Exercise 1 :

a) $$\int \frac{dx}{e^{-x}\sqrt{e^{2x}-1}} \overset{t=e^x}{=} \int \frac{dt}{\sqrt{t^2-1}} = \cosh^{-1}(e^x)+c.$$

b)

$$\begin{aligned}\int \frac{\tan x}{\sqrt{4-\cos^4 x}}dx &\overset{t=\frac{1}{2}\cos^2(x)}{=} -\frac{1}{4}\int \frac{dt}{t\sqrt{1-t^2}} \\ &= \frac{1}{4}\text{sech}^{-1}(\frac{1}{2}\cos^2(x))+c.\end{aligned}$$

c)

$$\begin{aligned}\lim_{x\longrightarrow 0} \frac{1}{x}-\frac{1}{\sin x} &= \lim_{x\longrightarrow 0} \frac{\sin x - x}{x\sin x} \\ &= \lim_{x\longrightarrow 0} \frac{\cos x - 1}{\sin x + x\cos x} \\ &= \lim_{x\longrightarrow 0} \frac{-\sin x}{2\cos x - x\sin x} = 0.\end{aligned}$$

Exercise 2 :

a)

$$\begin{aligned}\int e^{2x}\sin x dx &\overset{u=e^{2x},v'=\sin x}{=} -e^{2x}\cos x + 2\int e^{2x}\cos x dx \\ &\overset{u=e^{2x},v'=\cos x}{=} -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x dx\end{aligned}$$

Then $\int e^{2x}\sin x dx = -\frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x + c.$

b)

$$\begin{aligned}\int \sec^4 x\tan^7 x dx &\overset{u=\tan x}{=} \int u^7(1+u^2)du \\ &= \frac{1}{8}\tan^8 x + \frac{1}{10}\tan^{10} x + c.\end{aligned}$$

c)

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{x^2-4}} &\overset{x=2\sec\theta}{=} \frac{1}{8}\int \cos^2\theta d\theta \\
&= \frac{1}{16}\int (1+\cos(2\theta))d\theta = \frac{1}{16}(\theta+\cos\theta\sin\theta)+c \\
&= \frac{1}{16}(\sec^{-1}(\frac{x}{2})+\frac{2\sqrt{x^2-4}}{x^2})+c
\end{aligned}$$

Or

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{x^2-4}} &\overset{x^2-4=t^2}{=} \int \frac{dt}{(t^2+4)^2} \\
&= \frac{1}{16}\tan^{-1}(\frac{t}{2})+\frac{1}{16}\ln(\frac{t}{2})+c \\
&= \frac{1}{16}\tan^{-1}(\frac{\sqrt{x^2-4}}{2})+\frac{1}{16}\ln(\frac{\sqrt{x^2-4}}{2})+c
\end{aligned}$$

Exercise 3 :

a)

$$\begin{aligned}
\int \frac{6x^2+x+8}{x^3+4x}dx &= \int (\frac{2}{x}+\frac{4x+1}{x^2+4})dx \\
&= 2\ln|x|+2\ln(x^2+4)+\frac{1}{2}\tan^{-1}(\frac{x}{2})+c.
\end{aligned}$$

b)

$$\begin{aligned}
\int \frac{dx}{(x+1)^{\frac{5}{6}}-(x+1)^{\frac{1}{2}}} &\overset{x+1=t^6}{=} \int \frac{6t^5}{t^5-t^3}dt \\
&= 6\int \frac{t^2}{t^2-1}dt = 6t-3\ln\left|\frac{1+t}{1-t}\right|+c \\
&= 6(x+1)^{\frac{1}{6}}-3\ln\left|\frac{1+(x+1)^{\frac{1}{6}}}{1-(x+1)^{\frac{1}{6}}}\right|+c
\end{aligned}$$

c) $\displaystyle\int \frac{dx}{2+\cos x} \overset{t=\tan(\frac{x}{2})}{=} \int \frac{2dt}{3+t^2}dt = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{1}{\sqrt{3}}\tan(\frac{x}{2}))+ C$