d-Orthogonality of a generalization of both Laguerre and Hermite Polynomials

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Abstract

In this work, we give a unification and generalization of Laguerre and Hermite polynomials for which the orthogonal property is replaced by the d-orthogonality. We state some properties of these new polynomials.

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1 Introduction

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in the field of complex numbers $\mathbb{C}$. A polynomial sequence $\{P_n\}_{n\geq 0}$ in $\mathcal{P}$ is called a polynomial set if and only if $\deg P_n = n$ for all integer n.

The literature on unification and generalization of both Laguerre and Hermite polynomials contains several references but, as far as we know, only the following generalization $\{\Phi_n\}_{n\geq 0}$ introduced in [3] has a property related to an orthogonality notion.

$$\Phi_{n(m+1)+k}(x) = (-1)^n n! x^k L_n^{(\alpha+2k/(m+1))}(x^{m+1}); n \geq 0; k = 0, \dots, m$$

which is orthogonal on the star $\left\{r \exp\left(\frac{2ik\pi}{m+1}\right); r \geq 0; k \in \{0, \dots, m\}\right\}$ for the weight function $\omega(x) = |x|^{\alpha(m+1)+m} \exp\left((-x)^{m+1}\right)$.

(i) If $\alpha = \frac{-m}{m+1}$, we obtain the Endl polynomial set [17], if moreover $m = 1$, we obtain the Hermite polynomial set.

(ii) If $m = 1$, we obtain the generalized Hermite polynomial set of Szegö-Chihara.

(iii) If $m = 0$, we obtain the Laguerre polynomial set.

In this work, we provide a second unification and generalization of both Laguerre and Hermite polynomials having the property of d-orthogonality, $d-$orthogonal polynomials are multiple orthogonal polynomials near the diagonal along the step-line [1]. That may be defined as follows in [22] [24].

Let $\mathcal{P}'$ be the algebraic dual of $\mathcal{P}$. We denote by $\langle u, f\rangle$ the action of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. Let d be a positive integer and let $\{P_n\}_{n\geq 0}$ be a polynomial set in $\mathcal{P}$. $\{P_n\}_{n\geq 0}$ is called a d-orthogonal polynomial set with respect to the d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ if it satisfies the following conditions:

$$\begin{cases} \langle \Gamma_k, P_m P_n\rangle = 0 \quad , \quad m > nd + k \ , n \geq 0, \\ \\ \langle \Gamma_k, P_n P_{nd+k}\rangle \neq 0 \ , \ n \geq 0, \end{cases} \tag{1.1}$$

for each integer $k \in \{0, 1, \ldots, d-1\}$.

Maroni [22] showed that the conditions (1.1) are equivalent to the fact that the polynomials P_n, $n \geq 0$, satisfy a $(d+1)$-order recurrence relation of type:

$$xP_n(x) = \beta_{n+1}P_{n+1}(x) + \sum_{k=0}^{d} \alpha_{k,n-d+k}P_{n-d+k}(x),$$

where $\beta_{n+1}\alpha_{0,n-d} \neq 0$

For the particular case $d = 1$, we meet the well known notion of orthogonality [15].

Notice by the way, that there are in the literature some $d-$orthogonal polynomial sets that generalize the Laguerre polynomial set in ([6], [9], [10], [11], [25]) and others that generalize Hermite polynomial set in ([5], [6],[7], [16], [20], [26]).

To introduce our new polynomial set and to summarize the content of the paper, we recall the following notions.

A d-orthogonal polynomial sequence $\{P_n\}_{n\geq 0}$ is called to be Hahn-classical if the polynomial set derivative $\{P'_{n+1}\}_{n\geq 0}$ is also $d-$orthogonal.

Let m be a non negative integer. A polynomial set $\{P_n\}_n$ is called m-symmetric if $P_n(wx) = w^n P_n(x)$ for all n, where $w = e^{\frac{2i\pi}{m+1}}$ is $(m+1)$-root of 1. For the particular case $m = 1$, we meet the well known notion of symmetric polynomial set [15].

Let $\{P_n\}_{n\geq 0}$ be a m-symmetric polynomial set, then there exist $(m+1)$ polynomial sets $\{P_n^k\}_{n\geq 0}$; $k = 0, 1, \ldots, m$, such that

$$P_{(m+1)n+k}(x) \;=\; x^k\, P_n^k(x^{m+1}); \qquad n \geq 0.$$

The polynomial sets $\{P_n^k\}_{n\geq 0}$; $k \in \{0, 1, \ldots, m\}$; are called *the components of* the $m-$ symmetric polynomial set $\{P_n\}_{n\geq 0}$.
For any polynomial set $\{P_n\}_{n\geq 0}$, the sequence of linear functionals $\{\mathscr{L}_n\}_{n\geq 0}$ defined by: $\langle \mathscr{L}_n, P_m\rangle = \delta_{n,m}$ is called the dual sequence of the polynomial set $\{P_n\}_{n\geq 0}$.

The generalized hypergeometric functions ${}_pF_q(z)$ with p numerator and q denominator parameters are defined by (see [21], for instance)

$${}_pF_q\left(\begin{array}{c}(a_p)\\(b_q)\end{array};z\right) := \sum_{m=0}^{\infty} \frac{[a_p]_m}{[b_q]_m}\frac{z^m}{m!}, \tag{1.2}$$

where (a_p) designates the set $\{a_1, \ldots, a_p\}$, $[a_p]_m = \prod_{i=1}^{p}(a_i)_m$ and $(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}$ the Pochhammer symbol and z being a complex variable.

In this paper we consider the following polynomial set $\{Q_n\}_{n\geq 0}$ defined by its generating function

$$\begin{aligned} G(x,t) &= e^{t^{m+1}}{}_0F_r\left(\begin{array}{c}-\\ \alpha_1+1, \ldots, \alpha_r+1\end{array};-xt\right) \\ &= \sum_{n=0}^{\infty} Q_n(m, r, (\alpha_r); x)t^n, \end{aligned} \tag{1.3}$$

where $\alpha_1, \ldots, \alpha_r$ are real numbers such that $\alpha_j \notin \mathbb{N}$.

The Laguerre and Hermite Polynomials may be expressed by the polynomials Q_n, indeed,

$$H_n\left(\frac{x}{2}\right) = (-\mathrm{i})^n n! Q_n(1, 0, -; -\mathrm{i}x),$$

and

$$L_n^{(\alpha)}(x) = (\alpha+1)_n Q_n(0, 1, \alpha; x)$$

since (see [18] for instance, pages 242-243),

$$e^{t}{}_{0}F_{1}\left(\begin{matrix} - \\ \\ \alpha+1 \end{matrix} ;-xt\right)=\sum_{n=0}^{+\infty}\frac{L_n^{(\alpha)}(x)}{(\alpha+1)_n}t^n$$

and

$$e^{-t^2}e^{2xt}=e^{-t^2}{}_{0}F_{0}\left(\begin{matrix} - \\ \\ - \end{matrix} ;2xt\right)=\sum_{n=0}^{+\infty}\frac{H_n(x)}{n!}t^n.$$

The main results in this paper are the following:

Theorem 1.1
Let $\alpha_1,\ldots,\alpha_r$ be real numbers such that $-\alpha_j \notin \mathbb{N}$, then the polynomial set $\{Q_n\}_{n\geq 0}$ is $m-$symmetric Hahn-classical $d-$orthogonal with $d=(r+1)(m+1)-1$ and satisfies the following $(d+1)$-order recurrence relation

$$xQ_n(x)=\gamma_0(n,m,(\alpha_r))Q_{n+1}(x)+\sum_{j=1}^{r+1}\gamma_j(n,m,(\alpha_r))Q_{n+1-j(m+1)}(x), \quad (1.4)$$

where $\gamma_0(n,m,(\alpha_r))=-n\prod_{j=1}^{r}(n+\alpha_j)\neq 0$ and $\gamma_{r+1}(n,m,(\alpha_r))=(-1)^r(m+1)^{r+1}$.

Theorem 1.2
The polynomials $\{Q_n\}_{n\geq 0}$ satisfy the following $d-$orthogonality identity

$$\int_0^{+\infty}\psi_s(x)x^nQ_p(x)dx=0, \qquad p>nd+s,\ n\geq 0, \quad (1.5)$$

where the ψ_k functions are given explicitly as Meijer $G-$functions, $d=(r+1)(m+1)-1$ and $0\leq s\leq d-1$.

The outline of the paper is as follows. Section 2 deals with the proof of Theorems 1.1 and 1.2. We derive the dual sequence of the sequence $(Q_n)_{n\geq 0}$, from which, we deduce a d-dimensional vector of functionals for which the d-orthogonality holds. In Section 3, we give the components of the $m-$symmetric polynomial set $(Q_n)_{n\geq 0}$. In the last Section, we discuss some questions arising in the $d-$orthogonal polynomial theory, where the polynomial set $(Q_n)_{n\geq 0}$ may be useful. From all the obtained results in this paper, we derive for $(m,r)=(0,1)$ (respectively, $(m,r)=(1,0)$) well known results for Laguerre (respectively, Hermite) polynomials.

2 The $d-$orthogonality of the sequence $\{Q_n\}_{n\geq 0}$.

In this section we prove our main results, Theorem 1.1 and Theorem 1.2.
Proof of Theorem 1.1
The polynomial set $\{Q_n\}_{n\geq 0}$ is m-symmetric since $G(\omega x,t) = G(x,\omega t)$.
We have also $\theta_x G_0(x,t) = \theta_t G_0(x,t)$, where $\theta_x = x\frac{d}{dx}$ and $\theta_t = t\frac{d}{dt}$. Then the generating function $G(x,t)$ satisfies the following identity

$$xG(x,t) = \hat{X}_t G(x,t), \tag{2.6}$$

with $\hat{X}_t = -e^{t^{m+1}}\left(D_t\prod_{j=1}^{r}(\theta_t+\alpha_j)\right)e^{-t^{m+1}}$.
Iterating the identity $(\theta_t+b)\left(t^n e^{-t^{m+1}}\right) = \left((n+b)t^n-(m+1)t^{n+m+1}\right)e^{-t^{m+1}}$, we obtain

$$\hat{X}_t\left(t^n\right) = \sum_{k=0}^{r+1}\gamma_k(n,m,(\alpha_r))t^{n-1+k(m+1)}, \tag{2.7}$$

with

$$\gamma_0(n,m,(\alpha_r)) = -n\prod_{j=1}^{r}(n+\alpha_j)\neq 0 \quad \text{and} \quad \gamma_{r+1}(n,m,(\alpha_r)) = -\left(-(m+1)\right)^{r+1}.$$

Replace (2.7) in (2.6) and compare the coefficients of t^n, we deduce that the polynomial set $(Q_n)_n$ satisfies a $(d+1)$-order recurrence relation of type (1.4) with $d+1 = (m+1)(r+1)$. From which the d-orthogonality of the polynomial set $\{Q_n\}_{n\geq 0}$ follows.
The Hahn-classical property of $\{Q_n\}_{n\geq 0}$ results from the following lemma.

Lemma 2.1
The polynomial set $\{Q_n\}_{n\geq 0}$ satisfies the following forward identity:

$$DQ_n\left(m,r,(\alpha_r);.\right) = \frac{-1}{\prod_{j=1}^{r}(\alpha_j+1)}Q_{n-1}\left(m,r,(\alpha_r+1);.\right)$$

This lemma may be deduced by the following identity

$$\frac{\partial}{\partial x}G(x,t) = \frac{-t}{\prod_{j=1}^{r}(\alpha_j+1)}e^{t^{m+1}}{}_0F_r\left(\begin{matrix} - \\ \alpha_1+2,\ldots,\alpha_r+2\end{matrix};-xt\right).$$

$\square$

In order to prove Theorem 1.2, we need the following two lemmas related to the dual sequence of the polynomial set $\{Q_n\}_{n\geq 0}$ and its integral representation.

Lemma 2.2
The dual sequence $\{\mathscr{L}_n\}_n$ of the polynomial set $\{Q_n\}_{n\geq 0}$ is given by:

$$\langle \mathscr{L}_j, x^n\rangle = \sum_{k=0}^{[\frac{n}{m+1}]} \frac{n!(-1)^{n+k}\prod_{i=1}^{r}(1+\alpha_i)_n}{k!}\delta_{j,n-k(m+1)}.$$

Proof
From (1.3), we have

$$ {}_0F_r\left(\begin{matrix} - \\ 1+\alpha_1,\ldots,1+\alpha_r \end{matrix};-xt\right) = e^{-t^{m+1}}\sum_{n=0}^{\infty} Q_n(x)t^n,$$

then

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{(-xt)^n}{n!\prod_{j=1}^{r}(\alpha_j+1)_n} &= \sum_{n=0}^{+\infty}\frac{(-t^{(m+1)})^n}{n!}\sum_{n=0}^{\infty}Q_n(x)t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{m+1}]}\frac{(-1)^k Q_{n-k(m+1)}(x)}{k!}\right)t^n,\end{aligned}$$

which leads to the inversion formula:

$$x^n = \sum_{k=0}^{[\frac{n}{m+1}]}\frac{n!(-1)^{n+k}\prod_{j=1}^{r}(1+\alpha_j)_n}{k!}Q_{n-k(m+1)}(x).$$

It results that the dual sequence of the sequence $\{Q_n\}_{n\geq 0}$ is given by:

$$\langle \mathscr{L}_j, x^n\rangle = \sum_{k=0}^{[\frac{n}{m+1}]} \frac{n!(-1)^{n+k}\prod_{i=1}^{r}(1+\alpha_i)_n}{k!}\delta_{j,n-k(m+1)}.$$

$\square$

To derive an integral representation of the dual sequence, we need the Meijer G-functions. So we recall the definition and some properties of these functions.

The Meijer G-function

$$G_{p,q}^{m,n}\left(z\,\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)$$

is defined via the Mellin-Barnes integral (the reciprocal formula of Mellin transform) by:

$$G_{p,q}^{m,n}\left(z\,\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^p\Gamma(a_j-s)}\,z^s\,ds.$$

This definition holds under the following assumptions: $m\in\{0,\ldots,q\}$ and $n\in\{0,\ldots,p\}$ and $a_k-b_j\notin\mathbb{N}$. It should be noted that for $j>n$, the product $\prod\limits_{i=j}^{n}a_i$ are assumed to be equal to 1. It follows then

$$\int_0^\infty x^{s-1}\,G_{p,q}^{m,n}\left(\eta x\,\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)dx=\frac{\eta^{-s}\prod_{j=1}^m\Gamma(b_j+s)\prod_{j=1}^n\Gamma(1-a_j-s)}{\prod_{j=m+1}^q\Gamma(1-b_j-s)\prod_{j=n+1}^p\Gamma(a_j+s)}.$$

Properties 2.3

$$G_{p,q}^{m,n}\left(z\,\middle|\begin{matrix}a_1,\ldots,a_{p-1},b_1\\ b_1,\ldots,b_q\end{matrix}\right)=G_{p-1,q-1}^{m-1,n}\left(z\,\middle|\begin{matrix}a_1,\ldots,a_{p-1}\\ b_2,\ldots,b_q\end{matrix}\right),\quad m,p,q\geq 1.\tag{2.8}$$

$$z^sG_{p,q}^{m,n}\left(z\,\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=G_{p,q}^{m,n}\left(z\,\middle|\begin{matrix}a_1+s,\ldots,a_p+s\\ b_1+s,\ldots,b_q+s\end{matrix}\right)\tag{2.9}$$

$$\frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)}\,{}_pF_q\left(\begin{matrix}(a_1,\ldots,a_p);\\ (b_1,\ldots,b_q);\end{matrix}\;z\right)=G_{p,q+1}^{1,p}\left(-z\,\middle|\begin{matrix}1-a_1,\ldots,1-a_p\\ 0,1-b_1,\ldots,1-b_q\end{matrix}\right)\tag{2.10}$$

For more details we refer the reader to ([23], pages 45-47).

Lemma 2.4
Let $s=k+p'(m+1)$, $n=k+p(m+1)$, $d=(r+1)(m+1)-1$, with $k\in\{0,\ldots,m\}$ and $p''=p-p'$, then the dual sequence $\{\mathscr{L}_n\}_n$ associated to the sequence $\{Q_n\}_{n\geq 0}$ is given by

$$\langle\mathscr{L}_s,t^n\rangle=\int_0^\infty\psi_s(m,r,\alpha_r;t)t^ndt,\qquad s\leq n,\tag{2.11}$$

where

$$\psi_s(m,r,\alpha_r;t) = Ct^{-(s+1)}G^{d+1,0}_{1,d+1}\left(\frac{t^{m+1}}{\xi}\,\middle|\,\begin{matrix}1\\ \Delta(m{+}1,s{+}1),[\Delta(m{+}1,\alpha_r{+}s{+}1)]\end{matrix}\right),$$

$C = \dfrac{(m+1)s!(-1)^{n+p''}\prod_{i=1}^{r}(\alpha_i+1)_s}{\prod_{j=0}^{m}\left[\prod_{i=1}^{r}\Gamma(\frac{\alpha_i+1+s+j}{m+1})\right]\Gamma(\frac{s+1+j}{m+1})}$, $\xi = (m+1)^{d+1}$ *and* $\Delta(m+1,\lambda)$ *the array of* $m+1$ *parameters*

$$\frac{\lambda}{m+1},\ \frac{\lambda+1}{m+1},\ldots,\frac{\lambda+m}{m+1}.$$

Proof

From the inversion formula: $x^n = \displaystyle\sum_{j=0}^{p}\frac{(-1)^{n+j}n!\prod_{i=1}^{r}(1+\alpha_i)_n}{j!}Q_{n-j(m+1)}$,

we deduce that $\langle \mathscr{L}_s, x^n\rangle = (-1)^{n+p''}\dfrac{n!}{p''!}\displaystyle\prod_{i=1}^{r}\frac{\Gamma(\alpha_i+1+n)}{\Gamma(\alpha_i+1)}$.

On the other hand, we have

$$\begin{aligned}\frac{\Gamma(\alpha_i+1+n)}{\Gamma(\alpha_i+1)} &= \frac{\Gamma(p''(m+1)+\alpha_i+1+s)}{\Gamma(\alpha_i+1+s)}\frac{\Gamma(\alpha_i+1+s)}{\Gamma(\alpha_i+1)}\\ &= (m+1)^{p''(m+1)}(\alpha_i+1)_s\prod_{j=0}^{m}\frac{\Gamma(p''+\frac{\alpha_i+1+s+j}{m+1})}{\Gamma(\frac{\alpha_i+1+s+j}{m+1})}\end{aligned}$$

and $\dfrac{\Gamma(n+1)}{\Gamma(p''+1)} = (m+1)^{p''(m+1)}\dfrac{s!}{p''!}\displaystyle\prod_{j=0}^{m}\frac{\Gamma(p''+\frac{s+1+j}{m+1})}{\Gamma(\frac{s+1+j}{m+1})}$. So

$$\begin{aligned}\langle \mathscr{L}_s, x^n\rangle &= C_s\frac{(m{+}1)^{p''(d+1)}}{p''!}\prod_{j=0}^{m}\left[\prod_{i=1}^{r}\Gamma(p''{+}\frac{\alpha_i{+}1{+}s{+}j}{m{+}1})\right]\Gamma(p''{+}\frac{s{+}1{+}j}{m{+}1})\\ &= C_s\int_0^\infty G^{(d+1),0}_{1,(d+1)}\left(\frac{t}{\xi}\,\middle|\,\begin{matrix}1\\ \Delta(m{+}1,s{+}1),[\Delta(m{+}1,\alpha_r{+}s{+}1)]\end{matrix}\right)t^{p''-1}dt,\end{aligned}$$

where $C_s = \dfrac{(-1)^{n+p''}s!\prod_{i=1}^{r}(\alpha_i+1)_s}{\prod_{j=0}^{m}\left[\prod_{i=1}^{r}\Gamma(\frac{\alpha_i+1+s+j}{m+1})\right]\Gamma(\frac{s+1+j}{m+1})}$, and $\xi = (m+1)^{d+1}$. The formula (2.11) results by the change of variable $t = u^{m+1}$.

$\square$

Proof of Theorem 1.2

Maroni [22] proved that if a polynomial set is d-orthogonal, it is d-orthogonal with respect to a d-dimensional functional vector defined by the d first linear functionals of the corresponding dual sequence.

The sequence $\{Q_n\}_{n\geq 0}$ is $d-$orthogonal with respect to the $d-$dimensional functional $\Gamma = (\mathscr{L}_0, \ldots, \mathscr{L}_{d-1})^t$, then

$$\int_0^{+\infty} \psi_s(x)x^n Q_p(x)dx = 0, \qquad p > nd+s,\ n\geq 0 \text{ and } s\in\{0,\ldots,d-1\}.$$

$\square$

The particular case $(m,r) = (0,1)$ corresponds to the dual sequence of Laguerre polynomials. That is, for $k \leq n$,

$$\langle \mathscr{L}_k, x^n\rangle = \frac{(-1)^k}{\Gamma(\alpha+1)}\int_0^\infty G_{1,2}^{2,0}\left(t\,\middle|\,\begin{matrix}1\\(k+1),(\alpha+1)\end{matrix}\right)t^{n-k-1}dt.$$

For $k=0$, we obtain the well known weight function for which the Laguerre polynomials are orthogonal by the use of the identities (2.8), (2.9) and (2.10). In fact, we have

$$\begin{aligned}\frac{1}{t\Gamma(\alpha+1)}G_{1,2}^{2,0}\left(t\,\middle|\,\begin{matrix}1\\1,(\alpha+1)\end{matrix}\right) &\overset{(2.8)}{=} \frac{1}{t\Gamma(\alpha+1)}G_{0,1}^{1,0}\left(t\,\middle|\,\begin{matrix}-\\(\alpha+1)\end{matrix}\right)\\ &\overset{(2.9)}{=} \frac{t^\alpha}{\Gamma(\alpha+1)}G_{0,1}^{1,0}\left(t\,\middle|\,\begin{matrix}-\\0\end{matrix}\right)\\ &\overset{(2.10)}{=} \frac{t^\alpha}{\Gamma(\alpha+1)}{}_0F_0\left(\begin{matrix}-\\-\end{matrix};-t\right) = \frac{t^\alpha e^{-t}}{\Gamma(\alpha+1)}.\end{aligned}$$

For the particular case $(m,r) = (1,0)$ and $k\leq n$, we have:

$$\langle \mathscr{L}_{2k}, x^{2n}\rangle = \frac{2(-1)^{n-k}2k!}{\Gamma(\frac{2k+1}{2})\Gamma(k+1)}\int_0^\infty G_{1,2}^{2,0}\left(\frac{t^2}{4}\,\middle|\,\begin{matrix}1\\\Delta(2,2k+1)\end{matrix}\right)t^{2n-2k-1}dt$$

and

$$\langle \mathscr{L}_{2k+1}, x^{2n+1}\rangle = \frac{2(-1)^{n-k+1}(2k+1)!}{\Gamma(\frac{2k+3}{2})\Gamma(k+1)} \int_0^\infty G^{2,0}_{1,2}\left(\frac{t^2}{4}\,\middle|\, \begin{matrix} 1 \\ \Delta(2,2k+2) \end{matrix}\right) t^{2n-2k-1}dt.$$

For $k=0$, the first one is reduced to

$$\langle \mathscr{L}_0, x^{2n}\rangle = \frac{2(-1)^n}{\sqrt{\pi}} \int_0^\infty G^{2,0}_{1,2}\left(\frac{t^2}{4}\,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 1 \end{matrix}\right) t^{2n-1}dt.$$

On other hand, we have

$$\begin{aligned} \frac{2}{t}G^{2,0}_{1,2}\left(\frac{t^2}{4}\,\middle|\, \begin{matrix} 1 \\ 1, \frac{1}{2} \end{matrix}\right) &\overset{(2.8)}{=} \left(\frac{t^2}{4}\right)^{\frac{-1}{2}} G^{1,0}_{0,1}\left(\frac{t^2}{4}\,\middle|\, \begin{matrix} - \\ \frac{1}{2} \end{matrix}\right) \\ &\overset{(2.9)}{=} G^{1,0}_{0,1}\left(\frac{t^2}{4}\,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right) \overset{(2.10)}{=} {}_0F_0\left(\begin{matrix} - \\ - \end{matrix}; -\frac{t^2}{4}\right) = e^{-\frac{t^2}{4}}. \end{aligned}$$

That leads to the weight function for which the orthogonality of Hermite polynomials holds.

□

3 Components

Next, we derive the components of the polynomial set $\{Q_n\}_{n\geq 0}$.

Proposition 3.1
The components of the m-symmetric polynomial set $\{Q_n(m, r, (\alpha_r); .)\}_{n\geq 0}$ are given by the following identity

$$Q_{n(m+1)+k}(m, r, (\alpha_r); x) = C_1 x^k Q_n\left(0, m(r+2), [a_{m,m(r+1)}]; C_2 x^{m+1}\right), \quad (3.12)$$

where $k \in \{0, \ldots, m\}$, $[a_{m,m(r+1)}] = [\Delta^*(m+1, k+1), (\Delta(m+1, (\alpha_r + k))]$, $C_1 = \dfrac{1}{k! \prod_{i=1}^r (\alpha_i + 1)_k}$ *and* $C_2 = \dfrac{(-1)^m}{(m+1)^{(r+1)(m+1)}}$.

To derive this result, we recall firstly the identity 2, Problem 7, page 213 in [23]

$$
{}_pF_q\left(\begin{matrix} a_1, & \dots & , a_p & \\ & & & ; x \\ \alpha_1, & \dots & & , \alpha_q \end{matrix}\right) = \sum_{k=0}^{m} \frac{(a_1)_k \dots (a_p)_k}{(\alpha_1)_k \dots (\alpha_q)_k} \frac{x^k}{k!} \tag{3.13}
$$

$$
{}_{(m+1)p}F_{(m+1)q+m}\left(\begin{matrix} (\Delta(m+1,(a_p+k))) & & \\ & & ; z \\ \Delta^*(m+1,k+1), & (\Delta(m+1,(\alpha_q+k))) & \end{matrix}\right),
$$

where $z = \dfrac{x^{m+1}}{(m+1)^{(1-p+q)(m+1)}}$, $\Delta(m+1,\lambda)$ the array of $m+1$ parameters

$$
\frac{\lambda}{m+1}, \frac{\lambda+1}{m+1}, \dots, \frac{\lambda+m}{m+1},
$$

$\Delta^*(m+1,k+1)$ is the array of only m parameters

$$
\frac{k+1}{m+1}, \frac{k+2}{m+1}, \dots, \frac{k+m+1}{m+1}, \quad k \in \{0,\dots,m\}
$$

where we omit the term $\dfrac{m+1}{m+1}$, $\Delta(m+1,(\alpha_q+k))$ is the array of $(m+1)q$ parameters $\Delta(m+1,\alpha_j+k)$, $j \in \{1,\dots,q\}$.
Now, from the generating function (1.3) and the identity (3.13), we deduce the desired result.

Remark 3.2
The identity (3.12) may also be deduced from a general result obtained in [2].

Remark 3.3
The identity (3.12) means that all the components are of Ben Cheikh-Douak type [6]. So all the components are Hahn-classical $((m+1)(r+1)-1)$-orthogonal.

Remark 3.4
The identity (3.12) , for $(m,r) = (1,0)$, leads to (see, for instance, [18] p. 253)

$$
H_{2n}(x) = (-1)^n n! 2^{2n} L_n^{-\frac{1}{2}}\left(x^2\right)
$$

$$
H_{2n+1}(x) = (-1)^n n! 2^{2n+1} x L_n^{\frac{1}{2}}\left(x^2\right).
$$

4 Concluding Remarks

In this section, we survey some questions arising in the $d-$orthogonal polynomial theory and related to the polynomial set introduced in this paper.

Remark 4.1
In d-orthogonality theory, there are two interesting characterization problems which deal respectively with generalized hypergeometric polynomials and Brenke type polynomials. Such problems were solved completely for $d = 1$ [18] [14] and partially for $d \geq 1$ [10] [8] [4]. A further particular case related to these two problems consists to find all d-Orthogonal polynomial sets having generating functions of the type

$$G(x,t) = {}_uF_v \left(\begin{matrix} (c_u) \\ \\ (d_v) \end{matrix} ; t^{m+1} \right) {}_0F_q \left(\begin{matrix} - \\ \\ (b_q) \end{matrix} ; xt \right)$$

or, equivalently, a generalized hypergeometric representation of the type

$$x^n {}_{(m+1)(q+1)+u}F_v \left(\begin{matrix} \Delta(m+1,-n), \Delta(m+1,-n-(b_q)), (c_u) \\ \\ (d_v) \end{matrix} ; \left(\tfrac{1}{x}\right)^{m+1} \right)$$

In a forthcoming work [13], we show that the only solution for this problem arises for $(q,u,v) = (r,0,0)$ where $d+1 = (m+1)(r+1)$. That is to say the polynomial set studied in this paper is the only solution of this characterization problem. The particular case $m = 0$ corresponds to the polynomials studied by the second author and Douak [6] and appearing in a problem of singular values of products of random matrices [19].

Remark 4.2
Askey-scheme [18] is a convenient graphical way to see the hierarchy of hypergeometric orthogonal polynomials. The families on top have four parameters. When you go down one level then you lose one parameter. In the bottom level (Hermite polynomials), no parameters are left. The arrows denote limit transitions. In order to construct similar table to Askey-scheme in the context of d-orthogonality, Ben Cheikh, Lamiri and Ouni [8], introduced, via characterization theorems, many d-orthogonal polynomial sequences generalizing the polynomial sets arising in the Askey-scheme. They obtained, for instance for $d \geq 3$:
d-Wilson and d-Racah polynomial sets with $d+3$ parameters,

d-continuous dual Hahn, d-dual Hahn and d-Hahn polynomial sets with $d+2$ parameters,
d-Laguerre-Meixner, d-Jacobi, d-Meixner and d-Krawtchouk polynomial sets with $d+1$ parameters,
d-Laguerre, d-Bessel-Jacobi and d-Charlier-Meixner with d parameters,
d-Bessel-Jacobi and d-Charlier-Meixner with r parameters, $2 \leq r \leq d-1$,
Humbert, d-Charlier I, d-Charlier II, d-Charlier III and d-Bessel polynomial sets with one parameter,
and d-Hermite (Gould-Hopper) polynomial set with 0 parameter.
It's then of interest to look for generalized hypergeometric d-orthogonal polynomial sequences ($d \geq 3$) with q parameters, $1 \leq q \leq d-1$. Some polynomial sets satisfying such conditions may be derived from the results obtained in this paper. In fact, if $d+1$ is not a prime integer, say $d+1=(m+1)(r+1)$, $(m,r) \neq (0,0)$, the polynomial set $Q_n(m,r,(\alpha_r),.)_{n\geq 0}$ is d-orthogonal with r parameters and, by symmetry, the polynomial set $Q_n(r,m,(\alpha_m),.)_{n\geq 0}$ is d-orthogonal with m parameters. That may be done for all couple (m,r) satisfying the above conditions.

Remark 4.3
There are also two open questions related to two properties for Hahn-classical d-orthogonal polynomial sets:
Question 1: Are all the components of a m-symmetric Hahn-classical d-orthogonal polynomial set Hahn-classical?
Question 2: Do all Hahn-classical d-orthogonal polynomials satisfy $(d+1)$-order differential equations?
All the known Hahn-classical d-orthogonal polynomial sets as well as this new one introduced in this paper satisfy these two properties. That suggests us to conjecture that these two open questions have affirmative answers.

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