Directional Derivatives

Linear Algebra and Vector Analysis

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Unit Vector

A vector with magnitude of exactly 1 has *unit length*.

- This is vector does not measure units like meters
- Unit vectors have no units!
- The important feature of a unit vector is its direction

If A is a vector (ai + bj + ck), its unit vector is given by;

$$\hat{u} = \frac{A}{|A|}$$

$$where \quad |A| = \sqrt{a^2 + b^2 + c^2}$$

$$\hat{u} = \frac{(ai + bj + ck)}{\sqrt{a^2 + b^2 + c^2}}$$

Example: Find unit vector of A= (3i+2j+6k)

$$Solution:$$

$$\hat{u} = \frac{A}{|A|}$$

$$where \quad |A| = \sqrt{a^2 + b^2 + c^2}$$

$$= \sqrt{3^2 + 2^2 + 6^2} \Rightarrow \sqrt{49} \Rightarrow 7$$

$$\hat{u} = \frac{(3i + 2j + 6k)}{7} \Rightarrow \frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k$$

The Gradient Vector

The gradient of vector *f* is denoted by

$$\nabla f \quad it \ is \ called \ "del \ f"$$

If *f* is a function of two variables *x* and *y*, then the gradient of *f* is the vector function *f* given by the following formula:

$$\nabla f(x,y,z) = \left[f_x(x,y), f_y(x,y),, f_z(x,y) \right]$$
$$= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Example: Find gradient of the following vector at (0, 1, 2)

$$f(x, y, z) = sinx + e^{xy} + z^2$$

Solution :

We have following formula to find gradient of a vector

$$\nabla f(x, y, z) = \left[\, f_x(x, y, z),\, f_y(x, y, z),\, f_z(x, y, z) \right] \Rightarrow \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

After partially derivating with respect to x, y, z we get

$$\nabla f(x, y, z) = \left(\cos x + ye^{xy}\right) i + \left(xe^{xy}\right) j + (2z) k$$

After inserting given values of x, y and z we get

$$\nabla f(0,1,2) = (2i + 0j + 4k)$$

Directional Derivative

Now we introduce a type of derivative, called a directional derivative. It enables us to find:

- The rate of change of a function of two or more variables in any specific direction.
- Formula to find directional derivative is given below;

$$D_{\mathbf{u}} f(x,y,z) = \left(f_x(x,y,z) + f_y(x,y,z) + f_z(x,y,z) \right) . \hat{u}$$

Example Find the directional derivative of the function at the point (2, –1) in the direction of the vector (***v*** = *2* ***i*** + *5* ***j***).

$$f(x,y) = x^2y^3 - 4y$$

Solution: Step-1, First we need to compute the gradient vector at (2, –1):

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\nabla f(x,y) = 2xy^3\,\mathbf{i} + (3x^2y^2 - 4)$$

Now we put values of x and y

$$\nabla f(2,-1) = -4\,\mathbf{i} + 8\,\mathbf{j}$$

Step-2, After calculating gradient vector, we need to compute unit vector of vector *v*.

$$|\mathbf{v}| = \sqrt{(2)^2 + (5)^2}$$

$$|\mathbf{v}| = \sqrt{29}$$

$$u = \frac{\mathbf{v}}{|\mathbf{v}|}$$

$$u = \frac{2}{\sqrt{29}}\mathbf{i} + \frac{5}{\sqrt{29}}\mathbf{j}$$

Continue…

Step-3, Find the directional derivative

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot u$$

Putting values of x and y, we get

$$D_{\mathbf{u}} f(2, -1) = \nabla f(2, -1) \cdot \mathbf{u}$$

$$= (-4\mathbf{i} + 8\mathbf{j}) \cdot \left(\frac{2}{\sqrt{29}} \mathbf{i} + \frac{5}{\sqrt{29}} \mathbf{j} \right)$$

$$D_{\mathbf{u}} f(2, -1) = \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}$$

Example: Find directional derivative of the following function at (1, 3, 0) in the direction of (v = i + 2 j – k).

$$f(x, y, z) = xsin(yz)$$

Solution: Step-1, First we need to compute the gradient vector at (1, 2,0):

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$= (\sin yz) i + (xz \cos yz) j + (xy \cos yz) k$$

At (1, 3, 0), gradient is:

$$\boxed{\nabla f(1, 3, 0) = 3k}$$

Step-2, After calculating gradient vector, we need to compute unit vector of vector

$$\text{v} = i + 2j - k$$

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Step-3, Find the directional derivative

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot u$$

Putting values of x, y and z, we get :

$$D_{\mathbf{u}} f(1, 3, 0) = \nabla f(1, 3, 0) \cdot \mathbf{u}$$

$$= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k} \right)$$

$$D_{\mathbf{u}} f(1, 3, 0) = 3\left(-\frac{1}{\sqrt{6}} \right) = -\sqrt{\frac{3}{2}}$$