DEFINITION 8 If A is a square matrix, then the *trace of A*, denoted by tr(A), is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

▶ **EXAMPLE 11 Trace of a Matrix**

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \text{tr}(B) = -1 + 5 + 7 + 0 = 11 \quad ◀$$

In the exercises you will have some practice working with the transpose and trace operations.

Concept Review

- Matrix
- Entries
- Column vector (or column matrix)
- Row vector (or row matrix)
- Square matrix
- Main diagonal
- Equal matrices

- Matrix operations: sum, difference, scalar multiplication
- Linear combination of matrices
- Product of matrices (matrix multiplication)
- Partitioned matrices
- Submatrices

- Row-column method
- Column method
- Row method
- Coefficient matrix of a linear system
- Transpose
- Trace

Skills

- Determine the size of a given matrix.
- Identify the row vectors and column vectors of a given matrix.
- Perform the arithmetic operations of matrix addition, subtraction, scalar multiplication, and multiplication.
- Determine whether the product of two given matrices is defined.
- Compute matrix products using the row-column method, the column method, and the row method.

- Express the product of a matrix and a column vector as a linear combination of the columns of the matrix.
- Express a linear system as a matrix equation, and identify the coefficient matrix.
- Compute the transpose of a matrix.
- Compute the trace of a square matrix.

Exercise Set 1.3

1. Suppose that $A, B, C, D,$ and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) BA (b) $AC + D$ (c) $AE + B$

(d) $AB + B$ (e) $E(A + B)$ (f) $E(AC)$

(g) E^TA (h) $(A^T + E)D$

2. Suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(3×1)	(3×6)	(6×2)	(2×6)	(1×3)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

(a) EA (b) AB^T (c) $B^T(A + E^T)$

(d) $2A + C$ (e) $(C^T + D)B^T$ (f) $CD + B^T E^T$

(g) $(BD^T)C^T$ (h) $DC + EA$

3. Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

In each part, compute the given expression (where possible).

(a) $D + E$ (b) $D - E$ (c) $5A$

(d) $-9D$ (e) $2B - C$ (f) $7E - 3D$

(g) $2(D + 5E)$ (h) $B - B$ (i) $\operatorname{tr}(D)$

(j) $\operatorname{tr}(D - E)$ (k) $2\operatorname{tr}(4B)$ (l) $\operatorname{tr}(A)$

4. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) $2A^T + C$ (b) $D^T - E^T$ (c) $(D - E)^T$

(d) $B^T + 5C^T$ (e) $\frac{1}{2}C^T - \frac{1}{4}A$ (f) $B - B^T$

(g) $2E^T - 3D^T$ (h) $(2E^T - 3D^T)^T$ (i) $(CD)E$

(j) $C(BA)$ (k) $\operatorname{tr}(DE^T)$ (l) $\operatorname{tr}(BC)$

5. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) AB (b) BA (c) $(3E)D$

(d) $(AB)C$ (e) $A(BC)$ (f) CC^T

(g) $(DC)^T$ (h) $(C^TB)A^T$ (i) $\operatorname{tr}(DD^T)$

(j) $\operatorname{tr}(4E^T - D)$ (k) $\operatorname{tr}(A^TC^T + 2E^T)$ (l) $\operatorname{tr}((E^TC)B)$

6. Using the matrices in Exercise 3, in each part compute the given expression (where possible).

(a) $(2D^T - E)A$ (b) $(4B)C + 2B$

(c) $(-AC)^T + 5D^T$ (d) $(BA^T - 2C)^T$

(e) $B^T(CC^T - A^TA)$ (f) $D^T E^T - (ED)^T$

7. Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

Use the row method or column method (as appropriate) to find

(a) the first row of AB. (b) the third row of AB.

(c) the second column of AB. (d) the first column of BA.

(e) the third row of AA. (f) the third column of AA.

8. Referring to the matrices in Exercise 7, use the row method or column method (as appropriate) to find

(a) the first column of AB. (b) the third column of BB.

(c) the second row of BB. (d) the first column of AA.

(e) the third column of AB. (f) the first row of BA.

9. Referring to the matrices in Exercise 7 and Example 9,

(a) express each column vector of AA as a linear combination of the column vectors of A.

(b) express each column vector of BB as a linear combination of the column vectors of B.

10. Referring to the matrices in Exercise 7 and Example 9,

(a) express each column vector of AB as a linear combination of the column vectors of A.

(b) express each column vector of BA as a linear combination of the column vectors of B.

11. In each part, find matrices A, x, and b that express the given system of linear equations as a single matrix equation $Ax = b$, and write out this matrix equation.

(a) $5x + y + z = 2$
    $2x + 3z = 1$
    $x + 2y = 0$

(b) $x_1 + x_2 - x_3 - 7x_4 = 6$
    $ - x_2 + 4x_3 + x_4 = 1$
    $4x_1 + 2x_2 + x_3 + 8x_4 = 0$

12. In each part, find matrices A, x, and b that express the given system of linear equations as a single matrix equation $Ax = b$, and write out this matrix equation.

(a) $x_1 - 2x_2 + 3x_3 = -3$
    $2x_1 + x_2 = 0$
    $ - 3x_2 + 4x_3 = 1$
    $x_1 + x_3 = 5$

(b) $3x_1 + 3x_2 + 3x_3 = -3$
    $-x_1 - 5x_2 - 2x_3 = 3$
    $ - 4x_2 + x_3 = 0$

13. In each part, express the matrix equation as a system of linear equations.

(a) $\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$

14. In each part, express the matrix equation as a system of linear equations.

(a) $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 15–16, find all values of k, if any, that satisfy the equation.

15. $\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$

16. $\begin{bmatrix} 2 & 2 & k \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$

In Exercises 17–18, solve the matrix equation for $a, b, c,$ and d.

17. $\begin{bmatrix} 3 & a \\ 1 & a+b \end{bmatrix} = \begin{bmatrix} b & c-2d \\ c+2d & 0 \end{bmatrix}$

18. $\begin{bmatrix} a-b & b+a \\ 3d+c & 2d-c \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$

19. Let A be any $m \times n$ matrix and let 0 be the $m \times n$ matrix each of whose entries is zero. Show that if $kA = 0$, either $k = 0$ or $A = 0$.

20. (a) Show that if AB and BA are both defined, then AB and BA are square matrices.

(b) Show that if A is an $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

21. Prove: If A and B are $n \times n$ matrices, then
$$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$$

22. (a) Show that if B is any matrix with a column of zeros and A is any matrix for which AB is defined, then AB also has a column of zeros.

(b) Find a similar result involving a row of zeros.

23. In each part, find a 6×6 matrix $[a_{ij}]$ that satisfies the stated condition. Make your answers as general as possible by using letters rather than specific numbers for the nonzero entries.

(a) $a_{ij} = 0$ if $i \neq j$ (b) $a_{ij} = 0$ if $i > j$

(c) $a_{ij} = 0$ if $i < j$

(d) $a_{ij} = 0$ if $|i - j| > 1$

24. Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

(a) $a_{ij} = i - j$ (b) $a_{ij} = (-1)^1 ij$

(c) $a_{ij} = \begin{cases} 0 & |i - j| \geq i \\ -1 & |i - j| < i \end{cases}$

25. Consider the function $y = f(x)$ defined for 2×1 matrices x by $y = Ax$, where
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Plot $f(x)$ together with x in each case below. How would you describe the action of f?

(a) $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (b) $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

(c) $x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ (d) $x = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

26. Let I be the $n \times n$ matrix whose entry in row i and column j is
$$\begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Show that $AI = IA = A$ for every $n \times n$ matrix A.

27. How many 3×3 matrices A can you find such that
$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \\ 0 \end{bmatrix}$$
for all choices of $x, y,$ and z?

28. How many 3×3 matrices A can you find such that
$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix}$$
for all choices of $x, y,$ and z?

29. A matrix B is said to be a *square root* of a matrix A if $BB = A$.

(a) Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) How many different square roots can you find of $A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$?

(c) Do you think that every 2×2 matrix has at least one square root? Explain your reasoning.

30. Let 0 denote a 2×2 matrix, each of whose entries is zero.

(a) Is there a 2×2 matrix A such that $A \neq 0$ and $AA = 0$? Justify your answer.

(b) Is there a 2×2 matrix A such that $A \neq 0$ and $AA = A$? Justify your answer.

Exercise Set 1.4

1. Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}, \quad a = 4, \quad b = -7$$

Show that

(a) $A + (B + C) = (A + B) + C$

(b) $(AB)C = A(BC)$ (c) $(a + b)C = aC + bC$

(d) $a(B - C) = aB - aC$

2. Using the matrices and scalars in Exercise 1, verify that

(a) $a(BC) = (aB)C = B(aC)$

(b) $A(B - C) = AB - AC$ (c) $(B + C)A = BA + CA$

(d) $a(bC) = (ab)C$

3. Using the matrices and scalars in Exercise 1, verify that

(a) $(B^T)^T = B$ (b) $(A + C)^T = A^T + C^T$

(c) $(bA)^T = bA^T$ (d) $(CA)^T = A^T C^T$

In Exercises 4–7, use Theorem 1.4.5 to compute the inverses of the following matrices.

4. $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ **5.** $B = \begin{bmatrix} 6 & 3 \\ -5 & -2 \end{bmatrix}$

6. $C = \begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$ **7.** $D = \begin{bmatrix} 0 & -3 \\ 7 & 2 \end{bmatrix}$

8. Find the inverse of

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

9. Find the inverse of

$$\begin{bmatrix} \tfrac{1}{2}(e^x + e^{-x}) & \tfrac{1}{2}(e^x - e^{-x}) \\ \tfrac{1}{2}(e^x - e^{-x}) & \tfrac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

10. Use the matrix A in Exercise 4 to verify that $(A^T)^{-1} = (A^{-1})^T$.

11. Use the matrix B in Exercise 5 to verify that $(B^T)^{-1} = (B^{-1})^T$.

12. Use the matrices A and B in Exercises 4 and 5 to verify that $(AB)^{-1} = B^{-1}A^{-1}$.

13. Use the matrices A, B, and C in Exercises 4–6 to verify that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

In Exercises 14–17, use the given information to find A.

14. $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ **15.** $(5A)^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$

16. $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ **17.** $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

18. Let A be the matrix

$$\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$$

In each part, compute the given quantity.

(a) A^3 (b) A^{-3} (c) $A^2 - 2A + I$

(d) $p(A)$, where $p(x) = x - 2$

(e) $p(A)$, where $p(x) = 2x^2 - x + 1$

(f) $p(A)$, where $p(x) = x^3 - 2x + 4$

19. Repeat Exercise 18 for the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

20. Repeat Exercise 18 for the matrix

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -2 & 0 \\ 5 & 0 & 2 \end{bmatrix}$$

21. Repeat Exercise 18 for the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & -3 & -1 \end{bmatrix}$$

In Exercises 22–24, let $p_1(x) = x^2 - 9$, $p_2(x) = x + 3$, and $p_3(x) = x - 3$. Show that $p_1(A) = p_2(A)p_3(A)$ for the given matrix.

22. The matrix A in Exercise 18.

23. The matrix A in Exercise 21.

24. An arbitrary square matrix A.

25. Show that if $p(x) = x^2 - (a + d)x + (ad - bc)$ and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $p(A) = 0$.

26. Show that if $p(x) = x^3 - (a + b + c)x^2 + (ab + ae + be - cd)x - a(be - cd)$ and

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

then $p(A) = 0$.