2\sinh x \cosh x$
4. $\cosh(2x) = 2\cosh^2 x - 1 = 2\sinh^2 x + 1 = \cosh^2 x + \sinh^2 x$
5. $1 - \tanh^2 x = sech^2\, x$
6. $\coth^2 x - 1 = csch^2 x$
7. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
8. $\tanh(2x) = \dfrac{2\tanh x}{1 + \tanh^2 x}$

4.2.2 Differentiating and Integrating Hyperbolic Functions

The derivations of the hyperbolic functions are listed in Theorem 4.2.2 .

Theorem 4.2.2

1. $\frac{d}{dx} \sinh x = \cosh x$
2. $\frac{d}{dx} \cosh x = \sinh x$
3. $\frac{d}{dx} \tanh x = sech^2\, x$
4. $\frac{d}{dx} \coth x = -csch^2\, x$
5. $\frac{d}{dx} sech\, x = -sech\, x\, \tanh x$
6. $\frac{d}{dx} csch\, x = -csch\, x\, \coth x$

Example 4.2.1 *Find the derivative of the following functions:*

1. $y = \sinh x^2$
2. $y = \sqrt{x} \cosh x$
3. $y = e^{\sinh x}$
4. $y = (x+1)\tanh^2 x^3$

Solution:

1. $y' = 2x\cosh(x^2)$.
2. $y' = \frac{1}{2\sqrt{x}}\, \cosh x + \sqrt{x}\, \sinh x$.
3. $y' = e^{\sinh x}\, \cosh x$.
4. $y' = \tanh^2(x^3) + 6x^2(x+1)\, \tanh(x^3)\, sech^2(x^3)$.

Example 4.2.2 *Find $\frac{dy}{dx}$ if $y = x^{\cosh x}$.*

Solution: Take the natural logarithm (ln) for both sides

$$\ln y = \cosh x \ \ln x \ .$$

Differentiate both sides

$$\frac{y'}{y} = \sinh x \ \ln x + \frac{\cosh x}{x} \ \Rightarrow y' = \left[\sinh x \ \ln x + \frac{\cosh x}{x}\right] x^{\cosh x} \ .$$

From the list of the derivation given in Theorem 4.2.2, we have the following list of integrals:

- $\int \sinh x \, dx = \cosh x + c$
- $\int \cosh x \, dx = \sinh x + c$
- $\int sech^2 \, x \, dx = \tanh x + c$
- $\int csch^2 x \, dx = -\coth x + c$
- $\int sech \ x \ \tanh x \, dx = -sech \ x + c$
- $\int csch \ x \ \coth x \, dx = -csch \ x + c$

Example 4.2.3 *Evaluate the following integrals:*

1. $\int \sinh^2 x \ \cosh x \, dx$

2. $\int e^{\cosh x} \ \sinh x \, dx$

3. $\int \tanh x \, dx$

4. $\int e^x \ sech \ x \, dx$

Solution:

1. $\int \sinh^2 x \ \cosh x \, dx$.

   Put $u = \sinh x \Rightarrow du = \cosh x \, dx$. By substitution, we have $\int u^2 \, du = u^3/3 + c$. This implies

   $$\int \sinh^2 x \cosh x \, dx = \frac{\sinh^3 x}{3} + c \ .$$

2. $\int e^{\cosh x} \sinh x \, dx$

   Put $u = \cosh x \Rightarrow du = \sinh x \, dx$. By substitution, we have $\int e^u \, du = e^u + c$. Hence,

   $$\int e^{\cosh x} \sinh x \, dx = e^{\cosh x} + c \ .$$

3. $\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$

   Let $u = \cosh x \Rightarrow du = \sinh x \, dx$. Then, we have $\int \frac{1}{u} \, du = \ln \mid u \mid + c$. This implies

   $$\int \tanh x \, dx = \ln \mid \cosh x \mid + c \ .$$

4. $\int e^x \ sech \ x \, dx = \int \frac{2e^x}{e^x + e^{-x}} \, dx = \int \frac{2e^{2x}}{e^{2x} + 1} \, dx$

   Put $u = e^{2x} \Rightarrow du = 2e^{2x} \, dx$. By substitution, we have $\int \frac{1}{u+1} \, du = \ln \mid u + 1 \mid + c = \ln \mid e^{2x} + 1 \mid + c$.

Exercise 2:

1 - 10 ■ Find the derivative of the following functions:

1. $y = \sinh(\sqrt{x^3})$
2. $y = \tanh(5x)$
3. $y = e^{-x}\cosh(x)$
4. $y = e^{\sinh(2x)}$
5. $y = \ln(\coth(x))$
6. $y = \sqrt{csch(x)}$
7. $y = \sinh(\tan x)$
8. $y = \cosh(e^{\sqrt{x}})$
9. $y = \tanh(\ln x)$
10. $y = \sqrt{x+1}\ (x)$

11 - 20 ■ Evaluate the following integrals:

11. $\int \frac{\sinh(\sqrt{x})}{\sqrt{x}}\,dx$
12. $\int \frac{\cosh(\ln x)}{x}\,dx$
13. $\int e^x \tanh(e^x)\,dx$
14. $\int (1+\tanh x)^3 sech^2\,x\,dx$
15. $\int \frac{e^{\sinh x}}{sech\ x}\,dx$
16. $\int \frac{sech\ x\ \tanh x}{1+sech\ x}\,dx$
17. $\int \sqrt{3+\cosh x}\ \sinh x\,dx$
18. $\int \frac{\tanh(\sqrt{x})\ \left(sech(\sqrt{x})+1\right)}{\sqrt{x}}\,dx$
19. $\int \frac{1}{\cosh^2 x\ \tanh x}\,dx$
20. $\int \frac{\ln(\coth x)}{\sinh^2 x}\,dx$

4.3 Inverse Hyperbolic Functions

In the first section of this chapter, we defined the inverse trigonometric functions. In analogical way, we define the inverse hyperbolic functions.

4.3.1 Definition and Properties

(1) The function $\sinh : \mathbb{R} \to \mathbb{R}$ is bijective (i.e., it is one-to-one and onto), so it has an inverse function

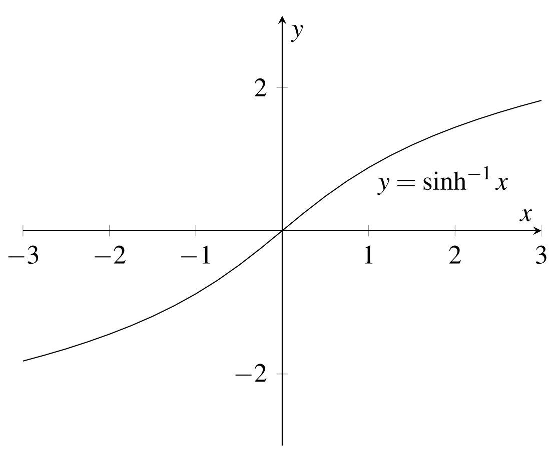
$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$$

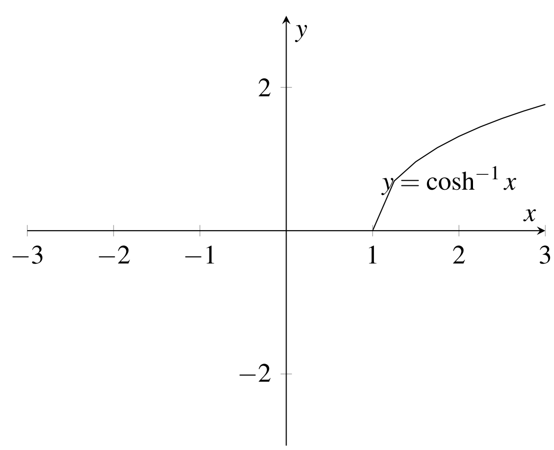
$$\sinh y = x \Leftrightarrow y = \sinh^{-1} x$$

(2) The function cosh is injective on $[0,\infty)$, so $\cosh : [0,\infty) \to [1,\infty)$ is bijective on $[0,\infty)$. It has an inverse function

$$\cosh^{-1} : [1,\infty) \to [0,\infty)$$

$$\cosh y = x \Leftrightarrow y = \cosh^{-1} x$$

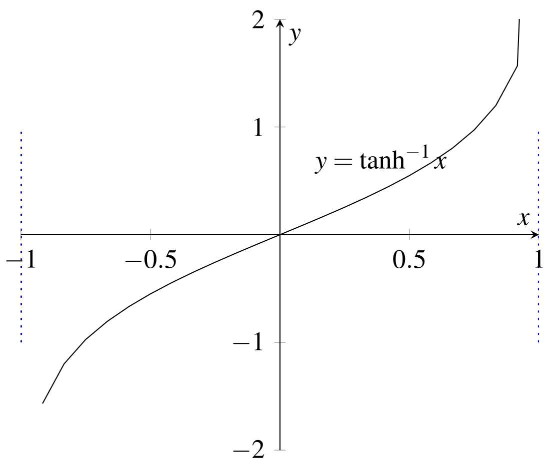




(3) The function $\tanh : \mathbb{R} \to (-1,1)$ is bijective, so it has an inverse function

$$\tanh^{-1} : (-1,1) \to \mathbb{R}$$

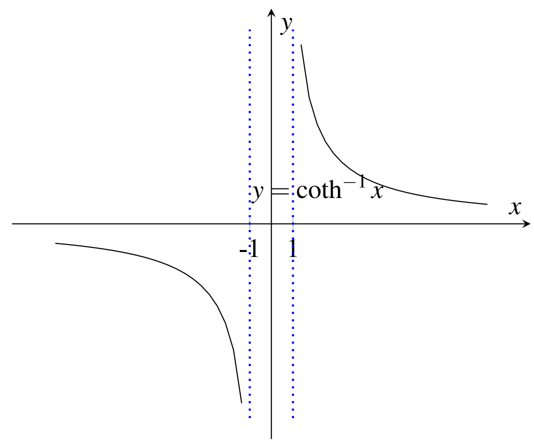
$$\tanh y = x \Leftrightarrow y = \tanh^{-1} x$$



(4) The function $\coth : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus [-1,1]$ is bijective, so it has an inverse function

$$\coth^{-1} : \mathbb{R} \setminus [-1,1] \to \mathbb{R} \setminus \{0\}$$

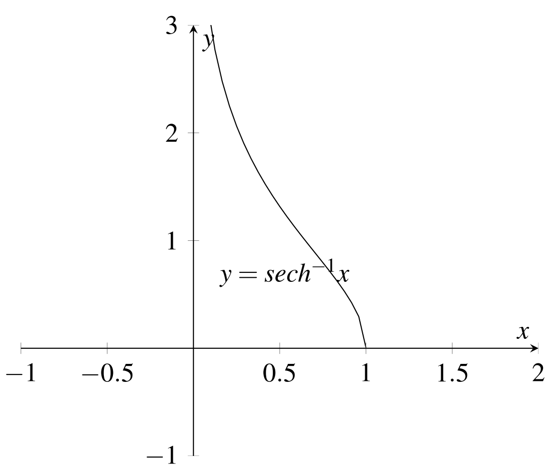
$$\coth y = x \Leftrightarrow y = \coth^{-1} x$$



(5) The function $sech : (0,1] \to [0,\infty)$ is bijective and the inverse function is

$$sech^{-1} : [0,\infty) \to (0,1]$$

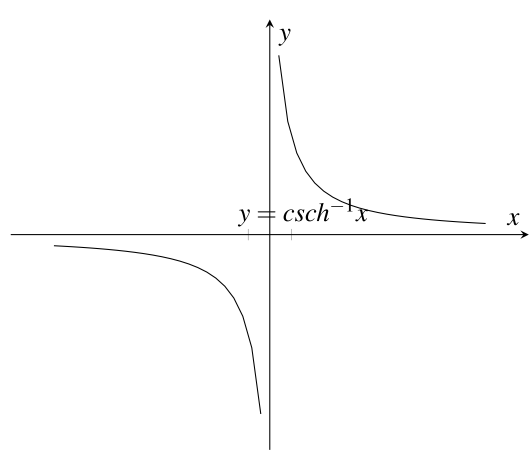
$$sechy = x \Leftrightarrow y = sech^{-1}\, x$$



(6) The function $csch : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is bijective and the inverse function is

$$csch^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$$

$$csch\; y = x \Leftrightarrow y = csch^{-1}\, x$$



Theorem 4.3.1

1. $\sinh^{-1}x = \ln(x+\sqrt{x^2+1}),\ \forall x \in \mathbb{R}$
2. $\cosh^{-1}x = \ln(x+\sqrt{x^2-1}),\ \forall x \in [1,\infty)$
3. $\tanh^{-1}x = \frac{1}{2}\ln\frac{1+x}{1-x},\ \forall x \in (-1,1)$
4. $\coth^{-1}x = \frac{1}{2}\ln\frac{x+1}{x-1},\ \forall x \in \mathbb{R}\setminus[-1,1]$
5. $sech^{-1}\,x = \ln(\frac{1+\sqrt{1-x^2}}{x}),\ \forall x \in \mathbb{R}\setminus[-1,1]$
6. $csch^{-1}x = \ln(\frac{1}{x}+\sqrt{1+(\frac{1}{x})^2}),\ \forall x \in \mathbb{R}\setminus\{0\}$

4.3.2 Differentiation and Integration

Theorem 4.3.2

1. $\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}$
2. $\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}},\ \forall x \in (1,\infty)$
3. $\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2},\ \forall x \in (-1,1)$
4. $\frac{d}{dx}\coth^{-1}x = \frac{1}{1-x^2},\ \forall x \in \mathbb{R}\setminus[-1,1]$
5. $\frac{d}{dx}sech^{-1}\,x = \frac{-1}{x\sqrt{1-x^2}},\ \forall x \in (0,1)$
6. $\frac{d}{dx}csch^{-1}\,x = \frac{-1}{|x|\sqrt{x^2+1}},\ \forall x \in \mathbb{R}\setminus\{0\}$

Example 4.3.1 *Find the derivative of the following functions:*

1. $y = \sinh^{-1}(\sqrt{x})$
2. $y = \tanh^{-1}(e^x)$
3. $y = \cosh^{-1}(4x^2)$
4. $y = \ln(\sinh^{-1}x)$
5. $y = csch^{-1}(4x)$
6. $y = x\ \tanh^{-1}(\frac{1}{x})$
7. $y = (\tanh^{-1}x)^2$
8. $y = e^x\ sech^{-1}x$

Solution:

1. $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}}\ \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x(x+1)}}$.
2. $y' = \frac{1}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}$.
3. $y' = \frac{8x}{\sqrt{16x^4-1}}$.
4. $y' = \frac{1}{\sinh^{-1}x}\ \frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}\ \sinh^{-1}x}$.
5. $y' = \frac{-1}{|4x|\sqrt{16x^2+1}}(4) = \frac{-1}{|x|\sqrt{16x^2+1}}$.
6. $y' = \tanh^{-1}(\frac{1}{x}) + x\,(\frac{1}{1-(\frac{1}{x})^2})(\frac{-1}{x^2}) = \tanh^{-1}(\frac{1}{x}) - \frac{x}{x^2-1}$.
7. $y' = 2\tanh^{-1}x\ \frac{1}{1-x^2} = \frac{2\tanh^{-1}x}{1-x^2}$
8. $y' = e^x\ sech^{-1}x - \frac{e^x}{x\sqrt{1-x^2}}$

From the list of the derivatives given in Theorem 4.3.2, we have the following list of integrals:

- $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a}+c.$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\frac{x}{a}+c,\ x>a.$
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\frac{x}{a}+c,\ |\,x\,|<a.$
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\coth^{-1}\frac{x}{a}+c,\ |\,x\,|>a.$
- $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}sech^{-1}\ \frac{|x|}{a}+c,\ |\,x\,|<a.$
- $\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}csch^{-1}\ \frac{|x|}{a}+c,\ |\,x\,|>a.$

Example 4.3.2 *Evaluate the following integrals:*

1. $\int \frac{1}{\sqrt{x^2-4}}\,dx$
2. $\int \frac{1}{\sqrt{4x^2+9}}\,dx$
3. $\int \frac{1}{\sqrt{e^{2x}+9}}\,dx$
4. $\int \frac{1}{x\sqrt{1-x^6}}\,dx$
5. $\int_0^1 \frac{1}{16-x^2}\,dx$
6. $\int_5^7 \frac{1}{16-x^2}\,dx$

Solution:

1. $\int \frac{1}{\sqrt{x^2-4}}\,dx = \cosh^{-1}(\frac{x}{2})+c$.

2. $\int \frac{1}{\sqrt{4x^2+9}}\,dx = \int \frac{1}{\sqrt{(2x)^2+9}}\,dx$.

   Put $u = 2x \Rightarrow du = 2dx \Rightarrow dx = \frac{dx}{2}$. By substitution, we have

$$\frac{1}{2}\int \frac{1}{\sqrt{u^2+9}}\,du = \frac{1}{2}\sinh^{-1}(\frac{u}{3})+c = \frac{1}{2}\sinh^{-1}(\frac{2x}{3})+c\ .$$

3. $\int \frac{1}{\sqrt{e^{2x}+9}}\,dx = \int \frac{1}{\sqrt{(e^x)^2+9}}\,dx$.

   Put $u = e^x \Rightarrow du = e^x dx$. By substituting the result into the integral, we have

$$\int \frac{1}{u\sqrt{u^2+9}}\,du = \frac{-1}{3}csch^{-1}(\frac{|\,u\,|}{3})+c = \frac{-1}{3}csch^{-1}(\frac{e^x}{3})+c\ .$$

4. $\int \frac{1}{x\sqrt{1-x^6}}\,dx = \int \frac{1}{x\sqrt{1-(x^3)^2}}\,dx$.

   Put $u = x^3 \Rightarrow du = 3x^2dx$. By substitution, we have

$$\frac{1}{3}\int \frac{1}{u\sqrt{1-u^2}}\,du = -\frac{1}{3}sech^{-1}(|\,u\,|)+c = -\frac{1}{3}sech^{-1}(|\,x^3\,|)+c\ .$$

5. Since the interval of the integral is sub-interval of $(-4,4)$, the value of the integral is $\tanh^{-1}$. Hence,

$$\int_0^1 \frac{1}{16-x^2}\,dx = \frac{1}{4}\Big[\tanh^{-1}\frac{x}{4}\Big]_0^1 = \frac{1}{4}\Big[\frac{1}{2}\ln(\frac{5}{3})-\frac{1}{2}\ln(1)\Big] = \frac{1}{8}\ln(\frac{5}{3})\ .$$

6. Since the interval of the integral is not sub-interval of $(-4,4)$, the value of the integral is $\coth^{-1}$. Hence,

$$\int_5^7 \frac{1}{16-x^2}\,dx = \frac{1}{4}\Big[\coth^{-1}\frac{x}{4}\Big]_5^7 = \frac{1}{8}\Big[\ln(11)-2\ln(3)\Big]\ .$$

Exercise 3:

1 - 6 ■ Find the derivative of the following functions:

1. $y = \sinh^{-1}(\tan x)$
2. $y = \cosh^{-1}(e^{\sqrt{x}})$
3. $y = \tanh^{-1}(\ln x)$
4. $y = \sqrt{x+1}\ csch^{-1}(x)$
5. $y = \tan x\ \tanh^{-1}(x)$
6. $y = (2x-1)^3\sinh^{-1}(\sqrt{x})$

7 - 14 ■ Evaluate the following integrals:

7. $\int \frac{1}{\sqrt{2x^2-2}}\,dx$
8. $\int \frac{e^x}{1-e^{2x}}\,dx$
9. $\int \frac{1}{x\sqrt{1-x^4}}\,dx$
10. $\int \frac{1}{\sqrt{x^2+25}}\,dx$
11. $\int \frac{1}{\sqrt{x^2-25}}\,dx$
12. $\int \frac{1}{\sec\ x(1-\sin^2 x)}\,dx$
13. $\int \frac{1}{x\sqrt{x^6+2}}\,dx$
14. $\int \frac{1}{\sqrt{4-e^{2x}}}\,dx$

Chapter 5

Techniques of Integration

5.1 Integration by Parts

Integration by parts is a method to transfer the original integral to an easier integral that can be evaluated. Practically, the integration by parts divides the original integral into two parts u and dv. Then, we try to find the du by deriving u and v by integrating dv.

Theorem 5.1.1 *If $u = f(x)$ and $v = g(x)$ such that $f'(x)$ and $g'(x)$ are continuous, then*

$$\int u\, dv = uv - \int v\, du\ .$$

Theorem 5.1.1 shows that the integration by parts transfers the integral $\int u\, dv$ into the integral $\int v\, du$ that should be easier than the original integral. The question here is, what we choose as $u(x)$ and what we choose as $dv = v'(x)\, dx$. It is useful to choose u as a function can be easily differentiated, and to choose dv as a function that can be easily integrated. This statement is clearly explained through the following examples.

Example 5.1.1 *Evaluate the following integral* $\int x\ \cos x\, dx$.

Solution:

Let $I = \int x\ \cos x\, dx$. Put $u = x$ and $dv = \cos x\, dx$. Hence,

$$u = x \Rightarrow du = dx\ ,$$
$$dv = \cos x\, dx \Rightarrow v = \int \cos x\, dx = \sin x\ .$$

Try to choose
$u = \cos x$ and $dv = x\, dx$
Do you have the same result?

From Theorem 5.1.1

$$I = x\ \sin x - \int \sin x\, dx = x\ \sin x + \cos x + c\ .$$

Example 5.1.2 *Evaluate the following integral* $\int x\, e^x\, dx$.

Solution:

Let $I = \int x\, e^x\, dx$. Put $u = x$ and $dv = e^x\, dx$. Hence,

$$u = x \Rightarrow du = dx\ ,$$
$$dv = e^x\, dx \Rightarrow v = \int e^x\, dx = e^x\ .$$

Try to choose $u = e^x$ and $dv = x$

From Theorem 5.1.1, $I = x\, e^x - \int e^x\, dx = x\, e^x - e^x + c$.

Remark 5.1.1

1. *Remember that when we consider the integration by parts, we want to have an easier integral. As we saw in Example 5.1.2, if we choose $u = e^x$ and $dv = x\, dx$, we have $\int \frac{x^2}{2}\, e^x\, dx$ which is more difficult than the original one.*
2. *When considering the integration by parts, we have to choose dv a function that can be integrated (see Example 5.1.3).*
3. *Sometimes we need to use the integration by parts two times as in Examples 5.1.4 and 5.1.5.*

Example 5.1.3 *Evaluate the following integral $\int \ln x\, dx$.*

Solution: Let $I = \int \ln x\, dx$. Let $u = \ln x$ and $dv = dx$. Hence,

$$u = \ln x \Rightarrow du = \frac{1}{x}\, dx\ ,$$
$$dv = dx \Rightarrow v = \int dx = x\ .$$

From Theorem 5.1.1,

$$I = x \ln x - \int x \frac{1}{x}\, dx = x \ln x - \int dx = x \ln x - x + c\ .$$

Example 5.1.4 *Evaluate the following integral $\int e^x \cos x\, dx$.*

Solution: Let $I = \int e^x \cos x\, dx$. Put $u = e^x$ and $dv = \cos x\, dx$.

$$u = e^x \Rightarrow du = e^x\, dx\ ,$$
$$dv = \cos x\, dx \Rightarrow v = \int \cos x\, dx = \sin x\ .$$

Hence, $I = e^x \sin x - \int e^x \sin x\, dx$.

The integral $\int e^x \sin x\, dx$ cannot be evaluated. Therefore, we use the integration by parts again where we assume $J = \int e^x \sin x\, dx$. Put $u = e^x$ and $dv = \sin x\, dx$. Hence,

$$u = e^x \Rightarrow du = e^x\, dx\ ,$$
$$dv = \sin x\, dx \Rightarrow v = \int \sin x\, dx = -\cos x\ .$$

This implies $J = -e^x \cos x + \int e^x \cos x\, dx$.

By substituting the result of J into I, we have

$$\begin{aligned} I &= e^x \sin x - \int e^x \sin x \, dx \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \\ &= e^x \sin x + e^x \cos x - I \, . \end{aligned}$$

This implies $2I = e^x \sin x + e^x \cos x \Rightarrow \int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + c$.

Example 5.1.5 *Evaluate the following integral* $\int x^2 e^x \, dx$*.*

Solution: Let $I = \int x^2 e^x \, dx$. Put $u = x^2$ and $dv = e^x \, dx$. Hence,

$$\begin{aligned} u = x^2 &\Rightarrow du = 2x \, dx \, , \\ dv = e^x dx &\Rightarrow v = \int e^x \, dx = e^x \, . \end{aligned}$$

This implies, $I = x^2 e^x - 2 \int x e^x \, dx$.

Now, we the integration by parts again for the integral $\int x e^x \, dx$. Let $J = \int x e^x \, dx$.

Put $u = x$ and $dv = e^x \, dx$. Hence,

$$\begin{aligned} u = x &\Rightarrow du = \; dx \, , \\ dv = e^x dx &\Rightarrow v = \int e^x \, dx = e^x \, . \end{aligned}$$

This implies $J = x e^x - \int e^x \, dx = x e^x - e^x$. By substituting the result into I, we have

$$I = x^2 e^x - 2(x e^x - e^x) + c = e^x (x^2 - 2x + 2) + c \, .$$

Example 5.1.6 *Evaluate the following integral* $\int_0^1 \tan^{-1} x \, dx$*.*

Solution:

Let $I = \int \tan^{-1} x \, dx$. Put $u = \tan^{-1} x$ and $dv = dx$. Hence,

$$\begin{aligned} u = \tan^{-1} x &\Rightarrow du = \frac{1}{x^2 + 1} \, dx \, , \\ dv = dx &\Rightarrow v = \int dx = x \, . \end{aligned}$$

From Theorem 5.1.1,

$$I = x \, \tan^{-1} x - \int \frac{x}{x^2 + 1} \, dx = x \, \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + c \, .$$

From this, $\int_0^1 \tan^{-1} x \, dx = \left[x \, \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) \right]_0^1 = (\tan^{-1}(1) - \frac{1}{2} \ln 2) - (0 - \frac{1}{2} \ln 1) = \frac{\pi}{4} - \ln \sqrt{2}$.

Exercise 1:

1 - 16 ■ Evaluate the following integrals:

1. $\int x^3 \ln x \, dx$
2. $\int_0^{\pi/3} \sin x \ln(\cos x) \, dx$
3. $\int \sin^{-1} x \, dx$
4. $\int x^3 \sqrt{4-x^2} \, dx$
5. $\int x \sin x \, dx$
6. $\int x^2 \cos x \, dx$
7. $\int e^x \sin 2x \, dx$
8. $\int_0^1 \tan^{-1} x \, dx$
9. $\int e^{2x} \cos x \, dx$
10. $\int (\ln x)^2 \, dx$
11. $\int \frac{\ln x}{x^2} \, dx$
12. $\int x \sin x \, \cos x \, dx$
13. $\int \frac{1}{x(\ln x)^3} \, dx$
14. $\int_0^1 x^2 e^x \, dx$
15. $\int x \tan^{-1} x \, dx$
16. $\int x e^{-x} \, dx$

5.2 Trigonometric Functions

5.2.1 Integration of Power of Trigonometric Functions

Form 1: $\int \sin^n x \, \cos^m x \, dx$.

This form of integrals is treated as follows:

1. If n is odd, we write $\sin^n x \, \cos^m x = \sin^{n-1} x \, \cos^m x \, \sin x$. Then, we use the identity $\cos^2 x + \sin^2 x = 1$ and the substitution $u = \cos x$.

2. If m is odd, we write $\cos^m x \, \sin^n x = \cos^{m-1} x \, \sin^n x \, \cos x$. Then, we use the identity $\cos^2 x + \sin^2 x = 1$ and the substitution $u = \sin x$.

3. If m and n are even, we use the identities $\cos^2 x = \frac{1+cos2x}{2}$ and $\sin^2 x = \frac{1-cos2x}{2}$.

Example 5.2.1 *Evaluate the following integrals:*

1. $\int \sin^3 x \, dx$

2. $\int \cos^4 x \, dx$

3. $\int \sin^5 x \cos^4 x \, dx$

4. $\int \sin^2 x \cos^2 x \, dx$

Solution:

1. $\int \sin^3 x \, dx$.

   We write $\sin^3 x = \sin^2 x \sin x = (1-\cos^2 x)\sin x$. This implies $\int \sin^3 x \, dx = \int (1-\cos^2 x)\sin x \, dx$.

   Put $u = \cos x$, this implies $du = -\, sinx \, dx$. By substitution, we have

$$\int (1-u^2) \, du = -u + \frac{u^3}{3} + c \Rightarrow \int \sin^3 x \, dx = -\cos x + \frac{\cos^3 x}{3} + c \, .$$

2. $\int \cos^4 x \, dx$.

We write $\cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$. This implies

$$\begin{aligned}\int \cos^4 x\, dx &= \int (\frac{1+\cos 2x}{2})^2\, dx \\ &= \frac{1}{4}\int 1+2\cos 2x+\cos^2 2x\, dx \\ &= \frac{1}{4}\int 1\, dx+\frac{1}{4}\int 2\cos 2x\, dx+\frac{1}{4}\cos^2 2x\, dx \\ &= \frac{1}{4}x+\frac{1}{4}\sin 2x+\frac{1}{8}\int 1+\cos 4x\, dx \\ &= \frac{1}{4}x+\frac{1}{4}\sin 2x+\frac{1}{8}(x+\frac{\sin 4x}{4})+c\ .\end{aligned}$$

3. $\int \sin^5 x\ \cos^4 x\, dx$

   We write $\sin^5 x\ \cos^4 x = \sin^4 x\ \cos^4 x\ \sin x = (1-\cos^2 x)^2\ \cos^4 x\ \sin x$.

   Let $u = \cos x \Rightarrow du = -\sin x\, dx$. Thus, the integral becomes

$$-\int (1-u^2)^2 u^4\, du = -\int u^4 - 2u^6 + u^8\, du = -(\frac{u^5}{5}-\frac{2}{7}u^7+\frac{u^9}{9})+c\ .$$

   This implies $\int \sin^5 x\cos^4 x\, dx = -\frac{\cos^5 x}{5}+\frac{2}{7}\cos^7 x-\frac{\cos^9 x}{9}+c$.

4. $\int \sin^2 x\cos^2 x\, dx$

   The integrand $\sin^2 x\ \cos^2 x = (\frac{1-\cos 2x}{2})(\frac{1+\cos 2x}{2}) = \frac{1-\cos^2 2x}{4} = \frac{sin^2 2x}{4} = \frac{1}{4}(\frac{1-\cos 4x}{2})$. Thus, the integral becomes

$$\frac{1}{8}\int 1-\cos 4x\, dx = \frac{1}{8}(x-\frac{sin 4x}{4})+c\ .$$

Form 2: $\int \tan^n x\ \ \sec^m x\, dx$.

This form is treated as follows:

1. If $n = 0$ and
   (a) $m = 1$, we write $\sec x = \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}$, then we use the substitution $u = \sec x\ \tan x$.
   (b) $m > 1$ is odd, we write $\sec^m x = \sec^{m-2} x\ \ \sec^2 x$, then we use the integration by parts.
   (c) m is even, we write $\sec^m x = \sec^{m-2} x\ \ \sec^2 x$, then we use the identity $\sec^2 x = 1+\tan^2 x$ and the substitution $u = \tan x$.
2. If $m = 0$ and
   (a) $n = 1$, we write $\tan x = \frac{\sin x}{\cos x}$, then we use the substitution $u = \cos x$.
   (b) n is odd or even , we write $\tan^n x = \tan^{n-2} x\ \ \tan^2 x$, then we use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \tan x$.
3. If n is even and m is odd, we use the identity $\tan^2 x = \sec^2 x - 1$ to change the integral to $\int \sec^r x\, dx$.
4. If $m \geq 2$ is even, we write $\tan^n x\ \sec^m x = \tan^n x\ \sec^{m-2} x\ \sec^2 x$, then we use the identity $\sec^2 x = 1+\tan^2 x$ and the substitution $u = \tan x$.

5. If n is odd and $m \geq 1$, we write $\tan^n x\ \sec^m x = \tan^{n-1} x\ \sec^{m-1} x\ \tan x\ \sec x$, then we use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \sec x$.

Example 5.2.2 *Evaluate the following integrals:*

1. $\int \tan^5 x\, dx$
2. $\int \tan^6 x\, dx$
3. $\int \sec^3 x\, dx$
4. $\int \tan^5 x\ \sec^4 x\, dx$
5. $\int \tan^4 x\ \sec^4 x\, dx$

Solution:

1. Write $\tan^5 x = \tan^3 x\ \tan^2 x = \tan^3 x\ (\sec^2 x - 1)$. Thus,

$$\begin{aligned}\int \tan^5 x\, dx &= \int \tan^3 x\ (\sec^2 x - 1)\ dx\\ &= \int \tan^3 x\ \sec^2 x\, dx - \int \tan^3 x\, dx\\ &= \frac{\tan^4 x}{4} - \int \tan x\ (\sec^2 x - 1)\ dx\\ &= \frac{\tan^4 x}{4} - \int \tan x\ \sec^2 x\, dx + \int \tan x\, dx\\ &= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec\ x| + c\ .\end{aligned}$$

2. Write $\tan^6 x = \tan^4 x\ \tan^2 x = \tan^4 x\ (\sec^2 x - 1)$. From this, the integral becomes

$$\begin{aligned}\int \tan^6 x\, dx &= \int \tan^4 x\ (\sec^2 x - 1)\ dx\\ &= \int \tan^4 x\ \sec^2 x\, dx - \int \tan^4 x\, dx\\ &= \frac{\tan^5 x}{5} - \int \tan^2 x\ (\sec^2 x - 1)\ dx\\ &= \frac{\tan^5 x}{5} - \int \tan^2 x\ \sec^2 x\, dx + \int \tan^2 x\, dx\\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int \sec^2 x - 1\, dx\\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c\ .\end{aligned}$$

3. Write $\sec^3 x = \sec x\ \sec^2 x$ and let $I = \int \sec x\ \sec^2 x\, dx$.

   We use the integration by part to evaluate the integral as follows:

$$u = \sec x \Rightarrow du = \sec x\ \tan x\, dx\ ,$$

$$dv = \sec^2 x\, dx \Rightarrow v = \int \sec^2 x\, dx = \tan x\ .$$

   Hence,

$$\begin{aligned}I &= \sec x\ \tan x - \int \sec x\ \tan^2 x\, dx\\ &= \sec x\ \tan x - \int \sec^3 x - \sec x\, dx\\ &= \sec x\ \tan x - I + \ln|\sec x + \tan x|\\ &= \frac{1}{2}(\sec x\ \tan x + \ln|\sec x + \tan x|) + c\ .\end{aligned}$$

4. Express the integrand $\tan^5 x\ \sec^4 x$ as follows

$$\tan^5 x\ \sec^4 x = \tan^5 x\ \sec^2 x\ \sec^2 x = \tan^5 x\ (\tan^2 x + 1)\ \sec^2 x\ .$$

This implies

$$\begin{aligned}\int \tan^5 x\ \sec^4 x\, dx &= \int \tan^5 x\ (\tan^2 x + 1)\ \sec^2 x\, dx \\ &= \int (\tan^7 x + \tan^5 x)\ \sec^2 x\, dx \\ &= \frac{\tan^8 x}{8} + \frac{\tan^6 x}{6} + c\ .\end{aligned}$$

5. Write $\tan^4 x\ \sec^4 x = \tan^4 x\ (\tan^2 x + 1)\ \sec^2 x$. The integral becomes

$$\begin{aligned}\int \tan^4 x\ \sec^4 x\, dx &= \int \tan^4 x\ (\tan^2 x + 1)\ \sec^2 x\, dx \\ &= (\tan^6 x + \tan^4 x)\ \sec^2 x\, dx \\ &= \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + c\ .\end{aligned}$$

Form 3: $\displaystyle\int \cot^n x\, csc^m x\, dx$.

This form of integrals is treated as the integral $\displaystyle\int \tan^n x\ \sec^m x\, dx$, except we use the identity $csc^2 x = 1 + \cot^2 x$.

Example 5.2.3 *Evaluate the following integrals:*

1. $\displaystyle\int \cot^3 x\, dx$ *2.* $\displaystyle\int \cot^4 x\, dx$ *3.* $\displaystyle\int \cot^5 x csc^4 x\, dx$

Solution:

1. Write $\cot^3 x = \cot x\ (\csc^2 x - 1)$. Then,

$$\begin{aligned}\int \cot^3 x\, dx &= \int \cot x\ (\csc^2 x - 1)\ dx \\ &= \int (\cot x\ \csc^2 x - \cot x)\ dx = \frac{-1}{2}\cot^2 x - \ln|\sin x| + c\ .\end{aligned}$$

2. The integrand can be expressed as $\cot^4 x = \cot^2 x(\csc^2 x - 1)$. Thus,

$$\begin{aligned}\int \cot^4 x\, dx &= \int \cot^2 x\ (\csc^2 x - 1)\ dx \\ &= \int \cot^2 x\, csc^2 x\, dx - \int \cot^2 x\, dx = \frac{-\cot^3 x}{3} + \cot x + x + c\ .\end{aligned}$$

3. Express the integrand as $\cot^5 x\ \csc^4 x = \csc^3 x\ \cot^4 x\ \csc x\ \cot x$. This implies

$$\begin{aligned}\int \cot^5 x\ \csc^4 x\, dx &= \int \csc^3 x\ \cot^4 x \csc x\ \cot x\, dx \\ &= \int \csc^3 x\ (\csc^2 x - 1)^2 \csc x\ \cot x\, dx \\ &= \int (\csc^7 x - 2\csc^5 x + \csc^3 x)\ \csc x\ \cot x\, dx \\ &= \frac{-\csc^8 x}{8} + \frac{\csc^6 x}{3} - \frac{\csc^4 x}{4} + c\ .\end{aligned}$$

5.2.2 Integration of Forms sin u x cos v x, sin u x sin v x and cos u x cos v x

We deal with these integrals by using the following formulas:

$$\sin ux\ \cos vx = \frac{1}{2}\left(\sin(u-v)\,x+\sin(u+v)\,x\right),$$
$$\sin ux\ \sin vx = \frac{1}{2}\left(\cos(u-v)\,x-\cos(u+v)\,x\right),$$
$$\cos ux\ \cos vx = \frac{1}{2}\left(\cos(u-v)\,x+\cos(u+v)\,x\right).$$

Example 5.2.4 *Evaluate the following integrals:*

1. $\int \sin 5x\ \sin 3x\, dx$
2. $\int \sin 7x\ \cos 2x\, dx$
3. $\int \cos 5x\ \sin 2x\, dx$
4. $\int \cos 4x\ \sin 6x\, dx$

Solution:

1. $\int \sin 5x\ \sin 3x\, dx$.

   From the formulas given above, we have $\sin 5x\ \sin 3x = \frac{1}{2}\left(\cos(2)x-\cos(8)x\right)$. Thus,

$$\int \sin 5x\ \sin 3x\, dx = \frac{1}{2}\int (\cos 2x-\cos 8x)\, dx = \frac{1}{4}\sin 2x-\frac{1}{16}\sin 8x+c\ .$$

2. $\int \sin 7x\ \cos 2x\, dx$.

   Since $\sin 7x\ \cos 2x = \frac{1}{2}\left(\sin 5x+\sin 9x\right)$, then

$$\int \sin 7x\ \cos 2x\, dx = \frac{1}{2}\int (\sin 5x+\sin 9x)\, dx = -\frac{1}{10}\cos 5x-\frac{1}{18}\cos 9x+c\ .$$

3. $\int \cos 6x\ \cos 4x\, dx$.

   Since $\cos 6x\ \cos 4x = \frac{1}{2}\left(\cos 2x+\cos 10x\right)$, then

$$\int \sin 6x\ \cos 4x\, dx = \frac{1}{2}\int (\cos 2x+\cos 10x)\, dx = \frac{1}{4}\sin 2x+\frac{1}{20}\sin 10x+c.$$

Exercise 2:

1 - 18 ■ Evaluate the following integrals:

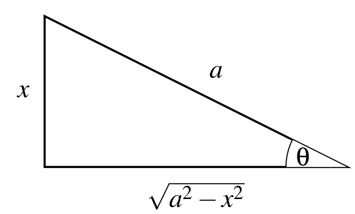
1. $\int \sin^2 x\cos^6 x\, dx$
2. $\int \sin^5 x\cos^2 x\, dx$
3. $\int \sin^3 x\cos^3 x\, dx$
4. $\int \cos^6(4x)\, dx$
5. $\int \tan^4 x\, dx$
6. $\int \cot^5 x\, dx$
7. $\int \frac{\sin^2\sqrt{x}}{\sqrt{x}}\, dx$
8. $\int \cot^2 x\ \csc^3 x\, dx$
9. $\int \cot^4 x\ \csc^2 x\, dx$
10. $\int \tan^3 x\ \sec^3 x\, dx$
11. $\int \tan^2 x\ \sec^2 x\, dx$
12. $\int \tan^2 x\ \sec^3 x\, dx$
13. $\int \sec^5 x\, dx$
14. $\int \tan^6 x\, dx$
15. $\int \sin 7x\ \cos 3x\, dx$
16. $\int \cos 4x\ \cos 3x\, dx$
17. $\int \sin 5x\ \sin 3x\, dx$
18. $\int \sin 3x\ \cos 5x\, dx$

5.3 Trigonometric Substitutions

In this section, we are going to study integrals consist of the following expressions $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$ where $a>0$. To get rid of the square roots, we convert them using substitutions rely on trigonometric functions. The result will be integrals involve power of trigonometric functions. The latter integrals will be evaluated by using the methods given in section 5.2.1. The conversion of the previous square roots is explained as follows:

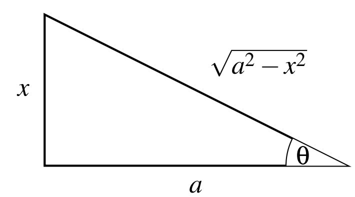
1. $\sqrt{a^2-x^2} = a\ \cos\theta$ if $x = a\ \sin\theta$.
   If $x = a\ \sin\theta$ where $\theta\in(-\pi/2,\pi/2)$, then

$$\begin{aligned}\sqrt{a^2-x^2} &= \sqrt{a^2-a^2\sin^2\theta}\\ &= \sqrt{a^2(1-\sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta}\\ &= a\ \cos\theta\ .\end{aligned}$$



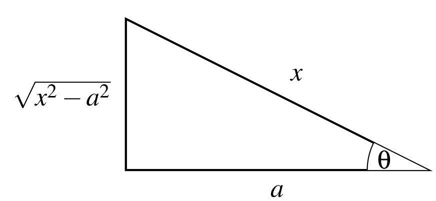
2. $\sqrt{a^2+x^2} = a\ \sec\theta$ if $x = a\ \tan\theta$.
   If $x = a\ \tan\theta$ where $\theta\in(-\pi/2,\pi/2)$, then

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+a^2\tan^2\theta}\\ &= \sqrt{a^2(1+\tan^2\theta)}\\ &= \sqrt{a^2\sec^2\theta}\\ &= a\ \sec\theta\ .\end{aligned}$$



3. $\sqrt{x^2-a^2} = a\ \tan\theta$ if $x = a\sec\theta$.
   If $x = a\ \sec\theta$ where $\theta\in[0,\pi/2)\cup[\pi,3\pi/2)$, then

$$\begin{aligned}\sqrt{x^2-a^2} &= \sqrt{a^2\sec^2\theta-a^2}\\ &= \sqrt{a^2(\sec^2\theta-1)}\\ &= \sqrt{a^2\tan^2\theta}\\ &= a\ \tan\theta\ .\end{aligned}$$



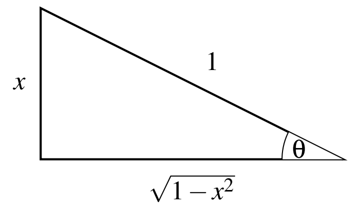
Example 5.3.1 *Evaluate the following integrals:*

1. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ *2.* $\displaystyle\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\,dx$ *3.* $\displaystyle\int \sqrt{x^2+9}\,dx$

Solution:

1. Let $x=\sin\theta$ where $\theta\in(-\pi/2,\pi/2)$, thus $dx=\cos\theta\ d\theta$. By substitution, we have

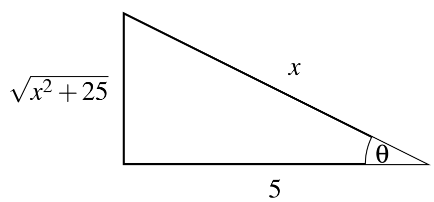
$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\,\cos\theta\,d\theta\\ &= \int \frac{\sin^2\theta\cos\theta}{\cos\theta}\,d\theta\\ &= \int \sin^2\theta\,d\theta\\ &= \frac{1}{2}\int 1-\cos 2\theta\,d\theta\\ &= \frac{1}{2}(\theta-\frac{1}{2}\sin 2\theta)+c\\ &= \frac{1}{2}(\theta-\sin\theta\,\cos\theta)+c\,.\end{aligned}$$



Now, we must return to the original variable x: $\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}(\sin^{-1}x - x\sqrt{1-x^2})+c$.

2. Let $x = 5\ \sec\theta$ where $\theta \in [0,\pi/2)\cup[\pi,3\pi/2)$, thus $dx = 5\ \sec\theta\ \tan\theta\ d\theta$. After substitution, the integral becomes

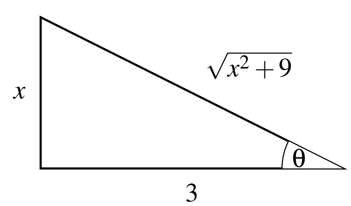
$$\begin{aligned}\int \frac{\sqrt{25\ \sec^2\theta-25}}{625\ \sec^4\theta}\,5\sec\theta\ \tan\theta\ d\theta &= \frac{1}{25}\int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta\\ &= \frac{1}{25}\int \sin^2\theta\ \cos\theta\ d\theta\\ &= \frac{1}{75}\sin^3\theta + c\,.\end{aligned}$$



We must return to the original variable x: $\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\,dx = \frac{1}{75}\left[\frac{(x^2-25)^{3/2}}{x^3}\right]_5^6 = \frac{1}{600}$.

3. Let $x = 3\tan\theta$ where $\theta\in(-\pi/2,\pi/2)$. This implies $dx = 3\sec^2\theta\ d\theta$. By substitution, we have

$$\begin{aligned}\int\sqrt{x^2+9}\,dx &= \int\sqrt{9\tan^2\theta+9}\,(3\sec^2\theta)\,d\theta\\ &= 9\int\sec^3\theta\,d\theta\\ &= \frac{9}{2}(\sec x\ \tan x + \ln|\sec x+\tan x|)\,.\end{aligned}$$



This implies

$$\int\sqrt{x^2+9}\,dx = \frac{9}{2}(\frac{x\sqrt{x^2+9}}{9}+\ln|\frac{\sqrt{x^2+9}+x}{3}|)+c\,.$$

Exercise 3:

1 - 16 ■ Evaluate the following integrals:

1. $\int \frac{1}{x^2\sqrt{x^2-16}}\,dx$
2. $\int \sqrt{9-x^2}\,dx$
3. $\int \frac{1}{(9x^2-1)^{\frac{3}{2}}}\,dx$
4. $\int \frac{1}{\sqrt{9+x^2}}\,dx$
5. $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$
6. $\int \frac{x^2}{(16-x^2)^2}\,dx$
7. $\int \frac{x^3}{\sqrt{1-x^8}}\,dx$
8. $\int \frac{\sec^2 x}{\sqrt{9+\tan^2 x}}\,dx$
9. $\int \frac{1}{x^4+2x^2+1}\,dx$
10. $\int \sqrt{x^2-16}\,dx$
11. $\int \sqrt{e^{2x}-25}\,dx$
12. $\int \frac{\cos x}{\sqrt{2-\sin^2 x}}\,dx$
13. $\int \frac{1}{\sqrt{1+x^2}}\,dx$
14. $\int \frac{1}{(1-x^2)^{\frac{5}{2}}}\,dx$
15. $\int e^x\sqrt{1-e^{2x}}\,dx$
16. $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

5.4 Integrals of Rational Functions

In this section, we study rational functions of the form $q(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials.

The practical steps to integrate the rational functions can be summarized as follows:

- Step 1: If degree of $g(x)$ is less than degree of $f(x)$, we do polynomial long-division; otherwise we move to step 2.

  From the long division shown on the right side, we have

  $$q(x) = \frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)} ,$$

  where $h(x)$ is called the quotient and $r(x)$ is called the remainder.

  $$\begin{array}{r|l} & \text{h(x)} \\ \hline g(x) & \text{f(x)} \\ & \dots \\ & \dots \\ \hline & \text{r(x)} \end{array}$$

- Step 2: Factor the denominator $g(x)$ into irreducible polynomials where the result is either linear or irreducible quadratic polynomials.

- Step 3: Find the partial fraction decomposition. This step depends on step 2 where if degree of $f(x)$ is less than the degree of $g(x)$, then the fraction $\frac{f(x)}{g(x)}$ can be written as a sum of partial fractions:

  $$q(x) = P_1(x) + P_2(x) + P_3(x) + \ldots + P_n(x) ,$$

  where each $P_i(x) = \frac{A}{(ax+b)^m}, m \in \mathbb{N}$ or $P_i(x) = \frac{Ax+B}{(ax^2+bx+c)^m}$ if $b^2-4ac<0$. The constants $A, B, \ldots$ are computed later.

- Step 4: Integrate the result of step 3.

Example 5.4.1 *Evaluate the following integral* $\int \frac{x+1}{x^2-2x-8}\,dx.$

Solution:

Factor the denominator $g(x)$ into irreducible polynomials : $g(x) = x^2-2x-8 = (x+2)(x-4)$. We need to find constants A and B such that

$$\frac{x+1}{x^2-2x-8} = \frac{A}{x+2} + \frac{B}{x-4} = \frac{Ax-4A+Bx+2B}{(x+2)(x-4)} .$$

Coefficients of the numerators:

$$A+B = 1 \rightarrow (1)$$
$$-4A+2B = 1 \rightarrow (2)$$

By doing some calculation, we have $A = \frac{1}{6}$ and $B = \frac{5}{6}$, thus

Multiply equation (1) by 4 and add the result to equation (2)

$$4A+4B = 4$$
$$-4A+2B = 1$$
$$- - - - - - - - - -$$
$$6B = 5$$

$$\int \frac{x+1}{x^2-2x-8}\,dx = \int \frac{1/6}{x+2}\,dx + \int \frac{5/6}{x-4}\,dx = \frac{1}{6}\ln|\,x+2\,| + \frac{5}{6}\ln|\,x-4\,| + c\,.$$

Example 5.4.2 *Evaluate the following integral* $\displaystyle\int \frac{2x^3-4x^2-15x+5}{x^2+3x+2}\,dx.$

Solution:

Since the degree of the denominator $g(x)$ is less than the degree of the numerator $f(x)$, we do polynomial long-division.
From the long division given on the right side, we have

$$q(x) = (2x-10) + \frac{11x+35}{x^2+3x+2}\,.$$

$$\begin{array}{r|rrrr}
 & 2x & -10 & & \\ \hline
x^2+3x+2 & 2x^3 & -4x^2 & -15x & +5 \\
 & 2x^3 & +6x^2 & +4x & \\ \hline
 & & -10x^2 & -19x & +5 \\
 & & -10x^2 & -30x & -20 \\ \hline
 & & & 11x & +25
\end{array}$$

Now, factor the denominator $g(x)$ into irreducible polynomials : $g(x) = x^2+3x+2 = (x+1)(x+2)$. Thus,

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2} = (2x-10) + \frac{A}{x+1} + \frac{B}{x+2} = (2x-10) + \frac{Ax+2A+Bx+B}{(x+1)(x+2)}\,,$$

and we need to find the constants A and B.

Coefficients of the numerators:

$$A+B = 11 \rightarrow \textcircled{1}$$
$$2A+B = 25 \rightarrow \textcircled{2}$$

Multiply equation $\textcircled{1}$ by -2 and add the result to equation $\textcircled{2}$

$$-2A-2B = -22$$
$$2A+B = 25$$
$$-\,-\,-\,-\,-\,-\,-\,-\,-$$
$$-B = 3$$

By doing some calculation, we have
$A = 24$ and $B = -13$, thus

$$\int q(x)\,dx = \int (2x-10)\,dx + \int \frac{14}{x+1}\,dx + \int \frac{-13}{x+2}\,dx$$
$$= x^2 - 10x + 14\ln|\,x+1\,| - 3\ln|\,x+2\,| + c\,.$$

Remark 5.4.1

1. *The number of constants $A, B, C, \ldots$ is equal to the degree of the denominator $g(x)$. Therefore, in the case of repeated factors of the denominator, we have to check the number of the constants and the degree of $g(x)$.*
2. *If the denominator $g(x)$ contains irreducible quadratic factors, the numerators of partial fractions should be polynomials of degree one.*

Example 5.4.3 *Evaluate the following integral* $\displaystyle\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}dx.$

Solution:

Since the denominator $g(x)$ has repeated factors, then

$$\frac{2x^2-25x-33}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-5} = \frac{A(x^2-4x-5)+B(x-5)+C(x^2+2x+1)}{(x+1)^2(x-5)}\,.$$

Coefficients of the numerators:

$$A + C = 2 \to \textcircled{1}$$
$$-4A + B + 2C = -25 \to \textcircled{2}$$
$$-5A - 5B + C = -33 \to \textcircled{3}$$

Hint: Multiply equation ② by 5 and add the result to equation ③ to have a new equation contains A and C.

By solving the system of equations, we have $A = 5$, $B = 1$ and $C = -3$. Hence,

$$\begin{aligned}\int \frac{2x^2 - 25x - 33}{(x+1)^2(x-5)}\,dx &= \int \frac{5}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx + \int \frac{-3}{x-5}\,dx \\ &= 5\ln|x+1| + \int (x+1)^{-2}\,dx - 3\ln|x-5| \\ &= 5\ln|x+1| - \frac{1}{(x+1)} - 3\ln|x-5| + c\,.\end{aligned}$$

Example 5.4.4 *Evaluate the following integral* $\int \frac{x+1}{x(x^2+1)} dx.$

Solution:

The denominator $g(x)$ is factoblack into irreducible polynomials, so

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2+1)}\,.$$

Coefficients of the numerators:

$$A + B = 0 \to \textcircled{1}$$
$$C = 1 \to \textcircled{2}$$
$$A = 1 \to \textcircled{3}$$

We have $A = 1$, $B = -1$ and $C = 1$. Hence,

$$\begin{aligned}\int \frac{x+1}{x(x^2+1)}\,dx &= \int \frac{1}{x}\,dx + \int \frac{-x+1}{x^2+1}\,dx \\ &= \ln|x| - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + c\,.\end{aligned}$$

Exercise 4:

1 - 20 ■ Evaluate the following integrals:

1. $\int \frac{1}{x(x-1)}\,dx$
2. $\int_0^2 \frac{1}{x^2+4x+3}\,dx$
3. $\int \frac{1}{x^2-4}\,dx$
4. $\int \frac{1}{x^2-x-2}\,dx$
5. $\int \frac{x+1}{x^2+8x+12}\,dx$
6. $\int \frac{x}{x^2+7x+12}\,dx$
7. $\int_1^5 \frac{x^2-1}{x^2+3x-4}\,dx$
8. $\int \frac{x^3}{x^2-25}\,dx$
9. $\int \frac{x}{x^2+7x+6}\,dx$
10. $\int \frac{1}{x^2+3x+9}\,dx$
11. $\int \frac{1}{(x-1)(x^2+1)}\,dx$
12. $\int \frac{x+2}{(x+1)(x^2-4)}\,dx$
13. $\int \frac{x^3+2x+1}{x^2-3x-10}\,dx$
14. $\int \frac{1}{x^2+1}\,dx$
15. $\int \frac{3x^2+3x-1}{x^3+x^2-x}\,dx$
16. $\int_0^{\pi/2} \frac{\sin x}{\cos^2 x - \cos x - 2}\,dx$
17. $\int \frac{2-x}{x^3+x^2}\,dx$
18. $\int_0^1 \frac{1}{1+e^x}\,dx$
19. $\int \frac{e^x}{e^{2x}-2e^x-15}\,dx$
20. $\int \frac{1}{x^4-x^2}\,dx$

5.5 Integrals of Quadratic Forms

In this section, we provide a new technique for integrals that contain irreducible quadratic expressions ax^2+bx+c where $b \neq 0$. This technique is completing square method: $a^2 \pm 2ab + b^2 = (a \pm b)^2$. Before starting presenting this method, we provide an example to remind the reader on how to complete the square.

Example 5.5.1 *The quadratic expression $x^2-6x+13$ is irreducible. To complete the square, we find $(\frac{b}{2})^2$, then add and substrate it as follows:*

$$x^2-6x+13 = \underbrace{x^2-6x+9}_{=(x-3)^2} - \underbrace{9+13}_{=4}$$

Remember:
If a quadratic polynomial has roots, it is reducible; otherwise it is irreducible.

Hence, $x^2-6x+13 = (x-3)^2+4$.

In the following examples, we use the previous idea to evaluate the integrals.

Example 5.5.2 *Evaluate the following integral* $\int \frac{1}{x^2-6x+13} dx$.

Solution:

The quadratic expression $x^2-6x+13$ is irreducible. By using the complete the square, we have

$$\int \frac{1}{x^2-6x+13} dx = \int \frac{1}{(x-3)^2+4} dx \,.$$

Let $u = x-3 \Rightarrow du = dx$. By substitution,

$$\int \frac{1}{u^2+4} du = \frac{1}{2}\tan^{-1}(\frac{u}{2}) + c = \frac{1}{2}\tan^{-1}(\frac{x-3}{2}) + c \,.$$

Example 5.5.3 *Evaluate the following integral* $\int \frac{x}{x^2-4x+8} dx$.

Solution:

Since the quadratic expression x^2-4x+8 is irreducible, we use the complete the square as follows:

$$\int \frac{x}{x^2-4x+8}\, dx = \int \frac{x}{(x-2)^2+4}\, dx$$

Let $u = x-2 \Rightarrow du = dx$. By substitution,

$$\begin{aligned}\int \frac{u+2}{u^2+4}\, du &= \int \frac{u}{u^2+4}\, du + \int \frac{2}{u^2+4}\, du \\ &= \frac{1}{2}\ln|\,u^2+4\,| + \tan^{-1}(\frac{u}{2}) \\ &= \frac{1}{2}\ln\big((x-2)^2+4\big) + \tan^{-1}(\frac{x-2}{2}) + c\end{aligned}$$

Example 5.5.4 *Evaluate the following integral* $\int \frac{1}{\sqrt{2x-x^2}} dx$.

Solution:

By completing the square, we have $2x-x^2 = -(x^2-2x) = -(x^2-2x+1-1) = 1-(x-1)^2$.

Thus,

$$\int \frac{1}{\sqrt{2x-x^2}}\, dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\, dx \,.$$

Let $u = x-1$, then $du = dx$. By substitution, the integral becomes

$$\int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1}(u) + c = \sin^{-1}(x-1) + c \,.$$

Example 5.5.5 *Evaluate the following integral* $\int \sqrt{x^2+2x-1}\, dx.$

Solution:

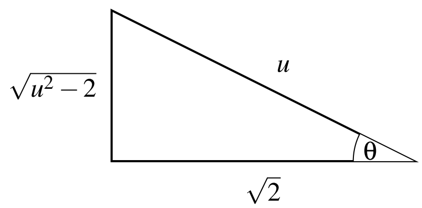
By completing the square, we have $x^2+2x-1=(x+1)^2-2$. Thus,

$$\int \sqrt{x^2+2x-1}\, dx = \int \sqrt{(x+1)^2-2}\ dx\ .$$

Let $u=x+1$, then $du=dx$. The integral becomes $\int \sqrt{u^2-2}\, du$.

Use the trigonometric substitutions, in particular assume $u=\sqrt{2}\ \sec\theta \Rightarrow du=\sqrt{2}\ \sec\theta\ \tan\theta\ d\theta$ where $\theta\in[0,\pi/2)\cup[\pi,3\pi/2)$. By substitution,

$$2\int \tan^2\theta\ \sec\theta = 2\int \sec^3\theta - \sec\theta\ d\theta\ .$$



From Example 5.2.2, we have $2\int \sec^3\theta-\sec\theta\ d\theta = \sec\theta\ \tan\theta - \ln|\sec\theta+\tan\theta|+c$. By returning to the variable u and then to x,

$$\int \sqrt{u^2-2}\, dx = \frac{u\sqrt{u^2-2}}{2} - \ln|\frac{u+\sqrt{u^2-2}}{\sqrt{2}}|+c = \frac{(x+1)\sqrt{(x+1)^2-2}}{2} - \ln|\frac{x+1+\sqrt{(x+1)^2-2}}{\sqrt{2}}|+c.$$

Exercise 5:

1 - 12 ■ Evaluate the following integrals:

1. $\int_0^1 \frac{1}{x^2+4x+5}\, dx$
2. $\int \frac{1}{x^2-6x+1}\, dx$
3. $\int \frac{2x+3}{x^2+2x-3}\, dx$
4. $\int \frac{x^2-2x+5}{2x-x^2}\, dx$
5. $\int_{-1}^0 \frac{1}{\sqrt{8+2x-x^2}}\, dx$
6. $\int \frac{1}{x^2+8x-9}\, dx$
7. $\int \frac{5}{\sqrt{1-4x-x^2}}\, dx$
8. $\int \frac{e^x}{e^{2x}+2e^x-1}\, dx$
9. $\int \frac{1}{\sqrt{6-6x-2x^2}}\, dx$
10. $\int \sqrt{x(2-x)}\, dx$
11. $\int \frac{\sec^2 x}{\tan^2 x-6\tan x+12}\, dx$
12. $\int \sqrt{8-2x-x^2}\, dx$

5.6 Miscellaneous Substitutions

We study in this section other three important substitutions needed in some cases.

5.6.1 Fractional Functions in sin x and cos x

The integrals that consist of rational expressions in $\sin x$ and $\cos x$ are treated by using the substitution $u=\tan(x/2)$, $-\pi<x<\pi$. This implies that $du=\frac{\sec^2(x/2)}{2}\, dx$. Since $\sec^2 x=\tan^2 x+1$, then $du=\frac{u^2+1}{2}\, dx$. Also,

$$\begin{aligned}\sin(x) = \sin 2(\frac{x}{2}) = 2\ \sin(\frac{x}{2})\ \cos(\frac{x}{2}) &= 2\frac{\sin(\frac{x}{2})}{\cos(\frac{x}{2})}\cos(\frac{x}{2})\ \cos(\frac{x}{2})\\ &= 2\tan(\frac{x}{2})\ \cos^2(\frac{x}{2})\\ &= \frac{2\tan(\frac{x}{2})}{\sec^2(\frac{x}{2})}\\ &= \frac{2u}{u^2+1}\ .\end{aligned}$$

For $\cos x$, we have

$$\cos(x) = \cos 2(\frac{x}{2}) = \cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})$$

We can find that

$$\cos(\frac{x}{2}) = \frac{1}{\sqrt{u^2+1}} \text{ and } \sin(\frac{x}{2}) = \frac{u}{\sqrt{u^2+1}} .$$

Use the identities:
$\sec^2(\frac{x}{2}) = \tan^2(\frac{x}{2}) + 1$
$\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2}) = 1$

This implies

$$\cos(x) = \frac{1-u^2}{1+u^2} .$$

The previous discussion can be summarized as follows:

For the integrals that contain rational expressions in $\sin x$ and $\cos x$, we assume

$$u = \tan(x/2), \quad du = \frac{u^2+1}{2}\, dx, \quad \sin(x) = \frac{2u}{u^2+1}, \quad \cos(x) = \frac{1-u^2}{1+u^2} .$$

Example 5.6.1 *Evaluate the following integrals:*

1. $\int \frac{1}{1+\sin x}\, dx$ *2.* $\int \frac{1}{2+\cos x}\, dx$ *3.* $\int \frac{1}{1+\sin x+\cos x}\, dx$

Solution:

1. $\int \frac{1}{1+\sin x}\, dx$

   Let $u = \tan \frac{x}{2}$, this implies $du = \frac{1+u^2}{2}$ and $\sin x = \frac{2u}{1+u^2}$. By substituting that into the integral, we have

$$\begin{aligned} \int \frac{1}{1+\frac{2u}{1+u^2}} \frac{2}{1+u^2}\, du &= 2\int \frac{1}{u^2+2u+1}\, du \\ &= 2\int (u+1)^{-2}\, du \\ &= \frac{-2}{u+1} + c = \frac{-2}{\tan\frac{x}{2}+1} + c . \end{aligned}$$

2. $\int \frac{1}{2+\cos x}\, dx$

   Let $u = \tan \frac{x}{2}$, this implies $du = \frac{1+u^2}{2}$ and $\cos x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned} \int \frac{1}{2+\frac{1-u^2}{1+u^2}} \frac{2}{1+u^2}\, du &= 2\int \frac{1}{u^2+3}\, du \\ &= \frac{2}{\sqrt{3}} \tan^{-1}(\frac{u}{\sqrt{3}}) + c = \frac{2}{\sqrt{3}} \tan^{-1}(\frac{\tan\frac{x}{2}}{\sqrt{3}}) + c . \end{aligned}$$

3. $\int \frac{1}{1+\sin(x)+\cos(x)}\, dx$

Assume $u = \tan\frac{x}{2}$, this implies $du = \frac{1+u^2}{2}$, $\sin x = \frac{2u}{1+u^2}$ and $\cos x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned}\int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\frac{2}{1+u^2}\,du &= \int \frac{2}{2+2u}\,du \\ &= \int \frac{1}{1+u}\,du \\ &= \ln\mid 1+u\mid +c = \ln\mid 1+\tan\frac{x}{2}\mid +c\;.\end{aligned}$$

5.6.2 Integrals of Fractional Powers

In the case of integrands that consist of fractional powers, it is better to use the substitution $u = x^{\frac{1}{n}}$ where n is the least common multiple of the denominators of the powers. To see this, we provide an example.

Example 5.6.2 *Evaluate the following integral* $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,dx$.

Solution:

Put $u = x^{\frac{1}{4}}$, we find $x = u^4$ and $dx = 4u^3du$. By substitution, we have

$$\begin{aligned}\int \frac{1}{u^2+u}4u^3\,du &= 4\int \frac{u^2}{u+1}\,du \\ &= 4\int u-1\,du + 4\int \frac{1}{1+u}\,du \\ &= 2u^2-4u+4\ln\mid u+1\mid +c \\ &= 2\sqrt{x}-4\sqrt[4]{x}+4\ln\mid \sqrt[4]{x}+1\mid +c\;.\end{aligned}$$

5.6.3 Integrals of $\sqrt[n]{f(x)}$

Here, we assume that the integrand is a function of form $\sqrt[n]{f(x)}$. To solve such integrals, it is useful to assume $u = \sqrt[n]{f(x)}$. This case differs from that given in the substitution method in Chapter 1 i. e., $\sqrt[n]{f(x)}\;f'(x)$ where the difference lies on existence of the derivative of $f(x)$.

Example 5.6.3 *Evaluate the following integral* $\int \sqrt{e^x+1}\,dx$

Solution:

Assume $u = \sqrt{e^x+1}$, this implies $du = \frac{e^x}{2\sqrt{e^x+1}}\,dx$ and $u^2 = e^x+1$. By substitution,

$$\begin{aligned}\int \frac{2u^2}{u^2-1}\,du &= \int 2\,du + 2\int \frac{1}{u^2-1}\,du \\ &= 2u + 2\int \frac{1}{u-1}\,du + 2\int \frac{1}{u+1}\,du \\ &= 2u + 2\ln\mid u-1\mid + 2\ln\mid u+1\mid +c \\ &= 2\sqrt{e^x+1} + 2\ln(\sqrt{e^x+1}-1) + 2\ln(\sqrt{e^x+1}+1) + c\;.\end{aligned}$$

Exercise 6:

1 - 12 ■ Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,dx$
2. $\int \frac{x^{1/2}}{1+x^{3/5}}\,dx$
3. $\int \frac{1}{\sqrt{\cos x+1}}\,dx$
4. $\int \frac{\sqrt{x}}{\sqrt{x}+4}\,dx$
5. $\int \frac{1}{1+3\sin x}\,dx$
6. $\int \frac{1}{3-\cos x}\,dx$
7. $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$
8. $\int \frac{x^{1/2}}{1+x^{1/4}}\,dx$
9. $\int \frac{1}{\sqrt{e^{2x}+1}}\,dx$
10. $\int \frac{1}{x^{1/2}-x^{3/5}}\,dx$
11. $\int \frac{1}{1-2\cos x}\,dx$
12. $\int \frac{1}{\sin x+\cos x}\,dx$

Chapter 6

Indeterminate Forms and Improper Integrals

6.1 Limit Rules

The limit is defined as the value of the function as the variable approaches to t value. A few examples are given below:

Example 6.1.1

1. $\lim_{x\to 2} 3 = 3$.
2. $\lim_{x\to 1} x = 1$.
3. $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$.
4. $\lim_{x\to 8} \sqrt{x} = 2\sqrt{2}$.

As you noted, the functions in the previous example are continuous. Meaning that, the limit is equal to the value of the function if it is continuous. Before discussing this issue deeply, let's see some general rules of the limits.

If $\lim_{x\longrightarrow c} f(x) = L$ and $\lim_{x\longrightarrow c} g(x) = M$, then

1. **Sum Rule:** $\lim_{x\longrightarrow c} \big(f(x)+g(x)\big) = \lim_{x\longrightarrow c} f(x) + \lim_{x\longrightarrow c} g(x) = L+M$.
2. **Difference Rule:** $\lim_{x\longrightarrow c} \big(f(x)-g(x)\big) = \lim_{x\longrightarrow c} f(x) - \lim_{x\longrightarrow c} g(x) = L-M$.
3. **Product Rule:** $\lim_{x\longrightarrow c} \big(f(x).g(x)\big) = \lim_{x\longrightarrow c} f(x) \times \lim_{x\longrightarrow c} g(x) = L\times M$.
4. **Constant Multiple Rule:** $\lim_{x\longrightarrow c} \big(k\ f(x)\big) = k\ \lim_{x\longrightarrow c} f(x) = k\,L$.
5. **Quotient Rule:** $\lim_{x\longrightarrow c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\longrightarrow c} f(x)}{\lim_{x\longrightarrow c} g(x)} = \frac{L}{M}$.

Example 6.1.2

1. $\lim_{x\to 0}(x^2-2x+1) = \lim_{x\to 0} x^2 - 2\lim_{x\to 0} x + \lim_{x\to 0} 1 = 0-0+1 = 1$.
2. $\lim_{x\to\pi} \sin x\ \cos x = 0$.
3. $\lim_{x\to 3^+} \frac{1}{(x-3)} = \frac{\lim_{x\to 3} 1}{\lim_{x\to 3}(x-3)} = \infty$.
4. $\lim_{x\to 1} \frac{x}{(x^2+1)} = \frac{\lim_{x\to 1} x}{\lim_{x\to 1}(x^2+1)} = \frac{1}{2}$.

6.2 Indeterminate Forms

In this section, we examine several situations, where a function is built up from other functions, but the limits of these functions are not sufficient to determine the overall limit. These situations are called indeterminate forms.

Case	Indeterminate Form
Quotient	$\frac{0}{0}$ and $\frac{\infty}{\infty}$
Product	$0.\infty$ and $0.(-\infty)$
Sum & Difference	$(-\infty)+\infty$ and $\infty-\infty$
Exponential	0^0, 1^∞, $1^{-\infty}$ and ∞^0

Table 6.2: List of the indeterminate forms.

Example 6.2.1

1. $\lim_{x\to 0}\frac{\sin x}{x}=\frac{0}{0}$
2. $\lim_{x\to\infty}\frac{e^x}{x}=\frac{\infty}{\infty}$
3. $\lim_{x\to 0^+}x^2\ln x=0.\infty$
4. $\lim_{x\to 1^+}\left(\frac{1}{x-1}-\frac{1}{\ln x}\right)=\infty-\infty$

Indeterminate Forms:

To treat such limits, students in previous courses were multiplying the function by a conjugate or using factoring method. In this course, we present a new method called L'Hopital Rule. Usually, this method is used for a fractional function where we calculate the derivative of the numerator and denominator.

L'Hopital Rule:

Theorem 6.2.1 *Suppose $f(x)$ and $g(x)$ are differentiable on an interval I and $c\in I$ where f and g may not be differentiable at c. If $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x=c$ and $g'(x)\neq 0$ for $x\neq c$, then*

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

if $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists or equals to ∞ or $-\infty$.

Remark 6.2.1

1. *L'Hopital rule works if $c=\pm\infty$ or when $x\to c^+$ or $x\to c^-$.*
2. *Sometimes, we need to apply L'Hopital rule twice.*

Example 6.2.2 *Use L'Hopital rule to find the following limits:*

1. $\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x^2-25}$.
2. $\lim_{x\to 0}\frac{\sin x}{x}$.
3. $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$.
4. $\lim_{x\to\infty}\frac{e^x}{x}$.

Solution:

1. $\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x^2-25}=\frac{0}{0}$ and this is an indeterminate form. By applying L'Hopital rule, we have

$$\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x^2-25}=\lim_{x\to 5}\frac{1}{4x\sqrt{x-1}}=\frac{1}{40}\ .$$

2. $\lim_{x\to 0}\frac{\sin x}{x}=\frac{0}{0}$. To treat this indeterminate form, we apply L'Hopital rule $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$.
3. $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\frac{\infty}{\infty}$ and this is an indeterminate form. Apply L'Hopital rule, $\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$.
4. $\lim_{x\to\infty}\frac{e^x}{x}=\frac{\infty}{\infty}$. By applying L'Hopital rule, we have $\lim_{x\to\infty}\frac{e^x}{x}=\lim_{x\to\infty}\frac{e^x}{1}=\infty$.

Example 6.2.3 *Use L'Hopital rule to find the following limits:*

1. $\lim_{x\to 0^+} x^2 \ln x$
2. $\lim_{x\to \frac{\pi}{4}} (1-\tan x)\ \sec(2x)$
3. $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$
4. $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$

Solution:

1. $\lim_{x\to 0^+} x^2 \ln x$.

   The limit is of the form $0.\infty$, so we cannot use the L'Hopital rule. However, if we rearrange the expression, we may able to use the L'Hopital rule. Meaning that, we need to rewrite the expression in a way enables us to apply the L'Hopital rule. Note that

   $$x^2 \ln x \ = \ \frac{\ln x}{\frac{1}{x^2}} .$$

   The limit of the new expression ($\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}}$) is of form $\frac{\infty}{\infty}$. Therefore, we can apply the L'Hopital rule:

   $$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0 . \qquad \text{L'Hopital rule}$$

2. $\lim_{x\to \frac{\pi}{4}} (1-\tan x)\ \sec(2x)$.

   The limit is of the form $0.\infty$, so we try to rewrite the function to apply the L'Hopital rule. We know that $\sec x = 1/\cos x$, thus

   $$(1-\tan x)\sec(2x) = \frac{(1-\tan x)}{\cos(2x)} .$$

   Now, the limit of the new expression is of form $\frac{0}{0}$. From the L'Hopital rule, we have

   $$\lim_{x\to \frac{\pi}{4}} (1-\tan x)\sec(2x) = \lim_{x\to \frac{\pi}{4}} \frac{(1-\tan x)}{\cos(2x)} = \lim_{x\to \frac{\pi}{4}} \frac{\sec^2 x}{2\sin 2x} \qquad = -\frac{(\sqrt{2})^2}{2} = -1 .$$

3. $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$.

   By substituting 1 into the function, we have the indeterminate form $\infty - \infty$. To treat this form, we write the function as a single fraction

   $$\frac{1}{x-1} - \frac{1}{\ln x} = \frac{\ln x - x + 1}{(x-1)\ln x} .$$

   The new expression takes the indeterminate form $\frac{0}{0}$. From the L'Hopital rule,

   $$\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{1-x}{x\ \ln x + x - 1}$$

   which is of form $\frac{0}{0}$. Therefore, we apply the L'Hopital rule again. This implies

   $$\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{-1}{\ln x + 2} = \frac{-1}{2} .$$

4. $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$.

   This limit is of form 1^∞. To treat this form, we assume that $y = (1+x)^{\frac{1}{x}}$. By taking ln for both sides, we have

   $$\ln y = \frac{1}{x} \ln(1+x)$$
   $$\Rightarrow \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{1}{x} \ln(1+x) = \lim_{x\to 0} \frac{\ln(1+x)}{x} = \frac{0}{0} .$$

By applying the L'Hopital rule, we have $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$.

Thus, $\lim_{x\to 0} \ln y = 1 \Rightarrow e^{\lim_{x\to 0} \ln y} = e^1 \Rightarrow \lim_{x\to 0} e^{(\ln y)} = e \Rightarrow \lim_{x\to 0} y = e \Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$.

Exercise 1:

1 - 14 ■ Find the following limits:

1. $\lim_{x\to 2} \frac{x^2-4x+4}{x-2}$
2. $\lim_{x\to 3} \frac{x^2-9}{x-3}$
3. $\lim_{x\to \pi^+} \frac{\cos x+\sin x}{\tan x}$
4. $\lim_{x\to 0} \frac{1-e^x}{x}$
5. $\lim_{x\to \pi/2^+} \tan x$
6. $\lim_{x\to 0^-} \frac{e^x-1}{x^2}$
7. $\lim_{x\to 0} (e^x+x)^{\frac{1}{x}}$
8. $\lim_{x\to \infty} \frac{x+2}{x-2}$
9. $\lim_{x\to 0^+} \frac{e^x-\ln(e^x)}{\ln x}$
10. $\lim_{x\to \pi/2} \frac{1-\sin x}{\cos x}$
11. $\lim_{x\to 1} \frac{\ln x}{\tan(\pi x)}$
12. $\lim_{x\to 0} \frac{\tan x}{x}$
13. $\lim_{x\to \infty} \frac{\ln(\ln x)}{\sqrt{x}}$
14. $\lim_{x\to 0} (\frac{1}{x^2})^x$

6.3 Improper Integrals

Definition 6.3.1 *The integral* $\int_a^b f(x)\,dx$ *is called a proper integral if*

1. *the interval* $[a,b]$ *is finite and closed, and*
2. $f(x)$ *is defined on* $[a,b]$.

If condition 1 or 2 is not satisfied, the integral is called **improper**. From this, we have two cases of the improper integrals.

6.3.1 Infinite Intervals

In this section, we study integrals of forms

$$\int_a^\infty f(x)\,dx,\ \int_{-\infty}^b f(x)\,dx,\ \int_\infty^{-\infty} f(x)\,dx\ .$$

Definition 6.3.2

1. *Let f be a continuous function on $[a,\infty)$. The improper integral $\int_a^\infty f(x)\,dx$ is defined as follows:*

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \ \text{ if the limit exists.}$$

2. *Let f be a continuous function on $(-\infty,b]$. The improper integral $\int_{-\infty}^b f(x)\,dx$ is defined as follows:*

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx \ \text{ if the limit exists.}$$

The previous integrals are convergent (or to converge) if the limit exists as a finite number i. e., the value of the integral is a finite number. However, if the limit does not exist or equals $\pm\infty$, the integral is called divergent (or to diverge).

3. *Let f be a continuous function on $\mathbb{R}$ and $a\in\mathbb{R}$. The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is defined as follows:*

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx\ .$$

The previous integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Example 6.3.1 *Determine whether the integral converges or diverges:*

1. $\int_0^\infty \frac{1}{(x+2)^2}\,dx$
2. $\int_0^\infty \frac{x}{1+x^2}\,dx$
3. $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution:

1. $\int_0^\infty \frac{1}{(x+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^2}\,dx$. The integral

$$\int_0^t \frac{1}{(x+2)^2}\,dx = \int_0^t (x+2)^{-2}\,dx = \Big[\frac{-1}{x+2}\Big]_0^t = -\big(\frac{1}{t+2}+\frac{1}{2}\big)\ .$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^2}\,dx = -\lim_{t\to\infty}\big(\frac{1}{t+2}+\frac{1}{2}\big) = -(0+\frac{1}{2}) = -\frac{1}{2}\ .$$

Therefore, the integral converges.

2. $\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx$. The integral

$$\int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2}\Big[\ln(1+x^2)\Big]_0^t = \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(1+t^2)\ .$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2}\lim_{t\to\infty}\ln(1+t^2) = \infty\ .$$

Therefore, the integral diverges.

3. $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\, dx + \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\, dx$. We know that, $\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$, so

$$\begin{aligned}\lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\, dx + \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2} &= \lim_{t\to-\infty} \left[0 - \tan^{-1}(t)\right] + \lim_{t\to\infty} \left[\tan^{-1}(t) - 0\right] \\ &= -\lim_{t\to-\infty} \tan^{-1}(t) + \lim_{t\to\infty} \tan^{-1}(t) \\ &= -(-\frac{\pi}{2}) + \frac{\pi}{2} = \pi\ .\end{aligned}$$

Therefore, the integral is convergent.

6.3.2 Discontinuous Integrands

Definition 6.3.3

1. *If f is continuous on $[a,b)$ and has an infinite discontinuity at b i.e., $\lim_{x\to b^-} f(x) = \pm\infty$, then*

$$\int_a^b f(x)\, dx = \lim_{t\to b^-} \int_a^t f(x)\, dx\ .$$

2. *If f is continuous on $(a,b]$ and has an infinite discontinuity at a i.e., $\lim_{x\to a^+} f(x) = \pm\infty$, then*

$$\int_a^b f(x)\, dx = \lim_{t\to a^+} \int_t^a f(x)\, dx\ .$$

In items 1 and 2, the integral is convergent if the limit exists as a finite number; otherwise the integral is divergent.

3. *If f is continuous on $[a,b]$ except at $c \in (a,b)$ such that $\lim_{x\to c^\pm} f(x) = \pm\infty$, the improper integral $\int_a^b f(x)\, dx$ is defined as follows:*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx\ .$$

The integral is convergent if the limit of the integrals on the right side exists as a finite number.

Example 6.3.2 *Determine whether the integral converges or diverges:*

1. $\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\, dx$ *2.* $\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\, dx$ *3.* $\int_{-3}^{1} \frac{1}{x^2}\, dx$

Solution:

1. Since $\lim_{x\to 4^-} \frac{1}{(4-x)^{\frac{3}{2}}} = \infty$ and the integrand is continuous on $[0,4)$, from Definition 6.3.3,

$$\begin{aligned}\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\, dx &= \lim_{t\to 4^-} \int_0^t (4-x)^{-\frac{3}{2}}\, dx \\ &= \lim_{t\to 4^-} \left[\frac{2}{\sqrt{4-x}}\right]_0^t \\ &= \lim_{t\to 4^-} \left(\frac{2}{\sqrt{4-t}} - 1\right) \\ &= \infty\ .\end{aligned}$$

Thus, the integral diverges.

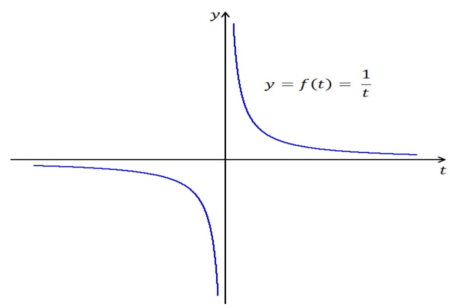
2. The limit $\lim_{x\to 0^+} \frac{\cos x}{\sqrt{\sin x}} = \infty$ and the integrand is continuous on $(0, \frac{\pi}{4}]$, thus

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\, dx &= \lim_{t\to 0^+} \int_t^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\, dx \\ &= 2\lim_{t\to 0^+} \left[\sqrt{\sin x}\right]_t^{\frac{\pi}{4}} \\ &= 2\lim_{t\to 0^+} \left(\frac{1}{\sqrt[4]{2}} - \sqrt{\sin t}\right) \\ &= \frac{2}{\sqrt[4]{2}}\,.\end{aligned}$$

The integral converges.

3. Since $\lim_{x\to 0^-} \frac{1}{x^2} = \lim_{x\to 0^+} \frac{1}{x^2} = \infty$ and the integrand is continuous on $[-3,0)\cup(0,1]$, then

$$\begin{aligned}\int_{-3}^{1} \frac{1}{x^2}\, dx &= \lim_{t\to 0^-} \int_{-3}^{t} \frac{1}{x^2} + \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2} \\ &= \lim_{t\to 0^-} \left[\frac{-1}{x}\right]_{-3}^{t} - \lim_{t\to 0^+} \left[\frac{-1}{x}\right]_t^1 \\ &= -\lim_{t\to 0^-} \left[\frac{1}{t} + \frac{1}{3}\right] + \lim_{t\to 0^+} \left[-1 + \frac{1}{t}\right] \\ &= \infty\,.\end{aligned}$$



The integral diverges.

Exercise 2:

1 - 16 ■ Determine whether the integral converges or diverges:

1. $\int_1^{\infty} \frac{1}{x}\, dx$
2. $\int_1^{\infty} \frac{1}{x^2}\, dx$
3. $\int_4^{\infty} \frac{1}{\sqrt{x}}\, dx$
4. $\int_{-\infty}^{0} e^x\, dx$
5. $\int_0^{\infty} e^x\, dx$
6. $\int_2^{\infty} \frac{1}{x-1}\, dx$
7. $\int_1^{2} \frac{1}{1-x}\, dx$
8. $\int_{-1}^{1} \frac{1}{x}\, dx$
9. $\int_0^{3} \frac{dx}{\sqrt{9-x^2}}$
10. $\int_0^{\infty} (1-x)e^{-x}\, dx$
11. $\int_0^{\infty} \frac{dx}{x^2+4}$
12. $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\, dx$
13. $\int_0^{\infty} \frac{1}{x-1}\, dx$
14. $\int_0^{\pi} \sec^2 x\, dx$
15. $\int_0^{2} \frac{1}{x^2+1}\, dx$
16. $\int_0^{\pi/2} \tan x\, dx$

Chapter 7

Application of Definite Integrals

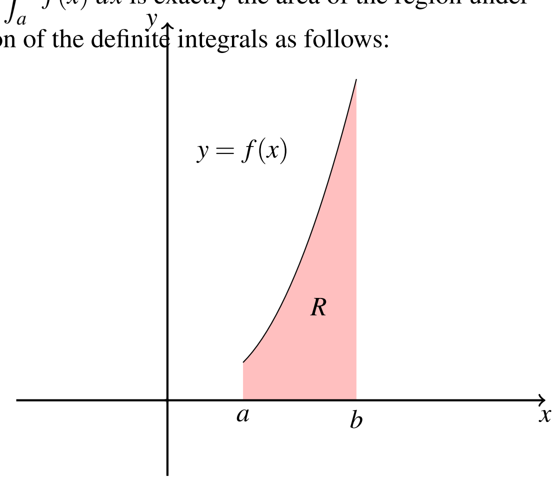
7.1 Areas

The definite integral can be used to calculate areas under graphs. The simplest case of this application is when we find the area by calculating a single definite integral.

In Chapter 2, we mentioned that if $f > 0$ $x \in [a,b]$, the definite integral $\int_a^b f(x)\,dx$ is exactly the area of the region under the graph of $f(x)$ from a to b. In more formally, we state this application of the definite integrals as follows:

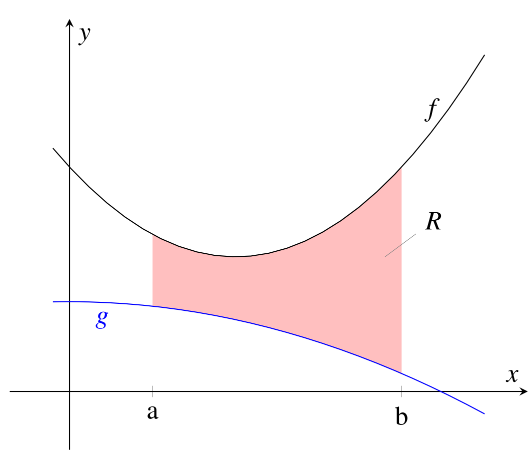
1. If $y = f(x)$ is a continuous function on $[a,b]$ and $f(x) \geq 0\ \forall x \in [a,b]$, the area of the region under the graph of $f(x)$ from $x = a$ to $x = b$ is given by the integral:

$$A = \int_a^b f(x)\,dx$$



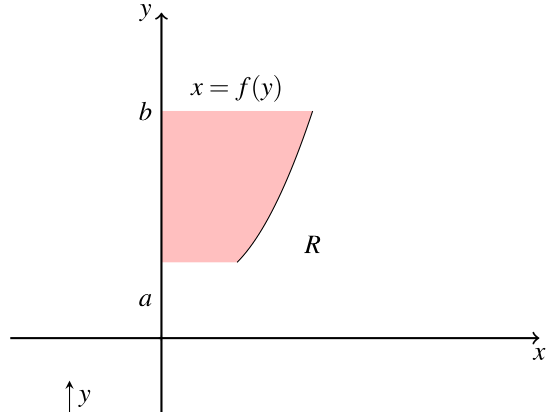
2. If $f(x)$ and $g(x)$ are continuous functions and $f(x) \geq g(x)$ for every $x \in [a,b]$, then the area A of the region bounded by the graphs of f and g is given by the integral:

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx$$



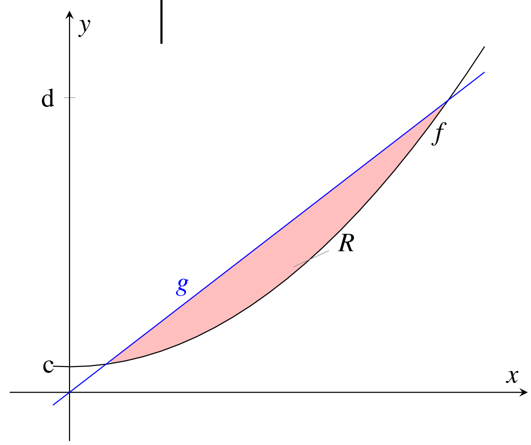
3. If $x = f(y)$ is a continuous function on $[c,d]$ and $f(y) \geq 0 \; \forall y \in [c,d]$, the area of the region under the graph of $f(y)$ from $y = c$ to $y = d$ is given by the integral:

$$A = \int_c^d f(y)\, dy$$



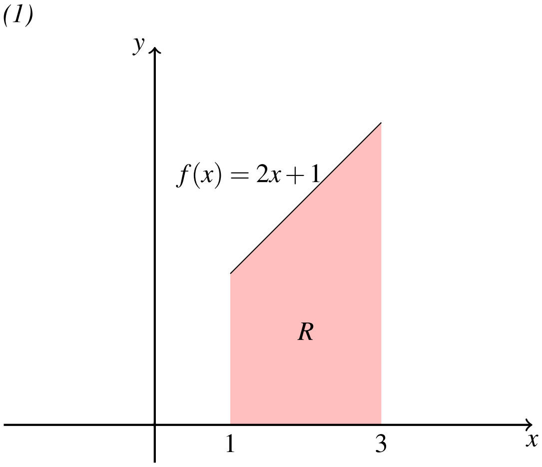
4. If $f(y)$ and $g(y)$ are continuous functions and $f(y) \geq g(y)$ for every $x \in [c,d]$, then the area A of the region bounded by the graphs of f and g is given by the integral:

$$A = \int_c^d \big(f(y) - g(y)\big)\, dy$$

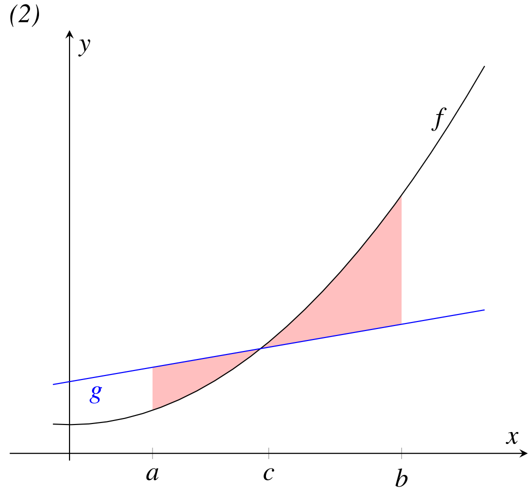


Example 7.1.1 *Express the area of the shaded region as a definite integral then find the area.*

(1)



(2)



Solution:

(1) Area : $A = \int_1^3 2x+1\; dx = \Big[x^2 + x\Big]_1^3 = \Big[(3^2+3) - (1^2+1)\Big] = 12 - 2 = 10$.

(2) We have two regions:

Region (1) : in the interval $[a,c]$
Upper graph: $y = g(x)$
Lower graph: $y = f(x)$
Area $A_1 = \displaystyle\int_a^c \big(g(x) - f(x)\big)\; dx.$

Region (2) : in the interval $[c,b]$
Upper graph: $y = f(x)$
Lower graph: $y = g(x)$
Area $A_2 = \displaystyle\int_c^b \big(f(x) - g(x)\big)\; dx.$

The total area is $A = A_1 + A_2$.

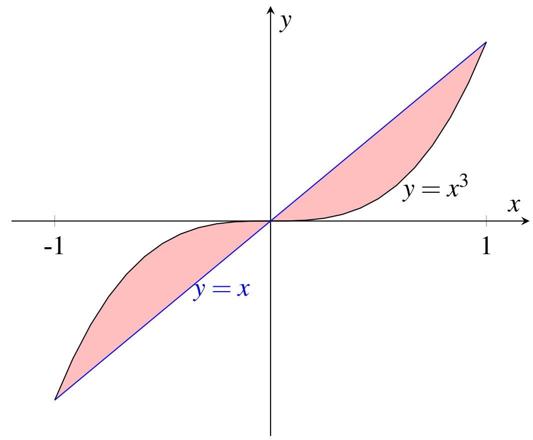
Example 7.1.2 *Sketch the region by the graphs of $y = x^3$ and $y = x$, then find its area.*

Solution:

The region bounded by the two curves is divided into two regions:

Region (1): in the interval $[-1, 0]$
Upper graph: $y = x^3$
Lower graph: $y = x$

$$A_1 = \int_{-1}^{0} x^3 - x\, dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = \left[0 - (\frac{1}{4} - \frac{1}{2})\right] = \frac{1}{4}\,.$$



Region (2): in the interval $[0, 1]$
Upper graph: $y = x$
Lower graph: $y = x^3$

$$A_2 = \int_{0}^{1} x - x^3\, dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \left[(\frac{1}{2} - \frac{1}{4}) - 0\right] = \frac{1}{4}\,.$$

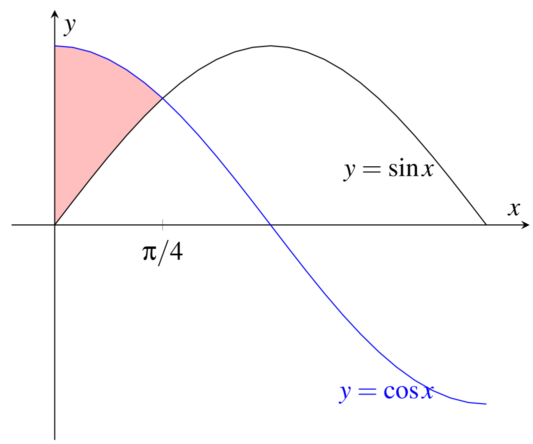
The total area is $A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Example 7.1.3 *Sketch the region by the graphs of $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{4}$, then find its area.*

Solution:

Note that, over the period $[0, \frac{\pi}{4}]$, the two curves intersect at $\frac{\pi}{4}$.

$$\begin{aligned} \text{Area: } A &= \int_0^{\frac{\pi}{4}} \left(\cos x - \sin x\right)\, dx \\ &= \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}} \\ &= \left[(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) - (1)\right] \\ &= 1 + \sqrt{2}\,. \end{aligned}$$

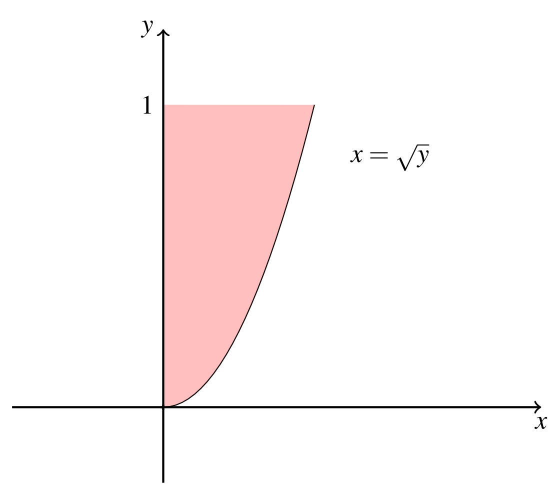


Example 7.1.4 *Sketch the region by the graphs of $x = \sqrt{y}$ from $y = 0$ and $y = 1$, then find its area.*

Solution:

The area bounded by the function $x = \sqrt{y}$ over the interval $[0, 1]$ is

$$\begin{aligned} A &= \int_0^1 \sqrt{y}\, dy \\ &= \frac{3}{2}\left[\, y\, \right]_0^1 \\ &= \frac{3}{2}\,. \end{aligned}$$



Example 7.1.5 *Sketch the region by the graphs of $x = 2y$, $x = \frac{y}{2} + 3$, then find its area.*

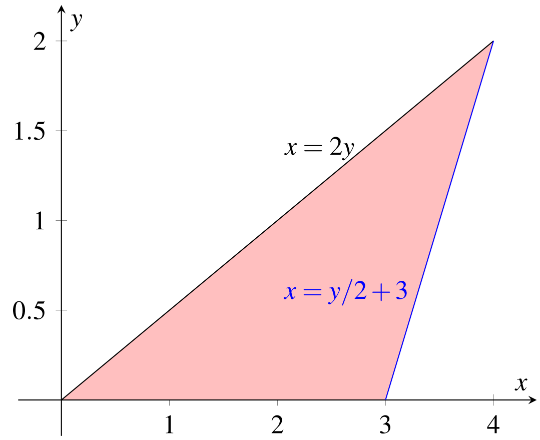
Solution:

First, we find the intersection points:

$$2y = \frac{y}{2} + 3 \Rightarrow 4y = y + 6 \Rightarrow y = 2\ .$$

The two curves intersect at $(4, 2)$.

$$\begin{aligned} \text{Area: } A &= \int_0^2 (\frac{y}{2} + 3 - 2y)\ dy \\ &= \int_0^2 (-\frac{3}{2}y + 3)\ dy \\ &= \Big[-\frac{3}{4}y^2 + 3y\Big]_0^2 \\ &= -3 + 6 \\ &= 3\ . \end{aligned}$$



Exercise 1:

1 - 27 ■ Sketch the region bounded by the graphs of the equations and find its area:

1. $y = \frac{x^2}{2},\ y = 0,\ x = 1,\ x = 3$
2. $y = x^3,\ x = 0,\ x = 2$
3. $y = x + 2,\ x = 1,\ x = 4$
4. $y = x^2 + 1,\ y = 0,\ x = 0,\ x = 2$
5. $y = x^3 + 1,\ y = 0,\ x = 0,\ x = 1$
6. $y = \sin x,\ x = 0,\ x = \pi$
7. $y = \tan x,\ x = \pi/4,\ x = \pi/3$
8. $y = -x,\ y = x + 1,\ x = 0$
9. $y = \sqrt{x},\ x + y = 2,\ y = 0$
10. $y = x^2,\ x = y - 2,\ y = 0$
11. $x = y^3,\ y = 0,\ y = 2,\ x = 0$
12. $x = \frac{y}{3},\ y = 1,\ y = 3,\ x = 0$
13. $x = (y+1)^2,\ y = 2,\ y = 5,\ x = 0$
14. $y = x^3 - 4x,\ y = 0,\ x = -2,\ x = 0$
15. $y = x^3,\ y = 2$
16. $y = x,\ y = 2x,\ y = -x + 2$
17. $y = \sqrt{x+1},\ x = 1,\ y = 0$
18. $x = y,\ x = y - 5,\ x = 0,\ x = 2$
19. $y = \sqrt{x-1},\ y = x,\ x = 1,\ x = 2$
20. $y = e^x,\ x = -2,\ x = 3$
21. $y = e^{x+1},\ x = 0,\ x = 1$
22. $y = \ln x,\ x = 1,\ x = 5$
23. $x = \sin y,\ y = 0,\ y = \pi/4$
24. $x = \sin y,\ x = \cos y,\ y = 0,\ y = \pi/4$
25. $y = \sin x,\ y = \cos x,\ x = -\pi/4,\ x = \pi/4$
26. $y = (x+1)^2 + 2,\ x = -2,\ x = 0$
27. $x = \ln y,\ x = 0,\ y = 1,\ y = e$

7.2 Solids of Revolution

Definition 7.2.1 *The solid of revolution (S) is a solid generated from rotating a region R about a line in the same plane where the line is called the axis of revolution.*

Example 7.2.1 *Let $f(x) \geq 0$ be continuous for every $x \in [a, b]$. Let R be a region bounded by the graph of f and x-axis form $x = a$ to $x = b$. Rotating the region R about x-axis generates a solid given in Figure 7.1 (right).*

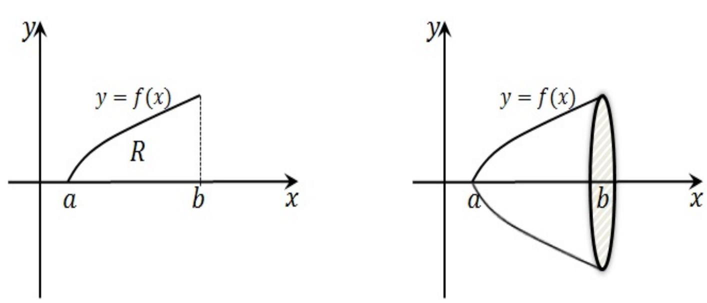


Figure 7.1: The figure on the left shows the region under the continuous curve $y = f(x)$ on the interval $[a, b]$. The figure on the right shows the solid S generated by rotating the region about the x-axis.

Example 7.2.2 *Let $f(x)$ be a constant function, as in Figure 7.2. The region R is a rectangle and rotating it about x-axis generates a circular cylinder.*

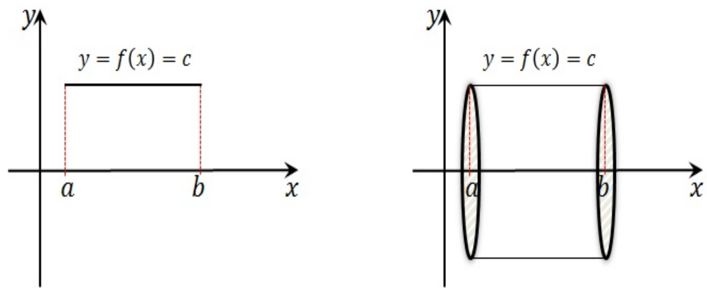


Figure 7.2: The figure on the left shows the region under the constant function $y = f(x) = c$ on the interval $[a,b]$. The figure on the right shows the circular cylinder generated by rotating the region about the x-axis.

Example 7.2.3 *Consider the region R bounded by the graph of $f(y)$ from $y = c$ to $y = d$ as in Figure 7.3 (left). Revolution of R about y-axis generates the revolution solid (right).*

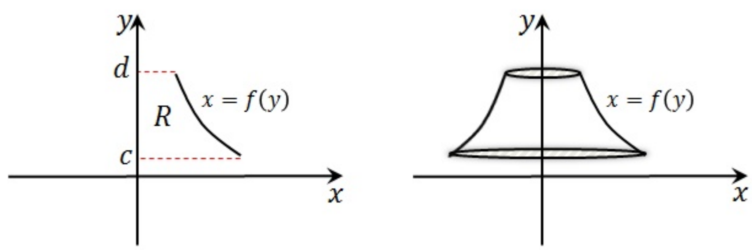


Figure 7.3: The figure on the left displays the region under the function $x = f(y)$ on the interval $[c,d]$. The figure on the right displays the solid S generated by rotating the region about the y-axis.

Exercise 2:

1 - 10 ■ Sketch the region R bounded by the graphs of the equations, then sketch the solid generated if R is revolved about about the specified axis.

1. $y = x^2, x = 1, y = 4$ about x-axis
2. $y = \sqrt{x}, x = 0, x = 9$ about x-axis
3. $y = \ln x, x = 0.5, x = e^3$ about x-axis
4. $y = e^x, x = -1, x = 5$ about x-axis
5. $y = \sin x, x = 0, x = \pi$ about x-axis
6. $y = \cos y, y = 0, y = \pi/2$ about y-axis
7. $y = e^{2x}, y = 0, y = 3$ about y-axis
8. $x = y + 1, y = -1, y = 5$ about y-axis
9. $y = x^2, y = x$ about x-axis
10. $y = \sqrt{x}, y = x$ about y-axis

7.3 Volumes of Solid of Revolution

In this section, we study three methods to evaluate the volume of the revolution solid known as the disk method, the washer method and the method of cylindrical shells.

7.3.1 Disk Method

Let f be continuous on $[a,b]$ and let R be the region bounded by the graphs, x-axis and the points $x = a,\ x = b$. Let S be the solid generated by revolving R about x-axis. Assume P is a partition of $[a,b]$ and $w_k \in [x_{k-1}, x_k]$. For each $[x_{k-1}, x_k]$, we form a rectangle, its high is $f(w_k)$ and its width is Δx_k.

The revolution of the rectangle about x-axis generates a circular disk as shown in Figure 7.4. Its radius and high are

$$r = f(w_k)\ , \quad h = \Delta x_k\ .$$

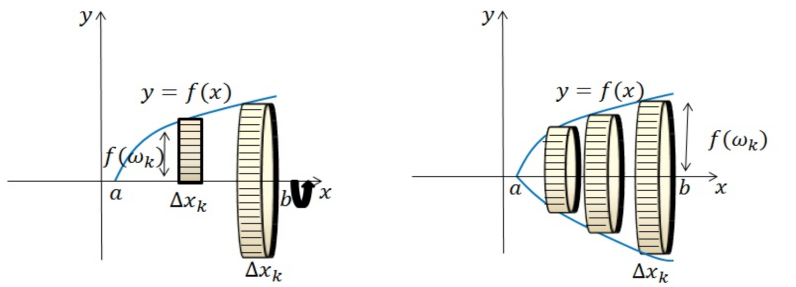


Figure 7.4: The figure on the left shows a continuous function f on $[a,b]$. The figure on the right shows a solid S generated by revolving R about x-axis.

From Figure 7.4, the volume of each circular disk is

$$V_k = \pi(f(w_k))^2\Delta x_k \ .$$

The sum of volumes of the circular disks approximately gives the volume of the solid of revolution:

$$V = \sum_{k=1}^{n} \Delta V_k = \lim_{n\to\infty} \sum_{k=1}^{n} \pi\left(f(w_k)\right)^2 \Delta x_k = \pi \int_a^b \left[f(x)\right]^2 dx \ .$$

Similarly, we find the volume of the solid of revolution about y-axis. Let f be continuous on $[c,d]$ and let R be the region bounded by the graphs, y-axis and the points $y = c,\ y = d$. Let S be the solid generated by revolving R about y-axis. Assume P is a partition of $[c,d]$ and $w_k \in [y_{k-1}, y_k]$. For each $[y_{k-1}, y_k]$, we form a rectangle, its high is $f(w_k)$ and its width is Δy_k.

The revolution of each rectangle about y-axis generates a circular disk as shown in 7.5. Its radius and high are

$$r = f(w_k)\ , \quad h = \Delta y_k \ .$$

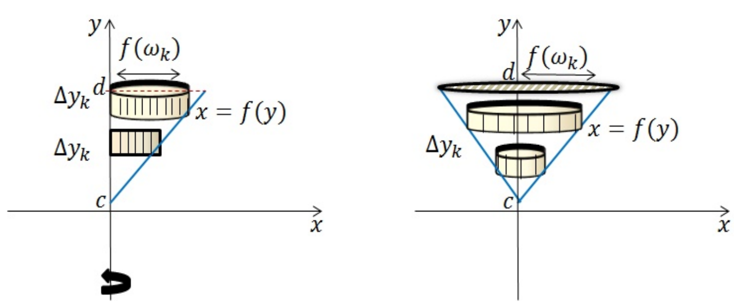


Figure 7.5: The figure on the left shows a continuous function f on $[c,d]$. The figure on the right presents a solid S generated by revolving R about y-axis.

The volume of the solid of revolution given in 7.5 (right) is approximately the sum of the volumes of circular disks:

$$V = \sum_{k=1}^{n} \Delta V_k = \lim_{n\to\infty} \sum_{k=1}^{n} \pi(f(w_k))^2\Delta y_k = \pi \int_c^d \left[f(y)\right]^2 dy \ .$$

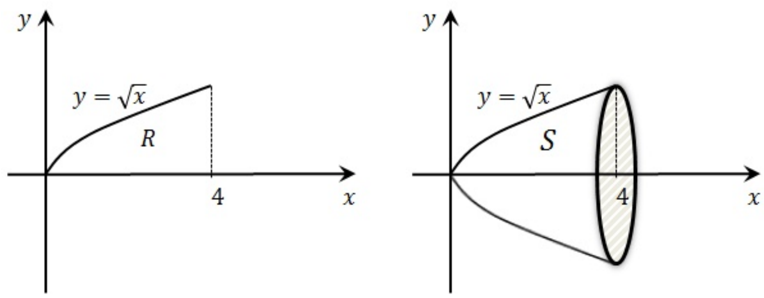
The volume of the solid of revolution by the disk method can be summarized in the following theorem:

Theorem 7.3.1

1. *If V is the volume of the solid of revolution determined by rotating the continuous function $f(x)$ on the interval $[a,b]$ about the x-axis, then*
$$V = \pi \int_a^b [f(x)]^2 \, dx \, .$$
2. *If V is the volume of the solid of revolution determined by rotating the continuous function $f(y)$ on the interval $[c,d]$ about the y-axis, then*
$$V = \pi \int_c^d [f(y)]^2 \, dy \, .$$

Example 7.3.1 *Sketch the region R bounded by the graphs of the equations $y = \sqrt{x}$, $x = 4$, $y = 0$. Then, find the volume of the solid generated if R is revolved about x-axis.*

Solution:

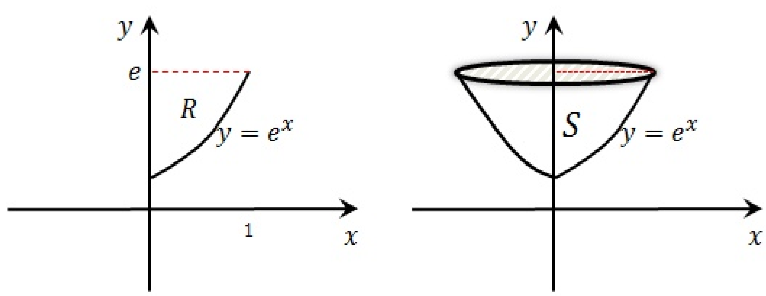


The previous figure shows the solid generated from revolving the region R about x-axis. Since the rotation is about x-axis, we have a vertical disk with radius $y = \sqrt{x}$ and thickness dx.

Thus, the volume of the solid S is
$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \frac{\pi}{2} \Big[x^2\Big]_0^4 = \frac{\pi}{2} \Big[16 - 0\Big] = 8\pi \, .$$

Example 7.3.2 *Sketch the region R bounded by the graphs of the equations $y = e^x$, $y = e$ and $x = 0$. Then, find the volume of the solid generated if R is revolved about y-axis.*

Solution:

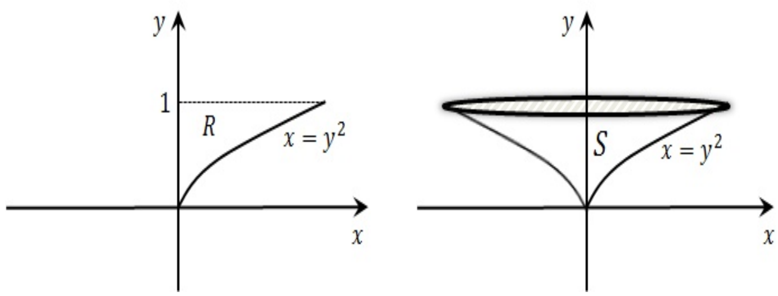


The previous figure shows the region R and the solid S generated by revolving the region about y-axis. Since the revolution of R is about y-axis, then we need to rewrite the function to become $x = f(y)$.
$$y = e^x \Rightarrow \ln y = \ln e^x \Rightarrow x = \ln y = f(y) \, .$$

Now, we have a horizontal disk with radius $x = \ln y$ and thickness dy. Thus, the volume of the solid S is
$$V = \pi \int_1^e (\ln y)^2 \, dy = \Big[2y + y \, (\ln y)^2 - 2y \, \ln y\Big]_1^e = e - 2 \, .$$

Example 7.3.3 *Let $x = y^2$ on the interval $[0,1]$. Rotate the region around the y-axis and find the volume of the resulting solid.*

Solution:

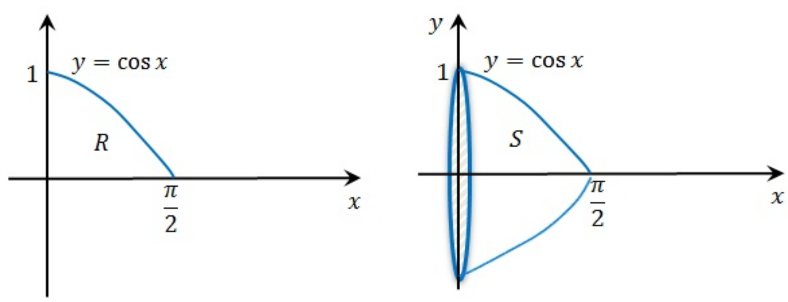


Since the revolution of R is about y-axis, we have a horizontal disk with radius $x = y^2$ and thickness dy. Thus, the volume of the solid S is

$$V = \pi \int_0^1 (y^2)^2 \, dy = \frac{\pi}{5}\Big[y^5\Big]_0^1 = \frac{\pi}{5}\Big[1-0\Big] = \frac{\pi}{5} \, .$$

Example 7.3.4 *Sketch the region R bounded by the graphs of the equations $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$. Then, find the volume of the solid generated if R is revolved about x-axis.*

Solution:



The region R and the solid S generated by revolving the region about x-axis is provided in the figure. Thus, the disk to evaluate the volume of the generated solid S is vertical where the radius is $y = \cos x$ and the thickness is dx:

$$V = \pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} 1 + \cos 2x \, dx = \frac{\pi}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{4} \, .$$

7.3.2 Washer Method

The washer method is a generalization of the disk method for a region between two functions $f(x)$ and $g(x)$. Let R be a region bounded by the graphs of $f(x)$ and $g(x)$ from $x = a$ to $x = b$ such that $f(x) > g(x)$ (see Figure 7.6). The volume of the solid S generated by rotating the area bounded by the graphs of the two functions around x-axis can be found by calculating the difference between the two solids generated by rotating the regions under f and g:

$$\begin{aligned} V &= \int_a^b [f(x)]^2 \, dx - \int_a^b [g(x)]^2 \, dx, \\ &= \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx \, . \end{aligned}$$

Similarly, let R be a region bounded by the graphs of $f(y)$ and $g(y)$ such that $f(y) > g(y)$ from $y = c$ to, $y = d$ (see Figure

7.7). The volume of the solid S generated from rotating the area bounded by the graphs of f and g around y-axis is

$$V = \int_c^d [f(y)]^2\, dy - \int_c^d [g(y)]^2\, dy,$$
$$= \int_c^d \left([f(y)]^2 - [g(y)]^2\right) dy\,.$$

Theorem 7.3.2 summarizes the washer method.

Theorem 7.3.2 1. *If V is the volume of the solid of revolution determined by rotating the continuous functions $f(x)$ and $g(x)$ such that $f > g$ on the interval $[a,b]$ about the x-axis, then*

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx\,.$$

2. *If V is the volume of the solid of revolution determined by rotating the continuous functions $f(y)$ and $g(y)$ such that $f > g$ on the interval $[c,d]$ about the y-axis, then*

$$V = \pi \int_c^d \left([f(y)]^2 - [g(y)]^2\right) dy\,.$$

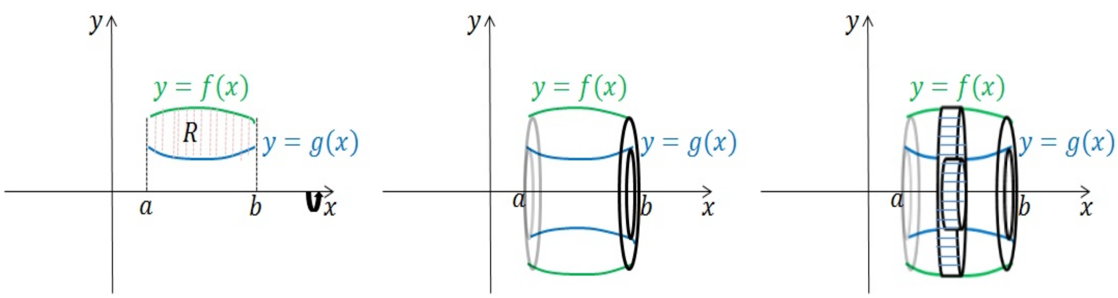


Figure 7.6: The volume by the washer method for the solid S generated from rotating the area around x-axis.

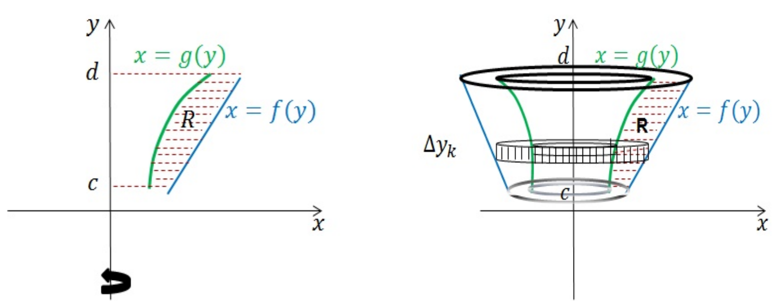


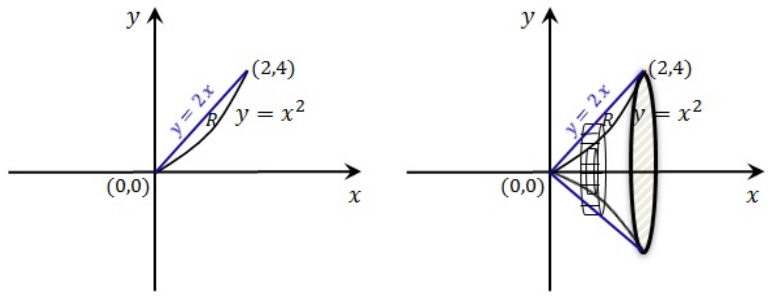
Figure 7.7: The volume by the washer method for the solid S generated from rotating the area around y-axis.

Example 7.3.5 *Let R be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving of the bounded region about x-axis.*

Solution:

Let $f(x) = x^2$ and $g(x) = 2x$. First, we find the intersection points:

$$f(x) = g(x) \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0 \text{ or } x = 2\,.$$



Substitute $x=0$ into $f(x)$ or $g(x)$ gives the same value $y=0$. Similarly, substitute $x=2$, we have $y=2$. Thus, the two curves f and g intersect in two points $(0,0)$ and $(2,4)$.

The figure shows the region R and the solid generated from revolving that region about x-axis. A vertical rectangle generates a washer where

the outer radius: $y_1 = 2x$,

the inner radius: $y_2 = x^2$ and

the thickness: dx .

The volume of the washer is

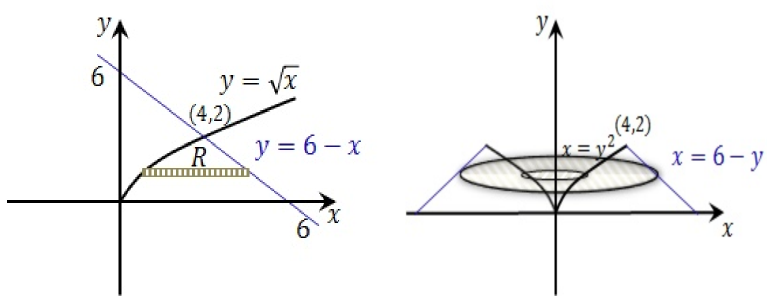
$$dV = \pi\left[2x - x^2\right]\, dx\, .$$

Thus, the volume of the solid over the interval $[0,2]$ is

$$V = \pi \int_0^2 \left[(2x)^2 - (x^2)^2\right]\, dx = \pi \int_0^2 4x^2 - x^4\, dx = \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \pi\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{64}{15}\pi\, .$$

Example 7.3.6 *Consider a region R bounded by the graphs $y = \sqrt{x}$, $y = 6 - x$ and x-axis. Rotate this region about y-axis and find the volume of the generated solid.*

Solution:



The two curves $y = \sqrt{x}$ and $y = 6 - x$ intersect in one point $(4,2)$. The region R is shown in the figure. The revolution of that region generates a solid S. Since the rotation is about y-axis, first, we need to rewrite the functions as $x = f(y)$ and $x = g(y)$. Thus, $x = y^2$ and $x = 6 - y$. Second, a horizontal rectangle generates a washer where

$y = \sqrt{x} \Rightarrow x = y^2 = f(y)$
and $y = 6 - x$
$\Rightarrow x = 6 - y = g(y)$

the outer radius: $x_1 = 6 - y$,

the inner radius: $x_2 = y^2$ and

the thickness: dy .

The volume of the washer is

$$dV = \pi\left[(6-y)^2 - (y^2)^2\right]\, dy\, .$$

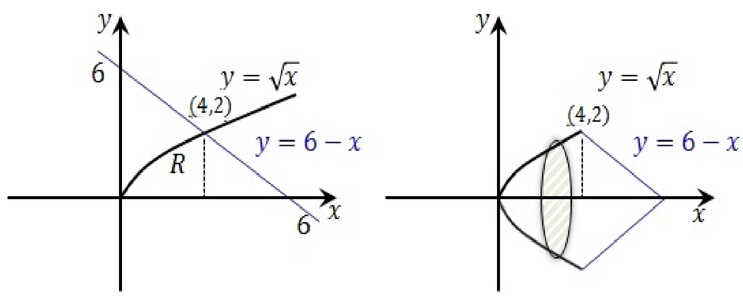
The volume of the solid over the interval $[0,2]$ is

$$V = \pi \int_0^2 \left[(6-y)^2 - (y^2)^2\right] dy = \pi\Big[-\frac{(6-y)^3}{3} - \frac{y^5}{5}\Big]_0^2 = \pi\Big[\big(-\frac{64}{3} - \frac{32}{5}\big) - \big(-\frac{216}{3} - 0\big)\Big] = \frac{664}{15}\pi\,.$$

Example 7.3.7 *Reconsider the same region as in Example 7.3.6 enclosed by the curves $y = \sqrt{x}$, $y = 6-x$ and x-axis. Now rotate this region about the x-axis instead and find the resulting volume.*

Solution:

From the figure, we find that the solid is made up of two separate functions and each requires its own integral. Meaning that, we use the disk method to evaluate the volume of the solid generated by each function:



$$\begin{aligned} V &= \pi \int_0^4 (\sqrt{x})^2\, dx + \pi \int_4^6 (6-x)^2\, dx \\ &= \pi \int_0^4 x\, dx + \pi \int_4^6 (6-x)^2\, dx \\ &= \frac{\pi}{2}\Big[x^2\Big]_0^4 - \frac{\pi}{3}\Big[(6-x)^3\Big]_4^6 \\ &= (8\pi - 0) - (0 - \frac{8\pi}{3}) = \frac{32}{3}\pi\,. \end{aligned}$$

Note: we used the substitution method to do the second integral (with $u = 6-x$ and $du = dx$)

The revolution of a region is not always around x-axis or y-axis. It could be around a line parallels x-axis or y-axis.

Remark 7.3.1

1. *If the axis of revolution is a line $y = y_0$, the volume is as the case when the region revolves around x-axis.*
2. *If the axis of revolution is a line $x = x_0$, the volume is as the case when the region revolves around y-axis.*

The difference between the revolution of the region about axis and the line $y = y_0$ or $x = x_0$ is in calculating the inner and the outer radius.

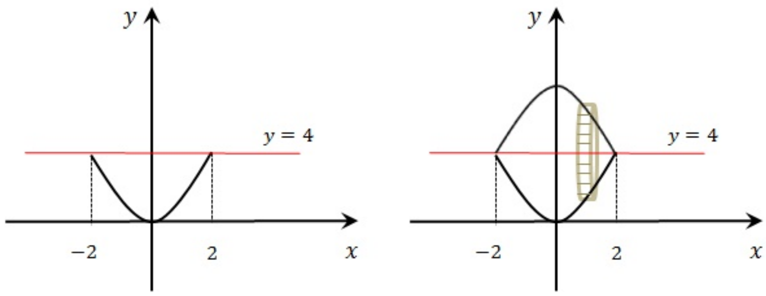
The following examples illustrate the previous remark.

Example 7.3.8 *Evaluate the volume of the solid generated by revolution of the bounded region by graphs of the functions $y = x^2$ and $y = 4$ if the revolution is about the given line:*

(a) $y = 4$ *(b)* $x = 2$

Solution:

(a)



Here, we have a vertical circular disk:

the radius of the disk: $4-y=4-x^2$, and

the thickness: dx .

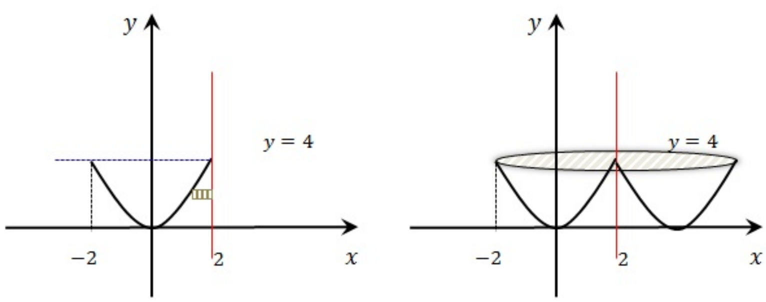
The volume of the disk is

$$dV = \pi(4-x^2)^2\ dx\ .$$

The volume of the solid over the interval $[-2,2]$ is

$$V = \pi\int_{-2}^{2}(4-x^2)^2\ dx = \pi\int_{-2}^{2} 16-8x^2+x^4\ dx = \pi\Big[16x-\frac{8}{3}x^3+\frac{x^5}{5}\Big]_{-2}^{2} = \frac{512}{15}\pi\ .$$

(b)



In this case, a horizontal rectangle will generate a washer where

the outer radius: $2+\sqrt{y}$,

the inner radius: $2-\sqrt{y}$ and

the thickness: dy .

The volume of the washer is

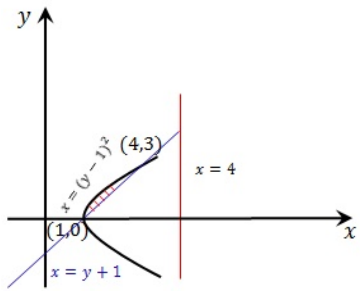
$$dV = \pi\big[(2+\sqrt{y})^2-(2-\sqrt{y})^2\big]\ dy = 8\pi\sqrt{y}\ dy\ .$$

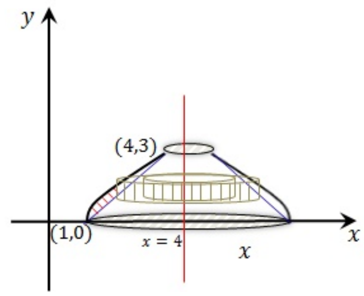
The volume of the solid over the interval $[0,4]$ is

$$V = \pi\int_{0}^{4} 8\sqrt{y}\ dx = \frac{16\pi}{3}\ \Big[\ y^{\frac{3}{2}}\ \Big]_{0}^{4} = \frac{128}{3}\pi\ .$$

Example 7.3.9 *Sketch the region R bounded by the graphs of the equations $x=(y-1)^2$ and $x=y+1$. Then, find the volume of the solid generated if R is revolved about $x=4$.*

Solution:





First, we find the intersection points:

$$(y-1)^2 = y+1 \Rightarrow y^2 - 2y + 1 = y + 1 \Rightarrow y^2 - 3y = 0 \Rightarrow y = 0 \text{ or } y = 3 \,.$$

Thus, the two curves intersect in two points $(1,0)$ and $(4,3)$. The figure shows the region R and the solid S. A horizontal rectangle generates a washer where

the outer radius: $4-(y-1)^2$,

the inner radius: $4-(y+1) = 3-y$ and

the thickness: dy .

The volume of the washer is

$$dV = \pi\left[(4-(y-1)^2)^2 - (3-y)^2\right]\, dy \,.$$

Thus, the volume of the solid over the interval $[0,3]$ is

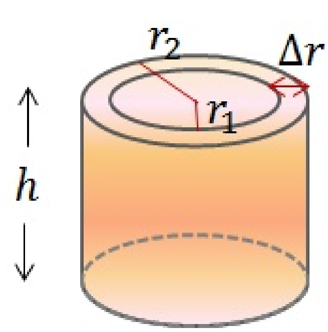
$$\begin{aligned} V = \pi\int_0^3 (4-(y-1)^2)^2 - (3-y)^2\, dy &= \pi\Big(\int_0^3 16\, dy - 8\int_0^3 (y-1)^2\, dy + \int_0^3 (y-1)^4\, dy - \int_0^3 (3-y)^2\, dy\Big) \\ &= \pi\Big[16y - \frac{8(y-1)^3}{3} + \frac{(y-1)^5}{5} + \frac{(3-y)^3}{3}\Big]_0^3 = \frac{108}{15}\pi \,. \end{aligned}$$

7.3.3 Method of Cylindrical Shells

The method of cylindrical shells sometimes easier than the washer method. This is because solving equations for one variable in terms of another is not sometimes simple (i. e., solving x in terms of y and versa visa). For example, the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. By the washer method, we would have to solve the cubic equation for x in terms of y and this is not simple.

In the washer method, we assume that the rectangle from each sub-interval is vertical to the axis of the revolution, but in the method of cylindrical shells, the rectangle is parallel to the axis of the revolution.

As shown in the next figure, let
r_1 be the inner radius of the shell,
r_2 be the outer radius of the shell,
h be high of the shell,
$\Delta r = r_2 - r_1$ be the thickness of the shell,
$r = \frac{r_1+r_2}{2}$ be the average radius of the shell.



The volume of the cylindrical shell is

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi(\frac{r_2 + r_1}{2})h(r_2 - r_1) \\ &= 2\pi r h \Delta r \ . \end{aligned}$$

Now, consider the graph given in Figure 7.8. The revolution of the region R about y-axis generates a solid given in the same figure. Let P be a partition of the interval $[a,b]$ and let w_k be the midpoint of $[x_{k-1}, x_k]$.

The revolution of the rectangle about y-axis generates a cylindrical shell where

the high = $f(w_k)$,

the average radius = w_k and

the thickness = Δx_k .

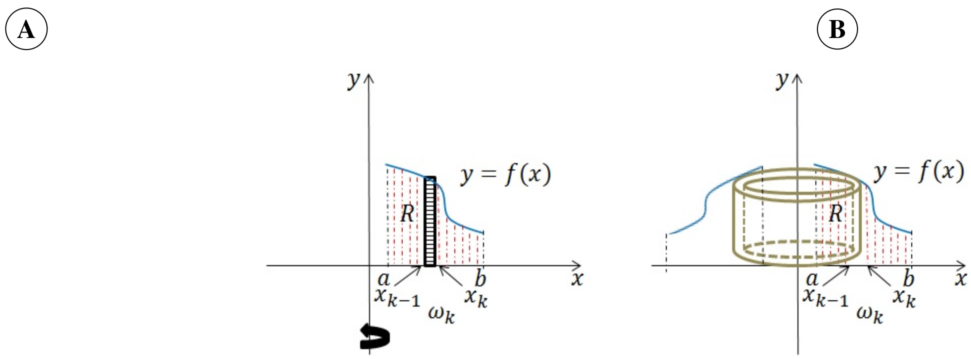


Figure 7.8: The volume by the method of cylindrical shells for the solid S generated by rotating the region around y-axis.

Hence, the volume of the cylindrical shell

$$V_k = 2\pi w_k f(w_k)\Delta x_k \ .$$

To evaluate the volume of the whole solid, we sum the volume of all cylindrical shells. This means

$$V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} w_k f(w_k)\Delta x_k \ .$$

From Riemann sum

$$\lim_{n\to\infty} \sum_{k=1}^{n} w_k f(w_k)\Delta x_k = \int_a^b x f(x)\, dx$$

and this implies

$$V = 2\pi \int_a^b x f(x)\, dx \ .$$

Similarly, if the revolution of the region is about x-axis, the volume of the solid of revolution is

$$V = 2\pi \int_c^d y f(y)\, dy \ .$$

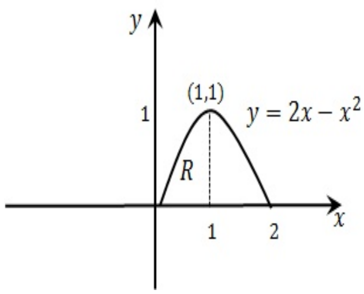
The volume by the method of cylindrical shells can be summarized as follows:

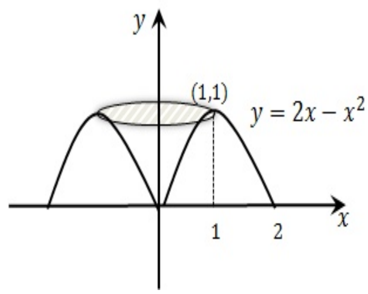
Theorem 7.3.3

1. *If V is the volume of the solid of revolution determined by rotating the continuous function $f(x)$ on the interval $[a,b]$ about the y-axis, then*
$$V = 2\pi \int_a^b x f(x)\, dx \, .$$
2. *If V is the volume of the solid of revolution determined by rotating the continuous function $f(y)$ on the interval $[c,d]$ about the x-axis, then*
$$V = 2\pi \int_c^d y f(y)\, dy \, .$$

Example 7.3.10 *Sketch the region R bounded by the graphs of the equations $y = 2x - x^2$ and $x = 0$. Then, by the method of cylindrical shells, find the volume of the solid generated if R is revolved about y-axis.*

Solution:





Since the revolution is about y-axis, the rectangle is vertical where

the high: $y = 2x - x^2$,

the average radius: x,

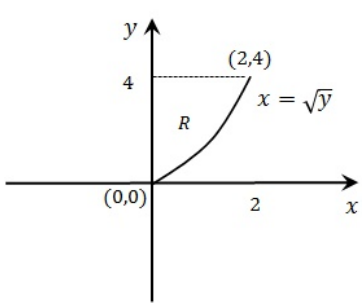
the thickness: dx .

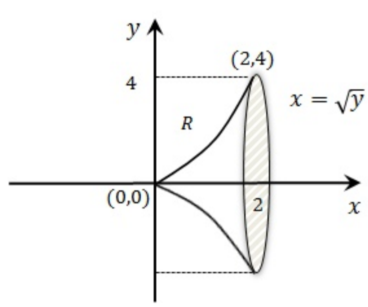
The volume of a cylindrical shell
$$dV = 2\pi x(2x - x^2)\, dx \, .$$

Thus, the volume of the solid is
$$V = 2\pi \int_0^2 x(2x - x^2)\, dx = 2\pi \int_0^2 2x^2 - x^3\, dx = 2\pi \Big[\frac{2x^3}{3} - \frac{x^4}{4}\Big]_0^2 = 2\pi\Big(\frac{16}{3} - \frac{16}{4}\Big) = \frac{8\pi}{3} \, .$$

Example 7.3.11 *Sketch the region R bounded by the graphs of the equations $x = \sqrt{y}$, $x = 2$ and y-axis. Then, find the volume of the solid generated if R is revolved about x-axis.*

Solution:





Since the revolution is about x-axis, the rectangle is horizontal where

the high: $x = \sqrt{y}$,

the average radius: y and

the thickness: dy .

The volume of the cylindrical shell

$$dV = 2\pi\, y\, \sqrt{y}\, dy\ .$$

Thus, the volume of the solid is

$$V = 2\pi \int_0^4 y\sqrt{y}\, dy = 2\pi \int_0^4 y^{\frac{3}{2}}\ dy = \frac{4\pi}{5}\left[y^{\frac{5}{2}}\right]_0^4 = \frac{4\pi}{5}\left[32-0\right] = \frac{128\pi}{5}\ .$$

Exercise 3:

1 - 8 ■ Sketch the region R bounded by the graphs of the equations and find the volume of the solid generated if R is revolved about the x-axis:

1. $y = x+1, x = 0, x = 1$
2. $y = x^2+1,\ x = 0,\ x = 2$
3. $y = x^3,\ x = 0,\ x = 2$
4. $y = \sqrt{x},\ x = 0,\ x = 4$
5. $y = \sqrt{x},\ x = y$
6. $y = \sin x,\ x = 0,\ x = \pi/2$
7. $y = 1-x^2,\ y = x^2$
8. $y = x^3+1,\ y = x+1$

9 - 16 ■ Sketch the region R bounded by the graphs of the equations and find the volume of the solid generated if R is revolved about the y-axis:

9. $y = x^2,\ y = 1,\ y = 4$
10. $y = \sqrt{x},\ y = 0,\ y = 3$
11. $x = \cos y,\ y = 0,\ y = \pi/2$
12. $x = \ln y,\ y = 1,\ y = e$
13. $y = x,\ y = (x-1)^2+1$
14. $y = e^x,\ x = 1,\ x = 2,\ y = 0$
15. $xy = 4,\ x+y = 5$
16. $y = x^2,\ y^2 = 8x$

17 - 26 ■ Set up evaluate an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis:

17. $y = x^2,\ y = 1$, about $x = 1$
18. $y = x^2,\ y = 1$, about x-axis
19. $y = x^2,\ x = y^2$ about $y = -1$
20. $y = \sqrt{x-1},\ y = 0,\ x = 5$ about $x = 5$
21. $y = x^2,\ x = 0,\ y = 1,\ y = 4$ about $y = 1$
22. $y = x-x^2,\ y = 0$ about $x = 2$
23. $y = x^2,\ y = 0,\ x = 1,\ x = 2$ about $x = 1$
24. $y = x^2,\ y = 0,\ x = 1,\ x = 2$ about $x = 4$
25. $y = \sqrt{x-1},\ y = 0,\ x = 5$ about $y = 3$
26. $y = x^4,\ y = \sin(\frac{\pi x}{2})$ about $x = -1$

27 - 35 ■ Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis. Sketch the region and a typical shell.

27. $x = 1+y^2,\ x = 0,\ y = 1,\ y = 2$ about x-axis
28. $x = \sqrt{y},\ x = 0,\ y = 1$ about x-axis
29. $y = x^3,\ y = 8,\ x = 0$ about x-axis
30. $y = \frac{1}{x},\ x = 1,\ x = 2$ about y-axis
31. $y = x^2,\ y = 0,\ x = 1$ about y-axis
32. $y = x^2,\ y = x$ about x-axis
33. $y = \sin x,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{4}$ about y-axis
34. $y = x^2+x,\ y = 0$ about y-axis
35. $y = x+\frac{4}{x},\ y = 5$ about $x = -1$

7.4 Arc Length and Surfaces of Revolution

7.4.1 Arc Length

Let $y = f(x)$ be a smooth function on $[a,b]$. Assume $P = \{x_0, x_1, \ldots, x_n\}$ is a regular partition of the interval $[a,b]$ and let $y_0, y_1, \ldots, y_n$ be the points on the curve as shown in the following figure.

The distance between any two points (x_{k-1}, y_{k-1}) and (x_k, y_k) in the curve is

$$
\begin{aligned}
d(y_{k-1}, y_k) &= \sqrt{(\triangle x_k)^2 + (\triangle y_k)^2} \\
&= \sqrt{(\triangle x_k)^2 + (f(x_k) - f(x_{k-1}))^2} \\
&= (\triangle x_k)\sqrt{1 + \frac{(f(x_k) - f(x_{k-1}))^2}{(\triangle x_k)^2}} \\
&= \frac{b-a}{n}\sqrt{1 + \Big[\frac{f(x_k) - f(x_{k-1})}{\triangle x_k}\Big]^2}
\end{aligned}
$$

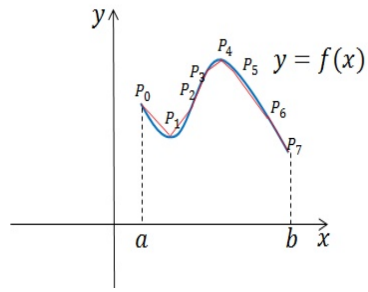


Figure 7.9: The length of the arc of $f(x)$ from $x = a$ to $x = b$.

From the conditions of the mean value theorem of differential calculus for the function f on $[x_{k-1}, x_k]$, we have

$$f'(c_i) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

for some $c_i \in (x_{k-1}, x_k)$. Thus, the distance between (x_{k-1}, y_{k-1}) and (x_k, y_k) is

$$d(y_{k-1}, y_k) = \frac{b-a}{n}\sqrt{1 + \left[f'(c_i)\right]^2} .$$

The sum of the distances is

$$\frac{b-a}{n}\left[\sqrt{1 + \left[f'(c_1)\right]^2} + \sqrt{1 + \left[f'(c_2)\right]^2} + ... + \sqrt{1 + \left[f'(c_n)\right]^2}\right] .$$

The previous sum is a Riemann sum for the function $\sqrt{1 + \left[f'(x_i)\right]^2}$ from a to b where for a better approximation, we let n be large enough. From this, the arc length is

$$L(f) = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

Similarly, let $x = g(y)$ be a smooth function on $[c, d]$. The length of the arc of g from $y = c$ to $y = d$ is

$$L(g) = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\, dy .$$

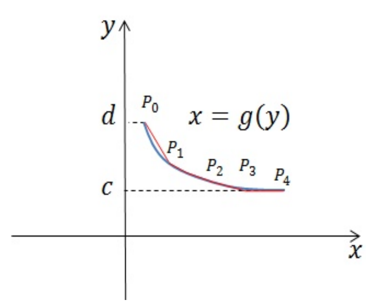


Figure 7.10: The length of the arc of $g(y)$ from $y = c$ to $y = d$.

Theorem 7.4.1

1. *Let $y = f(x)$ be a smooth function on $[a, b]$. The length of the arc of f is*

$$L(f) = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx .$$

2. *Let $x = g(y)$ be a smooth function on $[c, d]$. The length of the arc of g is*

$$L(g) = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\, dy .$$

Example 7.4.1 *Find the arc length of the graph of the given equation from A to B:*

1. $y = 5 - \sqrt{x^3}$; $A(0,5),\ B(4,-3)$
2. $x = 4y$; $A(0,0),\ B(1,4)$

Solution:

(1) If $y = 5 - \sqrt{x^3} \Rightarrow f'(x) = -\frac{3}{2}x^{\frac{1}{2}} \Rightarrow (f'(x))^2 = \frac{9}{4}x \Rightarrow 1 + (f'(x))^2 = \frac{4+9x}{4} \Rightarrow \sqrt{1+(f'(x))^2} = \frac{\sqrt{4+9x}}{2}$.

The length of the curve is

$$L = \frac{1}{2}\int_0^4 \sqrt{4+9x}\,dx = 3\Big[(4+9x)^{\frac{3}{2}}\Big]_0^4 = 3\Big[40^{\frac{3}{2}} - 4^{\frac{3}{2}}\Big] = \frac{8\Big[10\sqrt{10}-1\Big]}{3}\ .$$

(2) If $x = 4y \Rightarrow g'(y) = 4 \Rightarrow (g'(y))^2 = 16 \Rightarrow 1 + (g'(y))^2 = 17 \Rightarrow \sqrt{1+(g'(y))^2} = \sqrt{17}$.

The length of the curve is

$$L = \sqrt{17}\int_0^4 dy = \sqrt{17}\Big[\ y\ \Big]_0^4 = \sqrt{17}\Big[4-0\Big] = 4\sqrt{17}\ .$$

Example 7.4.2 *Find the arc length of the graph of the given equation over the indicated interval:*

1. $y = \cosh x$; $0 \le x \le 2$
2. $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$; $-2 \le y \le -1$

Solution:

(1) If $y = \cosh x \Rightarrow f'(x) = \sinh x \Rightarrow (f'(x))^2 = \sinh^2 x \Rightarrow 1 + (f'(x))^2 = 1 + \sinh^2 x \Rightarrow \sqrt{1+(f'(x))^2} = \cosh x$.

The length of the curve is

$$L = \int_0^2 \cosh x\,dx = \Big[\sinh x\Big]_0^2 = \sinh 2 - \sinh 0 = \sinh 2\ .$$

(2) If $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2} \Rightarrow g'(y) = \frac{1}{2}(y^3 - \frac{1}{y^3}) \Rightarrow (g'(y))^2 = \frac{(y^6-1)^2}{4y^6} \Rightarrow 1 + (g'(y))^2 = \frac{4y^6+y^{12}-2y^6+1}{y^6}$.

This implies

$$1 + (g'(y))^2 = \frac{y^{12}+2y^6+1}{y^6} \Rightarrow \sqrt{1+(g'(y))^2} = \sqrt{\frac{(y^6+1)^2}{y^6}} = \frac{y^6+1}{y^3}.$$

Since $y < 0$ over $[-2,-1]$, the length of the curve is

$$L = -\int_{-2}^{-1} y^3 + y^{-3}\,dy = -\Big[\frac{y^4}{4} - \frac{1}{2y^2}\Big]_{-2}^{-1} = \frac{35}{8}\ .$$

7.4.2 Surfaces of Revolution

Definition 7.4.1 *The surface of revolution is generated by rotating the curve of a continuous function about an axis.*

Let $y = f(x) \ge 0$ be a smooth function on the interval $[a,b]$. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of the interval $[a,b]$ and $y_0, y_1, ..., y_n$ be the points on the curve as shown in Figure 7.11. If D_k is a frustum cone generated from rotating the subinterval $[x_{k-1}, x_k]$ about x-axis. The area of a frustum cone with radii r_1 and r_2 and slant length ℓ is $S.A = \pi(r_1 + r_2)\ell$.

From this, the surface of D_k is

$$S.A(D_k) = \pi[f(x_k) + f(x_{k-1})] \triangle \ell_k$$

where $\triangle\ell_k$ is the length of the subinterval $[y_{k-1}, y_k]$ i.e., $\triangle\ell_k = \sqrt{(\triangle x_k)^2 + (f(x_k) - f(x_{k-1}))^2}$.

From the intermediate value theorem, there exists $\omega_k \in (x_{k-1}, x_k)$ such that

$$f(x_k) - f(x_{k-1}) = f'(\omega_k) \triangle x_k \ .$$

This implies $\triangle\ell_k = \triangle x_k \sqrt{1 + [f'(\omega_k)]^2}$.

For n large, $f(x_k) \approx f(x_{k-1}) \approx f(\omega_k)$ and this means

$$S.A = \sum_{k=1}^{n} 2\pi f(\omega_k) \sqrt{1 + [f'(\omega_k)]^2} \triangle x_k \ .$$

From Riemann sum,

$$S.A = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi f(\omega_k) \sqrt{1 + [f'(\omega_k)]^2} \triangle x_k = 2\pi \int_a^b |\, f(x) \,| \sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_a^b |\, y \,| \sqrt{1 + (\frac{dy}{dx})^2}\, dx \ .$$

If the rotation is about y-axis, then

$$S.A = 2\pi \int_a^b |\, x \,| \sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_a^b |\, x \,| \sqrt{1 + (\frac{dy}{dx})^2}\, dx \ .$$

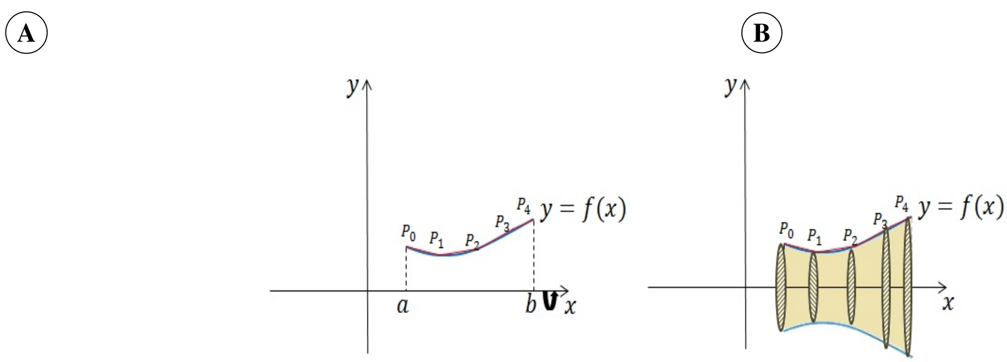


Figure 7.11: The surface of revolution generated by rotating the curve of a continuous function about x-axis.

Similarly, if $x = g(y)$ is a smooth function on $[c, d]$. The surface area $S.A$ generated by revolution the curve of g about y-axis from $y = c$ to $y = d$ is

$$S.A = 2\pi \int_c^d |\, g(y) \,| \sqrt{1 + \left[g'(y)\right]^2}\, dy = 2\pi \int_c^d |\, x \,| \sqrt{1 + (\frac{dx}{dy})^2}\, dy \ .$$

If the rotating is about x-axis, then

$$S.A = 2\pi \int_c^d |\, y \,| \sqrt{1 + \left[g'(y)\right]^2}\, dy = 2\pi \int_c^d |\, y \,| \sqrt{1 + (\frac{dx}{dy})^2}\, dy \ .$$

Theorem 7.4.2

1. *Let $y = f(x)$ be a smooth function on $[a,b]$.*
   - *If the rotating is about x-axis,*

$$S.A = 2\pi \int_a^b |\, y \,| \sqrt{1 + \left(f'(x)\right)^2}\, dx .$$

   - *If the rotating is about y-axis,*

$$S.A = 2\pi \int_a^b |\, x \,| \sqrt{1 + \left(f'(x)\right)^2}\, dx .$$

2. *Let $x = g(y)$ be a smooth function on $[c,d]$. The surface area of revolution about y-axis is*
   - *If the rotating is about y-axis,*

$$S.A = 2\pi \int_c^d |\, x \,| \sqrt{1 + \left(g'(y)\right)^2}\, dy .$$

   - *If the rotating is about x-axis,*

$$S.A = 2\pi \int_c^d |\, y \,| \sqrt{1 + \left(g'(y)\right)^2}\, dy .$$

Example 7.4.3 *Find the surface area generated by revolving the curve of the function $\sqrt{4-x^2}$, $-2 \le x \le 2$ around x-axis.*

Solution:

We use the formula $S.A = 2\pi \int_a^b |\, f(x) \,| \sqrt{1 + (f'(x))^2}\, dx$.

$$\text{If } y = \sqrt{4-x^2} \Rightarrow f'(x) = \frac{-2x}{2\sqrt{4-x^2}} \Rightarrow (f'(x))^2 = \frac{x^2}{4-x^2} \Rightarrow 1 + (f'(x))^2 = \frac{4}{4-x^2} \Rightarrow \sqrt{1+(f'(x))^2} = \frac{2}{\sqrt{4-x^2}} .$$

The area of the revolution surface is $S.A = 2\pi \int_{-2}^{2} \sqrt{4-x^2}\, \frac{2}{\sqrt{4-x^2}}\, dx = 4\pi\big[2+2\big] = 16\pi$.

Example 7.4.4 *Find the surface area generated by revolving the curve of the function $x = y^3$ on the interval $[0,1]$ around y-axis.*

Solution:

We use the formula $S.A = 2\pi \int_c^d |\, f(y) \,| \sqrt{1 + (f'(y))^2}\, dy$.

$$\text{If } x = y^3 \Rightarrow g'(y) = 3y^2 \Rightarrow (g'(y))^2 = 9y^4 \Rightarrow 1 + (g'(y))^2 = 1 + 9y^4 \Rightarrow \sqrt{1+(g'(y))^2} = \sqrt{1+9y^4} .$$

The area of the revolution surface is $S.A = 2\pi \int_0^1 y^3 \sqrt{1+9y^4}\, dy = \frac{\pi}{27}\left[(1+9y^4)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{27}\left[10\sqrt{10} - 1\right]$.

Exercise 4:

1 - 13 ■ Find the arc length of the graph:

1. $y = \ln x,\ 1 \le x \le 3$
2. $y = e^x,\ 0 \le x \le 1$
3. $y = x^2 + 1,\ 1 \le x \le 3$
4. $y = \sqrt{x},\ 1 \le x \le 4$
5. $y = \frac{1}{2}x^2,\ 0 \le x \le 1$
6. $y = \ln(\cos x),\ \pi/4 \le x \le \pi/3$
7. $x = \frac{2}{3}(y-1)^{\frac{3}{2}},\ 1 \le y \le 2$
8. $x = \sqrt{4-y^2},\ 0 \le y \le 1$
9. $x = 4 - 2y,\ 0 \le y \le 2$
10. $x = cosh y,\ 1 \le y \le 3$
11. $x = \frac{y^2}{3},\ 1 \le y \le 4$
12. $x = y^2,\ 0 \le y \le 1$
13. $x = ln(\sec y),\ 0 \le y \le \frac{\pi}{4}$

14 - 24 ■ Find the area of the surface generated by revolving the curve about the specified axis:

14. $y=\sqrt{4-x^2},\ -1\leq x\leq 1$ about x-axis
15. $y=x^2,\ 1\leq x\leq 2$ about y-axis
16. $y=e^x,\ 0\leq x\leq 1$ about x-axis
17. $y=\ln x,\ 1\leq x\leq 3$ about y-axis
18. $y=\sin x,\ 0\leq x\leq \pi/2$ about x-axis
19. $x=e^y,\ 1\leq y\leq 2$ about y-axis
20. $9x=y+18,\ 0\leq x\leq 2$ about x-axis
21. $y=x^3,\ 0\leq x\leq 2$ about x-axis
22. $y=\cos 2x,\ 0\leq x\leq \pi/6$ about x-axis
23. $y=\sqrt[3]{x},\ 1\leq y\leq 2$ about y-axis
24. $y=1-x^2,\ 0\leq x\leq 1$ about y-axis

Chapter 8

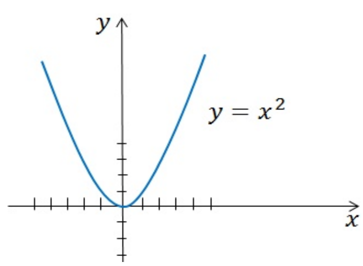
Parametric Equations and Polar Coordinates

8.1 Parametric Equations of Plane Curves

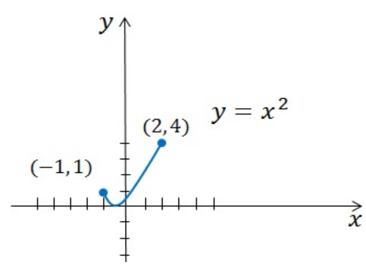
In this section, we rather than considering only functions $y = f(x)$, it is sometimes convenient to view both x and y as functions of a third variable t (called a parameter). The resulting equations $x = f(t)$ and $y = g(t)$ are called parametric equations. Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve.

Example 8.1.1

Let $y = f(x) = x^2$. The function is continuous and its graph given in the following figure:



If we consider the interval $-1 \leq x \leq 2$, then we have



Now, let $x = t$ and $y = t^2$ for $-1 \leq t \leq 2$. We have the same graph where the last equations are called parametric equations for the curve C.

> **Remark 8.1.1**
> 1. *The parametric equations give the same graph of $y = f(x)$.*
> 2. *The parametric equations give the orientation of C.*
> 3. *To find the parametric equations, we introduce a third variable t. Then, we rewrite x and y as functions of t. The result is the parametric equations:*
>
> $$x = f(t) \text{ parametric equation for } x,$$
> $$y = g(t) \text{ parametric equation for } y.$$

Example 8.1.2 *Write the curve given by $x(t) = 2t + 1$ and $y(t) = 4t^2 - 9$ as $y = f(x)$.*

Solution:

Since $x = 2t+1$, then $t = (x-1)/2$. This implies $y(t) = 4t^2 - 9 = 4\left(\frac{x-1}{2}\right)^2 - 9 \Rightarrow y = x^2 - 2x - 8$.

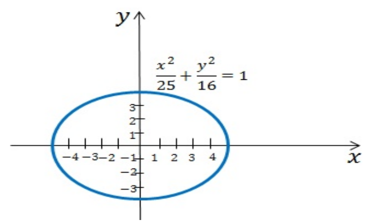
Example 8.1.3 *Sketch and identify the curve defined by the parametric equations* $x = 5\cos t,\ \ y = 2\sin t,\ \ \ 0 \le t \le 2\pi$.

Solution:

Let's first find the equation in x and y. Since $x = 5\cos t$ and $y = 2\sin t$, then $\cos t = x/5$ and $\sin t = y/2$.

We know that

$$\cos^2 t + \sin^2 t = 1$$
$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{4} = 1$$



Thus, the graph of the parametric equations is an ellipse.

Example 8.1.4 *For the following curve* $x = \sin t,\ \ y = \cos t,\ \ \ 0 \le t \le 2\pi$,

1. *find an equation in x and y whose graph contains the points on the curve,*
2. *sketch the graph of C,*
3. *indicate the orientation.*

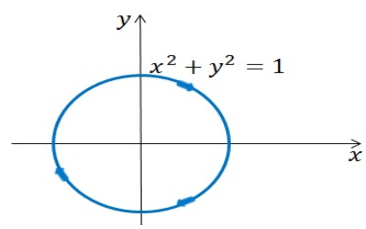
Solution:

1. We know that $\cos^2 t + \sin^2 t = 1$. This implies

$$x^2 + y^2 = 1\ .$$

Therefore, the graph of the parametric equations is a circle.

2.



3. The orientation can be indicated as follows:

t	0	$\frac{\pi}{2}$	π
x	0	1	0
y	1	0	-1
(x,y)	$(0,1)$	$(1,0)$	$(0,-1)$

Now, if $x = f(t)$ and $y = g(t)$ are parametric equations for the curve C. We are going to study slope of the tangent line at a point and second derivative.

8.1.1 Slope of the Tangent Line

Suppose f and g are differentiable functions. We want to find the tangent line at a point on the parametric curve $x = f(t)$, $y = g(t)$ where y is also a differentiable function of x. From the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\ \frac{dx}{dt}\ .$$

If $dx/dt \neq 0$, we can solve for dy/dx:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ if } \frac{dx}{dt} \neq 0$$

Remark 8.1.2

- *If $dy/dt = 0$ such that $dx/dt \neq 0$, the curve has a horizontal tangent.*
- *If $dx/dt = 0$ such that $dy/dt \neq 0$, the curve has a vertical tangent.*

Example 8.1.5 *Find the slope of the tangent line to the curve at the indicated value:*

1. $x = t + 1$, $y = t^2 + 3t$; *at* $t = -1$.
2. $x = t^3 - 3t$, $y = t^2 - 5t - 1$; *at* $t = 2$.
3. $x = \sin t$, $y = \cos t$; *at* $t = \frac{\pi}{4}$.

Solution:

1. $y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t+3}{1} = 2t + 3.$

   The slope of the tangent line at $t = -1$ is $\frac{dy}{dx} = 1$.

2. $y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-5}{3t^2-3}.$

   The slope of the tangent line at $t = 2$ is $\frac{dy}{dx} = \frac{-1}{9}$.

3. $y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t}{\cos t} = -\tan t.$

   The slope of the tangent line at $t = \frac{\pi}{4}$ is $\frac{dy}{dx} = -1$.

Example 8.1.6 *Find the equations of the tangent line and the vertical line at $t = 2$ to the curve $x = 2t, \;\; y = t^2 - 1$.*

Solution:

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t \; .$$

The slope of the tangent line at $t = 2$ is $m = 2$. Thus, the slope of the vertical line is $\frac{-1}{m} = \frac{-1}{2}$. At $t = 2$, we have $(x_0, y_0) = (4, 3)$. Therefore, the tangent line is

$$y - 3 = 2(x - 4)$$

and the vertical line is

$$y - 3 = -\frac{1}{2}(x - 4) \; .$$

Remember:
$y - y_0 = m(x - x_0)$

Example 8.1.7 *Find the points on the curve C at which the tangent line is either horizontal or vertical.*

1. $x = 1 - t$, $y = t^2$.
2. $x = t^3 - 4t$, $y = t^2 - 4$.

Solution:

1. Slope of the tangent line is $m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{-1} = -2t$.

For the horizontal tangent line, the slope $m = 0$. This implies $-2t = 0$ and then, $t = 0$. If $t = 0$, $x = 1$ and $y = 0$. Thus, the graph of C has horizontal tangent line at the point $(1,0)$.

For the vertical tangent line, the slope $\frac{-1}{m} = 0$. This implies $\frac{1}{2t} = 0$, but this equation cannot be solved i.e., we cannot find value for t to satisfy $\frac{1}{2t} = 0$. Therefore, there is no vertical line.

2. Slope of the tangent line is $m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2-4}$.

   For the horizontal tangent line, the slope $m = 0$. This implies $\frac{2t}{3t^2-4} = 0$ and this is acquiblack if $t = 0$. If $t = 0$, $x = 0$ and $y = -4$. Thus, the graph of C has a horizontal tangent line at the point $(0,-4)$.

   For the vertical tangent line, the slope $\frac{-1}{m} = 0$. This implies $\frac{-3t^2+4}{2t} = 0$ and this is acquiblack if $t = \pm\frac{2}{\sqrt{3}}$. If $t = \frac{2}{\sqrt{3}}$, $x = \frac{-16}{3\sqrt{3}}$ and $y = \frac{-8}{3}$. Also, if $t = \frac{-2}{\sqrt{3}}$, $x = \frac{16}{3\sqrt{3}}$ and $y = \frac{-8}{3}$. Thus, the graph of C has a vertical tangent line at the points $(\frac{-16}{3\sqrt{3}}, \frac{-8}{3})$ and $(\frac{16}{3\sqrt{3}}, \frac{-8}{3})$.

8.1.2 Second Derivative in a Parametric Form

If we want to find the second derivative of a parametric curve $x = f(t)$, $y = g(t)$ where f and g are differentiable functions, we use the following formula:

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Example 8.1.8 *Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the indicated value:*

1. $x = t$, $y = t^2 - 1$ *at* $t = 1$.
2. $x = \cos t$, $y = \sin t$ *at* $t = \frac{\pi}{3}$.

Solution:

1. $\frac{dy}{dt} = 2t$ and $\frac{dx}{dt} = 1$. Thus, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2t$. At $t = 1$, $\frac{dy}{dx} = 2(1) = 2$.

   $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = 2$.

2. $\frac{dy}{dt} = \cos t$ and $\frac{dx}{dt} = -\sin t$. Thus, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\tan x$. At $t = \frac{\pi}{3}$, $\frac{dy}{dx} = -\frac{1}{\sqrt{3}}$.

   $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\sin t}{-\sin t} = 1$.

8.1.3 Arc Length and Surface Area of Revolution

In the previous chapter, we study how to calculate the arc length of a smooth function f on an interval $[a,b]$. We concluded that the arc length of f is

$$L(f) = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx .$$

Let the curve C has the parametric equations $x = f(t)$ and $y = g(t)$ where $a \le t \le b$. Assume f' and g' are continuous, then

$$f'(x) = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} .$$

From this, $1+\left[f'(x)\right]^2 = 1+\left[\frac{dy/dt}{dx/dt}\right]^2 = \frac{(dx/dt)^2+(dy/dt)^2}{(dx/dt)^2}$. Thus,

$$\sqrt{1+\left[f'(x)\right]^2}\,dx = \sqrt{\frac{(dx/dt)^2+(dy/dt)^2}{(dx/dt)^2}}\,dx = \frac{\sqrt{(dx/dt)^2+(dy/dt)^2}}{dx/dt}\,dx = \sqrt{(dx/dt)^2+(dy/dt)^2}\,dt\ .$$

We conclude that the length of the curve $x = f(t)$, $y = g(t)$ where $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$$

In the following, we find the formula to evaluate the surface area of revolution of parametric curves. Let the curve C has the parametric equations $x = f(t)$, $y = g(t)$ and $a \le t \le b$ such that f' and g' are continuous. From the previous chapter, we know that if the rotation is about y-axis, then

$$S.A = 2\pi\int_a^b |\,x\,|\sqrt{1+[f'(x)]^2}\,dx = 2\pi\int_a^b |\underbrace{x}_{=f(t)}|\underbrace{\sqrt{1+\left[f'(x)\right]^2}\,dx}_{=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt}\ .$$

Thus, the surface area of revolution about y-axis is

$$S.A = 2\pi\int_a^b |\,f(t)\,|\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\ .$$

Similarly, we can find that if the rotating is about x-axis, then

$$S.A = 2\pi\int_a^b |\,g(t)\,|\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\ .$$

Theorem 8.1.1 *Let the curve C has the parametric equations $x = f(t)$, $y = g(t)$ and $a \le t \le b$ such that f' and g' are continuous.*

1. *Arc length:*

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt\ .$$

2. *Surface area of revolution:*
   - *if the revolution is about x-axis,*

$$S.A = 2\pi\int_a^b |\,y\,|\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\ .$$

   - *if the revolution is about y-axis,*

$$S.A = 2\pi\int_a^b |\,x\,|\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt\ .$$

Example 8.1.9 *Find the arc length of the curve $x = e^t\cos t,\ \ y = e^t\sin t,\ \ 0 \le t \le \frac{\pi}{2}$.*

Solution:

First, we find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = e^t\cos t - e^t\sin t \Rightarrow (\frac{dx}{dt})^2 = (e^t\cos t - e^t\sin t)^2\ ,$$
$$\frac{dy}{dt} = e^t\sin t + e^t\cos t \Rightarrow (\frac{dy}{dt})^2 = (e^t\sin t + e^t\cos t)^2\ .$$

Thus,

$$(\frac{dx}{dt})^2+(\frac{dy}{dt})^2 = e^{2t}\cos^2 t - 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\sin^2 t + 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t = e^{2t}+e^{2t} = 2e^{2t} \ .$$

Therefore, the arc length of the curve is

$$L = \sqrt{2}\int_0^{\frac{\pi}{2}} e^t \ dt = \sqrt{2}\left[e^t\right]_0^{\frac{\pi}{2}} = \sqrt{2}\left(e^{\frac{\pi}{2}}-1\right) .$$

Example 8.1.10 *Find the surface area of revolution of the curve* $x = 3\cos t, \ \ y = 3\sin t, \ \ 0 \le t \le \frac{\pi}{3}$ *around x-axis.*

Solution:

We use the formula $S.A = \int_a^b g(t)\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2} \ dt$ since the rotation is about x-axis. We find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as follows:

$$\frac{dx}{dt} = -3\sin t \Rightarrow (\frac{dx}{dt})^2 = 9\sin^2 t \ ,$$

$$\frac{dy}{dt} = 3\cos t \Rightarrow (\frac{dx}{dt})^2 = 9\cos^2 t \ .$$

Thus,

$$(\frac{dx}{dt})^2+(\frac{dy}{dt})^2 = 9(\sin^2 t+\cos^2 t) = 9 \ .$$

This implies

$$S.A = 18\pi\int_0^{\frac{\pi}{3}} \sin t \ dt = -18\pi\left[\cos t\right]_0^{\frac{\pi}{3}} = -18\pi\left[\frac{1}{2}-1\right] = 9\pi$$

Exercise 1:

1 - 8 ■ Curve C is given parametrically. Find an equation in x and y, then sketch the graph and indicate the orientation:

1. $x=t, y=2t+1, 1\le t\le 3$
2. $x=\cos 2t, y=\sin t, 0\le t\le \pi/2$
3. $x=2t, y=(2t)^2, -1\le t\le 1$
4. $x=1+\cos t, y=1+\sin t, 0\le t\le 2\pi$
5. $x=\ln t, y=e^t, 1\le t\le 4$
6. $x=3\cos t, y=3\sin t, 0\le t\le 2\pi$
7. $x=3t+2, y=t-1, -1\le t\le 5$
8. $x=t, y=t^3, 1\le t\le 3$

9 - 16 ■ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the indicated values:

9. $x=t^2, y=t^3+1$ at $t=1$
10. $x=t/3, y=t^3/2$ at $t=2$
11. $x=\sqrt{t^3}, y=2t+1$ at $t=1$
12. $x=t^2+1, y=1-t^3$ at $t=3$
13. $x=e^t, y=e^{-t}+1$ at $t=0$
14. $x=t+\cos t, y=\sin t$ at $t=\pi/4$
15. $x=t\cos t, y=t\sin t$ at $t=0$
16. $x=\sqrt[3]{t}, y=t^2$ at $t=1$

17 - 24 ■ Find the slope of the tangent line to the curve at the indicated value:

17. $x=2t, y=(2t)^2$ at $t=1$
18. $x=\sqrt{t^3}, y=2t+1$ at $t=2$
19. $x=t^2+1, y=1-t^3$ at $t=3$
20. $x=\cos 2t, y=\sin t$ at $t=\pi/3$
21. $x=3t+2, y=t-1$ at $t=1$
22. $x=t+\cos t, y=\sin t$ at $t=\pi/6$
23. $x=t, y=t^3$, at $t=1$
24. $x=\sqrt[3]{t}, y=t^2$ at $t=5$

25 - 30 ■ Find the points on the curve C at which the tangent line is either horizontal or vertical.

25. $x=t, y=t^3, t\in\mathbb{R}$
26. $x=4t, y=t^2, t\in\mathbb{R}$
27. $x=\ln t, y=e^t, t>0$
28. $x=t^2, y=t^3-3t, t\in\mathbb{R}$
29. $x=3t^2-6t, y=\sqrt{t}, t\ge 0$
30. $x=1-\sin t, y=2\cos t, t\in\mathbb{R}$

31 - 38 ■ Find the length of the curve:

31. $x=3t+2, y=t-1, -1\le t\le 3$
32. $x=3t^2, y=2t^3, 0\le t\le 2$
33. $x=t, y=t^2, 1\le t\le 4$
34. $x=\sin t, y=\cos t, \pi/6\le t\le \pi/4$
35. $x=\ln t, y=t, 1\le t\le 4$
36. $x=1+\cos t, y=1+\sin t, 0\le t\le \pi$
37. $x=3\cos t, y=3\sin t, 0\le t\le \pi/4$
38. $x=t^2, y=t^3, 0\le t\le 1/2$

39 - 46 ■ Find the area of the surface generated by revolving the curve about the specified axis:

39. $x=t^2, y=t, 0\le t\le 1$ about $x-axis$
40. $x=e^t\cos t, y=e^t\sin t, 0\le t\le \frac{\pi}{2}$ about $x-axis$
41. $x=t, y=t^2, 1\le t\le 4$ about $y-axis$
42. $x=t, y=\sqrt{t}, 0\le t\le 2$ about $x-axis$
43. $x=t^2, y=t, 0\le t\le 2$ about $x-axis$
44. $x=1+\cos t, y=1+\sin t, 0\le t\le \pi$ about $y-axis$
45. $x=\sin^2 t, y=\cos^2 t, 0\le t\le \pi/2$ about $y-axis$
46. $x=3t^2, y=t, 0\le t\le 2$ about $x-axis$

8.2 Polar Coordinates System

Previously, we used Cartesian coordinates to determine points (x, y) as shown in Figure 8.1 (left). In this section, we are going to study a new coordinate system called a polar coordinate.

Definition 8.2.1 *The polar coordinate system is a two-dimensional coordinate system in which each point P on a plane is determined by a distance r from a fixed point O that is called the pole (or origin) and an angle θ from a fixed direction.*

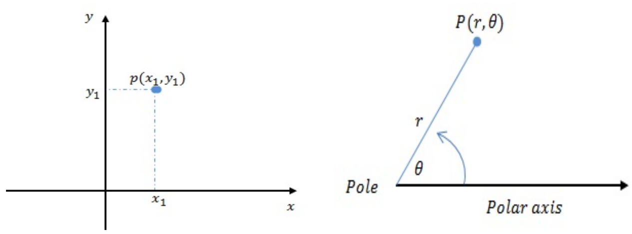
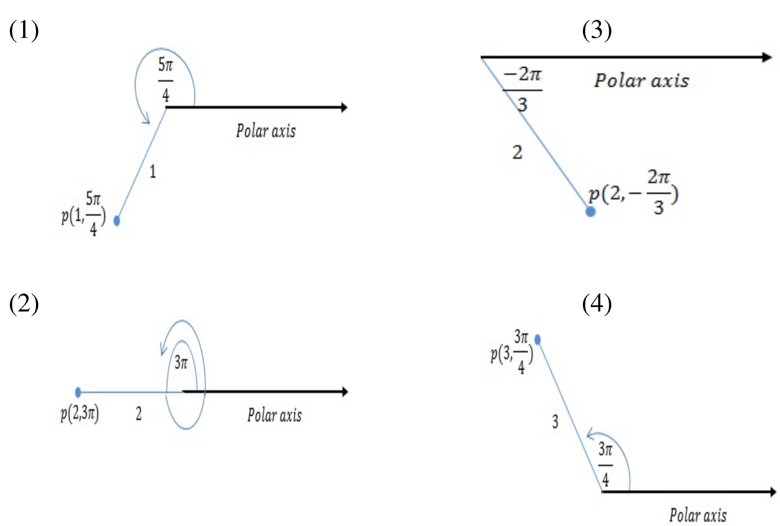


Figure 8.1: The Cartesian coordinate on the left and the polar coordinate on the right.

Example 8.2.1 *Plot the points whose polar coordinates are given:*

1. $(1, 5\pi/4)$
2. $(2, 3\pi)$
3. $(2, -2\pi/3)$
4. $(3, 3\pi/4)$

Solution:

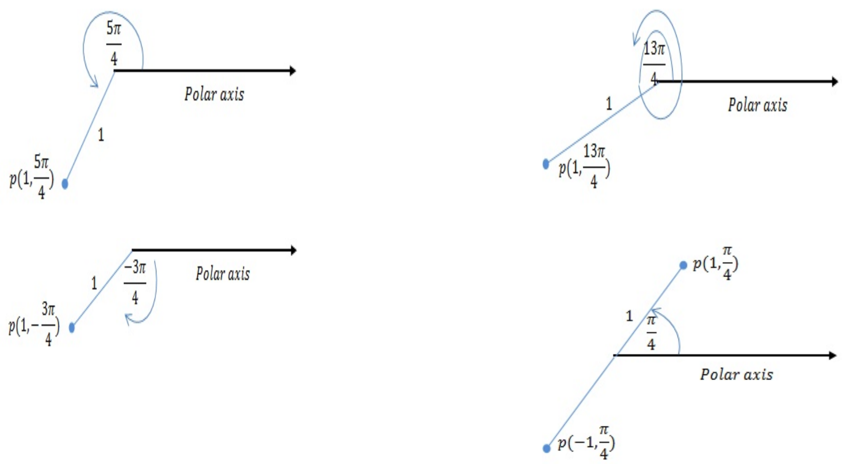


Remark 8.2.1

1. *From the definition, the point P in the polar coordinate system is represented by the ordeblack pair* (r,θ) *where r,* θ *are called polar coordinates.*
2. *In the polar coordinates* (r,θ)*, if* $r>0$*, the point* (r,θ) *lies in the same quadrant as* θ*; if* $r<0$*, it lies in the quadrant on the opposite side of the pole. Meaning that, the polar coordinates* (r,θ) *and* $(-r,\theta)$ *lie in the same line through the pole O and at the same distance* $\mid r \mid$ *from O, but on opposite sides of O.*
3. *In the Cartesian coordinate system, every point has only one representation, but in the polar coordinate system each point has many representations. The following formula gives all representations of each point* $P(r,\theta)$ *in the polar coordinate system*

$$(r,\theta+2n\pi)=(r,\theta)=(-r,\theta+(2n+1)\pi) \ \ n\in\mathbb{Z}\ .$$

Example 8.2.2 *In Example 8.2.1, the point* $(1,5\pi/4)$ *could be written as* $(1,-3\pi/4)$*,* $(1,13\pi/4)$ *or* $(-1,\pi/4)$*:*



8.2.1 Relationship between Rectangular and Polar Coordinates

Let (x,y) be a rectangular coordinate and (r,θ) be a polar coordinate. Let the pole at the origin point and polar axis on x-axis, and the line $\theta=\frac{\pi}{2}$ on y-axis as shown in Figure 8.2.

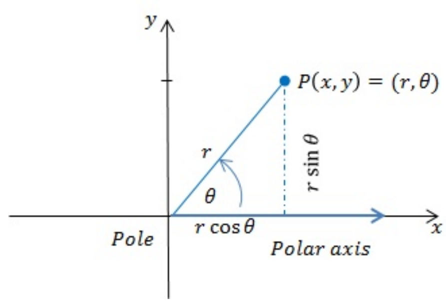


Figure 8.2: The relationship between the rectangular and polar coordinates.

From the triangle OAP

$$\cos\theta=\frac{x}{r}\Rightarrow x=r\cos\theta\ ,$$

$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta \ .$$

Thus,

$$\begin{aligned} x^2+y^2 &= (r\cos\theta)^2+(r\sin\theta)^2, \\ &= r^2(\cos^2\theta+\sin^2\theta) \ . \end{aligned}$$

This implies, $x^2+y^2=r^2$ and $\tan\theta = \frac{y}{x}$.

The previous relationships can be summarized as follows:

$$x = r\cos\theta, \ y = r\sin\theta, \ \tan\theta = \frac{y}{x}, \ x^2+y^2 = r^2$$

Example 8.2.3 *Convert the points from the polar coordinates to the rectangular coordinates:*

1. $(1,\pi/4)$
2. $(2,\pi)$
3. $(2,-2\pi/3)$
4. $(4,3\pi/4)$

Solution:

1. $r=1$ and $\theta=\frac{\pi}{4}$.

$$x = r\cos\theta = (1)\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \ ,$$

$$y = r\sin\theta = (1)\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \ .$$

Hence, $(x,y)=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$.

2. $r=2$ and $\theta=\pi$.

$$x = r\cos\theta = 2\cos(\pi) = -2 \ ,$$

$$y = r\sin\theta = 2\sin(\pi) = 0 \ .$$

Thus, $(x,y)=(-2,0)$.

3. $r=2$ and $\theta=\frac{-2\pi}{3}$.

$$x = r\cos\theta = 2\cos(\frac{-2\pi}{3}) = -1 \ ,$$

$$y = r\sin\theta = 2\sin(\frac{-2\pi}{3}) = -\sqrt{3} \ .$$

Thus, $(x,y)=(-1,-\sqrt{3})$.

4. $r=4$ and $\theta=\frac{3\pi}{4}$.

$$x = r\cos\theta = 4\cos(\frac{3\pi}{4}) = -2\sqrt{2} \ ,$$

$$y = r\sin\theta = 4\sin(\frac{3\pi}{4}) = 2\sqrt{2} \ .$$

This implies $(x,y)=(-2\sqrt{2},2\sqrt{2})$.

Example 8.2.4 *Convert the points from the rectangular coordinates to polar coordinates:*

1. $(5,0)$
2. $(2\sqrt{3},-2)$
3. $(0,2)$
4. $(1,1)$

Solution:

1. $x=5$ and $y=0$

$$\begin{aligned} &\Rightarrow r^2 = x^2+y^2 = 5^2+0^2 \\ &\Rightarrow r = 5 \ . \end{aligned}$$

Also, $\tan\theta = \frac{y}{x} = \frac{0}{5} = 0 \Rightarrow \theta = 0$.

Thus, $(r,\theta)=(5,0)$. Remember, in the polar coordinate system each point has many representations (Remark 8.2.1).

2. $x=2\sqrt{3}$ and $y=-2$

$$\begin{aligned} &\Rightarrow r^2 = x^2+y^2 = (2\sqrt{3})^2+(-2)^2 \\ &\Rightarrow r = 4 \ . \end{aligned}$$

Also, $\tan\theta = \frac{y}{x} = \frac{-2}{2\sqrt{3}} = \frac{-1}{\sqrt{3}} \Rightarrow \theta = \frac{5\pi}{6}$.

Thus, $(r,\theta) = (4, \frac{5\pi}{6})$.

3. $x = 0$ and $y = 2$

$$\begin{aligned} &\Rightarrow r^2 = x^2 + y^2 = 0^2 + 2^2 \\ &\Rightarrow r = 2 \ . \end{aligned}$$

Also, $\tan\theta = \frac{y}{x} = \infty \Rightarrow \theta = \frac{\pi}{2}$.

This implies $(r,\theta) = (2, \frac{\pi}{2})$.

4. $x = 1$ and $y = 1$

$$\begin{aligned} &\Rightarrow r^2 = x^2 + y^2 = 1^2 + 1^2 \\ &\Rightarrow r = \sqrt{2} \ . \end{aligned}$$

Also, $\tan\theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}$.

This implies, $(r,\theta) = (\sqrt{2}, \frac{\pi}{4})$.

Example 8.2.5 *Convert the rectangular equation to the polar form:*

1. $x = 7$

2. $y = -3$

3. $x^2 + y^2 = 4$

4. $y^2 = 9x$

Solution:

1. $x = 7 \Rightarrow r\cos\theta = 7.$

2. $y = -3 \Rightarrow r\sin\theta = -3.$

3. $x^2 + y^2 = 4$

$$\begin{aligned} x^2 + y^2 = 4 &\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta = 4 \\ &\Rightarrow r^2(\cos^2\theta + \sin^2\theta) = 4 \\ &\Rightarrow r^2 = 4 \\ &\Rightarrow r = 2 \ . \end{aligned}$$

4. $y^2 = 9x$

$$\begin{aligned} y^2 = 9x &\Rightarrow r^2\sin^2\theta = 9r\cos\theta \\ &\Rightarrow r\sin^2\theta = 9\cos\theta \\ &\Rightarrow r = 9\cot\theta\csc\theta \ . \end{aligned}$$

Example 8.2.6 *Convert the polar equation to the rectangular form:*

1. $r = 3$

2. $r = \sin\theta$

3. $r = 6\cos\theta$

4. $r = \sec\theta$

Solution:

1. $r = 3 \Rightarrow \sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9.$

2. $r = \sin\theta \Rightarrow r = \frac{y}{r} \Rightarrow r^2 = y \Rightarrow x^2 + y^2 = y \Rightarrow x^2 + y^2 - y = 0.$

3. $r = 6\cos\theta \Rightarrow r = 6\ \frac{x}{r} \Rightarrow r^2 = 6x \Rightarrow x^2 + y^2 - 6x = 0.$

4. $r = \sec\theta \Rightarrow r = \frac{1}{\cos\theta} \Rightarrow r\cos\theta = 1 \Rightarrow x = 1.$

8.2.2 Tangent Line to a Polar Curve

Let $r = f(\theta)$ be a polar curve where f' is continuous at (r_0, θ_0). Then,

$$x = f(\theta)\cos\theta\ , \quad y = f(\theta)\sin\theta\ .$$

From chain rule, we have

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta\ ,$$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta\ .$$

If $\frac{dx}{d\theta} \neq 0$ at $\theta = \theta_0$, the slope of the tangent line to the graph of $r = f(\theta)$ at (r_0, θ_0) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r_0\cos\theta_0 + \sin\theta_0(dr/d\theta)}{-r_0\sin\theta_0 + \cos\theta_0(dr/d\theta)}\ .$$

Remark 8.2.2

1. *If* $\frac{dy}{d\theta} = 0$ *such that* $\frac{dx}{d\theta} \neq 0$*, the curve has a horizontal tangent line.*
2. *If* $\frac{dx}{d\theta} = 0$ *such that* $\frac{dy}{d\theta} \neq 0$*, the curve has a vertical tangent line.*

Example 8.2.7 *Find the slope tangent of the curve* $r = \sin\theta$ *at* $\theta = \frac{\pi}{4}$.

Solution:

$$x = r\cos\theta \Rightarrow x = \sin\theta\cos\theta \Rightarrow \frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta\ ,$$

$$y = r\sin\theta \Rightarrow x = \sin^2\theta \Rightarrow \frac{dy}{d\theta} = 2\sin\theta\cos\theta\ .$$

$$\frac{dy}{dx} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{d\theta} = 1$ and $\frac{dx}{d\theta} = 0$. Thus, the curve has a vertical tangent line.

Example 8.2.8 *Find the points on the curve* $r = 2 + 2\cos\theta$ *for* $0 \leq \theta \leq 2\pi$ *at which tangent lines are either horizontal or vertical.*

Solution:

$$x = r\cos\theta = 2\cos\theta + 2\cos^2\theta \Rightarrow \frac{dx}{d\theta} = -2\sin\theta - 4\cos\theta\sin\theta\ ,$$

$$y = r\sin\theta = 2\sin\theta + 2\cos\theta\sin\theta \Rightarrow \frac{dy}{d\theta} = 2\cos\theta - 2\sin^2\theta + 2\cos^2\theta\ .$$

For the horizontal tangent line,

$$\frac{dy}{d\theta} = 0 \Rightarrow 2\cos\theta - 2\sin^2\theta + 2\cos^2\theta = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0 \Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0\ .$$

This implies $\theta = \pi$, $\theta = \pi/3$, or $\theta = 5\pi/3$. Therefore, the tangent line is horizontal at $(0, \pi)$, $(3, \pi/3)$ or $(3, 5\pi/3)$.

For the vertical tangent line,

$$\frac{dx}{d\theta} = 0 \Rightarrow \sin\theta(2\cos\theta + 1) = 0\ .$$

This implies $\theta = 0$, $\theta = \pi$, $\theta = 2\pi/3$, $\theta = 4\pi/3$ or $\theta = 2\pi$. However, we have to ignore $\theta = \pi$ and $\theta = 2\pi$ since at these values $dy/d\theta = 0$. Therefore, the tangent line is vertical at $(4, 0)$, $(1, 2\pi/3)$, or $(1, 4\pi/3)$.

8.2.3 Graphs in Polar Coordinates

Before starting sketching polar curves, it is important to know when the polar curves are symmetric about the polar axis, the vertical line $\theta = \frac{\pi}{2}$, or about the pole.

➤ **Symmetry in Polar Coordinates**

Theorem 8.2.1
1. ***Symmetry about the polar axis.***
The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if replacing (r,θ) with $(r,-\theta)$ or with $(-r,\pi-\theta)$ does not change the equation.
2. ***Symmetry about the vertical line $\theta = \frac{\pi}{2}$.***
The graph of $r = f(\theta)$ is symmetric with respect to the vertical line if replacing (r,θ) with $(r,\pi-\theta)$ or with $(-r,-\theta)$ does not change the equation.
3. ***Symmetry about the pole $\theta = 0$.***
The graph of $r = f(\theta)$ is symmetric with respect to the pole if replacing (r,θ) with $(-r,\theta)$ or with $(r,\theta+\pi)$ does not change the equation.

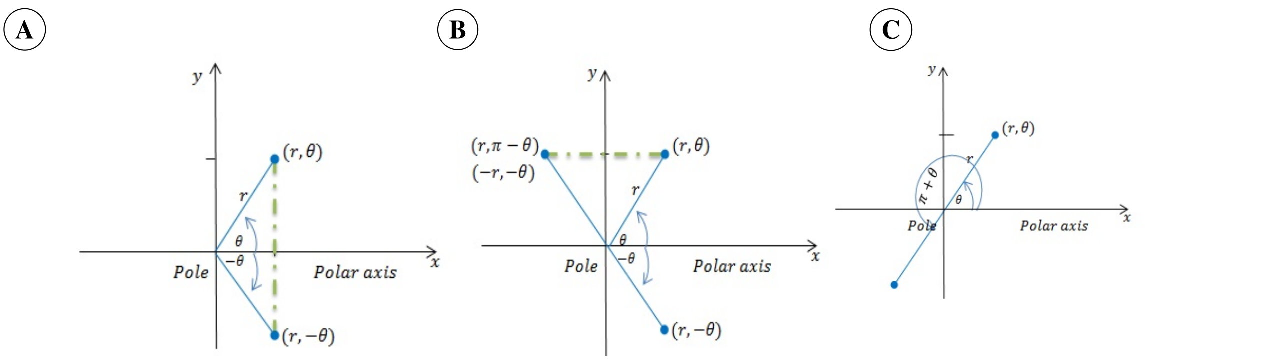


Figure 8.3: Symmetry in Polar Coordinates: (A) symmetry about the polar axis, (B) symmetry about the vertical line $\theta = \frac{\pi}{2}$, and (C) symmetry about the pole $\theta = 0$.

Example 8.2.9 1. *The graph of $r = 4\cos\theta$ is symmetric about the polar axis since*

$$\cos(-\theta) = \cos\theta\ .$$

2. *The graph of $r = 4\sin\theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since*

$$\sin(\pi-\theta) = \sin\theta \ \ \textit{and} \ \ -r\sin(-\theta) = r\sin\theta\ .$$

3. *The graph of $r^2 = a^2\sin 2\theta$ is symmetric about the pole since*

$$(-r)^2 = a^2\sin 2\theta,$$
$$\Rightarrow r^2 = a^2\sin 2\theta\ .$$

Also,

$$r^2 = a^2\sin[2(\pi+\theta)],$$
$$= a^2\sin(2\pi+2\theta),$$
$$= a^2\sin 2\theta\ .$$

➤ Sketch of Polar Curves

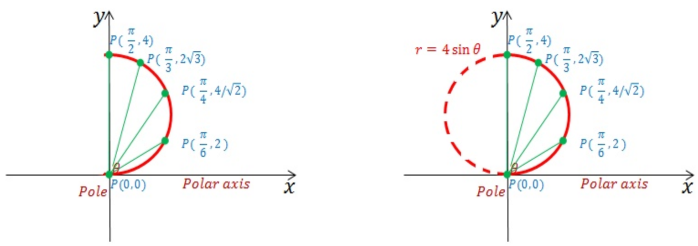
Here, we take two examples to explain how to plot polar curves.

Example 8.2.10 *Sketch the graph of* $r = 4\sin\theta$.

Solution:

Note that, $r = 4\sin\theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since $\sin(\pi-\theta) = \sin\theta$. Therefore, we restrict our attention to the interval $[0,\pi/2]$. The following table displays some solution of $r = 4\sin\theta$:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	2	$4/\sqrt{2}$	$2\sqrt{3}$	4

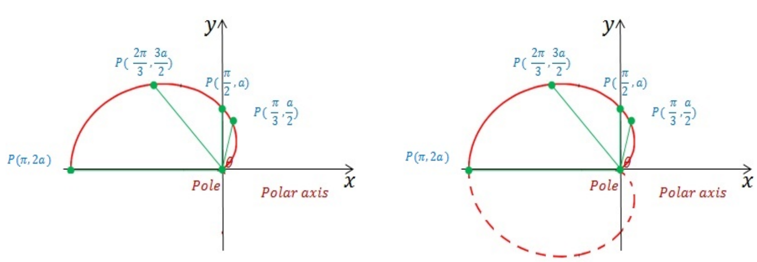


Example 8.2.11 *Sketch the graph of* $r = a(1-\cos\theta)$ *where* $a > 0$.

Solution:

The equation is symmetric about the polar axis since $\cos(-\theta) = \cos\theta$. Therefore, we restrict our attention to the interval $[0,\pi]$. The following table display some solution of the equation $r = a(1-\cos\theta)$:

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$a/2$	a	$3a/2$	$2a$



➤ Some Special Polar Graphs

• Lines in polar coordinates

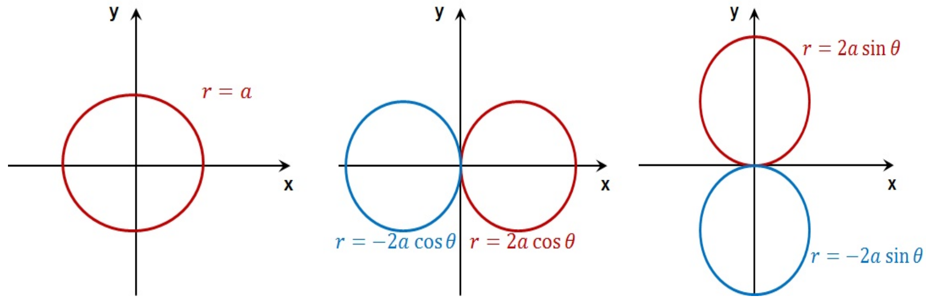
1. The polar equation of a straight line $ax+bx=c$ is $r = \frac{c}{a\cos\theta+b\sin\theta}$.
2. The polar equation of a vertical line $x = k$ is $r = k\sec\theta$. Put $r = k\sec\theta \Rightarrow r = \frac{k}{\cos\theta}$.

3. The polar equation of a horizontal line $y = k$ is $r = k\csc\theta$. Put $r = k\csc\theta \Rightarrow r = \frac{k}{\sin\theta}$.

4. The polar equation of a line that passes the origin point and makes an angle θ_0 is $\theta = \theta_0$.

• **Circles in polar coordinates**

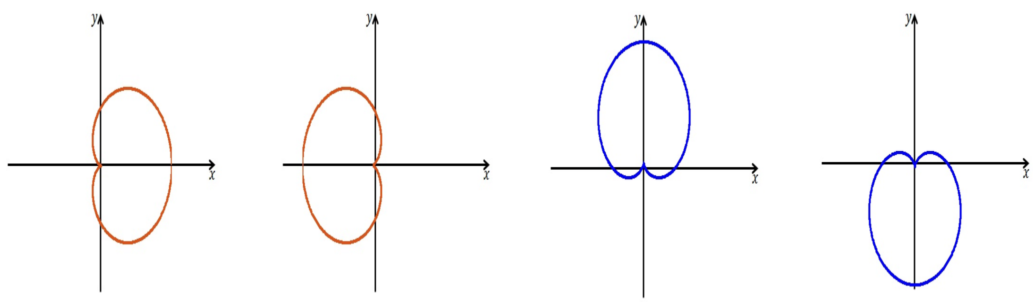
1. The circle equation its center at O and radius a is $r = a$.
2. The circle equation its center at $(a,0)$ and radius $|a|$ is $r = 2a\cos\theta$.
3. The circle equation its center at $(0,a)$ and radius $|a|$ is $r = 2a\sin\theta$.



• **Cardioid**

1. $r = a(1 \pm \cos\theta)$ 2. $r = a(1 \pm \sin\theta)$

$r = a(1+\cos\theta)$ $r = a(1-\cos\theta)$ $r = a(1+\sin\theta)$ $r = a(1-\sin\theta)$

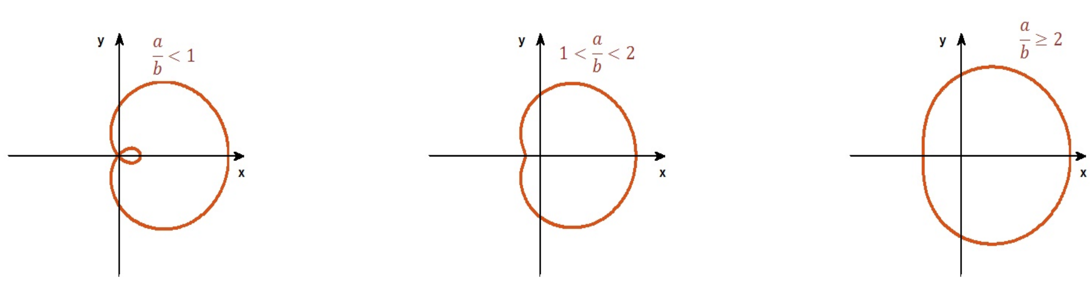


• **Limacons**

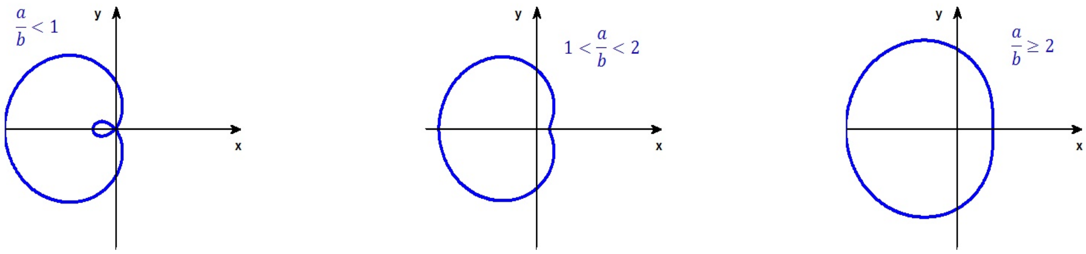
$r = a \pm b\cos\theta$ OR $r = a \pm b\sin\theta$

1. $r = a \pm b\cos\theta$

(a) $r = a + b\cos\theta$

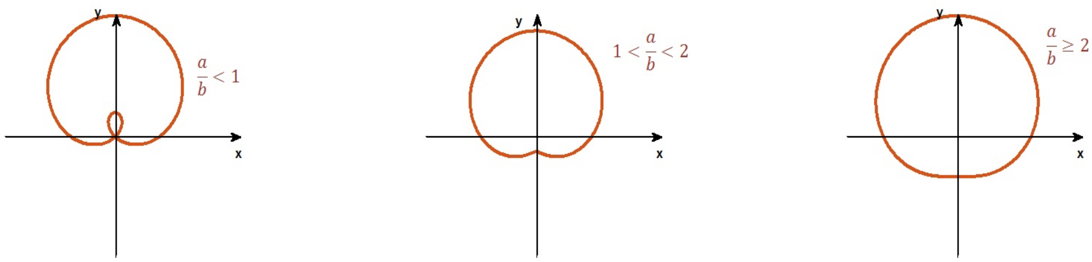


(b) $r = a - b\cos\theta$

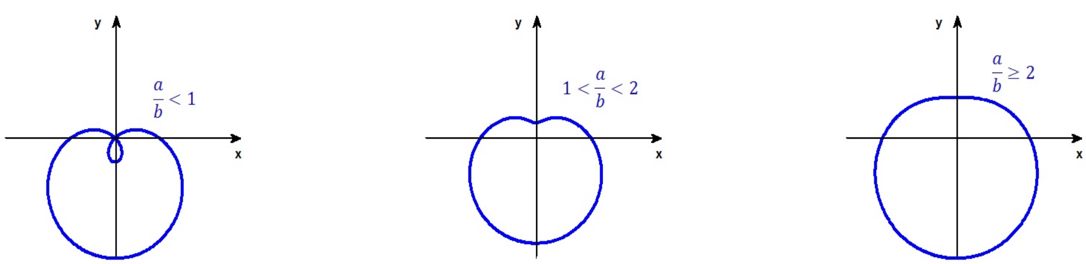


2. $r = a \pm b\sin\theta$

(a) $r = a + b\sin\theta$



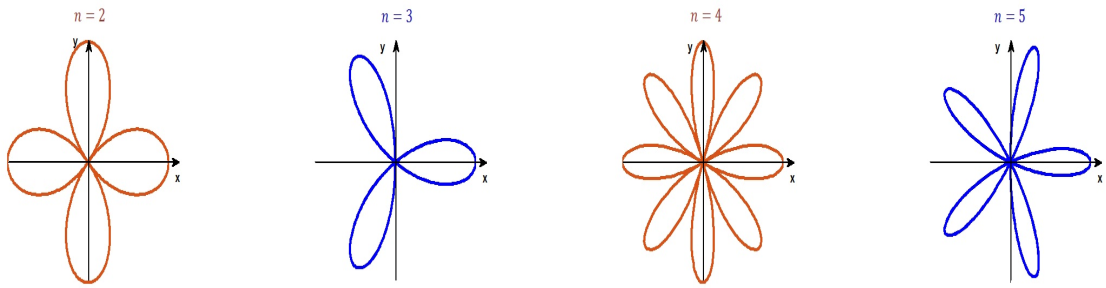
(b) $r = a - b\sin\theta$



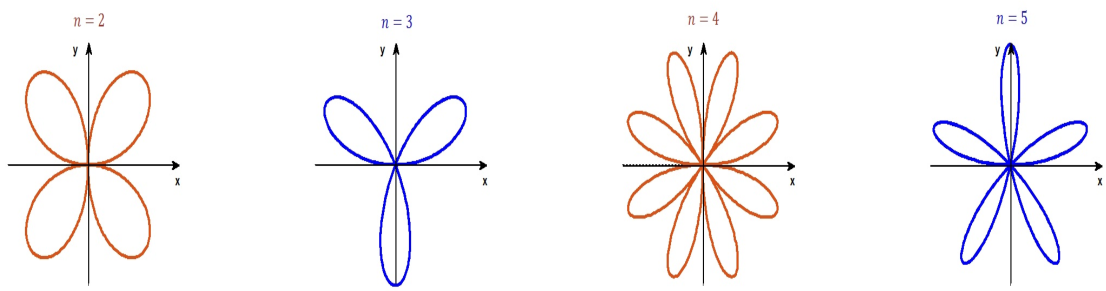
• **Roses**

1. $r = a\ \cos(n\theta)$ 2. $r = a\ \sin(n\theta)$ where $n \in \mathbb{N}$.

1. $r = a\ \cos(n\theta)$



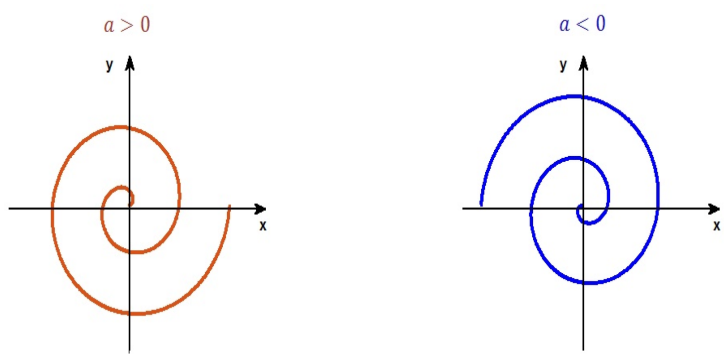
2. $r = a\ \sin(n\theta)$



Note: If n is odd, there are n petals. If n is even, there are $2n$ petals.

• **Spiral of Archimedes**

$r = a\,\theta$



Exercise 2:

1 - 8 ■ Find the corresponding rectangular coordinates for the following polar coordinates:

1. $(1, \frac{\pi}{2})$
2. $(-1, \frac{\pi}{2})$
3. $(2, \frac{\pi}{4})$
4. $(3, \pi)$
5. $(\frac{1}{2}, \frac{3\pi}{2})$
6. $(-3, 2\pi)$
7. $(7, \frac{3\pi}{4})$
8. $(3, \frac{\pi}{6})$

9 - 16 ■ Find the corresponding polar coordinates for the following rectangular coordinates:

9. $(1, 1)$
10. $(0, 2)$
11. $(1, -1)$
12. $(\sqrt{3}, 3)$
13. $(2, \sqrt{2})$
14. $(3, 0)$
15. $(4, 2)$
16. $(-3, -3)$

17 - 24 ■ Convert the rectangular equations to the polar form and vice versa:

17. $x = 9$
18. $x^2 + y^2 = 1$
19. $r = csc\theta$
20. $r = 2\cos\theta$
21. $x^2 = 3y$
22. $x^2 - y^2 = 16$
23. $r = \frac{3}{1-\sin\theta}$
24. $r = 3 - 2\sin\theta$

25 - 28 ■ Sketch the graph of the polar equations:

25. $r = \sec\theta$
26. $r = 4\cos\theta$
27. $r = 2 + 2\sin\theta$
28. $r = 3 + 2\cos\theta$

29 - 33 ■ Find the slope tangent of the curves at θ and then find the points on the curve at which the tangent lines are either horizontal or vertical:

29. $r = 2\sin\theta$ at $\theta = \frac{\pi}{3}$
30. $r = 3 + 2\cos\theta$ at $\theta = \frac{\pi}{4}$
31. $r = \cos 7\theta$ at $\theta = \frac{\pi}{2}$
32. $r = 1 + \sin\theta$ at $\theta = \frac{\pi}{4}$
33. $r = 1 - \cos\theta$ at $\theta = \frac{\pi}{6}$

8.3 Area in Polar Coordinates

Let $r = f(\theta)$ be a continuous function on the interval $[\alpha, \beta]$ such that $0 \le \alpha \le \beta \le 2\pi$. Let $f(\theta) > 0$ over that interval and R be a polar region bounded by the polar equations $r = f(\theta)$, $\theta = \alpha$ and $\theta = \beta$ as shown in Figure 8.4.

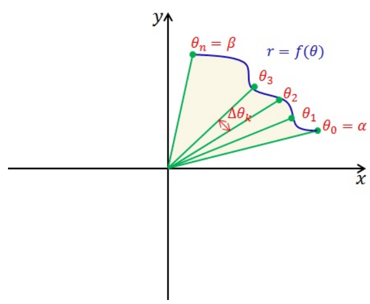


Figure 8.4: Area in polar coordinates.

To find the area of R, we assume $P = \{\theta_1, \theta_2, ..., \theta_n\}$ is a regular partition of the interval $[\alpha, \beta]$. Consider the interval $[\theta_{k-1}, \theta_k]$ where $\triangle\theta_k = \theta_k - \theta_{k-1}$. By choosing $\omega_k \in [\theta_{k-1}, \theta_k]$, we have a circular sector where its angle and radius are $\triangle\theta_k$ and $f(\omega_k)$, respectively. The area between θ_{k-1} and θ_k can be approximated by the circular sector (see Figure 8.4).

The area of the circular sector is $\frac{[f(\omega_k)]^2 \triangle\theta_k}{2}$, thus the area of R is $A = \sum_{k=1}^{n} \frac{1}{2}[f(\omega_k)]^2 \triangle \theta_k$. For $n \to \infty$, we have from Riemann sum

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta$$

Similarly, assume f and g are continuous on the interval $[\alpha, \beta]$ such that $f(\theta) > g(\theta)$. The area bounded by the curves of f and g on the interval $[\alpha, \beta]$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(\left[f(\theta)\right]^2 - \left[g(\theta)\right]^2\right) \, d\theta$$

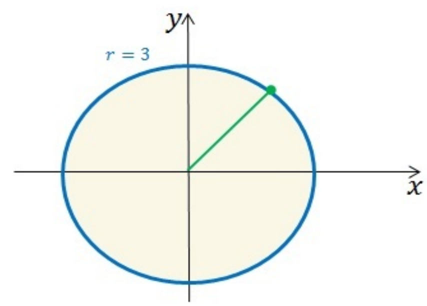
Example 8.3.1 *Find the area of the region bounded by the graph of the polar equation*

1. $r = 3$
2. $r = 2\cos\theta$
3. $r = 4\sin\theta$
4. $r = 6 - 6\sin\theta$

Solution:

1. From the figure, the area is

$$A=\frac{1}{2}\int_0^{2\pi}3^2\,d\theta=\frac{9}{2}\int_0^{2\pi}d\theta=\frac{9}{2}\Big[\,\theta\,\Big]_0^{2\pi}=9\pi\,.$$

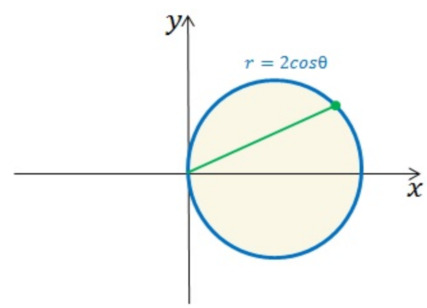


Note that, one can evaluate the area in the first quadrant and multiply the result by 4 to find the area of the whole region i.e.,

$$A=4\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}}3^2\,d\theta\Big)=18\int_0^{\frac{\pi}{2}}d\theta=18\Big[\,\theta\,\Big]_0^{\frac{\pi}{2}}=9\pi\,.$$

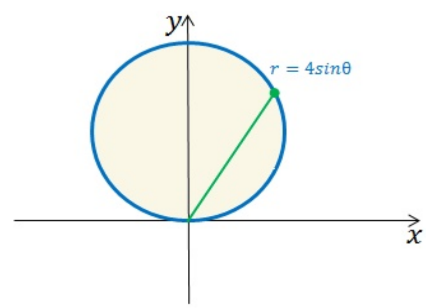
2. We find the area of the half circle and multiply the result by 2 as follows:

$$\begin{aligned}A=2\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}}(2\cos\theta)^2\,d\theta\Big)&=\int_0^{\frac{\pi}{2}}4\cos^2\theta\,d\theta\\&=2\int_0^{\frac{\pi}{2}}1+\cos2\theta\,d\theta\\&=2\Big[\theta+\frac{\sin2\theta}{2}\Big]_0^{\frac{\pi}{2}}\\&=2\Big[\frac{\pi}{2}-0\Big]\\&=\pi\,.\end{aligned}$$



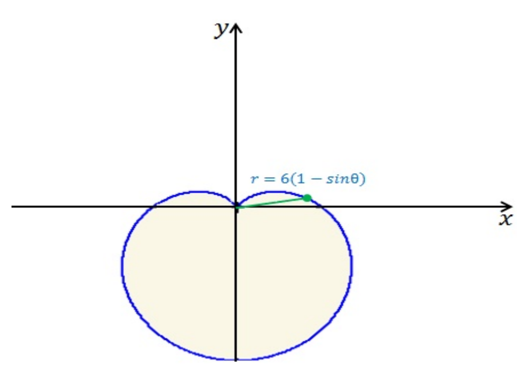
3. The area of the region is

$$\begin{aligned}A=\frac{1}{2}\int_0^{\pi}(4\sin\theta)^2\,d\theta=\frac{16}{4}\int_0^{\pi}1-\cos2\theta\,d\theta\\&=4\Big[\theta-\frac{\sin2\theta}{2}\Big]_0^{\pi}\\&=4\Big[\pi-0\Big]\\&=4\pi\,.\end{aligned}$$



4. The area of the region is

$$\begin{aligned}A&=\frac{1}{2}\int_0^{2\pi}36(1-\sin\theta)^2\,d\theta\\&=18\int_0^{2\pi}1-2\sin\theta+\sin^2\theta\,d\theta\\&=18\Big[\theta+2\cos\theta+\frac{\theta}{2}-\frac{\sin2\theta}{4}\Big]_0^{2\pi}\\&=18\Big[(2\pi+2+\pi)-2\Big]\\&=54\pi\,.\end{aligned}$$



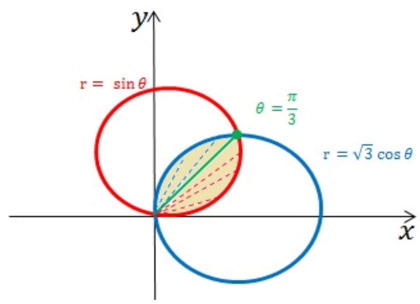
Example 8.3.2 *Find the area of the region that is inside the graphs of the equations* $r=\sin\theta$ *,* $r=\sqrt{3}\cos\theta$*.*

Solution:

First, we find the intersection point of the two curves.

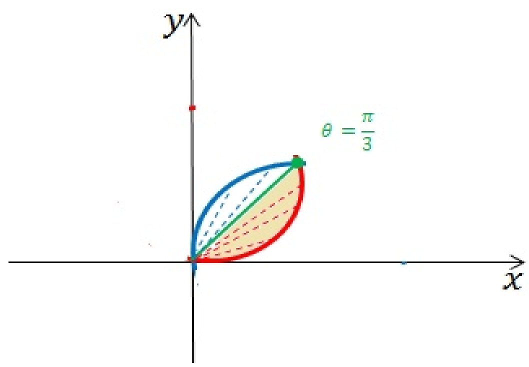
$$\sin\theta = \sqrt{3}\cos\theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}\ .$$

From the figure, the region is divided into two small regions: below and above the line $\frac{\pi}{3}$.



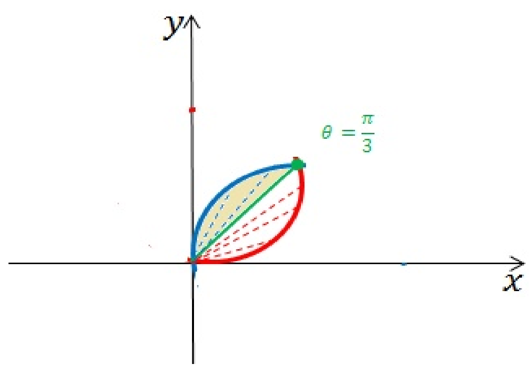
(1) Area of the region below the line $\frac{\pi}{3}$:

$$\begin{aligned} A_1 &= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\theta\, d\theta \\ &= \frac{1}{4}\int_0^{\frac{\pi}{3}} 1-\cos 2\theta\, d\theta \\ &= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2}\right] \\ &= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] \end{aligned}$$



(2) Area of the region above the line $\frac{\pi}{3}$:

$$\begin{aligned} A_2 &= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\theta)^2\, d\theta \\ &= \frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 1+\cos 2\theta\, d\theta \\ &= \frac{3}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \frac{3}{4}\left[(\frac{\pi}{2}-0) - (\frac{\pi}{3}+\frac{\sqrt{3}}{4})\right] \\ &= \frac{3}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right] \end{aligned}$$



Total area $= A_1 + A_2 = \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$.

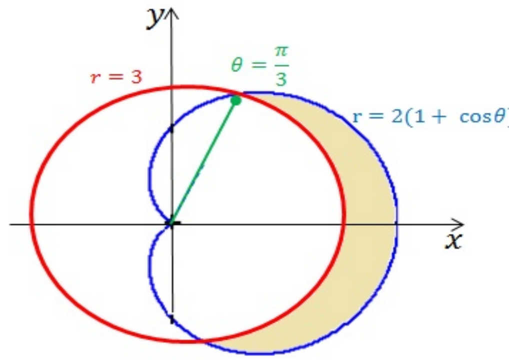
Example 8.3.3 *Find the area of the region that is outside the graph $r = 3$ and inside the graph $r = 2+2\cos\theta$.*

Solution:

As shown in the figure, we find the area in the first quadrant and then we double the result to find the area of the whole region.

The intersection point of the two curves in the first quadrant is

$$2+2\cos\theta = 3 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$



$$\begin{aligned}
\text{Area: } A &= 2\Big(\frac{1}{2}\int_0^{\frac{\pi}{3}} 4(1+\cos\theta)^2 - 9\, d\theta\Big) \\
&= \int_0^{\frac{\pi}{3}} 4(1+2\cos\theta+\cos^2\theta) - 9\, d\theta \\
&= \int_0^{\frac{\pi}{3}} 8\cos\theta + 4\cos^2\theta - 5\, d\theta \\
&= \Big[8\sin\theta + \sin 2\theta - 3\theta\Big]_0^{\frac{\pi}{3}} \\
&= \frac{9}{2}\sqrt{3} - \pi .
\end{aligned}$$

Exercise 3:

1 - 8 ■ Find the area of the region bounded by the graph of the polar equation:

1. $r = 4\sin\theta$
2. $r = 1+\sin\theta$
3. $r = 5$
4. $r = 2\cos\theta$
5. $r = 6(1+\sin\theta)$
6. $r = 2(1-\cos\theta)$
7. $r = 3\cos 3\theta$
8. $r = 3+2\sin\theta$

9 - 18 ■ Find the area of the region bounded by the graph of the polar equations:

9. inside $r = 1+\cos\theta$ and outside $r = 3\cos\theta$
10. inside $r = 2+2\cos\theta$ and outside $r = 3$
11. outside $r = 2-2\cos\theta$ and inside $r = 4$
12. inside both graphs $r = 1+\cos\theta$ and $r = 1$
13. inside $r = 1+\sin\theta$ and outside $r = 1$
14. inside both graphs $r = 2\cos\theta$ and $r = 2\sin\theta$
15. outside $r = 3$ and inside $r = -6\cos\theta$
16. inside both graphs $r = \cos\theta$ and $r = -\sin\theta$
17. between the graphs $r = 1+\sin\theta$ and inside $r = 3\sin\theta$
18. inside both graphs $r = 2$ and $r = 2+2\sin\theta$

19 - 20 ■ Find the area bounded by the graph of the polar equation:

19. $r = 1-\cos\theta$ in the first quadrant
20. $r = 1+\sin\theta$ and $r = 3\sin\theta$ in the second quadrant

8.3.1 Arc Length and Surface of Revolution in Polar Coordinates

8.3.2 Arc Length in Polar Coordinates

Let the polar function $r = f(\theta)$, $\alpha \le \theta \le \beta$ be smooth. We know that

$$x = f(\theta)\cos\theta \ \text{ and } \ y = f(\theta)\sin\theta, \ \ \alpha \le \theta \le \beta .$$

Thus,

$$\begin{aligned}
(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 &= \big(f'(\theta)\cos\theta - f(\theta)\sin\theta\big)^2 + \big(f'(\theta)\sin\theta + f(\theta)\cos\theta\big)^2 \\
&= \big(f'(\theta)\big)^2\cos^2\theta - 2f(\theta)f'(\theta)\cos\theta\sin\theta + \big(f(\theta)\big)^2\sin^2\theta \\
&+ \big(f'(\theta)\big)^2\sin^2\theta + 2f(\theta)f'(\theta)\cos\theta\sin\theta + \big(f(\theta)\big)^2\cos^2\theta \\
&= \big(f'(\theta)\big)^2\big[\cos^2\theta + \sin^2\theta\big] + \big(f(\theta)\big)^2\big[\sin^2\theta + \cos^2\theta\big] \\
&= \big(f'(\theta)\big)^2 + \big(f(\theta)\big)^2 .
\end{aligned}$$

From Section 7.4 in the previous chapter, the arc length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$$

Example 8.3.4 *Find the length of the curve:*

1. $r = 2$

2. $r = 2\sin\theta$

3. $r = e^{-\theta}$ *where* $0 \leq \theta \leq 2\pi$

4. $r = 2 - 2\cos\theta$

Solution:

1. $r^2 + (\frac{dr}{d\theta})^2 = 4$. Thus,
$$L = \int_0^{2\pi} \sqrt{4}\, d\theta = 2\Big[\ \theta\ \Big]_0^{2\pi} = 4\pi\ .$$

2. $r^2 + (\frac{dr}{d\theta})^2 = 4\sin^2\theta + 4\cos^2\theta = 4(\sin^2\theta + \cos^2\theta) = 4$. This implies
$$L = \int_0^{\pi} \sqrt{4}\, d\theta = 2\Big[\ \theta\ \Big]_0^{\pi} = 2\pi\ .$$

3. $r^2 + (\frac{dr}{d\theta})^2 = e^{-2\theta} + e^{-2\theta} = 2e^{-2\theta}$
$$L = \int_0^{2\pi} \sqrt{2e^{-2\theta}}\ d\theta = \sqrt{2}\int_0^{2\pi} e^{-\theta}\ d\theta = \sqrt{2}\big[1 - e^{-2\pi}\big]\ .$$

4. $r^2 + (\frac{dr}{d\theta})^2 = 4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta = 8 - 8\cos\theta = 8(1 - \cos\theta)$
$$L = \int_0^{2\pi} \sqrt{8(1-\cos\theta)}\ d\theta = 2\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos\theta}\ d\theta\ .$$

   We know that $\cos^2 v = \frac{1+cos2v}{2}$. If $v = \frac{\theta}{2}$, then $\cos^2\frac{\theta}{2} = \frac{1+cos\theta}{2}$. Thus,
$$L = 4\int_0^{2\pi} \sqrt{\cos^2\frac{\theta}{2}}\ d\theta = 8\int_0^{2\pi} \frac{1}{2}\cos\frac{\theta}{2}\ d\theta = 8\big[\sin\frac{\theta}{2}\big]_0^{\pi} = 8\ .$$

8.3.3 Surface of Revolution in Polar Coordinates

Let the polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ be smooth. Then,

$$x = f(\theta)\cos\theta\ \text{ and }\ y = f(\theta)\sin\theta,\ \ \alpha \leq \theta \leq \beta\ .$$

From Section 7.4 in the previous chapter, we have the following:

1. the surface area of revolution about the polar axis (x-axis) is
$$S.A = 2\pi\int_{\alpha}^{\beta} f(\theta)\sin\theta\sqrt{\big(f(\theta)\big)^2 + \big(f'(\theta)\big)^2}\ d\theta\ .$$

   This implies

$$S.A = 2\pi\int_{\alpha}^{\beta} r\sin\theta\sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$$

2. the surface area of revolution about the line $\theta = \frac{\pi}{2}$ (y-axis) is

$$S.A = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta \sqrt{\left(f(\theta)\right)^2 + \left(f'(\theta)\right)^2}\, d\theta \,.$$

This implies

$$\boxed{S.A = 2\pi \int_{\alpha}^{\beta} r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Example 8.3.5 *For the curve C: $r = 2\sin\theta$, find the area of the surface generated by revolving the curve C about*

1. *the polar axis.*
2. *the line $\theta = \frac{\pi}{2}$.*

Solution:

1. We use the formula $S = 2\pi \int_{\alpha}^{\beta} r\sin\theta \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$.

$$r^2 + (\frac{dr}{d\theta})^2 = 4\sin^2\theta + 4\cos^2\theta = 4(\sin^2\theta + \cos^2\theta) = 4 \,.$$

Thus,

$$S.A = 2\pi \int_0^{\pi} 2\sin^2\theta \sqrt{4}\, d\theta = 4\pi \int_0^{\pi} (1 - \cos 2\theta)\, d\theta = 4\pi\big[\theta - \frac{\sin 2\theta}{2}\big]_0^{\pi} = 4\pi\big[\pi - 0\big] = 4\pi^2 \,.$$

2. We use the formula $S = 2\pi \int_{\alpha}^{\beta} r\cos\theta \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$. Thus

$$S.A = 2\pi \int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta\sqrt{4}\, d\theta = -\frac{8\pi}{2}\big[\cos 2\theta\big]_0^{\frac{\pi}{2}} = -4\pi\big[0 - 1\big] = 4\pi \,.$$

Exercise 4:

1 - 6 ■ Find the length of the curve:

1. $r = 3\cos\theta$
2. $r = \sin\theta$
3. $r = 2(1 - \cos\theta)$
4. $r = 3$
5. $r = 3 + 2\cos\theta$
6. $r = \cos 4\theta$

7 - 12 ■ Find the area of the surface generated by revolving the graph of the equation about the polar axis:

7. $r = 1 + \cos\theta$
8. $r = \cos\theta$
9. $r = (2 - 3\cos\theta)$
10. $r = 4$
11. $r = 3\cos 3\theta$
12. $r = 6(1 + \cos\theta)$

13 - 18 ■ Find the area of the surface generated by revolving the graph of the equation about the line $\theta = \frac{\pi}{2}$:

13. $r = 1 + \sin\theta$
14. $r = 2$
15. $r = (1 - \sin\theta)$
16. $r = 2(1 + \sin\theta)$
17. $r = 4\cos 4\theta$
18. $r = \sin\theta$