Generalization of Posner’s Theorems

1AHMED A. M. KAMAL and 2KHALID H. AL-SHAALAN
1Department of Mathematics, Faculty of Sciences, Cairo University, Giza, Egypt
1Department of Mathematics, College of Science, King Saud University, Riyadh 2455, Kingdom of Saudi Arabia
2Science Department, King Abdul-Aziz Military Academy, Kingdom of Saudi Arabia
1aamkamal_9@hotmail.com, 2khshaalan@gmail.com

Abstract. In this paper we generalize Posner’s first theorem to a 3-prime near-ring with a \((\sigma, \tau)\)-derivation. We prove that a prime ring with a non-zero \((\sigma, \tau)\)-derivation is commutative if \(\sigma(x)d(x) = d(x)\tau(x)\) for all \(x \in U\) where \(U\) is a suitable subset of \(R\). Also, we generalize Posner’s second theorem completely to a prime ring with a \((\sigma, \sigma)\)-derivation and partially to a prime ring with a \((\sigma, \tau)\)-derivation.

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1. Introduction

Throughout this paper \(R\) will be a ring or a left near-ring. \(Z(R)\) will be its multiplicative center and \(\sigma, \tau\) two endomorphisms from \(R\) to \(R\). We say that \(R\) is prime (3-prime for near-rings) if, for all \(x, y \in R\), \(xRy = \{0\}\) implies \(x = 0\) or \(y = 0\). We say that \(U\) is a semigroup right (left) ideal of \(R\), if \(U\) is a non-empty subset of \(R\) satisfies \(UR \subseteq U\) (\(RU \subseteq U\)). We say that \(U\) is a semigroup ideal if it is both a semigroup right and left ideal. For all \(x, y \in R\), we write \([x, y] = xy - yx\) for the multiplicative commutator, \([x, y]_{\sigma, \tau} = \sigma(x)y - y\tau(x)\) and \((x, y) = x + y - x - y\) for the additive commutator. A map \(d: R \rightarrow R\) is called a \((\sigma, \tau)\)-derivation if \(d\) is additive and \(d(xy) = \sigma(x)d(y) + d(x)\tau(y)\) for all \(x, y \in R\). If \(\tau = 1_R\), then \(d\) is called a \(\sigma\)-derivation. An element \(x \in R\) is called a (right) zero divisor in \(R\) if there exists a non-zero element \(y \in R\) such that \(xy = 0\) (\(yx = 0\)). A zero divisor is either a left or a right zero divisor. By an integral near-ring, we mean a near-ring without non-zero divisors of zero. A near-ring \(R\) is called a constant near-ring, if \(xy = y\) for all \(x, y \in R\) and is called a zero-symmetric near-ring, if \(0x = 0\) for all \(x \in R\). For any group \((G, +)\), \(M_\theta(G)\) denotes the near-ring of all zero preserving maps from \(G\) to \(G\) with the two operations of addition and composition of maps. An abelian near-ring \(R\) is a near-ring such that \((R, +)\) is abelian. We refer the reader to the books of Meldrum [15] and Pilz [17] for basic results of near-ring theory and its applications.

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In this paper we use the commutator \([x,y]_{\sigma,\tau}\) to mean \(\sigma(x)y - y\tau(x)\), but its usual form is \(x\sigma(y) - \tau(y)x\) with using that \(d(xy) = d(x)\sigma(y) + \tau(x)d(y)\) for all \(x,y \in R\). According to the last form, Argac, Kaya and Kisir showed in [1] that a prime ring \(R\) admits a non-zero \((\sigma, \tau)\)-derivation such that \([d(x),x]_{\sigma,\tau} = 0\) for all \(x \in I\) if and only if \(R\) is commutative and \(\sigma = \tau\), where \(I\) is a non-zero right ideal of \(R\). They also showed that a prime ring \(R\) of characteristic not 2 admits a non-zero \((\sigma, \tau)\)-derivation such that \([d(x),x]_{\sigma,\tau} \in C_{\sigma,\tau}\) for all \(x \in I\) if and only if \(R\) is commutative and \(\sigma = \tau\), where \(C_{\sigma,\tau} = \{x \in R : x\sigma(y) = \tau(y)x\} \text{ for all } y \in R\). Also, Ashraf and Rehman showed in Theorem 1 in [2] that a 2-torsion free prime ring \(R\) is commutative if \(R\) admits a non-zero \((\sigma, \tau)\)-derivation such that \([d(x),x]_{\sigma,\tau} = 0\) for all \(x \in R\). In [3], Aydin had extended that theorem to \([d(x),x]_{\sigma,\tau} \in C_{\sigma,\tau}\) for all \(x \in R\). All above papers used that \(\sigma\) and \(\tau\) are automorphisms on \(R\). In the literature of studying commutativity of rings and near-rings, there are also some works studied the commutativity of rings and near-rings without the use of derivations, for example see [5] and [6]. Also, see [16] for subcommutativity in near-rings.

In Section 2 we give some well-known results and we add some new auxiliary results on a near-ring \(R\) admitting a non-zero \((\sigma, \tau)\)-derivation \(d\), which will be useful in the sequel. In Section 3 we study the problem of Posner for the composition of two derivations, in the more general case the composition of a \((\sigma, \tau)\)-derivation and an \((\alpha, \beta)\)-derivation, where \(\alpha\) is an automorphism and, \(\sigma, \beta\) and \(\tau\) are epimorphisms on a near-ring \(R\). Consequently, we generalize Posner’s first theorem for \((\sigma, \tau)\)-derivations in Theorem 3.1 which generalizes results due to K. I. Beidar, Y. Fong and X. K. Wang; O. Golbasi and M. S. Samman.

Section 4 is devoted to study Posner’s second theorem using \((\sigma, \tau)\)-derivations on prime rings. Consequently, we generalize Lemma 3 of [18] to \((\sigma, \tau)\)-derivations on prime rings. In Theorem 4.4 we study Posner’s second theorem using \((\sigma, \tau)\)-derivations on prime rings. Theorem 4.5 is a generalization of Posner’s second theorem to \((\sigma, \sigma)\)-derivations on prime rings, where \(\sigma\) is an epimorphism on \(R\). In the last of this section we study the condition \(d(x^2) \in Z(R)\) for all \(x \in R\), where \(d\) is a non-zero \((\sigma, \tau)\)-derivation on a prime ring \(R\).

2. Preliminaries and some results

We need the following lemmas:

**Lemma 2.1.** [10, Lemma 1] An additive mapping \(d\) on a near-ring \(R\) is a \((\sigma, \tau)\)-derivation if and only if \(d(xy) = d(x)\tau(y) + \sigma(x)d(y)\), for all \(x,y \in R\).

**Lemma 2.2.** [10, Lemma 2] Let \(R\) be a near-ring with a \((\sigma, \tau)\)-derivation \(d\) such that \(\tau\) is an epimorphism. Then \(R\) satisfies the partial distributive law, \((\sigma(x)d(y) + d(x)\tau(y))c = \sigma(x)d(y)c + d(x)\tau(y)c\) and \((d(x)\tau(y) + \sigma(x)d(y))c = d(x)\tau(y)c + \sigma(x)d(y)c\) for all \(x,y,c \in R\).

**Lemma 2.3.** [7, Lemma 1.2(iii)] Let \(R\) be a 3-prime near-ring and \(x \in Z(R) - \{0\}\). If either \(yx\) or \(xy\) in \(Z(R)\), then \(y \in Z(R)\).

**Lemma 2.4.** [9, Lemma 3(i),(ii)] Let \(R\) be a 3-prime near-ring and \(x \in Z(R) - \{0\}\). Then \(x\) is not a zero divisor in \(R\).

**Lemma 2.5.** [10, Lemma 3] Let \(d\) be a non-zero \((\sigma, \tau)\)-derivation on a 3-prime near-ring \(R\).

(i) If \(d(R)x = \{0\}\) and \(\tau\) is onto, then \(x = 0\).
(ii) If \( xd(R) = \{0\} \), then \( R \) is zero-symmetric and \( \sigma \) is onto, then \( x = 0 \).

**Lemma 2.6.** [13, Proposition 2.7] A near-ring \( R \) is zero-symmetric if and only if \( R \) admits a \((\sigma, \tau)\)-derivation \( d \) such that \( \sigma, \tau \) are endomorphisms and \( \tau \) is either one-to-one or onto.

**Lemma 2.7.** Let \( R \) be a near-ring with a \((\sigma, \tau)\)-derivation \( d \) such that \( 2R = \{0\} \) and \( \sigma, \tau \) commute with \( d \). Then \( d^2 \) is a \((\sigma^2, \tau^2)\)-derivation on \( R \).

**Proof.** For all \( x, y \in R \), we have \( d^2(x + y) = d^2(x) + d^2(y) \) since \( d \) is an additive mapping on \( R \). Now, for all \( x, y \in R \) we get

\[
d^2(xy) = d(d(xy)) = d(\sigma(x)d(y) + d(x)\tau(y))
\]

\[
= \sigma^2(x)d^2(y) + d\sigma(x)\tau d(y) + \sigma d(x)d\tau(y) + d^2(x)\tau^2(y)
\]

\[
= \sigma^2(x)d^2(y) + d\sigma(x)d\tau(y) + d\sigma(x)d\tau(y) + d^2(x)\tau^2(y)
\]

Thus, \( d^2(xy) = \sigma^2(x)d^2(y) + d^2(x)\tau^2(y) \) for all \( x, y \in R \) and \( d^2 \) is a \((\sigma^2, \tau^2)\)-derivation on \( R \).

**Lemma 2.8.** [7, Lemma 1.3(iii)] Let \( R \) be a 3-prime near-ring with a non-zero semigroup right ideal \( U \) of \( R \). If there exists \( x \in R \) which centralizes \( U \), then \( x \in Z(R) \). Moreover, if \( R \) is a prime ring and \( U \) is a semigroup left ideal, then \( x \in Z(R) \).

**Lemma 2.9.** [11, Lemma 4] Let \( R \) be a 3-prime near-ring with a \((\sigma, \tau)\)-derivation \( d \).

(i) If \( R \) is zero-symmetric and \( U \) is a non-zero semigroup right ideal of \( R \) such that \( \sigma \) is an epimorphism, \( \sigma(U) \neq \{0\} \) and \( d(U) = \{0\} \), then \( d = 0 \).

(ii) If \( U \) is a non-zero semigroup left ideal of \( R \) such that \( \tau \) is an epimorphism, \( \tau(U) \neq \{0\} \) and \( d(U) = \{0\} \), then \( d = 0 \).

**Lemma 2.10.** [7, Lemma 1.5] Let \( R \) be a 3-prime near-ring with a non-zero semigroup right (left) ideal \( U \) such that \( U \subseteq Z(R) \). Then \( R \) is a commutative ring.

**Lemma 2.11.** [7, Lemma 1.4] Let \( R \) be a 3-prime near-ring with a non-zero semigroup ideal \( U \). If \( x, y \in R \) and \( xUy = \{0\} \), then \( x = 0 \) or \( y = 0 \).

**Lemma 2.12.** [13, Corollary 4.6] Let \( R \) be a 3-prime near-ring with a non-zero \((\sigma, \tau)\)-derivation \( d \) such that one of \( \sigma, \tau \) is either a monomorphism or an epimorphism. If \( d(R) \subseteq Z(R) \), then \( R \) is a commutative ring.

**Lemma 2.13.** [13, Theorem 5.4] Let \( R \) be a 3-prime near-ring with a non-zero \((\sigma, \tau)\)-derivation \( d \) such that \( \tau \) is an automorphism and \( d(xy) = d(yx) \) for all \( x, y \in R \). Then \( R \) is a commutative ring.

**Lemma 2.14.** [13, Theorem 5.9] Let \( R \) be a 3-prime near-ring with a non-zero \((\sigma, \tau)\)-derivation \( d \) such that \( d(xy) = -d(yx) \) for all \( x, y \in R \). If \( \tau \) is an automorphism on \( R \), then \( R \) is a commutative ring of characteristic 2.

3. **Posner’s first theorem**

In this section we generalize Posner’s first theorem for \((\sigma, \tau)\)-derivations on near-rings. We need the following two lemmas to prove the first theorem in this section.
Lemma 3.1. Let $R$ be a near-ring with a $(\sigma, \tau)$-derivation $d$ and $\theta$ be any endomorphism of $R$. Then

(i) $\theta d$ is a $(\theta \sigma, \theta \tau)$-derivation on $R$.  
(ii) $d \theta$ is a $(\sigma \theta, \tau \theta)$-derivation on $R$.

Proof. (i) Clearly the composition of two additive mappings on $R$ is an additive mapping. Now, for all $x, y \in R$, we have

\[
\theta d(xy) = \theta(d(xy)) = \theta(\sigma(x)d(y) + d(x)\tau(y)) = \theta\sigma(x)\theta d(y) + \theta d(x)\theta \tau(y)
\]

and then $\theta d$ is a $(\theta \sigma, \theta \tau)$-derivation on $R$.

(ii) The proof is similar to (i). \hfill \Box

Lemma 3.2. Let $R$ be a near-ring with a non-zero $(\sigma, \tau)$-derivation $d$. Suppose one of the following two conditions holds:

(i) $R$ is a 3-prime near-ring and $\tau$ is onto, or
(ii) There exists $a \in R$ such that $d(a)$ is not a left zero divisor in $R$ and $\tau$ is either one-to-one or onto.

Then $nR = \{0\}$ if and only if $nd(R) = \{0\}$.

Proof. Clearly if $nR = \{0\}$, then $nd(R) = \{0\}$. Conversely, suppose $nd(R) = \{0\}$. Then $0 = nd(b) = d(nb)$ for all $b \in R$. Now, for all $x, y \in R$

\[
0 = d(n(yx)) = d(y(nx)) = \sigma(y)d(nx) + d(y)\tau(nx) = d(y)\tau(nx).
\]

If $R$ is 3-prime and $\tau$ is onto, then $d(R)\tau(nx) = \{0\}$ implies $\tau(nx) = 0$ for all $x \in R$ by Lemma 2.5(i). It follows that $\{0\} = \tau(nR) = n\tau(R) = nR$. If there exists $a \in R$ such that $d(a)$ is not a left zero divisor in $R$, then $d(a)\tau(nx) = 0$ and then $\tau(nx) = 0$ for all $x \in R$. Therefore $\tau(nR) = \{0\}$. If $\tau$ is onto, then by the same way above $nR = \{0\}$ and if $\tau$ is one-to-one, then $\tau(nR) = \{0\}$ implies $nR = \{0\}$.

The conditions “$\tau$ is onto” in Lemma 3.2(i) and “$\tau$ is either one-to-one or onto” in Lemma 3.2(ii) are not redundant as the following example shows.

Example 3.1. Let $(R, +)$ be the additive abelian group $(\mathbb{Z}_4, +)$ and define the multiplication to make $R$ a constant near-ring. Then $R$ is 3-prime. Suppose $\tau = 0$ and $\sigma$ is any endomorphism on $R$, then any additive mapping $d$ on $R$ is a $(\sigma, \tau)$-derivation. Define $d : R \rightarrow R$ by $d(\bar{x}) = \bar{x} + \bar{x}$ for all $\bar{x} \in R$. Then $d(\bar{x} + \bar{y}) = \bar{x} + \bar{y} + \bar{x} + \bar{y} = \bar{x} + \bar{x} + \bar{y} = d(\bar{x}) + d(\bar{y})$ for all $\bar{x}, \bar{y} \in R$ and $d$ is an additive endomorphism of $R$. So $d$ is a $(\sigma, \tau)$-derivation on $R$. Also, $d(\bar{T}) = \bar{T} + \bar{\bar{T}} = \bar{T}$ is not a left zero divisor in $R$ by the definition of the multiplication. Observe that $d(2\bar{x}) = d(\bar{x} + \bar{x}) = \bar{x} + \bar{x} + \bar{x} = 4\bar{x} = 0$ for all $\bar{x} \in R$. Thus, $2d(R) = \{0\}$. But $2R \neq \{0\}$ as $2(\bar{T}) = \bar{T} + \bar{\bar{T}} = \bar{T} \neq 0$.

The following theorem generalizes Theorem 1.1 of [4], Theorem 2.5 of [11] and the main Theorem of [19].

Theorem 3.1. Let $R$ be a 3-prime near-ring with a $(\sigma, \tau)$-derivation $d$ and an $(\alpha, \beta)$-derivation $D$ such that $\alpha$ commutes with $\beta$, $\alpha$ is an automorphism, $\sigma, \beta, \tau$ are epimorphisms and $\alpha, \beta, \tau$ commute with $D$. If $dD$ is a $(\sigma \alpha, \tau \beta)$-derivation, then one of the following statements holds:

(i) $d = 0$
(ii) $D = 0$
(iii) $2R = \{0\}$.  

Proof. Since \( \tau \) is an epimorphism, we have \( R \) is zero-symmetric by Lemma 2.6. As \( dD \) is a \( (\sigma\alpha, \tau\beta) \)-derivation, so \( dD(ab) = \sigma\alpha(a)dD(b) + dD(a)\tau\beta(b) \) for all \( a, b \in R \). On the other hand, \( d \) is a \( (\sigma, \tau) \)-derivation and \( D \) is an \( (\alpha, \beta) \)-derivation. Thus, \( dD(ab) = d(\alpha(a)D(b) + D(a)\beta(b)) = \sigma\alpha(a)dD(b) + d(\alpha(a))\tau\beta(b) + \sigma(D(a))d(\beta(b)) + dD(a)\tau\beta(b) \). Comparing the previous two equations, we get

\[
d(\alpha(a))\tau(D(b)) + \sigma(D(a))d(\beta(b)) = 0 \quad \text{for all} \quad a, b \in R.
\]

Replace \( a \) by \( ac \) where \( c \in R \). So using the partial distributive law (Lemma 2.2), we have for all \( a, b, c \in R \)

\[
0 = d(\alpha(ac))\tau D(b) + \sigma(D(ac))d(\beta(b)) = d(\alpha(a)\alpha(c))\tau D(b) + \sigma(D(ac))d(\beta(b))
\]

\[
= d\alpha(a)\tau\alpha(c)\tau D(b) + \sigma\alpha(a)d\alpha(c)\tau D(b) + \sigma(\alpha(a))D(c) + D(\alpha(\beta(c)))d(\beta(b))
\]

\[
= d\alpha(a)\tau\alpha(c)\tau D(b) + \sigma\alpha(a)d\alpha(c)\tau D(b) + \sigma(\alpha(a))\tau D(c) + \sigma(D(a))\tau D(\beta(c))d(\beta(b)).
\]

Notice that \( \sigma D \) is a \( (\sigma\alpha, \sigma\beta) \)-derivation by Lemma 3.1. Since \( \sigma\beta \) is onto, we can use the partial distributive law to obtain

\[
0 = d\alpha(a)\tau\alpha(c)\tau D(b) + \sigma\alpha(a)d\alpha(c)\tau D(b) + \sigma\alpha(a)\sigma D(c)d(\beta(b))
\]

\[
+ \sigma D(a)\sigma\beta(c)d(\beta(b))
\]

\[
= d\alpha(a)\tau D(c)\tau D(b) + \sigma\alpha(a)d\alpha(c)\tau D(b) + \sigma\alpha(a)\sigma D(c)d(\beta(b))
\]

\[
+ \sigma D(a)\sigma\beta(c)d(\beta(b))
\]

for all \( a, b, c \in R \). By using (3.1) with \( c \) instead of \( a \), we get for all \( a, b, c \in R \)

\[
d(\alpha(a))\tau\alpha(c)\tau D(b) + \sigma D(a)\sigma\beta(c)d(\beta(b)) = 0.
\]

As \( \alpha \) is bijective, we obtain \( d\alpha(a)\tau(r)\tau D(b) + \sigma D(a)\sigma\beta(\alpha^{-1}(r))d(\beta(b)) = 0 \) for all \( a, b, r \in R \) where \( r = \alpha(c) \). Taking \( r = D(t) \) where \( t \in R \), we obtain \( d\alpha(a)\tau D(t)\tau D(b) + \sigma D(a)\sigma\beta\alpha^{-1}(D(t))d(\beta(b)) = 0 \) for all \( a, b, t \in R \). Since \( \beta\alpha^{-1} \) commutes with \( D \), we have

\[
d(\alpha(a))\tau D(t)\tau D(b) + \sigma D(a)\sigma D(\beta\alpha^{-1}(t))d(\beta(b)) = 0.
\]

Replacing \( a \) by \( \beta\alpha^{-1}(t) \) in equation (3.1), we deduce that \( \sigma D(\beta\alpha^{-1}(t))d(\beta(b)) = -d(\alpha(\beta\alpha^{-1}(t)))\tau D(b) \). Since \( \alpha \) and \( \beta \) commute, we have \( \sigma D(\beta\alpha^{-1}(t))d(\beta(b)) = -d(\beta(t))\tau D(b) \) for all \( t, b \in R \). Therefore, (3.3) becomes \( 0 = d\alpha(a)\tau D(t)\tau D(b) + \sigma D(a)(-d(\beta(t))\tau D(b)) \) which means

\[
d\alpha(a)\tau D(t)\tau D(b) = \sigma D(a)d(\beta(t))\tau D(b) \quad \text{for all} \quad a, b, t \in R.
\]

Replacing \( b \) by \( tk \) in (3.1) where \( t, k \in R \), we have

\[
0 = d(\alpha(a))\tau D(tk) + \sigma D(a)d(\beta(tk)) = d(\alpha(a))\tau D(tk) + \sigma D(a)d(\beta(t)\beta(k))
\]

\[
= d\alpha(a)\tau D(t)\beta(k) + \alpha(t)D(k) + \sigma D(a)(\sigma\beta(t)d(\beta(k)) + d(\beta(t))\tau D(k))
\]

\[
= d\alpha(a)\tau D(t)\tau D(k) + \sigma D(a)d(\beta(t))\tau D(k) + \sigma D(\alpha)\sigma\beta(t)d(\beta(k)) + \sigma D(a)d(\beta(t))\tau D(k)
\]

\[
= d\alpha(a)\tau D(t)\tau D(k) + \sigma D(a)d(\beta(t))\tau D(k)
\]

as \( d\alpha(a)\tau D(tk) + \sigma D(a)d(\beta(tk)) = 0 \) by (3.2). Then \( d\alpha(a)\tau D(t)\tau(r) + \sigma D(a)d(\beta(t))\tau(r) = 0 \) for all \( a, t, r \in R \), since \( \beta \) is onto. Taking \( r = D(b) \) where \( b \in R \) in the last equation, we obtain

\[
d\alpha(a)\tau D(t)\tau D(b) + \sigma D(a)d\beta(t)\tau D(b) = 0 \quad \text{for all} \quad a, b, t \in R.
\]
Substituting (3.4) in (3.5) and using $\tau D = D\tau$, we get for all $a, b, t \in R$
$$0 = d(\alpha(a))D\tau(t)D\tau(b) + d(\alpha(a))\tau D\tau(t)D\tau(b) = d(\alpha(a))D(\tau(t))(2D(\tau(b))).$$
Since $\alpha$ and $\tau$ are onto, we have $d(R)D(R)(2D(R)) = \{0\}$. Suppose $d \neq 0$. So $D(R)(2D(R)) = \{0\}$ by Lemma 2.5(i). If $D \neq 0$, then $2D(R) = \{0\}$ by Lemma 2.5(i) and hence $2R = \{0\}$ by Lemma 3.2(i).

The following corollary generalizes [20, Corollary 1].

**Corollary 3.1.** Let $R$ be a $3$-prime near-ring such that $2R \neq \{0\}$ with a $(\sigma, \tau)$-derivation $d$ such that $\sigma$ commutes with $\tau$, $\sigma$ is an automorphism, $\tau$ is an epimorphism and $\sigma, \tau$ commute with $d$. If $d^2$ is a $(\sigma^2, \tau^2)$-derivation, then $d = 0$.

The conditions $2R = \{0\}$ in Theorem 3.1 and $2R \neq \{0\}$ in Corollary 3.1 are essential as the following example shows.

**Example 3.2.** Let $R = \mathbb{Z}_2[x]$. Then $R$ is an integral domain which means that $R$ is a commutative prime ring. Also, we have $2R = \{0\}$. If we take $d$ to be the usual derivative on $R = \mathbb{Z}_2[x]$, then $d$ is a $(1_R, 1_R)$-derivation on $R$ which is non-zero. But $d^2$ is also a $(1_R, 1_R)$-derivation on $R = \mathbb{Z}_2[x]$ by Lemma 2.7.

The following result generalizes [12, Proposition 4.8].

**Proposition 3.1.** Let $R$ be a near-ring with a $(\sigma, \tau)$-derivation $d$ and an $(\alpha, \beta)$-derivation $D$ such that $\alpha$ commutes with $\beta$, $\alpha$ is an automorphism, $\sigma, \beta, \tau$ are epimorphisms and $\alpha, \beta, \tau$ commute with $D$. If $dD$ is a $(\sigma\alpha, \tau\beta)$-derivation and there exist $x_o, y_o \in R$ such that $d(x_o), D(y_o)$ are not left zero divisors in $R$, then $2R = \{0\}$.

**Proof.** By the same way of the proof of Theorem 3.1, we will deduce that $d(R)D(R)(2D(R)) = \{0\}$. Since $d(x_o)$ is not a left zero divisor in $R$, we have $D(R)(2D(R)) = \{0\}$. Again, as $D(y_o)$ is not a left zero divisor in $R$, so $2D(R) = \{0\}$ which implies that $2R = \{0\}$ by Lemma 3.2(ii).

## 4. Posner’s second theorem

In this section we generalized Posner’s second theorem for $(\sigma, \tau)$-derivations.

**Lemma 4.1.** Let $R$ be a near-ring with a multiplicative epimorphism $\theta$. If $U$ is a non-zero semigroup right (left) ideal of $R$, then $\theta(U)$ is a semigroup right (left) ideal of $R$. Moreover, if $\theta$ is a multiplicative automorphism on $R$ then $\theta(U)$ is a non-zero semigroup right (left) ideal of $R$.

**Proof.** Let $U$ be a non-zero semigroup right ideal of $R$ and $x \in R$. Since $\theta$ is onto, there exists $r \in R$ such that $\theta(r) = x$. Thus, $\theta(u)x = \theta(u)\theta(r) = \theta(ur) \in \theta(U)$ for all $u \in U$. Hence, $\theta(U)$ is a semigroup right ideal of $R$. If $\theta$ is a multiplicative automorphism, then $\theta(U) = \{0\}$ implies $U = \{0\}$, a contradiction. The proof is similar for semigroup left ideals.

The following result generalizes [2, Theorem 1] and [18, Lemma 3].

**Theorem 4.1.** Let $R$ be a prime ring with a non-zero $(\sigma, \tau)$-derivation $d$ such that $\sigma$ or $\tau$ is an automorphism and $\sigma(x)d(x) = d(x)\tau(x)$ for all $x \in U$, where $U$ is a non-zero semigroup ideal of $R$ which is closed under addition. Then $R$ is a commutative ring.
Proof. Suppose \( \tau \) is an automorphism. \( U \) is closed under addition implies \( \sigma(x + y)d(x + y) = d(x + y)\tau(x + y) \) for all \( x,y \in U \). So \( \sigma(x)d(x) + \sigma(x)d(y) + \sigma(y)d(x) + \sigma(y)d(y) = d(x)\tau(x) + d(x)\tau(y) + d(y)\tau(x) + d(y)\tau(y) \). Using \( \sigma(x)d(x) = d(x)\tau(x) \) and \( \sigma(y)d(y) = d(y)\tau(y) \), we get

\[
(4.1) \quad \sigma(x)d(y) + \sigma(y)d(x) = d(x)\tau(y) + d(y)\tau(x) \quad \text{for all} \quad x,y \in U.
\]

Adding \( d(x)\tau(y) + \sigma(y)d(x) \) to both sides of (4.1), we have \( \sigma(x)d(y) + d(x)\tau(y) + 2\sigma(y)d(x) = d(y)\tau(x) + d(x)\tau(y) \) which means \( d(xy) + 2\sigma(y)d(x) = d(ay) + 2d(x)\tau(y) \) and then for all \( x,y \in U \), we get

\[
(4.2) \quad d(xy) - d(yx) = 2d(x)\tau(y) - 2\sigma(y)d(x) = 2d(x)\tau(y) - \sigma(y)d(x).
\]

Replacing \( y \) by \( xy \) in (4.2) and using \( \sigma(x)d(x) = d(x)\tau(x) \) for all \( x \in U \), we have

\[
d(xxy) - d(xy) = 2d(x)\tau(x) - \sigma(x)\sigma(y)d(x) = 2(\sigma(x)d(x)\tau(y) - \sigma(x)\sigma(y)d(x)) = \sigma(x)(d(xy) - d(yx)),
\]

On the other hand, we have

\[
d(xxy) - d(xy) = d(xy - yx) = \sigma(x)(d(xy) - d(yx)) + d(x)\tau(xy - yx).
\]

Comparing the last equations, we obtain \( d(x)\tau(xy - yx) = 0 \), for all \( x,y \in U \). Thus, we have the following

\[
(4.3) \quad d(x)\tau(x)\tau(y) = d(x)\tau(y)\tau(x) \quad \text{for all} \quad x,y \in U.
\]

Replacing \( y \) by \( yz \) and using (4.3), we get \( d(x)\tau(y)\tau(z)\tau(xy) = d(x)\tau(xy)\tau(xy)\tau(xy) = d(x)\tau(y)\tau(z)\tau(xy) \) for all \( x,y,z \in U \). So \( d(x)\tau(y)\tau(z)\tau(xy) - \tau(z)\tau(xy) = 0 \). Thus, \( d(x)\tau(U)\tau(y)\tau(z)\tau(xy) \) (4.1) for all \( x,z \in U \). Using Lemma 4.1 and Lemma 2.11, we have for all \( x \in U \)

\[
d(xxy) - d(xy) = \sigma(x)(d(xy) - d(yx)) + d(x)\tau(xy - yx).
\]

for all \( x,y \in U \) since \( d(\alpha x) - d(xa) = 0 \) for all \( x \in U \). On the other hand, \( d(xy) - d(ayx) = d(axy) - d(axy) = 0 \).

Comparing the last two equations, we get \( d(a)\tau(xy) - \tau(y)\tau(xy) = 0 \) and then \( d(a)\tau(\tau(xy) - \tau(y)\tau(xy) = 0 \) for all \( x,y \in U \). Putting \( x \tau(xy) \) instead of \( x \) where \( x \in U \), we get \( d(a)\tau(x)\tau(\tau(xy) = d(a)\tau(x)\tau(\tau(xy) = d(a)\tau(x)\tau(\tau(xy) \) for all \( x,y,z \in U \). Therefore, \( d(\alpha x)\tau(\tau(xy) - \tau(y)\tau(xy) = 0 \) for all \( x,y,z \in U \). Thus, \( a \in Z(R) \) by Lemma 2.8. Replacing \( y \) by \( ay \) in (4.2), we get \( d(\alpha ax) - d(\alpha ayx) = 2d(x)\tau(a)\tau(\tau(xy) - \sigma(a)\sigma(y)d(x)) \) for all \( x,y \in U \). But from (4.1), we have \( \sigma(x)d(x) + \sigma(a)d(x) - d(a)\tau(x) = d(x)\tau(a) \). Substituting this in the last equation and using (4.2) and \( a \in Z(R) \), it will be

\[
d(\alpha ax) - d(\alpha ax) = 2(\sigma(a)d(x)\tau(\tau(xy) - \sigma(x)d(x))d(x)) + 2(\sigma(x)d(x)\tau(\tau(xy) - \sigma(x)d(x))d(x)) + 2(\sigma(x)d(\alpha ax) - d(\alpha ax)\tau(x))\tau(x) = \sigma(a)(d(\alpha ax) - d(\alpha ax)\tau(x))\tau(x) = \sigma(a)(d(\alpha ax) - d(\alpha ax))\tau(\tau(xy) - \tau(y)\tau(xy) = 0 \).
Corollary 4.1. Let $R$ be a prime ring with a non-zero \( \sigma \) such that \( d = \sigma \) and \( \tau = 0 \). Then $R$ is a commutative ring.

It is not true to replace the condition “$\sigma(x)d(x) = d(x)d(x)$” in Theorem 4.1 by “$\sigma(x)d(x) = d(x)x$” as the following example shows.

Example 4.1. Let $R$ be the prime ring $M_2(\mathbb{Z}_2)$. Take $d = \tau$ is the identity map on $R$ and $\sigma = 0$ (or $d = \sigma$ is the identity map on $R$ and $\tau = 0$). Then $d$ is a non-zero $(\sigma, \tau)$-derivation on $R$. Clearly that $d(x)x = xd(x) = x^2$ for all $x \in R$. But $R$ is not commutative.

Corollary 4.1. Let $R$ be a prime ring with a non-zero $\sigma$-derivation $d$ such that $\sigma(x)d(x) = d(x)x$ for all $x \in U$ where $U$ is a non-zero semigroup ideal of $R$ which is closed under addition. Then $R$ is a commutative ring.

Lemma 4.2. Let $R$ be an abelian near-ring with a non-zero $(\sigma, \tau)$-derivation $d$ such that $\sigma$ and $\tau$ are epimorphisms. Then $d(\text{dist}(R)) \subseteq \text{dist}(R)$, where $\text{dist}(R)$ is the set of distributive elements of $R$.

Proof. For all $x, y \in R, s \in \text{dist}(R)$, we have $d((x+y)s) = d(xs + ys)$. That means $\sigma(x + y)d(s) + d(x+y)\tau(s) = \sigma(x)d(s) + d(x)\tau(s) + \sigma(y)d(s) + d(y)\tau(s)$. Since $\tau$ is onto, we get $\tau(s) \in \text{dist}(R)$. It follows that $(\sigma(x) + \sigma(y))d(s) + d(x)\tau(s) + d(y)\tau(s) = \sigma(x)d(s) + \sigma(y)d(s) + d(x)\tau(s) + d(y)\tau(s)$ and hence $(\sigma(x) + \sigma(y))d(s) = \sigma(x)d(s) + \sigma(y)d(s)$. So $d(s) \in \text{dist}(R)$.

Theorem 4.2. Let $R$ be an integral near-ring with a non-zero $(\sigma, \tau)$-derivation $d$ such that $\sigma$ and $\tau$ are automorphisms and $\sigma(x)d(x) = d(x)\tau(x)$ for all $x \in R$. Then $d$ is a $(\sigma, \sigma)$-derivation on $\text{dist}(R)$ and either $d(\text{dist}(R)) = 0$ or $\text{dist}(R)$ is a commutative ring. Moreover, if $d(\text{dist}(R)) \neq 0$, then $\sigma(s) = \tau(s)$ for all $s \in \text{dist}(R)$.

Proof. For all $x, y \in R$, we have $d(x(x+y)) = d(x^2 + xy)$. So

\[
d(x(x+y)) = \sigma(x)d(x+y) + d(x)\tau(x+y)
\]

Thus, $0 = \sigma(x)d(x+y) + d(x)\tau(x+y) = \sigma(x)d(x + y) + d(x)\tau(x + y) = \sigma(x)d(x) + \sigma(x)d(y) + d(x)\tau(y) + \sigma(x)d(x) + d(x)\tau(y) = \sigma(x)d(x) + \sigma(x)d(y) + d(x)\tau(y).

As $d(x)\tau(x) = \sigma(x)d(x)$. On the other hand

\[
d(x^2 + xy) = d(x^2) + d(xy) = \sigma(x)d(x) + d(x)\tau(x) + \sigma(x)d(y) + d(x)\tau(y)
\]

Thus, $0 = \sigma(x)d(y) + \sigma(x)d(x) = \sigma(x)d(x) + \sigma(x)d(y)$ for all $x, y \in R$. Since $R$ is without zero divisors and $\sigma$ is an automorphism, either $x = 0$ or $d(y + x - y - x) = 0$ for all $0 \neq x \in R$ and for all $y \in R$. But if $x = 0$, then $d(y + x - y - x) = d(y - y) = d(0) = 0$. So $d((x, y)) = 0$ for all $x, y \in R$. Since $z(x, y) = (zx, zy)$ for all $x, y, z \in R$, we have $d((z(x, y)) = 0$ and then $0 = d(z(x, y)) = \sigma(z)d((x, y)) + d(z)\tau(x, y) + d(z)\tau(x, y)$. Since $d \neq 0$, there exists $z \in R$ such that $d(z) \neq 0$ and then $\tau(x, y) = 0$ for all $x, y \in R$. It follows that $(R, +)$ is an abelian group. So $R$ is an abelian near-ring. Thus, $\text{dist}(R)$ is a subnear-ring of $R$ which is an integral ring. Also, $\text{dist}(R) \subseteq \text{dist}(R)$ by Lemma 4.2. Therefore, $d(\text{dist}(R)) = 0$ or $\text{dist}(R)$ is a commutative ring by Theorem 4.1. Now, if $d(\text{dist}(R)) = 0$, then $d$ is a $(\sigma, \sigma)$-derivation on $\text{dist}(R)$. Suppose that $d(\text{dist}(R)) \neq 0$. So $\sigma(s)d(s) = d(s)\tau(s)$ for all $s \in \text{dist}(R)$. Thus, $d(s)(\sigma(s) - \tau(s)) = 0$ and either $d(s) = 0$ or $\sigma(s) = \tau(s)$. That means if
$d(s) \neq 0$, then $\sigma(s) = \tau(s)$. Since $d(\operatorname{dist}(R)) \neq 0$, there exists $t \in \operatorname{dist}(R)$ such that $d(t) \neq 0$. So for all $s \in \operatorname{dist}(R) - \{0\}$ such that $d(s) = 0$, we get $\sigma(ts)d(ts) = d(ts)\tau(ts)$. It follows that $\sigma(t)\sigma(s)d(t)\tau(s) = d(t)\tau(s)\tau(t)\tau(s)$. As $\operatorname{dist}(R)$ is a commutative integral ring, $\tau$ is an automorphism and $\sigma(t) = \tau(t)$ where $d(t) \neq 0$ and $t \in \operatorname{dist}(R)$, we have $\sigma(s) = \tau(s)$ for all $s \in \operatorname{dist}(R)$. Also, $\sigma$ is an automorphism on $R$ implies that $\sigma$ is an automorphism on $\operatorname{dist}(R)$.

Therefore, $d$ is a non-zero $(\sigma, \sigma)$-derivation on $\operatorname{dist}(R)$.

The following result generalizes [1, Theorem 1].

**Theorem 4.3.** Let $R$ be a prime ring with a non-zero $(\sigma, \tau)$-derivation $d$ such that $\sigma, \tau$ are epimorphisms and $\sigma(x)d(x) = d(x)\tau(x)$ for all $x \in U$ where $U$ is a non-zero right (left) ideal of $R$. Then $\tau(U) = \{0\}$ or $\sigma(U) = \{0\}$ or ($R$ is a commutative ring and $\sigma = \tau$).

**Proof.** Suppose $U$ is a non-zero right ideal. The first part of the proof is similar to the first part of the proof of Theorem 4.1 up to equation (4.3)

$$d(x)\tau(x)\tau(y) = d(x)\tau(y)\tau(x) \quad \text{for all} \quad x, y \in U.$$ 

Replacing $y$ by $yz$ and using (4.3), we have $d(x)\tau(y)\tau(x)\tau(z) = d(x)\tau(x)\tau(y)\tau(z) = d(x)\tau(y)\tau(z)\tau(x)$ for all $x, y, z \in U$, which means $d(x)\tau(y)\tau(x)\tau(z) - \tau(z)\tau(x)\tau(y) = 0$. Thus, $d(x)\tau(U)$ and $\tau(U)\tau(x)$ are subgroups of $U$. By Lemma 4.1, either $\tau(U) = \{0\}$ or $d(x)\tau(U) = \{0\}$ or $\tau(x)\tau(z) = \tau(z)\tau(x)$ for all $x, z \in U$. Let $A = \{x \in U : d(x)\tau(U) = \{0\}\}$ and $B = \{x \in U : \tau(xz) = \tau(zx) \text{ for all } z \in U\}$. Then $A$ and $B$ are subgroups of $(U, +)$ and $A \cup B = U$. Thus, $A = U$ or $B = U$. In other words, $d(x)\tau(U) = \{0\}$ or $\tau(U) \subseteq Z(R)$. Suppose $d(U)\tau(U) = \{0\}$.

Then (4.1) will be $\sigma(x)d(y) + \sigma(y)d(x) = 0$ for all $x, y \in U$. Since $d(x) = \sigma(x)d(y), d(yx) = \sigma(y)d(x)$, we have

$$d(xy) + d(yx) = 0 \quad \text{for all} \quad x, y \in U. \tag{4.4}$$

Replacing $x, y$ by $z, (xy + yz)$ respectively in (4.4), we get $d(z(xy + yz)) + (xy + yz)z = 0$ for all $x, y, z \in U$. It follows that

$$0 = \sigma(z)d(xy + yz) + d(z)\tau(xy + yz) + \sigma(xy + yz)d(z) + d(xy + yz)\tau(z) \tag{4.5}$$

for all $x, y, z \in U$. Observe that $d(xy + yz)\tau(z) = d(z)\tau(xy + yz) = 0$ from $d(U)\tau(U) = \{0\}$ and $\sigma(z)d(xy + yz) = 0$ from (4.4). Thus, (4.5) will be $\sigma(xy + yz)d(z) = 0$. Replacing $y$ by $yz$, it yields $0 = \sigma(yz + yz)d(z) = \sigma(x)\sigma(y)\sigma(z)d(z) + \sigma(y)\sigma(y)\sigma(z)d(z) = \sigma(y)\sigma(z)d(x)d(z)$ for all $x, y, z \in U$ since $\sigma(z)d(z) = d(z)\tau(z) = 0$. Replacing $y$ by $yr$ where $r \in R$, we get $\sigma(y)\sigma(r)\sigma(z)d(x)d(z) = 0$. As $R$ is prime and $\sigma$ is onto, either $\sigma(U) = \{0\}$ or $\sigma(z)d(x)d(z) = 0$ for all $x, z \in U$. If $\sigma(U) \neq \{0\}$, then $\sigma(z)d(x)d(z) = 0$ for all $x, z \in U$. Putting $xy$ instead of $x$, we conclude that $\sigma(z)\sigma(x)d(x)d(z) = 0$ and then for every $z \in U$ either $d(z) = 0$ or $\sigma(z)\sigma(x) = \sigma(zx) = 0$. Let $A = \{u \in U : d(u) = 0\}$ and $B = \{u \in U : \sigma(ux) = 0 \text{ for all } x \in U\}$. So $A$ and $B$ are subgroups of $(U, +)$. Moreover, $U = A \cup B$. Thus, either $A = U$ or $B = U$. If $A = U$, then $d(U) = \{0\}$ and hence $d = 0$ by Lemma 2.9(i), a contradiction with the hypothesis. If $B = U$, then $\sigma(U^2) = \{0\}$ which implies $\sigma(U)\sigma(U) = \{0\}$.

But $\sigma(U)$ is a non-zero semigroup right ideal of $R$ by Lemma 4.1 and $\sigma(U) \neq \{0\}$. So $\sigma(U)\sigma(U) = \{0\}$, a contradiction. Hence, $d(U)\tau(U) \neq \{0\}$ if $\sigma(U) \neq \{0\}$. Therefore, $\tau(U) \subseteq Z(R)$. But $\tau(U) \neq \{0\}$ is a non-zero semigroup right ideal of $R$, so $R$ is a commutative ring by Lemma 2.10. It follows that $\sigma(x)d(x) = d(x)\tau(x)$ implies $d(x)(\sigma(x) - \tau(x)) = 0$ for all $x \in U$. Since $R$ is a commutative prime ring, it doesn’t have non-zero zero divisors by Lemma 2.4. Thus, either $d(x) = 0$ or $\sigma(x) = \tau(x).$
Let $A = \{x \in U | d(x) = 0\}$ and $B = \{x \in U | \sigma(x) = \tau(x)\}$. Then $A$ and $B$ are subgroups of $U$ whose union is $U$. As $d(U) \neq 0$, we have $B = U$ and $\sigma(x) = \tau(x)$ for all $x \in U$. Hence, $\sigma(ux) = \tau(ux)$ for all $u \in U$ and $x \in R$. That implies $\sigma(u)(\sigma(x) - \tau(x)) = 0$. Since $\sigma(U) \neq \{0\}$, we get $\sigma(x) = \tau(x)$ for all $x \in R$ and $\sigma = \tau$. The proof when $U$ is a non-zero left ideal is similar.

If a 3-prime near-ring $R$ with a $(\sigma, \sigma)$-derivation $d$ such that $\sigma(x)d(x) = d(x)\sigma(x)$ for all $x \in R$, then $R$ need not be a ring as the following example shows:

**Example 4.2.** Let $R = I \times I$ as a set, where $I$ is any integral ring with identity which has at least three elements. Define the addition and the multiplication on $R$ by $(a,b) + (c,d) = (a+c, b+d)$ and $(a,b)(c,d) = (ac,bc + d)$ if $(a,b) \neq (0,0)$ and $(0,0)(c,d) = (0,0)$. Then $R$ is a zero-symmetric abelian near-ring with identity $(1,0)$ which is not a ring. Let $D$ be a non-zero derivation on $I$ and $\sigma$ the endomorphism defined on $R$ by $\sigma((a,b)) = (a,0)$ for all $(a,b) \in R$. Define $d : R \to R$ by $d((a,b)) = (Da,0)$. Then $d$ is a non-zero $(\sigma, \sigma)$-derivation on $R$ by simple calculations. Observe that $R$ is 3-prime. Indeed, assume that $(a,b)R(c,d) = (0,0)$ with $(a,b) \neq (0,0)$. If $a \neq 0$, then $(a,b)(1,0)(c,d) = (0,0)$. That means $(a,b)(c,d) = (ac,bc + d) = (0,0)$. Thus, $c = 0$ and hence $d = 0$. Now, suppose $a = 0$ and $b \neq 0$. It follows that $(0,0) = (0,b)(0,1)(c,d) = (0,1)(c,d) = (0,c+d)$ and then $c = -d$. It follows that $(0,0) = (0,b)(0,y)(-d,d) = (0,y)(-d,d) = (0,-yd + d) = (0,(-y+1)d)$ for all $y \in I - \{0\}$. If $d \neq 0$, then $y = 1$ and $I = \{0,1\}$ which is a contradiction with the number of elements of $I$. Therefore, $d = 0$ and $(c,d) = (0,0)$. Hence, $R$ is a 3-prime near-ring.

Now, choose $I$ to be the integral domain $R[x]$ where $R$ is the field of real numbers and choose $D$ to be usual derivative on $R[x]$. Observe that we have $\sigma(a,b)d((a,b)) = d((a,b))\sigma(a,b)$ for all $(a,b) \in R$, but $R$ is not a ring.

**Proposition 4.1.** Let $R$ be a prime ring.

(i) If $nx = 0$ for some $x \in R$ and a positive integer $n$, then either $nR = \{0\}$ or $x = 0$.

(ii) If $nR \neq \{0\}$ for some positive integer $n$ and $nx \in Z(R)$ for some $x \in R$, then $x \in Z(R)$.

**Proof.** (i) For all $y, z \in R$, we have $0 = yz(nx) = n(yzx) = (ny)zx$. From the primeness of $R$, we have either $nR = \{0\}$ or $x = 0$.

(ii) If $Z(R) = \{0\}$, then $nx = 0$ and hence $x = 0$ by using (i). If $Z(R) \neq \{0\}$, then there exists $z \in Z(R) - \{0\}$. Observe that $ny \neq 0$ for all $y \in R - \{0\}$ from (i). Now, $z(nx) \in Z(R)$. Observe that $z(nx) = n(zx) = (nz)x \in Z(R)$. But $nz \in Z(R) - \{0\}$. Therefore, $x \in Z(R)$ byLemma 2.3.

The following example shows that the hypothesis “prime ring” in Proposition 4.1 can’t be replaced by “3-prime near-ring”.

**Example 4.3.** Let $R = M_3(G)$, where $G$ is the abelian group $(\mathbb{Z}_4, +)$. Then $M_3(G)$ is 3-prime. Take $f \in M_3(G)$ such that $xf = 2x$ for all $x \in G$. Then $2f = 0$, but neither $2M_3(G) = \{0\}$ nor $f = 0$. Observe that $2f \in Z(M_3(G))$ and $2M_3(G) \neq \{0\}$, but $f \notin Z(M_3(G))$ since $fg \neq gf$, where $g \in M_3(G)$ is defined by $\{0, 1, 3\}g = \{0\}$ and $2g = 1$.

**Lemma 4.3.** Let $R$ be a ring and $\sigma$ and $\tau$ are endomorphisms of $R$. Then for all $x, y, z \in R$, we have the following relations:

(i) $[x, y \pm z]_{\sigma, \tau} = [x, y]_{\sigma, \tau} \pm [x, z]_{\sigma, \tau}$.

(ii) $[x \pm y, z]_{\sigma, \tau} = [x, z]_{\sigma, \tau} \pm [y, z]_{\sigma, \tau}$. 
We divide the proof into two cases:

(iii) \([xy, z]_{\sigma, \tau} = \sigma(x)[y, z]_{\sigma, \tau} + [x, z]_{\sigma, \tau} \tau(y)\).

(iv) \([x, yz]_{\sigma, \tau} = y[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau} z\).

**Proof.**

(i) For all \(x, y, z \in R\), we have \([x, y \pm z]_{\sigma, \tau} = \sigma(x)(y \pm z) = (y \pm z)\tau(x) = \sigma(x)y \pm \sigma(x)z - y\tau(x) = \sigma(x)y + \sigma(x)z - (\sigma(y) - \tau(x)) = [x, y]_{\sigma, \tau} \pm [x, z]_{\sigma, \tau} \).

(ii) For all \(x, y, z \in R\), we have \([x \pm y, z]_{\sigma, \tau} = \sigma(x \pm y)z - \tau(x \pm y) = \sigma(x)z \pm \sigma(y)z - \tau(x \pm y) = [x, y]_{\sigma, \tau} \pm [x, z]_{\sigma, \tau} \).

(iii) For all \(x, y, z \in R\), we have \([xy, z]_{\sigma, \tau} = \sigma(x)yz - y\tau(x) = \sigma(x)y + (y\sigma(x)y - xy) - y\tau(x) = \sigma(x)y\).

(iv) For all \(x, y, z \in R\), we have \([x, yz]_{\sigma, \tau} = \sigma(x)yz - y\tau(x) = \sigma(x)y + \tau(y) = \sigma(x)y\).

It is not true in general that \([x, yz]_{\sigma, \tau} = y[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau} z\) as the following example shows.

**Example 4.4.** Let \(R\) be a ring. Choose \(\sigma = 1_R\) and \(\tau = 0\). Then for all \(x, y, z \in R\), we have \([x, yz]_{\sigma, \tau} = \sigma(x)yz - y\tau(x) = xyz\) and \(y[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau} z = y(\sigma(x)z - \tau(x))\) as the following example shows.

**Lemma 4.4.** Let \(R\) be a ring with \((\sigma, \tau)\)-derivations \(d\) and \(D\). Then

(i) \([xy, z]_{\sigma, \tau} = \sigma(x)yz - y\tau(x) = \sigma(x)y + (y\sigma(x)y - xy) - y\tau(x) = \sigma(x)y\).

(ii) \([x, yz]_{\sigma, \tau} = \sigma(x)yz - y\tau(x) = \sigma(x)y + (y\sigma(x)y - xy) - y\tau(x) = \sigma(x)y\).

(iii) \([x, y]_{\sigma, \tau} = \sigma(x)y\).

Proof. (i) For all \(x, y, z \in R\), we have \([x, y]_{\sigma, \tau} = \sigma(x)y\).

(ii) For all \(x, y, z \in R\), we have \([x, y]_{\sigma, \tau} = \sigma(x)y\).

(iii) Clearly that \(d + D\) is additive mapping. Now,

\[
(d + D)(xy) = d(xy) + D(xy) = \sigma(x)(d(y) + D(y)) + \sigma(x)(d(x) + D(x))\tau(y) = \sigma(x)(d(y) + D(y)) + \sigma(x)(d(x) + D(x))\tau(y) = \sigma(x)(d + D)(y) + (d + D)(x)\tau(y).
\]

Therefore, \(d + D\) is also a \((\sigma, \tau)\)-derivation on \(R\).

**Theorem 4.4.** Let \(R\) be a prime ring with a non-zero \((\sigma, \tau)\)-derivation \(d\), \(\sigma\) and \(\tau\) are epimorphisms of \(R\). If \(\sigma(x)d(x) - d(x)\tau(x) \in Z(R)\), for all \(x \in R\), then \(R\) is a commutative ring or \(d(Z(R)) = \{0\}\).

Proof. Observe that \(\sigma(x)d(x) - d(x)\tau(x) \in Z(R)\) for all \(x \in R\). From \([x, y, d(x + y)]_{\sigma, \tau} \in Z(R)\) for all \(x, y \in R\) and using Lemma 4.3, we have \([x, d(x)]_{\sigma, \tau} + [x, d(y)]_{\sigma, \tau} + [y, d(x)]_{\sigma, \tau} + [y, d(y)]_{\sigma, \tau} \in Z(R)\). Using \([x, d(x)]_{\sigma, \tau} \in Z(R), [y, d(y)]_{\sigma, \tau} \in Z(R)\) and that \(Z(R)\) is a subring of \(R\), we get

\[
[x, d(x)]_{\sigma, \tau} + [y, d(y)]_{\sigma, \tau} \in Z(R) \text{ for all } x, y \in R.
\]

If \(Z(R) = \{0\}\), then \(\sigma(x)d(x) - d(x)\tau(x) = 0\) for all \(x \in R\) and hence \(R\) is a commutative ring by Theorem 4.3. So \(R = Z(R) = \{0\}\) and \(d = 0\), a contradiction. Therefore, \(Z(R) \neq \{0\}\). We divide the proof into two cases:

1. If \(Z(R) = \{0\}\), then \(\sigma(x)d(x) - d(x)\tau(x) = 0\) for all \(x \in R\) and hence \(R\) is a commutative ring by Theorem 4.3. So \(R = Z(R) = \{0\}\) and \(d = 0\), a contradiction. Therefore, \(Z(R) \neq \{0\}\). We divide the proof into two cases:

2. If \(Z(R) \neq \{0\}\), then \(\sigma(x)d(x) - d(x)\tau(x) \neq 0\) for all \(x \in R\) and hence \(R\) is not a commutative ring. Then there exists \(x \in R\) such that \(\sigma(x)d(x) - d(x)\tau(x) \neq 0\). Since \(\sigma(x)d(x) - d(x)\tau(x) \in Z(R)\), we have \(\sigma(x)d(x) - d(x)\tau(x) = 0\) for all \(x \in R\) and hence \(R\) is a commutative ring by Theorem 4.3. So \(R = Z(R) = \{0\}\) and \(d = 0\), a contradiction. Therefore, \(Z(R) \neq \{0\}\). We divide the proof into two cases:
(i) $R$ is not of characteristic 2. Then there exists $c \in Z(R) - \{0\}$ such that $[x, d(c)]_{\sigma, \tau} + [c, d(x)]_{\sigma, \tau} \in Z(R)$ for all $x \in R$ by (4.6). Write $d_1(x) = [x, d(c)]_{\sigma, \tau}$ and $d_2(x) = [c, d(x)]_{\sigma, \tau}$. Observe that $d_1$, $d_2$ and $d_1 + d_2$ are $(\sigma, \tau)$-derivations by Lemma 4.4. If $d_1 + d_2 \neq 0$, then $(d_1 + d_2)(R) \subseteq Z(R)$ implies that $R$ is a commutative ring by Lemma 2.12. If $d_1 + d_2 = 0$, then $[x, d(c)]_{\sigma, \tau} + [c, d(x)]_{\sigma, \tau} = 0$ for all $x \in R, c \in Z(R)$. It follows that $0 = [c, d(c)]_{\sigma, \tau} + [c, d(c)]_{\sigma, \tau} = 2[c, d(c)]_{\sigma, \tau}$, and hence $[c, d(c)]_{\sigma, \tau} = 0$ by Proposition 4.1(i). As $\sigma(x), \tau(x) \in Z(R)$, we obtain $[c, d(c)]_{\sigma, \tau} = d((\sigma(c) - \tau(c)) = 0$. Thus, for all $c \in Z(R)$, either $d(c) = 0$ or $\sigma(c) = \tau(c)$. If $\sigma(c) \neq \tau(c)$ and $d(c) = 0$ for some $c \in Z(R)$, then $d_1 = 0$ which implies $d_2 = 0$. Thus, $(\sigma(c) - \tau(c))d(x) = 0$ for all $x \in R$ and $d = 0$ by Lemma 2.4, a contradiction. So if $d(Z(R)) = \{0\}$, then $\sigma(a) = \tau(a)$ for all $a \in Z(R)$. If $d(c) \neq 0$ and $\sigma(c) = \tau(c)$ for some $c \in Z(R)$, then $d_2 = 0$. So $d_1(x) = \sigma(x)d(c) - d(c)\tau(x) = 0$ for all $x \in R$. If there exists $a \in Z(R)$ such that $\sigma(a) \neq \tau(a)$, then $d(c)(\sigma(a) - \tau(a)) = 0$ and $d(c) = 0$, a contradiction. So if $d(c) \neq 0$ for some $c \in Z(R)$, then $\sigma(a) = \tau(a)$ for all $a \in Z(R)$. Now, we have the following case: $d_1 = d_2 = 0, d(Z(R)) \neq \{0\}$ and $\sigma(a) = \tau(a)$ for all $a \in Z(R)$. Replacing $y$ in (4.6) by $zy$ and using Lemma 4.3(i), (iii) and (iv), we get for all $x, y, z \in R$

$$[x, d(zy)]_{\sigma, \tau} + [zy, d(x)]_{\sigma, \tau}$$

$$= [x, \sigma(z)d(y) + d(z)\tau(y)]_{\sigma, \tau} + [zy, d(x)]_{\sigma, \tau}$$

$$= [x, \sigma(z)d(y)]_{\sigma, \tau} + [x, d(z)d(y)]_{\sigma, \tau} + [zy, d(x)]_{\sigma, \tau}$$

$$= [\sigma(x), \tau(x) + x, d(z)d(y)]_{\sigma, \tau} + [zy, d(x)]_{\sigma, \tau}$$

$$= [\sigma(x)[x, d(y)]_{\sigma, \tau} + x, d(z)d(y)]_{\sigma, \tau} + [zy, d(x)]_{\sigma, \tau}$$

Putting $z = c \in Z(R)$, using $d_2 = 0$ and (4.6), we deduce that $[x, d(c)]_{\sigma, \sigma} \tau(y) + d(c) [x, \tau(y)]_{\sigma, \tau} \in Z(R)$ for all $x, y \in R$. Then $\sigma(x)d(c)\tau(y) - d(c)\sigma(x)\tau(y) + d(c)\sigma(x)\tau(y) - d(c)\tau(x)\tau(y) = \sigma(x)d(c)\tau(y) - d(c)\tau(x)\tau(y) \in Z(R)$ for all $x, y \in R$. Suppose $d(c) \neq 0$ for some $c \in Z(R)$ and assume that

$$\sigma(x)d(c)\tau(y) = d(c)\tau(x)\tau(y) \quad \text{for all} \quad x, y \in R. \quad (4.7)$$

Multiplying both sides by $\tau(z)$ from the right, we obtain

$$\sigma(x)d(c)\tau(y)\tau(z) = d(c)\tau(y)\tau(x)\tau(z) \quad \text{for all} \quad x, y, z \in R. \quad (4.8)$$

Replacing $y$ by $yz$ in (4.7), we have

$$\sigma(x)d(c)\tau(y)\tau(z) = d(c)\tau(y)\tau(z) \quad \text{for all} \quad x, y, z \in R. \quad (4.9)$$

From (4.8) and (4.9), we conclude $d(c)\tau(y)(\tau(z)\tau(x) - \tau(x)\tau(z)) = 0$ for all $x, y, z \in R$. Since $R$ is prime and $d(c) \neq 0$, we obtain that $R$ is commutative. Now, assume that $\tau(a) \neq 0$ for some $a \in R$ such that $\sigma(x)d(c)\tau(a) \neq d(c)\tau(a)\tau(x)$. It follows that $\delta(x) = \sigma(x)d(c)\tau(a) - d(c)\tau(a)\tau(x) \in Z(R)$ for all $x \in R$ is a non-zero inner $(\sigma, \tau)$-derivation and $R$ is a commutative ring by Lemma 2.12.

(ii) $R$ is of characteristic 2. Adding $d(x)\tau(y) + d(y)\tau(x) - d(x)\tau(y) - d(y)\tau(x) = 0$ to (4.6), we have $\sigma(x)d(y) + d(x)\tau(y) - 2d(x)d(y) + d(y)\tau(x) - 2d(y)d(x) \in Z(R)$ which means

$$d(xy + yx) \in Z(R) \quad \text{for all} \quad x, y \in R. \quad (4.10)$$
Now, suppose $d(Z(R)) \neq \{0\}$ and there exists $c \in Z(R) - \{0\}$ such that $d(c) \neq 0$. Replace $y$ by $yc$ in (4.10). Then $d((xy + yx) = d(c(xy + yx)) \in Z(R)$ for all $x, y \in R$. It follows that $\sigma(c)d(xy + yx) + d(\tau(xy + yx)) \in Z(R)$. Since $\sigma(c)d(xy + yx) \in Z(R)$, we have $d(c) \tau(xy + yx) \in Z(R)$ and then $d(c)(uv + vu) \in Z(R)$ for all $u, v \in R$ as $\tau$ is onto. Firstly, suppose that $d(c)(xy + yx) = 0$ for all $x, y \in R$. So $d(c)xy = d(c)yx$ for all $x, y \in R$. Replacing $x$ by $xz$ in the last equation, we get $d(c)xzy = d(c)yzx = d(c)xzy$ and hence $d(c)x(zy - yz) = 0$ for all $x, y, z \in R$. The primeness of $R$ and $d(c) \neq 0$ imply that $R$ is commutative. Now, suppose $d(c)(st + ts) \in Z(R) - \{0\}$ for some $s, t \in R$. Using $d(c)(xy + yx) \in Z(R)$ for all $x, y \in R$ and replacing $x$ by $[s, t]x$ and $y$ by $[s, t]y$, we have $d(c)([s, t]x[st]y + [s, t]y[s, t]x) \in Z(R)$. Thus, $d(c)[s, t](x)y + y[s, t]x) \in Z(R)$. Since $d(c)[s, t] \in Z(R) - \{0\}$, it is not a zero divisor by Lemma 2.4. It follows that $(x)[s, t]y + y[s, t]x \in Z(R)$ for all $x, y \in R$. Replacing $x$ by $c$ and putting $a = [s, t]$, we obtain $c(ay + ya) \in Z(R)$. Again, by Lemma 2.3, we have $ay + ya \in Z(R)$ for all $y \in R$. Define $a : R \to R$ by $d(y) = ay + ya$ for all $y \in R$. Then $a$ is an inner derivation on $R$ and $d(a) \subseteq Z(R)$. If $d(a)$ is non-zero, then $R$ is commutative by Lemma 2.12. If $d(a) = 0$, then $a = [s, t] \in Z(R) - \{0\}$. Using Lemma 2.3, we get $d(c) \in Z(R) - \{0\}$. Thus, $d(c)(xy + yx) \in Z(R)$ for all $x, y \in R$ implies $xy + yx \in Z(R)$ for all $x, y \in R$. If there exists $b \in R$ such that $by + yb \neq 0$ for some $y \in R$, then $d(b)$ is a non-zero derivation on $R$ and $d(b) \subseteq Z(R)$ which implies $R$ to be a commutative ring by Lemma 2.12 and hence $by + yb = 0$, a contradiction. Thus, $xy + yx = 0$ and then $R$ is a commutative ring.

**Corollary 4.2.** Let $R$ be a prime ring of characteristic 2 with a non-zero $(\sigma, \tau)$-derivation $d$ such that $\sigma$ and $\tau$ are automorphisms and commute with $d$. If $\sigma(x)d(x) + d(x)\tau(x) \in Z(R)$ for all $x \in R$, then $R$ is a commutative ring or $d^2 = 0$.

**Proof.** Using Theorem 4.4, $R$ is a commutative ring or $d(Z(R)) = \{0\}$. If $d(Z(R)) = \{0\}$, then $d^2(x) = d^2(y) \forall x, y \in R$ from (4.10) in the proof of Theorem 4.4. Using Lemma 2.7, $d^2$ is a $(\sigma^2, \tau^2)$-derivation on $R$. So by Lemma 2.13, $R$ is a commutative ring or $d^2 = 0$.

The following result generalizes Theorem 1 (in its part of derivations) of [14] and [8, Theorem 4].

**Theorem 4.5.** Let $R$ be a prime ring with a non-zero $(\sigma, \sigma)$-derivation $d$ such that $\sigma$ is an epimorphism and $\sigma(x)d(x) - d(x)\sigma(x) \in Z(R)$ for all $x \in U$, where $U$ is a non-zero right (left) ideal of $R$. Then $R$ is a commutative ring or $\sigma(U) = \{0\}$.

**Proof.** From $[x + y, d(x + y)]_{\sigma, \sigma} \in Z(R)$ for all $x, y \in U$, we have

$$[x, d(y)]_{\sigma, \sigma} + [y, d(x)]_{\sigma, \sigma} \in Z(R)$$

for all $x, y \in U$.

We divide the proof into two cases:

(i) $R$ is not of characteristic 2. Replacing $y$ in (4.11) by $x^2$ and using Lemma 4.3, we get

$$[x, d(x)]_{\sigma, \sigma} + [x, d(x)]_{\sigma, \sigma}$$

$$= [x, \sigma(x)d(x) + d(x)\sigma(x)]_{\sigma, \sigma} + [x, d(x)]_{\sigma, \sigma}$$

$$= [x, \sigma(x)d(x)]_{\sigma, \sigma} + [x, d(x)\sigma(x)]_{\sigma, \sigma} + \sigma(x)[x, d(x)]_{\sigma, \sigma} + [x, d(x)]_{\sigma, \sigma}\sigma(x)$$

$$= \sigma(x)[x, d(x)]_{\sigma, \sigma} + [x, d(x)]_{\sigma, \sigma}\sigma(x) + 2\sigma(x)[x, d(x)]_{\sigma, \sigma} = 4\sigma(x)[x, d(x)]_{\sigma, \sigma}$$

and hence $4\sigma(x)[x, d(x)]_{\sigma, \sigma} \in Z(R)$. It follows that $\sigma(x)[x, d(x)]_{\sigma, \sigma} \in Z(R)$ by Proposition 4.1(ii). If $[x, d(x)]_{\sigma, \sigma} \neq 0$, then $\sigma(x) \in Z(R)$ by using Lemma 2.3. But that means
Replacing \( y = 0 \), a contradiction. Thus, \([x, d(x)]_{\sigma, \sigma} = 0\) for all \( x \in U \). Therefore, \( R \) is a commutative ring or \( \sigma(U) = \{0\} \) by Theorem 4.3.

(ii) \( R \) is of characteristic 2. Using Lemma 4.3(ii), (iii) and \([x, d(x)]_{\sigma, \sigma} \in Z(R)\), we have for all \( x, y \in U \)

\[
[x + yx, d(x)]_{\sigma, \sigma} + [x^2, d(y)]_{\sigma, \sigma} = [x, d(x)]_{\sigma, \sigma} + [y, d(x)]_{\sigma, \sigma} + [x^2, d(y)]_{\sigma, \sigma}
\]

so

\[
\sigma(x)[y, d(x)]_{\sigma, \sigma} + [x, d(x)]_{\sigma, \sigma} \sigma(y) + \sigma(y)[x, d(x)]_{\sigma, \sigma} + [y, d(x)]_{\sigma, \sigma} \sigma(x)
\]

and consequently, we get

\[
[x + yx, d(x)]_{\sigma, \sigma} + [x^2, d(y)]_{\sigma, \sigma} = 0 \quad \text{for all} \quad x, y \in U.
\]

Using \( d(x) \sigma(y) + d(y) \sigma(x) - d(x) \sigma(y) - d(y) \sigma(x) = 0 \) for all \( x, y \in U \) and (4.11), we have

\[
\sigma(x)d(y) + d(x)\sigma(y) - 2d(x)\sigma(y) + \sigma(y)d(x) + d(y)\sigma(x) - 2d(y)\sigma(x) \in Z(R)
\]

and consequently, we get

\[
d(xy + yx) \in Z(R) \quad \text{for all} \quad x, y \in U.
\]

Replacing \( y \) by \( xy + yx \) in (4.12) and using (4.13), we have

\[
0 = [x(xy + yx) + (xy + yx)x, d(x)]_{\sigma, \sigma} + [x^2, d(xy + yx)]_{\sigma, \sigma}
\]

\[
= [xy + yx + yx + yxx, d(x)]_{\sigma, \sigma} + [xy + yxx, d(x)]_{\sigma, \sigma} - [xy + yxx, d(x)]_{\sigma, \sigma}.
\]

Replacing \( y \) by \( xy \) in the last equation and using Lemma 4.3(iii), we get

\[
0 = [xxyx + yxxx, d(x)]_{\sigma, \sigma} = [x(xxy + yxx), d(x)]_{\sigma, \sigma}
\]

\[
= \sigma(x)[xxy + yxx, d(x)]_{\sigma, \sigma} + [x, d(x)]_{\sigma, \sigma}\sigma(xxy + yxx)
\]

\[
= [x, d(x)]_{\sigma, \sigma}\sigma(xxy + yxx).
\]

If there exists \( a \in U \) such that \([a, d(a)]_{\sigma, \sigma} \neq 0\), then \( \sigma(U) \neq \{0\} \) and \( 0 = \sigma(a^2y + ya^2) = [a^2, d(y)]_{\sigma, \sigma} \) for all \( y \in U \). Thus, \( \sigma(a^2) \in Z(R) \) by Lemma 4.1 and Lemma 2.8. So Substituting \( x \) by \( a \) in (4.12), we get \([ay + ya, d(a)]_{\sigma, \sigma} = 0 \) for all \( y \in U \). Putting \( ay \) instead of \( y \), we obtain

\[
0 = [a(ay + ya), d(a)]_{\sigma, \sigma} = \sigma(a)[ay + ya, d(a)]_{\sigma, \sigma} + [a, d(a)]_{\sigma, \sigma}\sigma(ay + ya)
\]

\[
= [a, d(a)]_{\sigma, \sigma}\sigma(ay + ya).
\]

Since, \([a, d(a)]_{\sigma, \sigma} \) is not a zero divisor, we have \( \sigma(a)\sigma(y) - \sigma(y)\sigma(a) = 0 \) for all \( y \in U \). It follows that \( \sigma(a) \) centralizes \( \sigma(U) \neq \{0\} \). Lemma 4.1 and Lemma 2.8 implies \( \sigma(a) \in Z(R) \). But that implies \([a, d(a)]_{\sigma, \sigma} = 0\), a contradiction. Therefore, \([x, d(x)]_{\sigma, \sigma} = 0 \) for all \( x \in U \) and \( R \) is commutative by Theorem 4.3.

The proof when \( U \) is a non-zero left ideal of \( R \) is similar.

We finish this section by studying the commutativity of a prime ring \( R \) admitting a non-zero \((\sigma, \tau)\)-derivation \( d \) and satisfying the condition \( d(x^2) \in Z(R) \) for all \( x \in R \).
Proposition 4.2. Let \( R \) be a prime ring with a non-zero \( (\sigma, \tau) \)-derivation \( d \) such that \( \tau \) is an automorphism and \( d(x^2) = 0 \) for all \( x \in R \). Then \( R \) is a commutative ring of characteristic 2.

Proof. From \( d((x + y)^2) = 0 \), we have \( \sigma(x + y)d(x + y) = -d(x + y)\tau(x + y) \) for all \( x, y \in R \). So \( \sigma(x)d(x) + \sigma(x)d(y) + \sigma(y)d(x) + \sigma(y)d(y) = -d(x)\tau(x) - d(x)\tau(y) - d(y)\tau(x) - d(y)\tau(y) \). Using \( \sigma(x)d(x) = -d(x)\tau(x) \) and \( \sigma(y)d(y) = -d(y)\tau(y) \), we get \( \sigma(x)d(y) + \sigma(y)d(x) = -d(x)\tau(y) - d(y)\tau(x) \) and then
\[
d(xy) = -d(yx) \quad \text{for all } x, y \in R.
\]

Therefore, \( R \) is a commutative ring of characteristic 2 by Lemma 2.14.

Theorem 4.6. Let \( R \) be a prime ring with \( 2R \neq \{0\} \) and a non-zero \( (\sigma, \tau) \)-derivation \( d \) such that \( \sigma \) and \( \tau \) are automorphisms and \( d(x^2) \in Z(R) \) for all \( x \in R \). Then \( R \) is a commutative ring.

Proof. From \( d((x + y)^2) = \sigma(x + y)d(x + y) + d(x + y)\tau(x + y) \in Z(R) \) for all \( x, y \in R \), we have \( \sigma(x)d(x) + \sigma(x)d(y) + \sigma(y)d(x) + \sigma(y)d(y) + d(x)\tau(x) + d(x)\tau(y) + d(y)\tau(x) + d(y)\tau(y) \in Z(R) \). Using \( \sigma(x)d(x) + d(x)\tau(x) \in Z(R) \), \( \sigma(y)d(y) + d(y)\tau(y) \in Z(R) \), and that \( Z(R) \) is a subring of \( R \), we get \( \sigma(x)d(y) + d(x)\tau(y) + \sigma(y)d(x) + d(y)\tau(x) \in Z(R) \) for all \( x, y \in R \). It follows that \( d(xy) + d(yx) \in Z(R) \) for all \( x, y \in R \). If \( Z(R) = \{0\} \), then \( R \) is a commutative ring of characteristic 2 by Lemma 2.14 and then \( R = \{0\} \) and \( d = 0 \), a contradiction. So there exists \( c \in Z(R) - \{0\} \) such that \( d(cy) + d yc = 2d(cy) \in Z(R) \) for all \( y \in Z(R) \). Thus,
\[
d(cy) \in Z(R) \quad \text{for all } y \in R \quad \text{and for all } c \in Z(R) - \{0\}
\]
by Proposition 4.1(ii). It follows that \( d(ccc) = \sigma(c)d(cc) + d(c)\tau(cc) \in Z(R) \). Since \( \sigma(c)d(cc) \in Z(R) \), we have \( d(c)\tau(cc) \in Z(R) \) as \( Z(R) \) is a subring of \( R \). Using Lemma 2.3, Lemma 2.4 and \( \tau \) is an automorphism, we get that \( d(c) \in Z(R) \) for all \( c \in Z(R) - \{0\} \).

If \( d(Z(R)) \neq \{0\} \), then there exists \( c \in Z(R) - \{0\} \) such that \( d(c) \in Z(R) - \{0\} \). From (4.14), we have \( d(ccy) = \sigma(c)d(cy) + d(c)\tau(cy) \in Z(R) \). But \( \sigma(c)d(cy) \in Z(R) \), so \( d(c)\tau(cy) \in Z(R) \) for all \( y \in R \). Using that \( d(c), \tau(c) \in Z(R) - \{0\} \) and Lemma 2.3, we obtain \( \tau(R) \subseteq Z(R) \). Therefore, \( R \) is a commutative ring since \( \tau \) is onto.

If \( d(Z(R)) = \{0\} \), then for all \( c \in Z(R) - \{0\} \), (4.14) implies
\[
d(cy) = \sigma(c)d(y) + d(c)\tau(y) = \sigma(c)d(y) \in Z(R) \quad \text{for all } y \in R.
\]

Since \( \sigma \) is an automorphism, we have \( d(R) \subseteq Z(R) \) by Lemma 2.3. Therefore, \( R \) is a commutative ring by Lemma 2.12.

References


