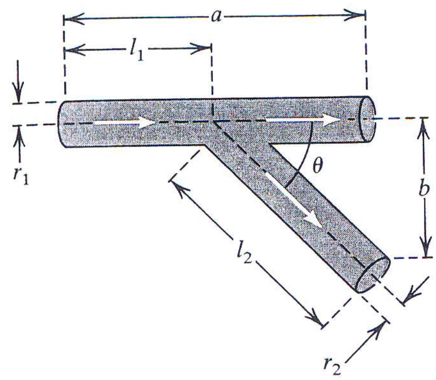
Exercise 70



Exer. 71–74: (a) Verify the correctness by differentiation. (b) Derive the formulas using the approach illustrated in Example 10.

71 Formula (i) of (6.40)

72 Formula (ii) of (6.40)

73 Formula (iii) of (6.40)

74 Formula (iv) of (6.40) (Hint for part (b): Verify first that if $g(x) = \ln|x + \sqrt{x^2 - 1}|$, then $g'(x) = 1/\sqrt{x^2 - 1}$.)

Exer. 75–78: Evaluate the integral.

75 $\int \sin^{-1} 2x\, dx$

76 $\int \cos^{-1} \frac{1}{3}x\, dx$

77 $\int x \tan^{-1}(x^2)\, dx$

78 $\int \frac{\sec^{-1}\sqrt{x}}{\sqrt{x}}\, dx$

c **Exer. 79–82: Approximate the arc length of the graph of the function between A and B. Use Simpson's rule or numerical integration provided on a calculator or a computer to ensure at least four correct decimal places.**

79 $y = \arcsin x$; $A(0, 0)$, $B\left(\frac{1}{2}, \frac{\pi}{6}\right)$

80 $y = \arccos x$; $A\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$, $B\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$

81 $y = \arctan x$; $A(0, 0)$, $B\left(\sqrt{3}, \frac{\pi}{3}\right)$

82 $y = \arctan x$; $A(-5, -\arctan 5)$, $B(5, \arctan 5)$

c **Exer. 83–84: Approximate the surface area generated if the graph of the function between A and B is revolved about the x-axis. Use Simpson's rule or numerical integration provided on a calculator or a computer to ensure at least four correct decimal places.**

83 $y = 4\arctan(x^2)$; $A(0, 0)$, $B(1, \pi)$

84 $y = \operatorname{arcsec} x$; $A(2, \operatorname{arcsec} 2)$, $B(10, \operatorname{arcsec} 10)$

6.8 HYPERBOLIC AND INVERSE HYPERBOLIC FUNCTIONS

The hyperbolic functions and their inverses, which we investigate in this section, are used to solve a variety of problems in the physical sciences and engineering.

HYPERBOLIC FUNCTIONS

Many of the advanced applications of calculus involve the exponential expressions

$$\frac{e^x - e^{-x}}{2} \quad \text{and} \quad \frac{e^x + e^{-x}}{2},$$

which define the hyperbolic functions. The properties of these expressions are similar in many ways to those of $\sin x$ and $\cos x$. Later in our discussion, we shall see why they are called the *hyperbolic sine* and the *hyperbolic cosine* of x.

Definition 6.41

The **hyperbolic sine function**, denoted by **sinh**, and the **hyperbolic cosine function**, denoted by **cosh**, are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

for every real number x.

We pronounce $\sinh x$ and $\cosh x$ as *sinch x* and *kosh x*, respectively.

The graph of $y = \cosh x$ may be found by **addition of y-coordinates**. Noting that $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$, we first sketch the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same coordinate plane, as shown with dashes in Figure 6.39. We then add the y-coordinates of points on these graphs to obtain the graph of $y = \cosh x$. Note that the range of cosh is $[1, \infty)$.

We may find the graph of $y = \sinh x$ by adding y-coordinates of the graphs of $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$, as shown in Figure 6.40.

Figure 6.39

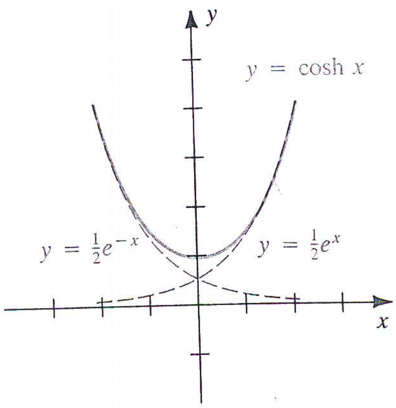
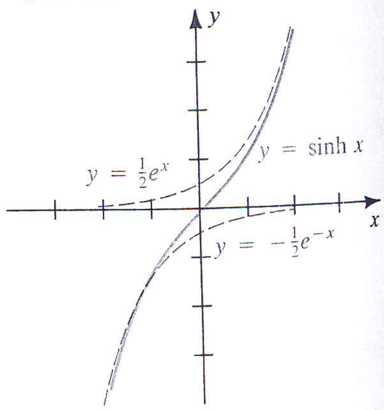


Figure 6.40



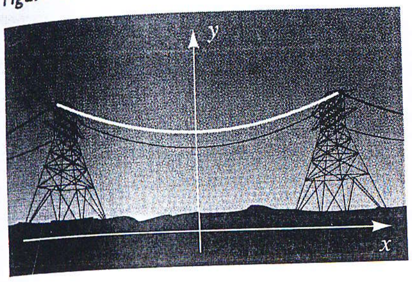
Some scientific calculators have keys that can be used to find values of sinh and cosh directly. We can also substitute numbers for x in Definition (6.41), as in the following illustration.

ILLUSTRATION

- $\sinh 3 = \dfrac{e^3 - e^{-3}}{2} \approx 10.0179$
- $\cosh 0.5 = \dfrac{e^{0.5} + e^{-0.5}}{2} \approx 1.1276$

The hyperbolic cosine function can be used to describe the shape of a uniform flexible cable, or chain, whose ends are supported from the same height. As illustrated in Figure 6.41, telephone or power lines may be strung between poles in this manner. The shape of the cable appears to be

Figure 6.41



a parabola, but is actually a **catenary** (after the Latin word for *chain*). If we introduce a coordinate system, as in Figure 6.41, we will later show that an equation corresponding to the shape of the cable is $y = a\cosh(x/a)$ for some real number a.

The hyperbolic cosine function also occurs in the analysis of motion in a resisting medium. If an object is dropped from a given height and if air resistance is disregarded, then the distance y that it falls in t seconds is $y = \frac{1}{2}gt^2$, where g is a gravitational constant. However, air resistance cannot always be disregarded. As the velocity of the object increases, air resistance may significantly affect its motion. For example, if the air resistance is directly proportional to the square of the velocity, then the distance y that the object falls in t seconds is given by

$$y = A\ln(\cosh Bt)$$

for constants A and B (see Exercise 42). Another application is given in Example 2 of this section.

Many identities similar to those for trigonometric functions hold for the hyperbolic sine and cosine functions. For example, if $\cosh^2 x$ and $\sinh^2 x$ denote $(\cosh x)^2$ and $(\sinh x)^2$, respectively, we have the following identity.

Theorem 6.42

$$\cosh^2 x - \sinh^2 x = 1$$

PROOF By Definition (6.41),

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}\\ &= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4}\\ &= \frac{4}{4} = 1.\end{aligned}$$

Theorem (6.42) is analogous to the identity $\cos^2 x + \sin^2 x = 1$. Other hyperbolic identities are stated in the exercises. To verify an identity, it is sufficient to express the hyperbolic functions in terms of exponential functions and show that one side of the equation can be transformed into the other, as illustrated in the proof of Theorem (6.42). The hyperbolic identities are similar to (but not always the same as) certain trigonometric identities—differences usually involve signs of terms.

If t is a real number, there is an interesting geometric relationship between the points $P(\cos t, \sin t)$ and $Q(\cosh t, \sinh t)$ in a coordinate plane. Let us consider the graphs of $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$, sketched in

Figure 6.42 $x^2 + y^2 = 1$

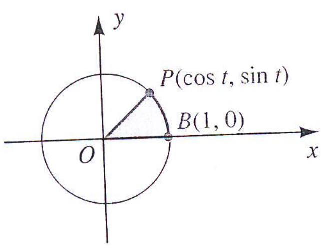
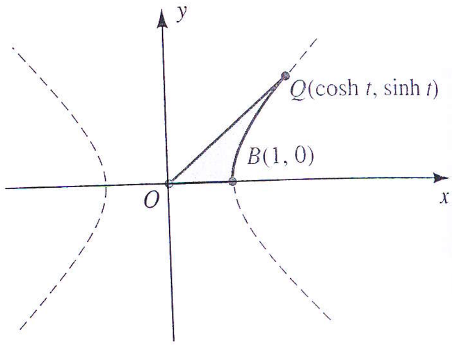


Figure 6.43 $x^2 - y^2 = 1$



Figures 6.42 and 6.43. The graph in Figure 6.42 is the unit circle with center at the origin. The graph in Figure 6.43 is a *hyperbola*. (Hyperbolas and their properties are discussed in the Precalculus Review Chapter.) Note first that since $\cos^2 t + \sin^2 t = 1$, the point $P(\cos t, \sin t)$ is on the circle $x^2 + y^2 = 1$. Next, by Theorem (6.42), $\cosh^2 t - \sinh^2 t = 1$, and hence the point $Q(\cosh t, \sinh t)$ is on the hyperbola $x^2 - y^2 = 1$. These are the reasons for referring to cos and sin as *circular* functions and to cosh and sinh as *hyperbolic* functions.

The graphs in Figures 6.42 and 6.43 are related in another way. If $0 < t < \pi/2$, then t is the radian measure of angle POB, shown in Figure 6.42. The area A of the shaded circular sector is $A = \frac{1}{2}(1)^2 t = \frac{1}{2}t$, and hence $t = 2A$. Similarly, if $Q(\cosh t, \sinh t)$ is the point in Figure 6.43, then $t = 2A$ for the area A of the shaded hyperbolic sector (see Exercise 33).

The impressive analogies between the trigonometric and the hyperbolic sine and cosine functions motivate us to define hyperbolic functions that correspond to the four remaining trigonometric functions. The **hyperbolic tangent, hyperbolic cotangent, hyperbolic secant,** and **hyperbolic cosecant functions,** denoted by tanh, coth, sech, and csch, respectively, are defined as follows.

Definition 6.43

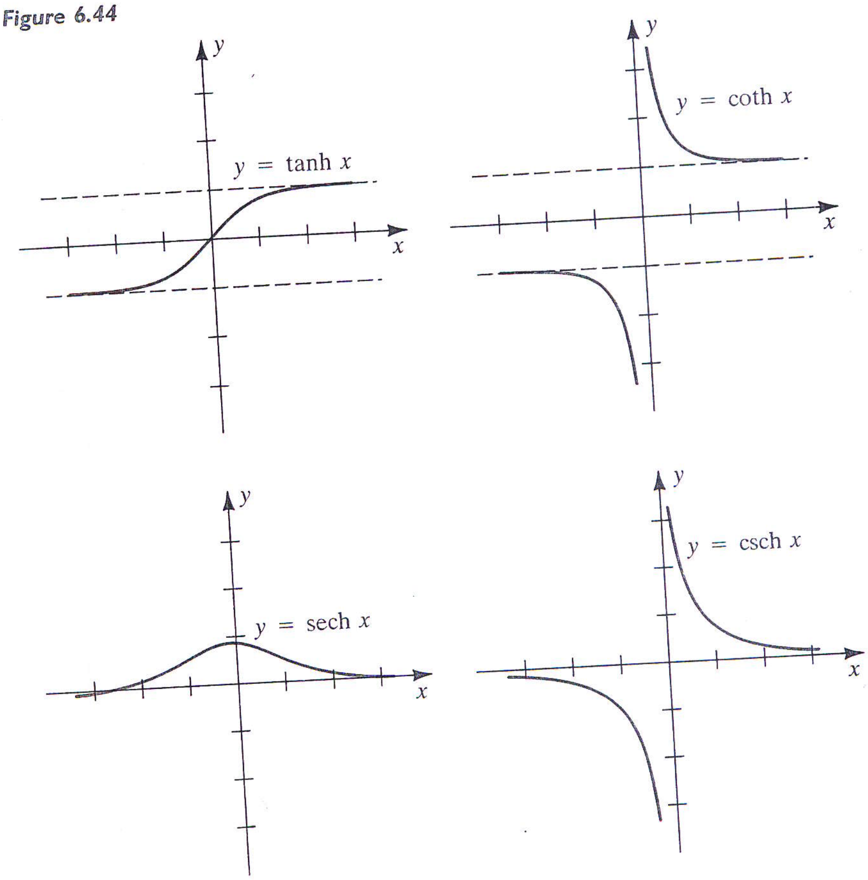
(i) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

(ii) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0$

(iii) $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(iv) $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}, \quad x \neq 0$

Figure 6.44



We pronounce the four function values in the preceding definition as *tansh x, cotansh x, setch x,* and *cosetch x*. Their graphs are sketched in Figure 6.44.

If we divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ (see (6.42)) by $\cosh^2 x$, we obtain

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}.$$

Using the definitions of $\tanh x$ and $\operatorname{sech} x$ gives us (i) of the next theorem. Formula (ii) may be obtained by dividing both sides of (6.42) by $\sinh^2 x$.

Theorem 6.44

(i) $1 - \tanh^2 x = \operatorname{sech}^2 x$ **(ii)** $\coth^2 x - 1 = \operatorname{csch}^2 x$

Note the similarities and differences between (6.44) and the analogous trigonometric identities.

Derivative formulas for the hyperbolic functions are listed in the next theorem, where $u = g(x)$ and g is differentiable.

Theorem 6.45

(i) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

(ii) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

(iii) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

(iv) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

(v) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

(vi) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

Figure

Basem

PROOF As usual, we consider only the case $u = x$. Since $(d/dx)(e^x) = e^x$ and $(d/dx)(e^{-x}) = -e^{-x}$,

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

and

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

To differentiate $\tanh x$, we apply the quotient rule as follows:

$$\begin{aligned}\frac{d}{dx}(\tanh x) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\ &= \frac{\cosh x (d/dx)(\sinh x) - \sinh x (d/dx)(\cosh x)}{\cosh^2 x}\\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x\end{aligned}$$

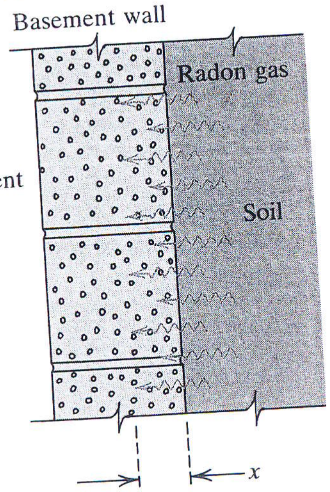
The remaining formulas can be proved in similar fashion. ■

EXAMPLE ■ 1 If $f(x) = \cosh(x^2 + 1)$, find $f'(x)$.

SOLUTION Applying Theorem (6.45)(i), with $u = x^2 + 1$, we obtain

$$\begin{aligned} f'(x) &= \sinh(x^2+1) \cdot \frac{d}{dx}(x^2+1) \\ &= 2x\sinh(x^2+1). \end{aligned}$$

Figure 6.45



EXAMPLE ▪ 2 Radon gas can readily diffuse through solid materials such as brick and cement. If the direction of diffusion in a basement wall is perpendicular to the surface, as illustrated in Figure 6.45, then the radon concentration $f(x)$ (in joules/cm^3) in the air-filled pores within the wall at a distance x from the outside surface can be approximated by

$$f(x) = A\sinh(qx) + B\cosh(qx) + k,$$

where the constant q depends on the porosity of the wall, the half-life of radon, and a diffusion coefficient; the constant k is the maximum radon concentration in the air-filled pores; and A and B are constants that depend on initial conditions. Show that $y = f(x)$ is a solution of the *diffusion equation*

$$\frac{d^2y}{dx^2} - q^2y + q^2k = 0.$$

SOLUTION Differentiating $y = f(x)$ twice gives us

$$\frac{dy}{dx} = qA\cosh(qx) + qB\sinh(qx)$$

and

$$\frac{d^2y}{dx^2} = q^2A\sinh(qx) + q^2B\cosh(qx).$$

Since $y = A\sinh(qx) + B\cosh(qx) + k$, we have

$$q^2y = q^2A\sinh(qx) + q^2B\cosh(qx) + q^2k.$$

Subtracting the expressions for d^2y/dx^2 and q^2y yields

$$\frac{d^2y}{dx^2} - q^2y = -q^2k$$

and hence

$$\frac{d^2y}{dx^2} - q^2y + q^2k = 0.$$

The integration formulas that correspond to the derivative formulas in Theorem (6.45) are as follows.

Theorem 6.46

(i) $\int \sinh u\, du = \cosh u + C$

(ii) $\int \cosh u\, du = \sinh u + C$

(iii) $\int \operatorname{sech}^2 u\, du = \tanh u + C$

(iv) $\int \operatorname{csch}^2 u\, du = -\coth u + C$

(v) $\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$

(vi) $\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$

EXAMPLE 3 Evaluate $\int x^2 \sinh x^3\, dx$.

SOLUTION If we let $u = x^3$, then $du = 3x^2\, dx$ and

$$\begin{aligned}\int x^2 \sinh x^3\, dx &= \tfrac{1}{3}\int (\sinh x^3)3x^2\, dx\\ &= \tfrac{1}{3}\int \sinh u\, du = \tfrac{1}{3}\cosh u + C = \tfrac{1}{3}\cosh x^3 + C.\end{aligned}$$

INVERSE HYPERBOLIC FUNCTIONS

We now investigate the inverses of the hyperbolic functions, which frequently occur in evaluating certain types of integrals. We will also see how an inverse hyperbolic function is used in the derivation of the equation for a hanging cable.

The hyperbolic sine function is continuous and increasing for every x and hence, by Theorem (6.6), has a continuous, increasing inverse function, denoted by $\sinh^{-1}$. Since $\sinh x$ is defined in terms of e^x, we might expect that $\sinh^{-1}$ can be expressed in terms of the inverse, ln, of the natural exponential function. The first formula of the next theorem shows that this is the case.

Theorem 6.47

(i) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

(ii) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$

(iii) $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}, \quad |x| < 1$

(iv) $\operatorname{sech}^{-1} x = \ln\dfrac{1+\sqrt{1-x^2}}{x}, \quad 0 < x \leq 1$

PROOF To prove (i), we begin by noting that

$$y = \sinh^{-1} x \quad \text{if and only if} \quad x = \sinh y.$$

The equation $x = \sinh y$ can be used to find an explicit form for $\sinh^{-1} x$. Thus, if

$$x = \sinh y = \frac{e^y - e^{-y}}{2},$$

then

$$e^y - 2x - e^{-y} = 0.$$

Multiplying both sides by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

Applying the quadratic formula yields

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}, \quad \text{or} \quad e^y = x \pm \sqrt{x^2 + 1}.$$

Since $x - \sqrt{x^2 + 1} < 0$ and e^y is never negative, we must have

$$e^y = x + \sqrt{x^2 + 1}.$$

The equivalent logarithmic form is

$$y = \ln(x + \sqrt{x^2 + 1});$$

that is,

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}).$$

Formulas (ii)–(iv) are obtained in similar fashion. As with trigonometric functions, some inverse functions exist only if the domain is restricted. For example, if the domain of cosh is restricted to the set of nonnegative real numbers, then the resulting function is continuous and increasing, and its inverse function $\cosh^{-1}$ is defined by

$$y = \cosh^{-1} x \quad \text{if and only if} \quad \cosh y = x, \quad y \geq 0.$$

Employing the process used for $\sinh^{-1} x$ leads us to (ii). Similarly,

$$y = \tanh^{-1} x \quad \text{if and only if} \quad \tanh y = x \quad \text{for} \quad |x| < 1.$$

Using Definition (6.43), we may write $\tanh y = x$ as

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x.$$

Solving for y gives us (iii).

Finally, if we restrict the domain of sech to nonnegative numbers, the result is a one-to-one function, and we define

$$y = \operatorname{sech}^{-1} x \quad \text{if and only if} \quad \operatorname{sech} y = x, \quad y \geq 0.$$

Again, introducing the exponential form leads to (iv). ■

In the next theorem, $u = g(x)$, where g is differentiable and x is suitably restricted.

Theorem 6.48

(i) $$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$

(ii) $$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$$

(iii) $$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$$

(iv) $$\frac{d}{dx}(\operatorname{sech}^{-1} u) = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

PROOF By Theorem (6.47)(i),

$$\begin{aligned}
\frac{d}{dx}(\sinh^{-1} x) &= \frac{d}{dx}(\ln(x+\sqrt{x^2+1})) \\
&= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) \\
&= \frac{\sqrt{x^2+1}+x}{(x+\sqrt{x^2+1})\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}.
\end{aligned}$$

This formula can be extended to $(d/dx)(\sinh^{-1} u)$ by applying the chain rule. The remaining formulas can be proved in similar fashion. ■

EXAMPLE 4 If $y = \sinh^{-1}(\tan x)$, find dy/dx.

SOLUTION Using Theorem (6.48)(i) with $u = \tan x$, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sqrt{\tan^2 x + 1}}\frac{d}{dx}\tan x = \frac{1}{\sqrt{\sec^2 x}}\sec^2 x \\
&= \frac{1}{|\sec x|}|\sec x|^2 = |\sec x|.
\end{aligned}$$

The following theorem may be verified by differentiating the right-hand side of each formula.

Theorem 6.49

(i) $$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \sinh^{-1}\frac{u}{a} + C, \quad a > 0$$

(ii) $$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \cosh^{-1}\frac{u}{a} + C, \quad 0 < a < u$$

(iii) $$\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C, \quad |u| < a$$

(iv) $$\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\operatorname{sech}^{-1}\frac{|u|}{a} + C, \quad 0 < |u| < a$$

If we use Theorem (6.47), then each of the integration formulas in the preceding theorem can be expressed in terms of the natural logarithm function. To illustrate,

$$\begin{aligned}\int \frac{1}{\sqrt{a^2+u^2}}\,du &= \sinh^{-1}\frac{u}{a} + C \\ &= \ln\left(\frac{u}{a} + \sqrt{\left(\frac{u}{a}\right)^2 + 1}\right) + C.\end{aligned}$$

We can show that if $a > 0$, then the last formula can be written as

$$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln(u + \sqrt{a^2+u^2}) + D,$$

where D is a constant. In Section 7.3, we shall discuss another method for evaluating the integrals in Theorem (6.49).

EXAMPLE ■ 5 Evaluate $\displaystyle\int \frac{1}{\sqrt{25+9x^2}}\,dx$.

SOLUTION We may express the integral as in Theorem (6.49)(i), by using the substitution

$$u = 3x, \qquad du = 3\,dx.$$

Since du contains the factor 3, we adjust the integrand by multiplying by 3 and then compensate by multiplying the integral by $\frac{1}{3}$ before substituting:

$$\begin{aligned}\int \frac{1}{\sqrt{25+9x^2}}\,dx &= \frac{1}{3}\int \frac{1}{\sqrt{5^2+(3x)^2}}\,3\,dx \\ &= \frac{1}{3}\int \frac{1}{\sqrt{5^2+u^2}}\,du \\ &= \frac{1}{3}\sinh^{-1}\frac{u}{5} + C \\ &= \frac{1}{3}\sinh^{-1}\frac{3x}{5} + C\end{aligned}$$

EXAMPLE 6 Evaluate $\int \frac{e^x}{16 - e^{2x}}\, dx$.

SOLUTION Substituting $u = e^x$, $du = e^x\, dx$ and applying Theorem (6.49)(iii) with $a = 4$, we have

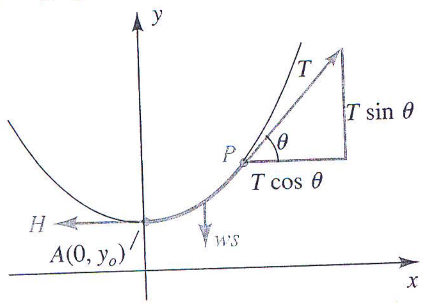
$$\begin{aligned}\int \frac{e^x}{16 - e^{2x}}\, dx &= \int \frac{1}{4^2 - (e^x)^2} e^x\, dx \\ &= \int \frac{1}{4^2 - u^2}\, du \\ &= \frac{1}{4}\tanh^{-1}\frac{u}{4} + C \\ &= \frac{1}{4}\tanh^{-1}\frac{e^x}{4} + C\end{aligned}$$

for $|u| < a$ (that is, $e^x < 4$).

We now consider how the hyperbolic cosine and the inverse hyperbolic sine functions are used in describing the shape of the curve along which a hanging cable lies. We first derive a differential equation for the function whose graph is the curve, and then we solve the differential equation.

Figure 6.41 shows a hanging cable in the form of a power line strung between two towers. A section of the cable is shown in Figure 6.46, where we have set up a coordinate system with the vertical y-axis running through the lowest point $A(0, y_0)$ of the cable. Consider a section of the cable running upward from A to a point P. Figure 6.46 also shows the forces acting on the cable: There is a horizontal tension H at the point A, a tangential tension T at the point P, and a downward gravitational force ws.

Figure 6.46
Hanging cable



The tangential tension can be resolved in a horizontal component $T\cos\theta$ and a vertical component $T\sin\theta$, where θ is the angle that the tangent line to the cable at P makes with the horizontal. (This angle is also the angle of inclination of the tangent line.) Thus, the derivative dy/dx at P is equal to $\tan\theta$. The force due to gravity is equal to the weight of the section of the cable, expressed as ws, where w is the weight per unit length and s is the length of the section.

Since the cable is not moving, the forces acting on any section of it must cancel out. Since the cable is not moving to the right or the left, the magnitude of the horizontal tension at point A equals the magnitude of the horizontal tension at point P:

$$T\cos\theta = H$$

But the cable is also stationary in the vertical direction, so the magnitude of the gravitational force equals the vertical tension at P:

$$T\sin\theta = ws$$

We can now write

$$\frac{ws}{H} = \frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{dy}{dx},$$

or simply,
$$\frac{dy}{dx} = \frac{ws}{H}.$$

If we differentiate this equation with respect to x and use Theorem (5.17), we obtain

$$\frac{d^2y}{dx^2} = \frac{w}{H}\frac{ds}{dx} = \frac{w}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Letting $a = w/H$ gives

$$\frac{d^2y}{dx^2} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

for some constant a, as the differential equation satisfied by the equation $y = f(x)$ of the curve formed by a hanging cable. We can now solve the differential equation to find an explicit expression for the function f.

EXAMPLE ■ 7 Solve the differential equation

$$\frac{d^2y}{dx^2} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

to find an explicit formula for the curve of a hanging cable.

SOLUTION The differential equation

$$\frac{d^2y}{dx^2} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

is a second-order differential equation, because it involves the second derivative of y with respect to x. We first reduce it to a first-order differential equation by the substitution

$$z = \frac{dy}{dx}$$

so that
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{dz}{dx}.$$

This result converts the original differential equation to a first-order equation

$$\frac{dz}{dx} = a\sqrt{1 + z^2}$$

in which the variables separate. Dividing each side by $\sqrt{1 + z^2}$ and integrating, we obtain

$$\int \frac{1}{\sqrt{1 + z^2}}\,dz = \int a\,dx.$$

The integrand on the left-hand side of the equation, $1/\sqrt{1 + z^2}$, is the derivative of the inverse hyperbolic sine of z. By Theorem (6.49)(i), we have

$$\sinh^{-1} z = ax + C \quad \text{and hence} \quad z = \sinh(ax + C).$$

Since $z = dy/dx$, the last equation becomes

$$\frac{dy}{dx} = \sinh(ax + C).$$

Because $(0, y_0)$ is the minimum point on the curve, the tangent line to the curve at $(0, y_0)$ is horizontal. Thus, at $x = 0$, $dy/dx = 0$, and

$$0 = \sinh(a \cdot 0 + C) = \sinh C.$$

Therefore, $$C = 0 \quad \text{and} \quad \frac{dy}{dx} = \sinh ax.$$

Thus, $$y = \int \sinh ax\, dx = \frac{1}{a}(\cosh ax) + C.$$

To find the constant of integration, we first use the fact that $y = y_0$ at $x = 0$:

$$y_0 = \frac{1}{a}(\cosh 0) + C = \frac{1}{a}(1) + C = \frac{1}{a} + C$$

Hence, if we choose our coordinate system so that $y_0 = 1/a$, we have $C = 0$ and the equation for the hanging cable is

$$y = \frac{1}{a}(\cosh ax).$$

Note that if the coordinate system has already been established in such a way that $y_0 \neq 1/a$, then the equation for the catenary has the more general form

$$y = \left(y_0 - \frac{1}{a}\right) + \frac{1}{a}\cosh ax.$$

Another commonly used form for the equation of the catenary is

$$y = b + a\cosh\left(\frac{x}{a}\right),$$

where a and b are constants and the lowest point on the curve occurs at $x = 0$.

EXAMPLE ■ 8 A cable television line hangs between two 30-ft poles that are 36 ft apart. At its lowest point, the cable is 16 ft above the level ground. Determine the height of the cable above a point on the ground that is 6 ft from the poles.

SOLUTION We use the form

$$y = b + a\cosh\left(\frac{x}{a}\right)$$

and determine first the values of the constants a and b. Since the lowest point occurs at $x = 0$, we have

$$16 = b + a\cosh\left(\frac{0}{a}\right) = b + a\cosh 0 = b + a,$$

so that $$b = 16 - a.$$

We also have $y = 30$ when $x = 18$ since the poles are 36 ft apart. Thus,

$$30 = b + a\cosh\left(\frac{18}{a}\right),$$

or $$b = 30 - a\cosh\left(\frac{18}{a}\right)$$

Equating the two expressions for b, we have

$$16 - a = 30 - a\cosh\left(\frac{18}{a}\right)$$

or, equivalently,

$$a - a\cosh\left(\frac{18}{a}\right) + 14 = 0.$$

We use Newton's method to solve for a, obtaining $a \approx 13.42$. So $b = 16 - a \approx 2.58$, and the equation for the catenary becomes

$$y = 2.58 + 13.42\cosh\left(\frac{x}{13.42}\right).$$

At a point 6 ft from one of the poles, we have $x = \pm 12$. When $x = \pm 12$, $y = 2.58 + 13.42\cosh(\pm 12/13.42) \approx 21.73$. Thus, at a point 12 ft from the lowest point on the cable television line, the height of the cable is approximately 21.73 ft.

The analysis we have seen for hanging cables also applies to the Gateway Arch to the West in St. Louis. All the internal forces are in equilibrium when a cable hangs freely. There are no transverse forces pushing the cable out of shape. Constructing an arch in the shape of an inverted hyperbolic cosine creates a structure for which there are also no transverse forces that might cause the arch to collapse. This inherent stability of the inverted catenary, along with its beauty, led Saarinen to choose it for his design of the Gateway Arch.

As with other functions that we have studied, we can gain an understanding of compositions of functions that use inverse hyperbolic functions as components by combining the techniques of calculus with the graphs that a graphing utility can display. The next example illustrates this process.

EXAMPLE 9 For the function $f(x) = \ln[\sinh^{-1}(x^2 + 1)]$,

(a) determine the domain of the function f

(b) find the derivative f'

(c) use a graphing utility to plot both the function and its derivative in the viewing window $-5 \le x \le 5, -1 \le y \le 1.5$

SOLUTION

(a) The function f is a composition of functions, requiring that we first add 1 to the square of x, then compute an inverse hyperbolic sine, and finally determine the natural logarithm of the resulting number. Since $x^2 + 1$ and the inverse hyperbolic sine are defined for all real numbers, the only step that may cause difficulty in computing $f(x)$ is that the natural logarithm is defined only for positive values.

We note first that by Theorem 6.47(i),

$$\sinh^{-1}(x^2+1) = \ln\left[(x^2+1) + \sqrt{(x^2+1)^2+1}\right] = \ln u,$$

where $u = x^2 + 1 + \sqrt{(x^2+1)^2+1}$. Now u is strictly positive and has its minimum value $1 + \sqrt{2}$ when $x = 0$. Hence,

$$\sinh^{-1}(x^2+1) \geq \ln(1+\sqrt{2}) \approx \ln 2.4142136 \approx 0.8813736.$$

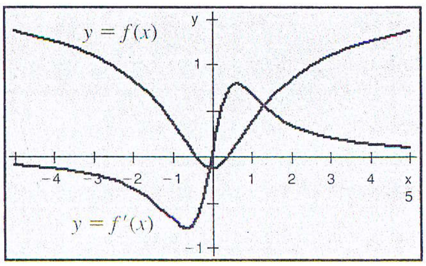
Since $\sinh^{-1}(x^2+1)$ is always positive, $f(x) = \ln(\sinh^{-1}(x^2+1))$ is defined for all real numbers x. Thus, the domain of f consists of all real numbers.

(b) We use the chain rule twice to find the derivative:

$$f'(x) = \frac{[\sinh^{-1}(x^2+1)]'}{\sinh^{-1}(x^2+1)} = \frac{\dfrac{1}{\sqrt{(x^2+1)^2+1}}(x^2+1)'}{\sinh^{-1}(x^2+1)}$$

$$= \frac{2x}{\sqrt{(x^2+1)^2+1}\,\sinh^{-1}(x^2+1)}$$

Figure 6.47
$-5 \leq x \leq 5,\ -1 \leq y \leq 1.5$



(c) We use a graphing utility to plot f and f' in the specified viewing window, as shown in Figure 6.47. From the figure, it appears that the graph of f is symmetric about the y-axis and the graph of f' is symmetric about the origin. We can confirm these observations by substituting into the expressions for the function and its derivative to find that $f(-x) = f(x)$ and that $f'(-x) = -f'(x)$. Using the trace feature, we find that f' has a maximum of 0.6580 at approximately $x = 0.7477$, where the graph of f has a point of inflection. By symmetry, there is also a point of inflection for f at $x = -0.7477$, where f' has a minimum.

EXERCISES 6.8

c Exer. 1–2: Approximate to four decimal places.

1 (a) $\sinh 4$ (b) $\cosh \ln 4$ (c) $\tanh(-3)$
(d) $\coth 10$ (e) $\operatorname{sech} 2$ (f) $\operatorname{csch}(-1)$

2 (a) $\sinh \ln 4$ (b) $\cosh 4$ (c) $\tanh 3$
(d) $\coth(-10)$ (e) $\operatorname{sech}(-2)$ (f) $\operatorname{csch} 1$

Exer. 3–14: Find $f'(x)$ if $f(x)$ is the given expression.

3 $\sinh 5x$

4 $\sinh(x^2+1)$

5 $\cosh(x^3)$

6 $\cosh^3 x$

7 $\sqrt{x}\tanh\sqrt{x}$

8 $\arctan \tanh x$

9 $\coth(1/x)$

10 $\coth x/\cot x$

11 $\dfrac{\operatorname{sech}(x^2)}{x^2+1}$

12 $\sqrt{\operatorname{sech} 5x}$

13 $\operatorname{csch}^2 6x$

14 $x \operatorname{csch} e^{4x}$

c Exer. 15–18: (a) Find the domain of the function. (b) Find $f'(x)$. (c) Plot f and f' in the indicated viewing window.

15 $f(x) = \cosh\sqrt{4x^2+3}$; $-3 \le x \le 3$, $-25 \le y \le 50$

16 $f(x) = \dfrac{1+\cosh x}{1-\cosh x}$; $-8 \le x \le 8$, $-5 \le y \le 2$

17 $f(x) = \dfrac{1}{\tanh x + 1}$; $-3 \le x \le 3$, $-25 \le y \le 50$

18 $f(x) = \ln|\tanh x|$; $-2 \le x \le 2$, $-10 \le y \le 10$

Exer. 19–30: Evaluate the integral.

19 $\int x^2 \cosh(x^3)\,dx$

20 $\int \dfrac{1}{\operatorname{sech} 7x}\,dx$

21 $\int \dfrac{\sinh\sqrt{x}}{\sqrt{x}}\,dx$

22 $\int x \sinh(2x^2)\,dx$

23 $\int \dfrac{1}{\cosh^2 3x}\,dx$

24 $\int \operatorname{sech}^2(5x)\,dx$

25 $\int \operatorname{csch}^2(\frac{1}{2}x)\,dx$

26 $\int (\sinh 4x)^{-2}\,dx$

27 $\int \tanh 3x \operatorname{sech} 3x\,dx$

28 $\int \sinh x \operatorname{sech}^2 x\,dx$

29 $\int \cosh x \operatorname{csch}^2 x\,dx$

30 $\int \coth 6x \operatorname{csch} 6x\,dx$

31 Find the points on the graph of $y = \sinh x$ at which the tangent line has slope 2.

32 Find the arc length of the graph of $y = \cosh x$ from $(0, 1)$ to $(1, \cosh 1)$.

33 If A is the region shown in Figure 6.43, prove that $t = 2A$.

34 The region bounded by the graphs of $y = \cosh x$, $x = -1$, $x = 1$, and $y = 0$ is revolved about the x-axis. Find the volume of the resulting solid.

35 The Gateway Arch to the West in St. Louis has the shape of an inverted catenary (see figure). Rising 630 ft at its center and stretching 630 ft across its base, the shape of the arch can be approximated by

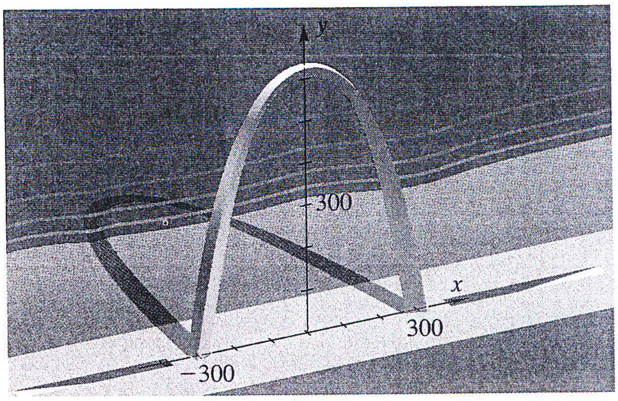
$$y = -127.7\cosh(x/127.7) + 757.7$$

for $-315 \le x \le 315$.

(a) Approximate the total open area under the arch.

(b) Approximate the total length of the arch.

Exercise 35



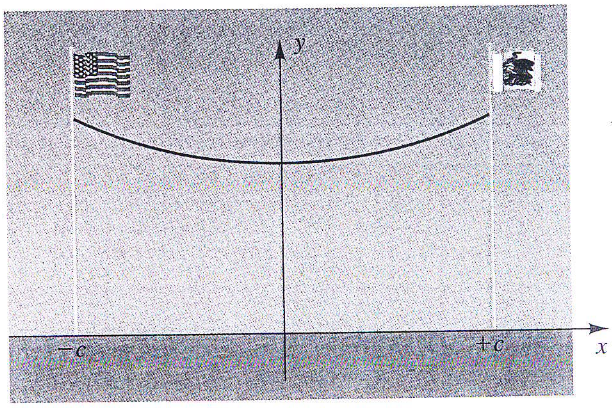
36 A uniform flexible cable supported by poles at $x = -c$ and $x = c$ takes the shape of the graph of the equation $y = b + a\cosh(x/a)$ for $-c \le x \le c$ (see figure).

(a) Find the height of the cable on the poles at each end.

(b) Find the height of the cable at its lowest point.

(c) Find the arc length of the cable hanging between the two poles.

Exercise 36



c Exer. 37–40: Refer to Exercise 36.

37 A power line is strung between two 21-ft poles that are 33 ft apart. At its lowest point, the cable is 16 ft above level ground. Find the arc length of the cable hanging between the two poles.

38 A rope 12 ft long is hung between two 5-ft high poles that are 10 ft apart. How high will the rope be off the level ground at its lowest point?

39 Two children pick up a 15-ft rope to play jump rope. Each child grasps the rope 6 in. from an end and holds the rope 3.5 ft above level ground. The two move together until the rope just touches the ground hanging

between their hands before they start to swing the rope. How far apart will they be?

40 A telephone line is to be strung across a city street between two 25-ft poles that are 30 ft apart. To allow large trucks to pass under the line, the lowest point should be at least 19 ft high. Find the arc length of the line between the two poles if it has a lowest point of exactly 19 ft.

41 If an object falls through the air toward the ground in such a way that the air resistance is proportional to the square of the velocity,

(a) show that position y of the object satisfies the differential equation

$$y'' = g - \alpha(y')^2$$

(b) make the change of variable

$$z = y'$$

as in Example 7 and solve the differential equation in part (a)

42 If a steel ball of mass m is released into water and the force of resistance is directly proportional to the square of the velocity, then the distance y that the ball travels in t seconds is given by

$$y = km \ln \cosh\left(\sqrt{\frac{g}{km}}\,t\right),$$

where g is a gravitational constant and $k > 0$. Show that y is a solution of the differential equation

$$m\frac{d^2y}{dt^2} + \frac{1}{k}\left(\frac{dy}{dt}\right)^2 = mg.$$

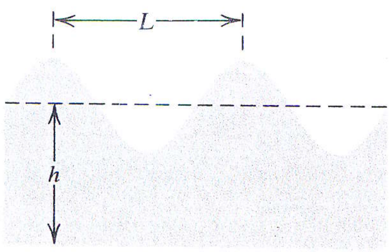
43 If a wave of length L is traveling across water of depth h (see figure), the velocity v, or *celerity*, of the wave is related to L and h by the formula

$$v^2 = \frac{gL}{2\pi}\tanh\frac{2\pi h}{L},$$

where g is a gravitational constant.

(a) Find $\lim_{h\to\infty} v^2$ and conclude that $v \approx \sqrt{gL/(2\pi)}$ in deep water.

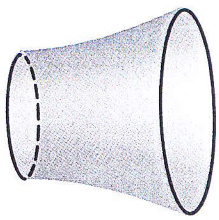
Exercise 43



(b) If $x \approx 0$ and f is a continuous function, then, by the mean value theorem (3.12), $f(x) - f(0) \approx f'(0)x$. Use this fact to show that $v \approx \sqrt{gh}$ if $h/L \approx 0$. Conclude that wave velocity is independent of wave length in shallow water.

44 A soap bubble formed by two parallel concentric rings is shown in the figure. If the rings are not too far apart, it can be shown that the function f whose graph generates this surface of revolution is a solution of the differential equation $yy'' = 1 + (y')^2$, where $y = f(x)$. If A and B are positive constants, show that $y = A\cosh Bx$ is a solution if and only if $AB = 1$. Conclude that the graph is a catenary.

Exercise 44



$\boxed{c}$ **45** Graph, on the same coordinate axes, $y = \tanh x$ and $y = \operatorname{sech}^2 x$ for $0 \le x \le 2$.

(a) Estimate the x-coordinate a of the point of intersection of the graphs.

(b) Use Newton's method to approximate a to three decimal places.

$\boxed{c}$ **46** Graph, on the same coordinate axes, $y = \cosh^2 x$ and $y = 2$.

(a) Set up integrals for estimating the centroid of the region R bounded by the graphs.

(b) Use Simpson's rule, with $n = 2$, to approximate the coordinates of the centroid of R.

Exer. 47–58: Verify the identity.

47 $\cosh x + \sinh x = e^x$

48 $\sinh(-x) = -\sinh x$

49 $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

50 $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

51 $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

52 $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

53 $\sinh 2x = 2 \sinh x \cosh x$

54 $\cosh 2x = \cosh^2 x + \sinh^2 x$

55 $\sinh^2 \dfrac{x}{2} = \dfrac{\cosh x - 1}{2}$

56 $\cosh^2 \dfrac{x}{2} = \dfrac{\cosh x + 1}{2}$

57 $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ for every positive integer n (*Hint:* Use Exercise 47.)

58 $(\cosh x - \sinh x)^n = \cosh nx - \sinh nx$ for every positive integer n

c **Exer. 59–60: Approximate to four decimal places.**

59 (a) $\sinh^{-1} 1$ (b) $\cosh^{-1} 2$
(c) $\tanh^{-1}(-\frac{1}{2})$ (d) $\operatorname{sech}^{-1} \frac{1}{2}$

60 (a) $\sinh^{-1}(-2)$ (b) $\cosh^{-1} 5$
(c) $\tanh^{-1} \frac{1}{3}$ (d) $\operatorname{sech}^{-1} \frac{3}{5}$

Exer. 61–68: Find $f'(x)$ if $f(x)$ is the given expression.

61 $\sinh^{-1} 5x$

62 $\sinh^{-1} e^x$

63 $\cosh^{-1} \sqrt{x}$

64 $\sqrt{\cosh^{-1} x}$

65 $\tanh^{-1}(-4x)$

66 $\tanh^{-1} \sin 3x$

67 $\operatorname{sech}^{-1} x^2$

68 $\operatorname{sech}^{-1} \sqrt{1 - x}$

c **Exer. 69–72: (a) Find the domain of the function. (b) Find $f'(x)$. (c) Plot f and f' in the indicated viewing window.**

69 $f(x) = \ln \cosh^{-1} 4x;\quad 0 \le x \le 10,\ 0 \le y \le 2$

70 $f(x) = \cosh^{-1} \ln 4x;\quad 0 \le x \le 10,\ 0 \le y \le 2$

71 $f(x) = \tanh^{-1}(x + 1);\quad -2 \le x \le 0,\ -3 \le y \le 5$

72 $f(x) = \tanh^{-1} x^3;\quad -1 \le x \le 1,\ -3 \le y \le 5$

Exer. 73–80: Evaluate the integral.

73 $\displaystyle\int \frac{1}{\sqrt{81 + 16x^2}}\,dx$

74 $\displaystyle\int \frac{1}{\sqrt{16x^2 - 9}}\,dx$

75 $\displaystyle\int \frac{1}{49 - 4x^2}\,dx$

76 $\displaystyle\int \frac{\sin x}{\sqrt{1 + \cos^2 x}}\,dx$

77 $\displaystyle\int \frac{e^x}{\sqrt{e^{2x} - 16}}\,dx$

78 $\displaystyle\int \frac{2}{5 - 3x^2}\,dx$

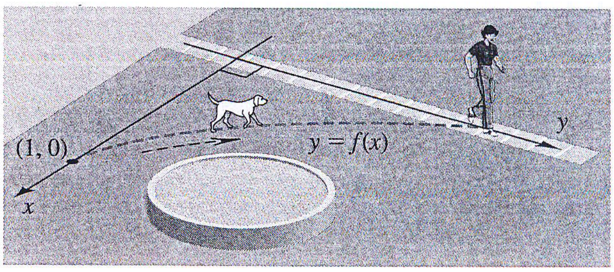
79 $\displaystyle\int \frac{1}{x\sqrt{9 - x^4}}\,dx$

80 $\displaystyle\int \frac{1}{\sqrt{5 - e^{2x}}}\,dx$

81 A point moves along the line $x = 1$ in a coordinate plane with a velocity that is directly proportional to its distance from the origin. If the initial position of the point is $(1, 0)$ and the initial velocity is 3 ft/sec, express the y-coordinate of the point as a function of time t (in seconds).

82 The rectangular coordinate system shown in the figure illustrates the problem of a dog seeking its master. The dog, initially at the point $(1, 0)$, sees its master at the point $(0, 0)$. The master proceeds up the y-axis at a constant speed, and the dog runs directly toward its master at all times. If the speed of the dog is twice that of the master, it can be shown that the path of the dog is given by $y = f(x)$, where y is a solution of the differential equation $2xy'' = \sqrt{1 + (y')^2}$. Solve this equation by first letting $z = dy/dx$ and solving $2xz' = \sqrt{1 + z^2}$, obtaining $z = \frac{1}{2}[\sqrt{x} - (1/\sqrt{x})]$. Finally, solve the equation $y' = \frac{1}{2}[\sqrt{x} - (1/\sqrt{x})]$.

Exercise 82



Exer. 83–86: Sketch the graph of the equation.

83 $y = \sinh^{-1} x$

84 $y = \cosh^{-1} x$

85 $y = \tanh^{-1} x$

86 $y = \operatorname{sech}^{-1} x$

Exer. 87–91: (a) Derive the formula. (b) and (c) Verify the formula.

87 (a) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}),\quad x \ge 1$

(b) $\dfrac{d}{dx}(\cosh^{-1} u) = \dfrac{1}{\sqrt{u^2 - 1}}\dfrac{du}{dx},\quad u > 1$

(c) $\displaystyle\int \frac{1}{\sqrt{u^2 - a^2}}\,du = \cosh^{-1}\frac{u}{a} + C,\quad 0 < a < u$

88 (a) $\tanh^{-1} x = \dfrac{1}{2}\ln\dfrac{1 + x}{1 - x},\quad |x| < 1$

(b) $\dfrac{d}{dx}(\tanh^{-1} u) = \dfrac{1}{1 - u^2}\dfrac{du}{dx},\quad |u| < 1$

(c) $\displaystyle\int \frac{1}{a^2 - u^2}\,du = \frac{1}{a}\tanh^{-1}\frac{u}{a} + C,\quad |u| < a$

89 (a) $\operatorname{sech}^{-1} x = \ln\dfrac{1 + \sqrt{1 - x^2}}{x},\quad 0 < x \le 1$

(b) $\dfrac{d}{dx}(\operatorname{sech}^{-1} u) = -\dfrac{1}{u\sqrt{1 - u^2}}\dfrac{du}{dx},\quad 0 < u < 1$

(c) $\displaystyle\int \frac{1}{u\sqrt{a^2 - u^2}}\,du = -\frac{1}{a}\operatorname{sech}^{-1}\frac{|u|}{a} + C,\quad 0 < |u| < a$