PHYSICS 505 – 1st Semester 2017-2018
1st HOMEWORK
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Hand in: Monday 16th October 2017 at 23:59.

1. The electron in the hydrogen atom is in the ground state. Calculate the probability to find it at distances smaller or equal to two Bohr radii.

Solution:

The wave function in the ground state is given by:

$$\Psi_{100} = R_{10}(r)Y_0^0(\theta,\phi) \qquad (1)$$

with:

$$R_{10}(r) = \frac{2}{\sqrt{a_0^3}}e^{-r/a_0} \quad \text{and} \quad Y_0^0 = \sqrt{\frac{1}{4\pi}}$$, so (1) becomes:

$$\Psi_{100} = \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0} \qquad (2)$$

Thus for the probability we have:

$$P(r \le a_0) = \int_0^{2a_0}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left|\Psi_{100}\right|^2 r^2\,dr\sin\theta\,d\theta\,d\phi = \frac{1}{\pi a_0^3}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\int_0^{2a_0}e^{-2r/a_0}r^2\,dr =$$

$$\frac{1}{\pi a_0^3}2\cdot 2\pi a_0^3\int_0^{2a_0}e^{-2r/a_0}\left(\frac{2r}{a_0}\right)^2 d\left(\frac{r}{a_0}\right) \underset{x=r/a_0}{=} 4\int_0^{2}e^{-2x}x^2\,dx = 4\cdot\frac{1}{4}(1-13e^{-4}) = 0.76 = 76\%$$

2. Prove the relation: $l_-\left|lm\right\rangle = \hbar\sqrt{l\left(l+1\right)-m\left(m-1\right)}\left|l,m-1\right\rangle$.

Solution:

Assume that: $l_-\left|lm\right\rangle = c_{lm}\left|l,m-1\right\rangle$

$$l_-\left|lm\right\rangle = c_{lm}\left|l,m-1\right\rangle \Rightarrow \left\| l_-\left|lm\right\rangle \right\| = \left|c_{lm}\right| \Rightarrow \left|c_{lm}\right|^2 = \left\| l_-\left|lm\right\rangle \right\|^2 \Rightarrow$$

$$\left|c_{lm}\right|^2 = \left(l_-\left|lm\right\rangle, l_-\left|lm\right\rangle\right) \Rightarrow \left|c_{lm}\right|^2 = \left(l_-\left|lm\right\rangle\right)^+ l_-\left|lm\right\rangle \Rightarrow$$

$$\left|c_{lm}\right|^2 = \left(l_-\left|lm\right\rangle, l_-\left|lm\right\rangle\right) \Rightarrow \left|c_{lm}\right|^2 = \left\langle lm\right| l_+ l_- \left|lm\right\rangle$$

But (see question 4)

$$l_+ l_- = \mathbf{l}^2 - l_z\left(l_z - \hbar\right)$$

Thus:

$$\left|c_{lm}\right|^2 = \left\langle lm\right| \mathbf{l}^2 - l_z\left(l_z - \hbar\right) \left|lm\right\rangle \Rightarrow \left|c_{lm}\right|^2 = \left\langle lm\right| \mathbf{l}^2 \left|lm\right\rangle - \left\langle lm\right| l_z^2 \left|lm\right\rangle + \left\langle lm\right| \hbar l_z \left|lm\right\rangle =$$

$$\left|c_{lm}\right|^2 = \hbar^2 l(l+1) - \hbar^2 m^2 + \hbar^2 m \Rightarrow \left|c_{lm}\right| = \hbar\sqrt{l(l+1) - m(m-1)}$$

3. Prove the relation $l_+ l_- = \mathbf{l}^2 - l_z\left(l_z - \hbar\right)$.

Solution:

We know that $l_\pm = l_x \pm i l_y$ thus

$$l_+ l_- = \left(l_x + i l_y\right)\cdot\left(l_x - i l_y\right) = l_x^2 - i l_x l_y + i l_y l_x + l_y^2 = l_x^2 + l_y^2 - i\left(l_x l_y - l_y l_x\right) =$$
$$l_x^2 + l_y^2 + l_z^2 - l_z^2 - i\left(l_x l_y - l_y l_x\right) = \mathbf{l}^2 - l_z^2 - i\left[l_x,\ l_y\right]$$

But we know that: $\left[l_x,\ l_y\right] = i\hbar l_z$ thus

$$l_+ l_- = \left(l_x + i l_y\right)\cdot\left(l_x - i l_y\right) = l_x^2 - i l_x l_y + i l_y l_x + l_y^2 = l_x^2 + l_y^2 - i\left(l_x l_y - l_y l_x\right) =$$
$$l_x^2 + l_y^2 + l_z^2 - l_z^2 - i\left(l_x l_y - l_y l_x\right) = \mathbf{l}^2 - l_z^2 - i(i\hbar l_z) = \mathbf{l}^2 - l_z^2 + \hbar l_z = \mathbf{l}^2 - l_z\left(l_z - \hbar\right)$$

4. Prove the relation $\left\langle l_y \right\rangle = 0$.

Solution:

We know that

$$l_y = \frac{1}{2i}\left(l_+ - l_-\right) \Rightarrow \left\langle l_y \right\rangle = \frac{1}{2i}\left\langle \left(l_+ - l_-\right)\right\rangle \Rightarrow$$

$$\left\langle Y_l^m \middle| l_y \middle| Y_l^m \right\rangle = \frac{1}{2i}\left\langle Y_l^m \middle| \left(l_+ - l_-\right) \middle| Y_l^m \right\rangle \Rightarrow$$

$$\left\langle Y_l^m \middle| l_y \middle| Y_l^m \right\rangle = \frac{1}{2i}\left\{\left\langle Y_l^m \middle| l_+ \middle| Y_l^m \right\rangle + \left\langle Y_l^m l_- Y_l^m \right\rangle\right\} \Rightarrow$$

$$\left\langle Y_l^m \middle| l_y \middle| Y_l^m \right\rangle = \frac{1}{2i}\left\{\hbar\sqrt{l\left(l+1\right) - m\left(m+1\right)}\underbrace{\left\langle Y_l^m | Y_l^{m+1} \right\rangle}_{0} + \hbar\sqrt{l\left(l+1\right) - m\left(m-1\right)}\underbrace{\left\langle Y_l^m | Y_l^{m-1} \right\rangle}_{0}\right\} \Rightarrow$$

$$\left\langle Y_l^m \middle| l_y \middle| Y_l^m \right\rangle = \frac{1}{2i}\left\{0+0\right\} = 0$$

You have to send your answers in pdf form (typed or in clearly handwritten form) in my email address (vlempesis@ksu.edu.sa) before the deadline. Do not forget to put your name (in English) and your ID number on it and on the name of the file for example: Homework 1 – Vasileios Lempesis 345678965.pdf