9

2. *Geometric series* (Geometric progression) denoted by G.P.

An example of a G.P. is the series:

$$1, 3, 9, 27, 81, \ \dots \ \text{etc.}$$

Here you see that any term can be written from the previous term by multiplying it by a constant factor 3. This constant factor is called the *common ratio* and is found by selecting any term and dividing it by the previous one.

$$\text{e.g. } 27 \div 9 = 3; \quad 9 \div 3 = 3; \text{ etc.}$$

A G.P. therefore has the form:

$$a, \ ar, \ ar^2, \ ar^3, \ ar^4, \dots \ \text{etc.}$$

where a = first term, r = common ratio.

So in the geometric series 5, −10, 20, −40, etc. the common ratio, r, is ....

10

$$\boxed{r = \frac{20}{-10} = -2}$$

The general geometric series is therefore:

$$a, \ ar, \ ar^2, \ ar^3, \ ar^4, \ \dots \ \text{etc.} \quad \dots\dots\dots\dots \quad \text{(iv)}$$

and you will remember that

(i) the n^{th} term $= ar^{n-1}$ .... (v)

(ii) the sum of the first n terms is given by

$$S_n = \frac{a(1-r^n)}{1-r} \quad \dots\dots\dots\dots \quad \text{(vi)}$$

Make a note of these items in your record book.

So, now you can do this one:

For the series $8, 4, 2, 1, \frac{1}{2}, \ \dots$ etc., find the sum of the first 8 terms.

Then on to frame 11.

11

$$\boxed{S_8 = 15\tfrac{15}{16}}$$

Since, for the series 8, 4, 2, 1, ... etc.

$$a = 8; \quad r = \frac{2}{4} = \frac{1}{2}; \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\therefore S_8 = \frac{8(1 - [\frac{1}{2}]^8)}{1 - \frac{1}{2}}$$

$$= \frac{8(1 - \frac{1}{256})}{1 - \frac{1}{2}} = 16.\frac{255}{256} = \frac{255}{16} = \underline{15\tfrac{15}{16}}$$

Now here is another example.

If the 5th term of a G.P. is 162 and the 8th term is 4374, find the series.

We have 5th term = 162 $\therefore a.r^4 = 162$

8th term = 4374 $\therefore a.r^7 = 4374$

$$\frac{ar^7}{ar^4} = \frac{4374}{162} \quad \therefore r^3 = 27 \quad \therefore r = 3$$

$\therefore a =$ ....

12

$$\boxed{a = 2}$$

for $ar^4 = 162$; $ar^7 = 4374$ and $r = 3$

$$\therefore a.3^4 = 162 \quad \therefore a = \frac{162}{81} \quad \therefore a = 2$$

$\therefore$ The series is: $\underline{2, 6, 18, 54, \ldots \text{ etc.}}$

Of course, now that we know the values of a and r, we could calculate the value of any term or the sum of a given number of terms. For this same series, find

(i) the 10th term

(ii) the sum of the first 10 terms.

When you have finished, turn to frame 13.

13

$$a = 2; \quad r = 3$$

(i) 10th term $= ar^9 = 2.3^9 = 2(19683) = \boxed{39366}$

(ii) $S_{10} = \dfrac{a(1-r^{10})}{1-r} = \dfrac{2(1-3^{10})}{1-3}$

$$= \frac{2(1-59049)}{-2} = \boxed{59048}$$

Geometric mean

The geometric mean of two given numbers P and Q is a number A such that P, A and Q form a G.P.

$$\frac{A}{P} = r \text{ and } \frac{Q}{A} = r$$

$$\therefore \frac{A}{P} = \frac{Q}{A} \quad \therefore A^2 = PQ \quad A = \sqrt{(PQ)}$$

So the geometric mean of 2 numbers is the square root of their product.

Therefore, the geom. mean of 4 and 25 is ....

14

$$A = \sqrt{(4 \times 25)} = \sqrt{100} = \boxed{10}$$

□ □

To insert 3 G.M's between two given numbers, P and Q means to insert 3 numbers, A, B, C, such that P, A, B, C, Q form a G.P.

Example. Insert 4 geometric means between 5 and 1215.

Let the means be A, B, C, D. Then 5, A, B, C, D, 1215 form a G.P.

$$\text{i.e. } a = 5 \text{ and } ar^5 = 1215$$

$$\therefore r^5 = \frac{1215}{5} = 243 \quad \therefore r = 3$$

$$\therefore A = 5.3 = 15$$
$$B = 5.9 = 45$$
$$C = 5.27 = 135$$
$$D = 5.81 = 405$$

The required geometric means are:

15, 45, 135, 405

Now here is one for you to do: Insert two geometric means between 5 and 8·64.

Then on to frame 15.

15

Required geometric means are 6·0, 7·2

For, let the means be A and B.

Then 5, A, B, 8·64 form a G.P.

$$\therefore\ a = 5;\quad \therefore\ ar^3 = 8{\cdot}64;\quad \therefore\ r^3 = 1{\cdot}728;\quad \therefore\ r = 1{\cdot}2$$

$$\left.\begin{aligned} A &= 5.1{\cdot}2 = 6 \\ B &= 5.1{\cdot}44 = 7{\cdot}20 \end{aligned}\right\}\ \text{Required means are } 6{\cdot}0 \text{ and } 7{\cdot}2$$

Arithmetic and geometric series are, of course, special kinds of series. There are other special series that are worth knowing. These consist of the series of the powers of the natural numbers. So let us look at these in the next frame.

16

Series of powers of the natural numbers

1. The series $1 + 2 + 3 + 4 + 5 + \ldots + n$ etc. $= \sum_{1}^{n} r$.

   This series, you will see, is an example of an A.P., where $a = 1$ and $d = 1$. The sum of the first n terms is given by:

$$\sum_{1}^{n} r = 1 + 2 + 3 + 4 + 5 + \ldots + n$$

$$= \frac{n}{2}(2a + \overline{n-1}\,d) = \frac{n(n+1)}{2}$$

$$\sum_{1}^{n} r = \frac{n(n+1)}{2}$$

So, the sum of the first 100 natural numbers is ....

Then on to frame 17.

17

$$\boxed{\sum_1^{100} r = 5050}$$

for $$r = \frac{100(101)}{2} = 50(101) = 5050$$

□□

2. That was easy enough. Now let us look at this one: To establish the result for the sum of n terms of the series $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + n^2$., we make use of the identity

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

We write this as $$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

Replacing n by $n - 1$, we get

$$n^3 - (n-1)^3 = 3(n-1)^2 + 3(n-1) + 1$$

and again $$(n-1)^3 - (n-2)^3 = 3(n-2)^2 + 3(n-2) + 1$$

and $$(n-2)^3 - (n-3)^3 = 3(n-3)^2 + 3(n-3) + 1$$

Continuing like this, we should eventually arrive at:

$$3^3 - 2^3 = 3.2^2 + 3.2 + 1$$

$$2^3 - 1^3 = 3.1^2 + 3.1 + 1$$

If we now add all these results together, we find on the left-hand side that all the terms disappear except the first and the last.

$$(n+1)^3 - 1^3 = 3\left\{n^2 + (n-1)^2 + (n-2)^2 + \ldots + 2^2 + 1^2\right\}$$
$$+ 3\left\{n + (n-1) + (n-2) + \ldots + 2 + 1\right\} + n(1)$$

$$= 3.\sum_1^n r^2 + 3\sum_1^n r + n$$

$$\therefore\ n^3 + 3n^2 + 3n + \cancel{1} - \cancel{1} = 3\sum_1^n r^2 + 3\sum_1^n r + n = 3\sum_1^n r^2 + 3\,\frac{n(n+1)}{2} + n$$

$$\therefore\ n^3 + 3n^2 + 2n = 3\sum_1^n r^2 + \frac{3}{2}(n^2 + n)$$

$$\therefore\ 2n^3 + 6n^2 + 4n = 6\sum_1^n r^2 + 3n^2 + 3n$$

$$6\sum_1^n r^2 = 2n^3 + 3n^2 + n$$

$$\therefore\ \sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6}$$

So, the sum of the first 12 terms of the series $1^2 + 2^2 + 3^2 + \ldots$ is ....

18

$$\sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\therefore \sum_1^{12} r^2 = \frac{12(13)(25)}{6} = 26(25) = \boxed{650}$$

3. The sum of the cubes of the natural numbers is found in much the same way. This time, we use the identity

$$(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1$$

We rewrite it as before

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

If we now do the same trick as before and replace n by $(n-1)$ over and over again, and finally total up the results we get the result

$$\sum_1^n r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Note in passing that $\sum_1^n r^3 = \left\{\sum_1^n r\right\}^2$

19

Let us collect together these last three results. Here they are:

1. $\sum_1^n r = \frac{n(n+1)}{2}$ .... (vii)

2. $\sum_1^n r^2 = \frac{n(n+1)(2n+1)}{6}$ .... (viii)

3. $\sum_1^n r^3 = \left\{\frac{n(n+1)}{2}\right\}^2$ .... (ix)

These are handy results, so copy them into your record book.

Now turn on to frame 20 and we can see an example of the use of these results.

20 *Example:* Find the sum of the series $\sum_{n=1}^{5} n(3+2n)$

$$S_5 = \sum_1^5 n(3+2n) = \sum_1^5 (3n+2n^2)$$
$$= \sum_1^5 3n + \sum_1^5 2n^2$$
$$= 3\sum_1^5 n + 2\sum_1^5 n^2$$
$$= \frac{3.5.6}{2} + 2.\ \frac{5.6.11}{6}$$
$$= 45 + 110$$
$$= \underline{155}$$

It is just a question of using the established results. Here is one for you to do in the same manner.

Find the sum of the series $\sum_{n=1}^{4} (2n+n^3)$

21

$$S_4 = \sum_1^4 (2n+n^3)$$
$$= 2\sum_1^4 n + \sum_1^4 n^3$$
$$= \frac{2.4.5}{2} + \left\{\frac{4.5}{2}\right\}^2$$
$$= \quad 20 + \ 100 \quad = \quad \boxed{120}$$

Remember

Sum of first n natural numbers $= \dfrac{n(n+1)}{2}$

Sum of squares of first n natural numbers $= \dfrac{n(n+1)(2n+1)}{6}$

Sum of cubes of first n natural numbers $= \left\{\dfrac{n(n+1)}{2}\right\}^2$