## Course Homepage

http://www.math.cmu.edu/~pikhurko/301/

## Instructor

## Oleg Pikhurko (Office: Wean Hall 7105)

Office Hours: Mon 14:00-15:00, Wen 13:05-14:00 and Thurs 13:30-15:30 on all class days for the university.

Normally, I will not answer queries by email. If you have any questions, please do one of the following:

- talk to me after a class;
- come to my office hours;
- post your question at the discussion board via http://www.cmu.edu/blackboard/ (I will be regularly checking it and answering all new queries);
- call (412) 268-9782 to schedule an individual appointment.


## Syllabus

The objective of the course is to introduce students with a previous experience in discrete mathematics (courses such as $15-251$ or 21-228) to more advanced combinatorial results and techniques. Also, we will consider applications to computer science and information theory.

I expect to cover the following topics:

- Counting techniques
- binomial coefficients and identities
- the Stirling numbers of the 2d kind
- occupancy problem
- the Shapley-Shubik power index
- Gray sequences
- Inclusion-exclusion
- Pigeonhole principle
- Ramsey's theorem
- Erdős-Szekeres' theorem
- Generating functions
- ordinary generating functions
- linear reccurences and the method of characteristic roots
- generalized binomial theorem
- expanding functions into power series
- exponential generating functions
- Coding theory
- error correction/detection and linear codes
- Hamming codes
- Hadamard codes
- Graph theory
- matchings, coverings, and the König-Egervary Theorem
- the Hungarian algorithm
- Eulerian chains in directed graphs with applications to De Bruijn sequences
- Probabilistic method


## Textbook

There is no required textbook for the course. If you come to all classes and take comprehensive lecture notes, you will have all material needed to do homework and to prepare for the exams. If you wish to read more on the topics covered in the course, I would recommend the following:

RT: R.Roberts and B.Tesman, Applied Combinatorics, 2d Edition, CRC Press, 2009.
This book covers much material from the course (but not everything) as well as many other important topics, including numerous applications and exercises.

## Grade

The TOTAL score consists of

- Weekly homework: $18 \%$
- Unannounced quizes: 7\%
- Three best test scores (our of four), with one test being worth $25 \%$

The final grade will be determined as follows:

A: TOTAL $\geq 85 \%$
B: $72 \% \leq$ TOTAL $<85 \%$
C: $60 \% \leq$ TOTAL $<72 \%$
D: $50 \% \leq$ TOTAL $<60 \%$

## Homework Rules

There will be weekly homework assignments. The homework problems will have different weights. The point value of each problem will be indicated.

You have to bring written solutions, on the due date (usually Friday) to the class.
You can work together on homework problems with your classmates but you have to write your solutions independently and on your own. Direct copying of someone else's homework is prohibited; any two submissions violating this rule may get score 0 .

## Quizes

There will be unannounced quizes during classes ( $5-15$ quizes). Their purpose is to test attendance and knowledge. If you are absent during a quiz (for whatever reason), you get score 0 for this quiz. No make-up quizes will be administered.

## Exams

The course will have four in-class exams. They will be given during the regular lecture hour on

- September 24 (Friday)
- October 18 (Monday)
- November 12 (Friday)
- December 3 (Friday)

All exams will be closed-book and closed-notes. No calculators will be permitted. In case you must miss an exam due to illness, family emergency, University-sponsored trip, or religious observance, please notify me as soon as possible. I may require documentation in order to excuse the absence. Failure to do so will result in score 0 .

## Blackboard

The course Blackboard is available via http://www.cmu.edu/blackboard/
It gives an access to the gradebook (under "Tools") and the discussion forums (under "Discussion Board"). You can use Blackboard's forum to post your questions. Also, other messages related to the course are welcome on the forum. In particular, if you see a question by somebody else and you know the answer or just want to add a comment, you are very welcome to post a follow-up message.

I will be checking the forum regularly and posting replies to any queries that still need answers.

## Returning Graded Material

I will be distributing all graded homeworks and exams by passing it after a class (and keeping all unclaimed copies in my office). If you prefer that your work is not distributed this way, please let me know and I will individually accommodate such requests.

## Homework 1. Due September 3

The following applies to all homework assignments:
The point value of each exercise is stated in the brackets. Attempt all questions. Please bring your written solutions to the class on the due date. Cooperation is permitted but you have to write your solutions independently and on your own.

Problem $1[1+1]$ In this exercise you are allowed to write the final answer as a product without computing it.
i) How many different 20 -digit numbers can be made of digits $1,2,3,4,5$ so that no two consecutive digits give 6 if added together.
ii) We have digits $0,1,2, \ldots, 9$ and we are allowed to use each digit only once. How many 10 -digit decimal numbers can be made? (Note: a number cannot start with 0 .)

Problem 2 [3] What is the number of permutations of $[n]$ such that 2 does not occur between 1 and 3? (For example, 1324 is allowed but 3421 is not.)

Problem 3 [3] Given integers $a$ and $b$, what is the number of words made of $a$ letters $A$ and $b$ letters $B$ such that the first and the last letter is $A$, and there are at least two letters $A$ between any two letters $B$ ? (For example, if $a=5$ and $b=2$, then there are three such words: $A A B A A B A, A B A A A B A$, and $A B A A B A A$.)

Problem 4 [2] How many integral solutions of

$$
x_{1}+x_{2}+x_{3}+x_{4}+x_{5}=36
$$

satisfy $x_{1} \geq-5, x_{2} \geq 0, x_{3} \geq 10, x_{4} \geq 10$, and $x_{5} \geq 13 ?$

Problem $5[\mathbf{1}+\mathbf{1}+\mathbf{3}]$ Let $m$ and $n$ be positive integers.
i) Show that the number of functions $f:[n] \rightarrow[m]$ such that $f(1) \leq f(2) \leq \ldots \leq f(n)$ is $\binom{m+n-1}{n}$.
ii) What is the number of functions $f:[n] \rightarrow[m]$ such that $f(1)<f(2)<\ldots<f(n)$ ?
iii) Suppose $n=2 k$ is even. What is the number of functions $f:[n] \rightarrow[m]$ such that

$$
f(1) \leq f(2) \leq \ldots \leq f(k)<f(k+1)<f(k+2)<\ldots<f(2 k) ?
$$

[Hint: consider the differences $f(i)-f(i-1)$.]

## Solution to Quiz 1

Question. Suppose we have Players 1, 2, 3 and 4 and the winning coalitions are

$$
12,13,14,123,124,134,1234,234
$$

(that is, Player 1 with at least one more players or all of Players 2,3 and 4 ). What is the Shapley-Shubik Power Index of Player 1?
Answer. Player 1 is the pivot if and only if at least one but no more than 2 other players preceed her, that is, she is in the 2 d or 3 d position. Since each position is equally likely, the answer is $2 / 4=1 / 2$.

## Bicameral Government with the President

Suppose that we have $n_{1}$ and $n_{2}$ members in Cameras $C_{1}$ and $C_{2}$ respectively. A bill is passed if and only if it gets more than half of voices from each camera and the president's approval. Let $B$ denote the president. Thus the winning coalitions are

$$
W=\left\{A_{1} \cup A_{2} \cup\{B\}: A_{1} \subseteq C_{1}, A_{2} \subseteq C_{2},\left|A_{1}\right|>n_{1} / 2,\left|A_{2}\right|>n_{2} / 2\right\}
$$

Let $n=n_{1}+n_{2}+1$. Assume $C_{1} \cup C_{2} \cup\{B\}=[n]$. Take a random permutation of $[n]$. Then power index of each player (in particular, the President) equals the probability of being pivotal.

Let Event $E$ state that the President is in position $k$ such that $k-1 \leq n-k$, that is, the number of people before him is at most the number of people after him. This happens with probability at least $1 / 2$ : indeed, consider a permutation and its reverse - for at least one of them $E$ occurs. If $E$ occurs, then the President is not pivotal. Thus

$$
p(B)=\operatorname{Prob}(B \text { is pivot }) \leq 1-\operatorname{Prob}(E) \leq 1 / 2
$$

Let us show that the President's power is at about $1 / 2$ when $n_{1}$ and $n_{2}$ are large.
Theorem (Shapley \& Shubik (1954)) For every $\varepsilon>0$ there is $N$ such that for all $n_{1}, n_{2}>N$, we have $1 / 2-\varepsilon \leq p(B)$.

Sketch of Proof. Let $\varepsilon>0$ be given and $n_{1}, n_{2}$ be large. Let $l=\left\lfloor\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)\left(n_{1}+n_{2}+1\right)\right\rfloor$. The probability of the Event $L$ that President comes after position $l$ is $\frac{n-l}{n} \geq \frac{1}{2}-\frac{\varepsilon}{2}$. For $i=1,2$, let $M_{i}$ be event that more than $n_{i} / 2$ members of the $i$-th camera are in positions 1 to $l$ (that is, the first $l$ players create majority in the $i$-th camera). The President is pivotal whenever $L \cap M_{1} \cap M_{3}$ occurs, that is, all three events happen. (But observe that the converse is not generally true.) Hence, if we can show that $\operatorname{Prob}\left(\operatorname{not} M_{i}\right)<\varepsilon / 6$, then we are done:

$$
\begin{aligned}
p(B) & =\operatorname{Prob}(B \text { is pivot }) \geq \operatorname{Prob}\left(L \cap M_{1} \cap M_{2}\right) \\
& \geq \operatorname{Prob}(L)-\operatorname{Prob}\left(\text { not } M_{1}\right)-\operatorname{Prob}\left(\operatorname{not} M_{2}\right) \geq 1 / 2-\varepsilon
\end{aligned}
$$

Let us estimate e.g. the probability $\operatorname{Prob}\left(\right.$ not $\left.M_{1}\right)$. Let $k$ be the number of people from $C_{1}$ that come in positions 1 to $l$. Then

$$
\operatorname{Prob}\left(\operatorname{not} M_{1}\right)=\sum_{k<n_{1} / 2} \frac{\binom{n_{1}}{k}\binom{n_{2}+1}{l-k} l!(n-l)!}{n!}=\sum_{k<n_{1} / 2} \frac{\binom{n_{1}}{k}\binom{n_{2}+1}{l-k}}{\binom{n}{l}}
$$

The last formula can be seen directly: the $l$ members that come in the first $l$ positions are drawn uniformly from all $\binom{n}{l}$ possible $l$-subsets of $[n]$.
This sum a special case of a hypergeometric sum. Unfortunately, no closed form is known but there are good estimates (called Chernoff's bounds). Here is just a sketch of how to estimate this sum directly.
Let $s_{k}$ be the $k$-th summand in the sum for $\operatorname{Prob}\left(\right.$ not $\left.M_{1}\right)$. The ratio

$$
\frac{s_{k+1}}{s_{k}}=\frac{(l-k)\left(n_{1}-k\right)}{(k+1)\left(n_{2}+2-l+k\right)}
$$

is a monotone decreasing function of $k$ and monotone increasing function of $l$. Thus, for every $k<\left\lfloor n_{1} / 2\right\rfloor$, the ratio is at most than its value when $k=n_{1} / 2-1$ and $l=\left(\frac{1}{2}+\frac{\varepsilon}{2}\right)\left(n_{1}+n_{2}+1\right)$, that is,

$$
\begin{aligned}
\frac{s_{k+1}}{s_{k}} & \geq \frac{n_{2} / 2+\varepsilon\left(n_{1}+n_{2}+1\right) / 2+2}{n_{2} / 2-\varepsilon\left(n_{1}+n_{2}+1\right) / 2+1 / 2} \geq \frac{n_{2} / 2+\varepsilon n_{2} / 2+2}{n_{2} / 2-\varepsilon n_{2} / 2+1 / 2} \\
& \geq \frac{1+\varepsilon}{1-\varepsilon}=1+\frac{2 \varepsilon}{1-\varepsilon} \geq 1+2 \varepsilon
\end{aligned}
$$

(Note that $n_{1}-k>k+1$ for $k=n_{1} / 2-1$.) This means that the summands increase at least as fast as a geometric progression of ratio $1+2 \varepsilon$, so the main contribution is made by the last few summands. Formally,

$$
\frac{\operatorname{Prob}\left(\operatorname{not} M_{1}\right)}{s_{\left\lfloor n_{1} / 2\right\rfloor}} \leq 1+\frac{1}{1+2 \varepsilon}+\frac{1}{(1+2 \varepsilon)^{2}}+\ldots=\frac{1+2 \varepsilon}{2 \varepsilon}
$$

Thus we have to show that $s_{\left\lfloor n_{1} / 2\right\rfloor} \leq(\varepsilon / 6) \times 2 \varepsilon /(1+2 \varepsilon)$. This can be done (calculations omitted) using Stirling's formula

$$
n!\approx \sqrt{2 \pi n}\left(\frac{n}{e}\right)^{n}
$$

an amazing formula relating $\pi=3.1415 \ldots$ and $e=2.7182 \ldots$

## Homework 2. Due September 10

Problem 6 [3] What is the sum $\sum_{i=k}^{n}\binom{n}{i} S(i, k)$, where $S(n, k)$ denotes the Stirling number of the 2 d kind?

Problem 7 [2] Prove by induction on $n$ that

$$
s(n, 2)=(n-1)!\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n-1}\right), \quad n \geq 2
$$

where $s(n, k)$ denotes the Stirling number of the 1 st kind.

Problem 8 [2] Suppose we have a bicameral government with 3 people in the first camera and $m$ people in the second. A winning coalition is formed if and only if we have at least two people from the first camera and all people from the second camera. Compute the ShapleyShubik power index of every player.

Problem 9 [2] Determine all those $n$ for which there is a set of winning coalitions on $[n]$ such that the Shapley-Shubik power index of Player 1 is exactly $1 / n!$.

Problem $10[\mathbf{1 + 2}]$ Let $n, s, t$ be positive integers with $s<t$. Determine the following sums:
i) $\quad \sum_{i=s}^{t}\binom{n+i}{i}$,
ii) $\quad \sum_{i=0}^{n} i(i-1)\binom{n}{i}$.

## On Roots of Polynomials

Since we have used the following result in class twice, let me provide its proof.
Theorem. A non-zero polynomial $P(x)=a_{n} x^{n}+\ldots+a_{1} x+a_{0}$ can have at most $n$ roots (counting their multiplicity).

Proof. We use induction on $n$ with $n=1$ being trivialy true. Suppose that $n \geq 2$ and the claim has been proved for all smaller values of $n$. Let $a$ be a root of $P$. (If $P$ has no roots, there is nothing to do.) Divide $P$ by $x-a$ with a reminder:

$$
\begin{equation*}
P(x)=(x-a) Q(x)+R(x) \tag{1}
\end{equation*}
$$

Here $R(x)$ has degree less than 1 , that is, $R(x)=r_{0}$ is a constant. Plugging $x=a$ into (1), we get $r_{0}=0$. Thus $P(x)=(x-a) Q(x)$, where the degree of $Q$ is at most $n-1$. By the induction assumption, $Q$ has at most $n-1$ roots. Hence, $P$ has at most $n$ roots, as required.

Corollary If two polynomials $S(x)$ and $T(x)$ coincide on infinitely many $x$, then they are identically equal.

Proof. The polynomial $S(x)-T(x)$ has infinitely many roots. By the above theorem, it is identically zero.

## An Open Problem

The Middle Two Layers Conjecture: For every integer $k \geq 1$ one can order $\binom{[2 k+1]}{k} \cup$ $\binom{[2 k+1]}{k+1}$ so that every two adjacent sets differ in exactly one vertex.

Known results: Savage and Shields (1999) verified it for all $k \leq 15$. (Note that for $k=16$ we have $\binom{2 k+1}{k}>10^{9}$.) Johnson (2004) proved that there is a cycle containing almost all sets, that is, $m(k) /\binom{2 k+1}{k} \rightarrow 0$ as $k \rightarrow \infty$, where $m(k)$ is the number of the sets that are not listed in Johnson's construction.

A correct solution of the conjecture (a proof or disproof) will earn Grade $\mathbf{A}$ in 21-301 automatically!

## Exam 1

Exam 1 will take place during the class on September 24. Since we will start promptly at 9:30am, please arrive about 10 minutes earlier.

All exams are closed book and notes. Everything that we covered in the lectures (including the lecture on Sep'17) is examinabile, unless I explicitly marked anything as non-examinable.

As a rough guide, we covered so far Sections 2.3-2.16 and 2.19 of the Roberts-Tesman book as well as the Stirling numbers of the 1st kind.

Please note that the term combination may be defined differently by different authors. In order to avoid any confusion, I will avoid using this term in the exams.

## Solution to Quiz 2

Question. In how many different ways can we color all pairs on $[n]$ with 2 colors?
Answer. There are $\binom{n}{2}$ pairs and for each we have 2 choices. Thus the answer is $2\binom{n}{2}$.

## Homework 3. Due September 17

Problem 11 [3] Let $m$ and $n$ be positive integers. What is the number of functions $f$ : $[n] \rightarrow[m]$ such that $|f(i)-f(j)| \geq 4$ for every distinct $i, j \in[n]$ ?

Problem 12 [2] Find a closed form for $\sum_{i=1}^{n} i^{4}$.
Your answer has to be a linear combination of at most five binomial coefficients. (You do not have to simplify this expression any further.)

Problem 13 [1] Consider the lexicographic order on $\binom{[n]}{r}$. Thus the first element is $[r]$ and the $\binom{n}{r}$-th element is $\{n-r+1, \ldots, n\}$. Which element comes in position $\binom{n-1}{r-1}$ ?

Problem 14 [2] Let $0 \leq r \leq n$. We want to list all binary sequences of length $n$ with exactly $r$ ones so that every two consecutive sequences differ in exactly two bits.

Here is a recursive construction due to Frank Gray. If $r=n$, let $G(n, r)=1 \ldots 1$. If $r=0$, let $G(n, r)=0 \ldots 0$. If $1 \leq r \leq n-1$. then define

$$
G(n, r)=0 G(n-1, r), 1 \operatorname{Reverse}(G(n-1, r-1))
$$

that is, we take the list $G(n-1, r)$ and prepend 0 to each binary sequence there, then we take the list $G(n-1, r-1)$ where we reverse the order of sequences (but not the order inside sequences!) and prepend 1 to each sequence. Here are some examples:

$$
\begin{aligned}
G(2,1) & =01,10 \\
G(3,1) & =001,010,100 \\
G(3,2) & =011,110,101 \\
G(4,1) & =0001,0010,0100,1000
\end{aligned}
$$

$$
\begin{aligned}
G(4,2) & =0011,0110,0101,1100,1010,1001 \\
G(4,3) & =0111,1101,1110,1011
\end{aligned}
$$

Prove by induction that $G(n, r)$ indeed lists every $n$-element sequence with $r$ ones exactly once and every two neighboring sequences differ in at most two places. If your inductive statement is stronger, state carefully all extra property(-ies) that you prove.

Problem $15[\mathbf{1 + 2}]$ Consider the Gray ordering $G(n, 3)$ of $0 / 1$-sequences of length $n$ with exactly three 1's.
i) Let $n=7$. Which sequence comes after 1000110 ?
ii) Let $n \geq 4$. Which sequence comes after $S=110^{n-3} 1$ (that is, $S$ consists of 11 , followed by $n-3$ zeros, followed by 1 )? Which sequence comes before $S$ ?

## Homework 4. Due September 24

Problem 16 [1] Prove that for any integers $m, n, k \geq 0$ we have

$$
\binom{m+n}{k}=\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i}
$$

 $\ldots<x_{n}$ in $\left[n^{2}+1\right]$ such that all pairs $\left\{x_{i-1}, x_{i}\right\}$ with $i \in[n]$ have the same color.

Problem 18 [3] What is the smallest $l$ such that for any $l$ numbers chosen from

$$
\{1,3,5, \ldots, 999,1001\}
$$

there are two such that one divides the other?

Problem 19 [3] Prove that in every coloring of $\binom{[17]}{2}$ with 3 colors, there is a monochromatic triangle (that is, three vertices such that the three edges spanned by them have the same color).

Problem 20 [3] The Erdős-Szekeres bound tells us that $R(4,3) \leq\binom{ 4+3-2}{4-1}=10$. Prove that in fact $R(4,3) \leq 9$.

## Extra Reference Material on Generating Functions

[GKP] R. L. Graham, D. E. Knuth, and O. Patashnik, Concrete mathematics: a foundation for computer science, Addison-Wesley Publ. Comp., 1989.
[N] I. Niven, Formal power series, Amer. Math. Monthly 76 (1969), 871-889.
[W] H. S. Wilf, Generatingfunctionology, Academic Press, 1990 (Avialable from the author's website).

The following useful tables are copied from [GKP].

Table 334 Generating function manipulations.

$$
\begin{aligned}
\alpha F(z)+\beta G(z) & =\sum_{n}\left(\alpha f_{n}+\beta g_{n}\right) z^{n} \\
z^{m} G(z) & =\sum_{n} g_{n-m} z^{n}, \quad \text { integer } m \geqslant 0 \\
\frac{G(z)-g_{0}-g_{1} z-\cdots-g_{m-1} z^{m-1}}{z^{m}} & =\sum_{n \geqslant 0} g_{n+m} z^{n}, \quad \text { integer } m \geqslant 0 \\
G(c z) & =\sum_{n} c^{n} g_{n} z^{n} \\
G^{\prime}(z) & =\sum_{n}(n+1) g_{n+1} z^{n} \\
z G^{\prime}(z) & =\sum_{n} n g_{n} z^{n} \\
\int_{0}^{z} G(t) d t & =\sum_{n \geqslant 1} \frac{1}{n} g_{n-1} z^{n} \\
F(z) G(z) & =\sum_{n}\left(\sum_{k} f_{k} g_{n-k}\right) z^{n} \\
\frac{1}{1-z} G(z) & =\sum_{r}\left(\sum_{n \leqslant r i} g_{k}\right) z^{n}
\end{aligned}
$$

| sequence | generating function | closed form |
| :---: | :---: | :---: |
| $\langle 1,0,0,0,0,0, \ldots\rangle$ | $\sum_{n \geqslant 0}[n=0] z^{n}$ | 1 |
| $\langle 0, \ldots, 0,1,0,0, \ldots\rangle$ | $\sum_{n \geqslant 0}[n=m] z^{n}$ | $z^{\text {m }}$ |
| $\langle 1,1,1,1,1,1, \ldots\rangle$ | $\sum_{n \geqslant 0} z^{n}$ | $\frac{1}{1-z}$ |
| $\langle 1,-1,1,-1,1,-1, \ldots\rangle$ | $\sum_{n \geqslant 0}(-1)^{n} z^{n}$ | $\frac{1}{1+z}$ |
| $\langle 1,0,1,0,1,0, \ldots\rangle$ | $\sum_{n \geqslant 0}[2 \backslash n] z^{n}$ | $\frac{1}{1-z^{2}}$ |
| $\langle 1,0, \ldots, 0,1,0, \ldots, 0,1,0, \ldots\rangle$ | $\sum_{n \geqslant 0}[m \backslash n] z^{n}$ | $\frac{1}{1-z^{m}}$ |
| $\langle 1,2,3,4,5,6, \ldots\rangle$ | $\sum_{n \geqslant 0}(n+1) z^{n}$ | $\frac{1}{(1-z)^{2}}$ |
| $\langle 1,2,4,8,16,32, \ldots\rangle$ | $\sum_{n \geqslant 0} 2^{n} z^{n}$ | $\frac{1}{1-2 z}$ |
| $\langle 1,4,6,4,1,0,0, \ldots\rangle$ | $\sum_{n \geqslant 0}\binom{4}{n} z^{n}$ | $(1+z)^{4}$ |
| $\left\langle 1, \mathrm{c},\binom{\right.$ c }{2},$\binom{$ c }{3},$\left.\ldots\right\rangle$ | $\sum_{n \geqslant 0}\binom{$ c }{$n} z^{n}$ | $(1+z)^{\text {c }}$ |
| $\left\langle 1, \mathrm{c},\binom{c+1}{2},\binom{c+2}{3}, \ldots\right\rangle$ | $\sum_{n \geqslant 0}\binom{c+n-1}{n} z^{n}$ | $\frac{1}{(1-z)^{\text {c }}}$ |
| $\left\langle 1, \mathrm{c}, \mathrm{c}^{2}, \mathrm{c}^{3}, \ldots\right\rangle$ | $\sum_{n \geqslant 0} c^{n} z^{n}$ | $\frac{1}{1-c z}$ |
| $\left\langle 1,\binom{m+1}{m},\binom{m+2}{m},\binom{m+3}{m}, \ldots\right\rangle$ | $\sum_{n \geqslant 0}\binom{m+n}{m} z^{n}$ | $\frac{1}{(1-z)^{m+1}}$ |
| $\left\langle 0,1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\rangle$ | $\sum_{n \geqslant 1} \frac{1}{n} z^{n}$ | $\ln \frac{1}{1-z}$ |
| $\left\langle 0,1,-\frac{1}{2}, \frac{1}{3},-\frac{1}{4}, \ldots\right\rangle$ | $\sum_{n \geqslant 1} \frac{(-1)^{n+1}}{n} z^{n}$ | $\ln (1+z)$ |
| $\left\langle 1,1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots\right\rangle$ | $\sum_{n \geqslant 0} \frac{1}{n!} z^{n}$ | $e^{z}$ |

## Homework 5. Due October 1

Problem 21 [3] Prove that in any group of at least 2 people, there are always 2 people who have the same number of acquaintances in the group. (We assume that if $A$ is an acquaintance of $B$, then $B$ is an acquaintance of $A$.)

Problem 22 [1] Let $n \geq 1$ be an integer divisible by 30 . How many numbers between 1 and $n$ are divisible by exactly two of the primes 2,3 , and 5 ?

Problem 23 [2] Count the number of symmetric $n \times n$ matrices such that each entry is 0 or 1 and no row consists entirely of zeros. (Your answer is allowed to be a single sum but not nested sums or products.)

Problem 24 [3] The classroom has $m$ chairs. On the first lecture there were $n$ students, where $n<m$. (Thus $m-n$ chairs remained empty.) When the same $n$ students came to the second lecture they decided to sit so that no person is in the same chair as on the first lecture.

In how many possible ways can a sitting plan for the second lecture be arranged? Write your answer as a sum, using the Principle of Inclusion-Exclusion.

Problem 25 [2] Suppose we have six sets $A_{1}, \ldots, A_{6} \subseteq A$. Express the number of elements that belong to at least 3 of the sets $A_{i}$ in terms of the parameters

$$
m_{j}=\sum_{1 \leq i_{1}<\ldots<i_{j} \leq 6}\left|A_{i_{1}} \cap \ldots \cap A_{i_{j}}\right|
$$

## Exam 2

Exam 2 will take place during the class on October 18. Since we will start promptly at 9:30am, please arrive in time.

All exams are closed book and notes. Everything that we covered in the lectures (from the beginning of the course to the lecture on Oct'8 inclusive) is examinabile, unless I explicitly marked anything as non-examinable.

As a rough guide, in addition to the material covered by Exam 1, we did Sections 5.1-5.4, 6.1-6.4, and 7.1-7.2 of the Roberts-Tesman book.

## Solution to Quiz 3

Question. Expand $\frac{1}{2 z+3}$ into powers of $z$.
Answer. We have

$$
\frac{1}{2 z+3}=\frac{1}{3} \cdot \frac{1}{1-(-2 / 3) z}=\sum_{n \geq 0} \frac{1}{3}\left(-\frac{2}{3}\right)^{n} z^{n}
$$

## Solution to Quiz 4

Question. Let $f_{i}=5 i+\sqrt{3}$. Find $F(z)=\sum_{i=0}^{\infty} f_{i} z^{i}$ in closed form.
Answer. The sequence $\left(f_{i}\right)$ is obtained from the sequence $(1,1, \ldots)$ by multiplying the entry indexed by $i$ by the polynomial $P(i)=5 i+\sqrt{3}$. Hence the answer is

$$
P\left(z \frac{d}{d z}\right)\left(\sum_{i \geq 0} z^{i}\right)=5 z\left(\frac{1}{1-z}\right)^{\prime}+\sqrt{3} \frac{1}{1-z}=\frac{5 z}{(1-z)^{2}}+\frac{\sqrt{3}}{1-z}
$$

## The Method of Characteristic Roots

Suppose that we have the following linear homogeneous recurrence:

$$
a_{n}=c_{1} a_{n-1}+c_{2} a_{n-2}+\ldots+c_{k} a_{n-k}, \quad n \geq k
$$

One way to solve it, is to consider the characteristic equation $P(\alpha)=0$, where

$$
P(\alpha)=\alpha^{k}-c_{1} \alpha^{k-1}-c_{2} \alpha^{k-2}-\ldots-c_{k}
$$

is the characteristic polynomial. Each (real or complex) root $\alpha_{i}$ of $P$ of multiplicity $m_{i}$ gives $m_{i}$ special solutions for $a_{n}$ :

$$
\alpha_{i}^{n}, \quad n \alpha_{i}^{n}, \quad \ldots, \quad n^{m_{i}-1} \alpha_{i}^{n} .
$$

Since $P$ as a degree- $k$ polynomial has $k$ roots (counting their multiplicity), we obtain $k$ special solutions. Any linear combination of them is a also a solution. To determine the correct $k$ coefficients, we have to use some other information about our sequence such as, for example, the initial values of $a_{0}, \ldots, a_{k-1}$.

## The Average Number of Comparisons in Quicksort

Let us analyze the following sorting procedure, called Quicksort. We pick some element $x_{1}$, compare it with the remaining elements $x_{2}, \ldots, x_{n}$, splitting them into two groups $L$ and $R$ : those that preceed $x$ and those that succeed $x$. Now repeat the procedure recursively inside each group. One advantage of this algorithm that it is possible to organize it so that we do not neeed any extra memory in addition to what is need for storing $x_{1}, \ldots, x_{n}$, see

```
http://en.wikipedia.org/wiki/Quicksort
```

for more details. Also, this procedure can be represented by an SOR tree, where $x_{1}$ is the root, $L$ is its left brach and $R$ its right (defined recursively).

We are interested in the total number of comparisons (that is, how many pairs of elements are compared until all $n$ elements are sorted).
Quicksort does not perform well in the worst case. For example, if $x_{1} \prec x_{2} \prec \ldots \prec x_{n}$, then the final tree is a path (going to the right all the time) and we used $\binom{n}{2}$ comparisons.
Let us analyze the average performance. Let $c_{n}$ be the average number of steps. Namely, we take a random ordering of the elements (with all $n$ ! orderings being equally likely), and let $c_{n}$ be the expected (mean) number of comparisons. For example, if $n=3$, and $x_{1}$ is between $x_{2}$ and $x_{3}$ in the $\prec$-order, then we use 2 comparisons; otherwise 3 comparisons are performed. Hence

$$
c_{3}=\frac{2+2+3+3+3+3}{3!}=\frac{16}{6}=\frac{8}{3} .
$$

Also, $c_{0}=0, c_{1}=0$ and $c_{2}=1$.
Let us write a recurrence for $c_{n}$. Let $x_{1}$ be the $j$-th element with respect to the unknown order $\prec$. Note that $x_{1}$ becomes the root of the whole tree so it is compared to every other element while the two branches at $x$ have sizes $j-1$ and $n-j$. If we look at the $j-1$ elements that made into the left branch, their ordering is uniform (that is, all $(j-1)$ ! permutations are
equally likely). So we use $c_{j-1}$ comparison on average to sort them. A similar claim applies to the right branch. So, for every $n \geq 1$ we have

$$
c_{n}=n-1+\frac{1}{n} \sum_{j=1}^{n}\left(c_{j-1}+c_{n-j}\right)=n-1+\frac{2}{n} \sum_{i=0}^{n} c_{i}
$$

Thus

$$
\begin{equation*}
n c_{n}=n(n-1)+2 \sum_{i=0}^{n-1} c_{i} \tag{2}
\end{equation*}
$$

Let $C(z)=\sum_{i \geq 0} c_{i} z^{i}$.
Note that $\sum_{n \geq 0} n(n-1) z^{n}=2 z^{2} /(1-z)^{3}$. Let us multiply $(2)$ by $z^{n}$ and add for all $n \geq 0$ :

$$
z C^{\prime}(z)=\frac{2 z^{2}}{(1-z)^{3}}+\frac{2 z C(z)}{1-z}
$$

We divide by $z$, obtaining

$$
\begin{equation*}
C^{\prime}(z)=\frac{2 z}{(1-z)^{3}}+\frac{2 C(z)}{1-z} \tag{3}
\end{equation*}
$$

Alternatively, we could have multiplied (2) by $z^{n-1}$ and added for all $n \geq 1$, getting the same formula (3) a bit faster.

Mathematica gives the following solution to the differential equation (3) given the initial value $C(0)=c_{0}=0:$

$$
C(z)=\frac{-2 z-2 \ln (1-z)}{(1-z)^{2}}
$$

whre $\ln$ denotes the natural logarithm. Note that $(\ln (1-z))^{\prime}=-1 /(1-z)=-\sum_{i \geq 0} z^{i}$. By integrating, we obtain that $\ln (1-z)=a_{0}-\sum_{i \geq 1} \frac{z^{i}}{i}$ for some $a_{0}$. From $\ln (1-0)=0$, we conclude that $a_{0}=0$. Thus $-2 z-2 \ln (1-z)=2 \sum_{n \geq 2} z^{n} / n$ and

$$
C(z)=2\left(\sum_{n \geq 2} \frac{z^{n}}{n}\right)\left(\sum_{m \geq 0}(m+1) z^{m}\right)=2 \sum_{n \geq 2} \sum_{i=2}^{n} \frac{n-i+1}{i} z^{n}
$$

Thus

$$
c_{n}=2 \sum_{i=2}^{n} \frac{n-i+1}{i}=-2(n-1)+2(n+1) \sum_{i=2}^{n} \frac{1}{i}
$$

Let us run a check: for $n=3$, this gives $2\left(\frac{2}{2}+\frac{1}{3}\right)=8 / 3$, as expected.
Here is a simple proof. Suppose that the unknown order is $y_{1} \prec \ldots \prec y_{n}$. For $i<j$, let the random variable $Y_{i j}$ be 1 if we compare $y_{i}$ and $y_{j}$ and 0 otherwise. Note that the probability of $Y_{i j}=1$ is exactly $\frac{2}{j-i+1}$. Indeed, consider for the first time when one of $y_{i}, y_{i+1}, \ldots, y_{j}$
becomes the pivot; then we compare $y_{i}$ and $y_{j}$ if and only if the pivot is $y_{i}$ or $y_{j}$ Then, by the linearily of expectation,

$$
c_{n}=\sum_{1 \leq i<j \leq n} \frac{2}{j-i+1}=\sum_{k=2}^{n}(n-k+1) \frac{2}{k} .
$$

where we group all pairs $i<j$ by $k=j-i+1$.
Let us try to determine the asymptotics of $c_{n}$. Note that

$$
\sum_{i=2}^{n} \frac{1}{i}=\int_{1}^{n} \frac{d x}{\lceil x\rceil}
$$

and, for every $x \geq 1$, we have $1 /(x+1) \leq 1 /\lceil x\rceil \leq 1 / x$. Hence,

$$
\ln (n+1)-\ln 2=\int_{1}^{n} \frac{d x}{x+1} \leq \sum_{i \geq 2} \frac{1}{i} \leq \int_{1}^{n} \frac{d x}{x}=\ln n
$$

Thus

$$
c_{n} \approx 2 n \ln n
$$

## Homework 6. Due October 8

Problem 26 [3] Let $n \geq k \geq 1$ be given. At a party, $n$ couples are to be seated on a round table, with seats numbered $1, \ldots, 2 n$ cyclically. (Thus the seats numbered $2 n$ and 1 are adjacent.) What is the number of ways to seat these $2 n$ people so that exactly $k$ couples sit together?

Problem 27 [1] Find the expansion of $1 /(2+z)^{3}$ into powers of $z$.
Problem 28 [1] Find the Taylor expansion around 0 of

$$
F(z)=\frac{1-8 z}{2 z^{2}+z-1}
$$

Problem $29[\mathbf{1}+\mathbf{1}]$ Let the sequence $f_{0}, f_{1}, \ldots$ satisfy $f_{0}=2, f_{1}=0$, and

$$
f_{n}=f_{n-1}-\frac{1}{4} f_{n-2}, \quad \text { for } n \geq 2
$$

i) Determine the generating function $F(z)=\sum_{n \geq 0} f_{n} z^{n}$.
ii) Find a formula for $f_{n}$. (You can use i) or proceed any other way.)

Problem 30 [2] Let the sequence $f_{i}$ be defined by $f_{0}=1, f_{1}=0, f_{2}=-5, f_{3}=0$, and $f_{n}=2 f_{n-1}-f_{n-2}$, for $n \geq 4$. (Note that this recurrence is not satisfied for $n=2$ or 3 .) Find the generating function $F(z)$ of this sequence.

## Homework 7. Due October 18

Problem $31[\mathbf{1}+\mathbf{1}]$ i) Suppose that some sequences $\left(g_{i}\right)_{i \geq 0}$ and $\left(f_{i}\right)_{i \geq 0}$ satisfy $g_{i}=i \cdot 2^{i} \cdot f_{i+2}$ for all $i \geq 0$. Find the generating function $G(z)=\sum_{i \geq 0} g_{i} z^{i}$, in term $F(z)=\sum_{i \geq 0} f_{i} z^{i}$.
ii) The same question if we know that $g_{i}=f_{0}+\sum_{j=0}^{i+1} f_{j} f_{i+1-j}$ for all $i \geq 0$.

Problem 32 [2] Compute

$$
\sum_{i \geq 1} \frac{i^{2}}{(i-1)!}
$$

Problem 33 [2] Show that if $2 n$ points are marked on a circle and if $a_{n}$ is the number of ways of joining them in pairs by $n$ non-intersecting chords, then $a_{n}=\frac{1}{n+1}\binom{2 n}{n}$ is the $n$-th Catalan number.

Problem $34[\mathbf{1}+\mathbf{1}+\mathbf{1}]$ Find the following generating functions in closed form. (You do not have to expand into partial fractions.)
i) $F(z)=\sum_{n \geq 0}\left(n^{2}-4\right) z^{n}$;
ii) $F(z)=\sum_{n \geq 0}(n+2) 5^{n / 2} z^{n}$;
iii) $S(z)=\sum_{n \geq 0} s_{n} z^{n}$, where $s_{n}=\sum_{i=1}^{n} i(i-1)$.

Problem $35[\mathbf{1}+\mathbf{3}]$ For $n \geq 3$, let $a_{n}$ be the number of binary strings $\varepsilon_{1} \varepsilon_{2} \ldots \varepsilon_{n}$ of length $n$ such that the substring 010 appears for the first time at the end of the sequence (that is, we have $\left.\min \left\{i \in[n-2]: \varepsilon_{i} \varepsilon_{i+1} \varepsilon_{i+2}=010\right\}=n-2\right)$.

For example, $a_{3}=1, a_{4}=2$, and $a_{5}=3$.
i) Show that for $n \geq 6$ we have

$$
2^{n-3}=a_{n}+a_{n-2}+a_{n-3} 2^{0}+a_{n-4} 2^{1}+\ldots+a_{3} 2^{n-6}
$$

(Note that $a_{n-1}$ is missing on the right-hand side!)
ii) Find the generating function $A(z)=\sum_{i \geq 3} a_{i} z^{i}$.

Problem $36[\mathbf{1}+\mathbf{2}]$ A simple, partly ordered, rooted tree (SPR tree) is a simple rooted tree in which the labels $L$ and $R$ are placed on the children of a vertex only if there are two children. Here are examples of a few different SPR trees:


Let $u_{n}$ count the number of SPR trees on $n$ vertices. For example, $u_{1}=1, u_{2}=1, u_{3}=2$, and $u_{4}=4$. (Also, it is convenient to assume that $u_{0}=0$.)
i) Show that for every $n \geq 1$ we have

$$
u_{n+1}=u_{n}+\sum_{j=1}^{n-1} u_{j} u_{n-j}
$$

ii) Find the generating function $U(z)=\sum_{n \geq 1} u_{n} z^{n}$.

## Remarks on the Exponential Formula

For more details on the Exponential Formula, see Section 5 in Herbert Wilf's book "Generatingfunctionology" that should be available for free from the author's webpage.

Here is one important special case. Let us use the same definitions as in class.
Let the sequence $h_{n}=\sum_{k=0}^{n} h_{n, k}$ count the total number of $H$-objects on $[n]$ (with $h_{0}=1$ ). Then its exponential generating function $H(z)=\sum_{n \geq 0} h_{n} z^{n} / n$ ! is just $H(z, 1)$, where

$$
H(z, y)=\sum_{n, k \geq 0} h_{n, k} \frac{z^{n}}{n!} y^{k}
$$

(Here, taking $y=1$ is legitimate: the coefficient $\sum_{k=0}^{n} h_{n, k} y^{k}$ at $z^{n} / n!$ in $H(z, y)$ is a finite sum.) Thus the Exponential Formula implies that

$$
H(z)=\mathrm{e}^{D(z)}
$$

## Homework 8. Due October 29

Problem $37[1+3]$ Let $a_{n}$ be the number of involutions of $[n]$ (that is permutations of [ $n$ ] such that every cycle has 1 or 2 elements). Let us agree that $a_{0}=1$ and let $A(z)=$ $\sum_{n=0}^{\infty} a_{n} z^{n} / n!$.
i) Show that $A(z)=\mathrm{e}^{z+z^{2} / 2}$.
ii) Find a simple closed form for

$$
\sum_{i=0}^{n}(-1)^{i}\binom{n}{i} a_{i} .
$$

Problem 38 [4] A graph is $k$-regular if every vertex has degree exactly $k$. Let $a_{n}$ be the number of 2 -regular graphs on vertex set $[n]$. (Thus the vertices are labeled.) Find a closed formula for the EGF $A(z)=\sum_{n \geq 0} a_{n} z^{n} / n$ !. (We do not allow loops nor multiple edges; in particular, $a_{1}=a_{2}=0$.)

Problem $39\left[\mathbf{2 + 2 + 1 + 1 ]}\right.$ Let $F(z)=\sum_{i \geq 0} f_{i} z^{i}$ and $G(z)=\sum_{i \geq 0} g_{i} z^{i}$ (resp. $E(z)=$ $\sum_{i \geq 0} f_{i} z^{i} / i$ ! and $H(z)=\sum_{i \geq 0} g_{i} z^{i} / i$ !) be the ordinary (resp. exponential) generating functions of the sequences $\left(f_{0}, f_{1}, \ldots\right)$ and ( $\left.g_{0}, g_{1}, \ldots\right)$. In the following, you have to write your answer in terms of $F(z)$ and $E(z)$.
i) Find $G$ and $H$ when $g_{i}=(i+2) f_{i+1}, i \geq 0$;
ii) Find $G$ and $H$ when $g_{i}=f_{i}+f_{i-1}$ for $i \geq 1$, and $g_{0}=f_{0}$;
iii) Find $G$ when $g_{i}=\sum_{j=0}^{i+1} f_{j}, i \geq 0$;
iv) Find $H$ when $g_{i}=\sum_{j=0}^{i+1}\binom{i+1}{j} f_{j}, i \geq 0$.

Problem 40 [3] Find $f_{k}$, the number of $k$-words from an alphabet $\{a, b, c, d, e\}$ in which $b$ occurs an odd number of times. Find the EGF of $f_{k}$ in a closed form.

## Solutions to Quiz 5

Question. Find a linear code $C \subseteq \mathbb{F}_{2}^{5}$ with 4 codewords and Hamming distance 3.
Answer. Up to a permutation of the 5 coordinates, there is only one such code

$$
C=\{00000,10101,01011,11110\} .
$$

(Note that, since $C$ is linear, we must have $00000 \in C$.) A generator matrix for $C$ is

$$
M=\left[\begin{array}{lllll}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1
\end{array}\right]
$$

## Exam 3

Exam 3 will take place during the class on November 12. Since we will start promptly at 9:30am, please arrive in time.

All exams are closed book and notes. Everything that we covered in the lectures (from the beginning of the course to the lecture on Nov'5 inclusive) is examinabile, unless I explicitly marked anything as non-examinable.

As a rough guide, in addition to the material covered by Exam 2, we did Sections 5.5, 10.110.5, and 12.1-12.7 of the Roberts-Tesman book.

Please note that there will be no office hours on November 11 (Thursday) (I will be in Princeton, giving a seminar talk on that day). I will have longer office hours on November 8 (Monday): 13:00-15:00.

## Homework 9. Due November 5

Problem $41[\mathbf{2 + 1}]$ Recall that an $(n, d)$-code $C$ is perfect if the balls of radius $\lfloor(d-1) / 2\rfloor$ around codewords perfectly partition $\mathbb{F}_{2}^{n}=\{0,1\}^{n}$.
i) Prove that no $(n, d)$-code with $d$ even and $n>d>0$ can be perfect.
ii) Prove that no perfect $(13,5)$-code exists.

Problem 42 [2] Let $C \subseteq \mathbb{F}_{2}^{n}$ be a linear code with a generator $k \times n$-matrix $M=[I G]$ and the corresponding parity check matrix $H=\left[G^{T}, I_{n-k}\right]$. Prove that if a vector $\mathbf{x} \in \mathbb{F}_{2}^{n}$ satisfies $H \mathbf{x}^{T}=\mathbf{0}$ then $x \in C$.

Problem $43[\mathbf{1 + 4 + 1}]$ Let $n=2^{p}-1$ and $\mathcal{H}_{p}$ be the Hamming code. (Thus its parity check matrix $H$ is an $p \times\left(2^{p}-1\right)$-matrix consisting of all possible non-zero binary columns.)
i) Prove that the Hamming distance $d\left(\mathcal{H}_{p}\right)$ is at most 3 .
ii) Let $a_{i}$ be the number of codewords of $\mathcal{H}_{p}$ of weight $i$, i.e. with exactly $i$ bits equal to 1 . (Thus $a_{0}=1, a_{1}=a_{2}=0$, and $a_{i}=0$ for all $i>n$.) Show that

$$
\binom{n}{i-1}=i a_{i}+a_{i-1}+(n-i+2) a_{i-2}
$$

for every $i \geq 2$. [Hint: Take all possible vectors $\mathbf{x} \in \mathbb{F}_{2}^{n}$ with exactly $i-1$ entries 1 and consider $H \mathbf{x}^{T}$.]
iii) Convert this recurrence into an equation about the generating function $A(z)=\sum_{i=0}^{n} a_{i} z^{i}$.

Problem $44[1+1+1+1]$ Let the code $C$ have the following generator matrix:

$$
M=\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0
\end{array}\right]
$$

i) How many codewords does $C$ have?
ii) What is the Hamming distance of $C$ ? (Carefully justify your answer.)
iii) Which of the following are codewords of $C$ ?

$$
\begin{aligned}
& A=111111 \\
& B=000000 \\
& C=101101
\end{aligned}
$$

iv) Suppose we received $D=100110$ and we know that at most one error occurred. What was the sent codeword?

Problem 45 [3] Prove that if two Hadamard $n \times n$-matrices $H$ and $G$ have the same first $n-1$ rows (that is, $H_{i j}=G_{i j}$, all $i \in[n-1], j \in[n]$ ) then their $n$-th rows are the same or differ by the factor -1 .

## Homework 10. Due November 12

Problem 46 [2] Deduce from the König Theorem that a bipartite graph with parts $A$ and $B$ has a matching $M$ that covers all but at most $d$ vertices of $A$ provided $|N(X)| \geq|X|-d$ for every $X \subseteq A$, where

$$
N(X)=\{y \in B: \exists x \in X x y \in E(G)\} .
$$

Problem 47 [3+1] Let $n \geq k \geq 0$ be given integers. We consider bipartite graphs $G$ with parts $A$ and $B$ such that $|A|=|B|=n$ and $\mu(G)=k$. (Recall that $\mu(G)$ is the maximum size of a matching in $G$.)
i) What is $f(n, k)$, the maximal possible number of edges in a such graph?
ii) What is $g(n, k)$, the minimal possible number of edges in a such graph?

Let $G$ be the complete bipartite graph with parts $A=\left\{a_{1}, \ldots, a_{m}\right\}$ and $B=\left\{b_{1}, \ldots, b_{n}\right\}$, where the weight of an edge $\left\{a_{i}, b_{j}\right\}$ is $c_{i j}$. Solve the following three problems using the Hungarian Algorithm (and include all intermediate steps).

Problem 48 [1] Let $m=n=5$. Find the minimum weight of a perfect matching in $G$ if

$$
\left(c_{i j}\right)=\left[\begin{array}{ccccc}
2 & 4 & 3 & 2 & 1 \\
3 & 4 & 4 & 5 & 2 \\
1 & 4 & 1 & 4 & 5 \\
3 & 8 & 5 & 3 & 8 \\
4 & 6 & 6 & 2 & 3
\end{array}\right] .
$$

Problem 49 [1] Let $m=n=4$. Find the minimum weight of a perfect matching in $G$ if

$$
\left(c_{i j}\right)=\left[\begin{array}{llll}
7 & 2 & 5 & 4 \\
6 & 5 & 4 & 3 \\
4 & 1 & 5 & 2 \\
2 & 1 & 1 & 2
\end{array}\right]
$$

Problem 50 [1] Let $m=4$ and $n=5$. Find the maximum weight of a matching (where some vertex of $B$ will be unmatched) if

$$
\left(c_{i j}\right)=\left[\begin{array}{ccccc}
9 & 8 & 7 & 6 & 5 \\
9 & 7 & 5 & 3 & 1 \\
9 & 6 & 3 & 0 & 3 \\
9 & 5 & 1 & 5 & 9
\end{array}\right] .
$$

## Extra Credit

If you wish to earn up to 5 homework points as extra credit, please submit (along with Homework 11) a joke, a cartoon, or a short funny story about any mathematical concepts that we have covered in the class (or about hedgehogs).

But note that if the Instructor appears in your submission, then your extra credit can be negative!

## Solutions to Quiz 6

Question. What is the number of vertices and edges in $G_{p, n}$ ?
Answer. Since vertices (resp. edges) are labeled by sequences of length $n-1$ (resp. $n$ ), the answer is $p^{n-1}$ (resp. $p^{n}$ ).

## Thanksgiving Break

Please note that there will be no office hours on November 24 and 25 .

## Homework 11. Due November 19

Problem $51[\mathbf{1 + 1 + 2}]$ Let $G=(V, E)$ be a graph.
i) Show that if there is $S \subseteq V$ such that $G-S$ (meaning that all vertices of $S$ as well as all edges intersecting $S$ are removed) has more than $|S|$ connectivity components with an odd number of vertices, then $G$ cannot have a perfect matching.
ii) Show that if $G$ is 3 -regular (i.e. each vertes has degree 3 ), then $|V|$ is even.
iii) Find a 3 -regular graph without a perfect matching.

Problem 52 [4] Let $G$ be a bipartite graph with parts $A$ and $B$. Let $f: A \rightarrow \mathbb{N}$ be a function from $A$ into non-negative integers. Using Hall's theorem prove that there are disjoint sets $B_{a} \subseteq B$ indexed by $a \in A$ such that $\left|B_{a}\right|=f(a)$ and $B_{a} \subseteq N(a)$ for every $a \in A$ if and only if for every $X \subseteq A$ we have

$$
|N(X)| \geq \sum_{a \in X} f(a) .
$$

Problem 53 [3] Let $G$ be a bipartite graph with parts $A$ and $B$. Suppose that every nonempty $X \subseteq A$ has at least $|X|+1$ neighbors in $B$. Prove that every edge of $G$ can be extended to an $A$-saturating matching.

## Some Hedgehog-Inspired Creativity



By Shufeng Han


By Fan Yang
Q. What happens when you fall in a hedge of hedgehogs?

A: You get poked by a perfect matching.
By Amos Yuen

## Exam 4

Exam 4 will take place during the class on December 3. Since we will start promptly at 9:30am, please arrive in time.

All exams are closed book and notes. Everything that we covered in the lectures (from the beginning of the course to the lecture on Nov'22 inclusive) is examinabile, unless I explicitly marked anything as non-examinable.
As a rough guide, in addition to the material covered by Exams 1-3, we did Sections 3.1-3.2, 11.3-11.4, 12.1-12.7 of the Roberts-Tesman book and parts of Chapters 1-4 of Alon-Spencer's "The Probabilistic Method".

## Homework 12. Due December 1

Problem 54 [2] Let an integer $n \geq 2$ be given. What is the maximum number of perfect matchings in a 2 -regular graph with $4 n$ vertices?

Problem 55 [1] Given a weakly connected digraph $D$, does there always exist a directed walk that uses each arc of $D$ at least once?

Problem 56 [2] Let $G=(V, E)$ be a graph and for a partition $V=A \cup B$ let $e(A, B)$ be the number of edges going across. Prove that if $e(A, B)$ cannot be strictly increased by moving a vertex across, then $e(A, B) \geq e(G) / 2$, that is, at least half of edges go across.

Problem 57 [3] If $G=(V, E)$ has $2 n$ vertices and $e$ edges, then there is a partition $V=A \cup B$ with $|A|=|B|=n$ and $e(A, B) \geq \frac{e n}{2 n-1}$. (Note that $\frac{n}{2 n-1}>\frac{1}{2}$.)

Problem 58 [4] A graph $G=(V, E)$ is $k$-existentially closed ( $k$-E.C. for short) if $v(G) \geq k$ and for every disjoint $A, B \subseteq V$ with $|A|+|B|=k$, there is $x \in V \backslash(A \cup B)$ such that $x$ is connected to every vertex of $A$ but no vertex of $B$. Prove that for every $k$ there is a $k$-E.C. graph.

Problem $59[\mathbf{1}+\mathbf{3}]$ Let $G=(V, E)$ be a graph and $\alpha(G)$ be the largest size of an independent set in $G$ (that is, $A \subseteq V$ such that no edge lies inside $A$ ). Let $S$ be a random subset of $V$ obtained by choosing each vertex of $G$ independently with probability $p$, and let $F=G[S]$ be the graph induced by $S$. Consider the random variables $X=v(F)$ and $Y=e(F)$.
i) Show that $\alpha(F) \geq X-Y$.
ii) By calculating $E(X-Y)$ and selecting the value of $p$ appropriately, deduce that $\alpha(G) \geq$ $n^{2} / 4 m$ provided that $m \geq n / 2$, where $n=|V|$ and $m=|E|$.

