2.5: if ϕ_1, ϕ_2 and ϕ_3 are solutions of Equation (2.4), prove that:

$$\begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{vmatrix} = 0.$$

* ملاحظة مهمة:
حل الطالبة على السبورة
خاطئ.

case (1): if for any two solutions ϕ_i, ϕ_j are linearly dependent s.t. $i \neq j$:

i.e. $\phi_i = \alpha \phi_j$

$$\Rightarrow \begin{vmatrix} \phi_i & \phi_j & \phi_k \\ \phi_i' & \phi_j' & \phi_k' \\ \phi_i'' & \phi_j'' & \phi_k'' \end{vmatrix} = \begin{vmatrix} \alpha\phi_j & \phi_j & \phi_k \\ \alpha\phi_j' & \phi_j' & \phi_k' \\ \alpha\phi_j'' & \phi_j'' & \phi_k'' \end{vmatrix}$$

$$= \alpha \begin{vmatrix} \phi_j & \phi_j & \phi_k \\ \phi_j' & \phi_j' & \phi_k' \\ \phi_j'' & \phi_j'' & \phi_k'' \end{vmatrix} = 0.$$

case (2): if for any two solutions (say ϕ_1, ϕ_2) are linearly independent solutions:

i.e. $\phi_3 = c_1\phi_1 + c_2\phi_2$.

$$\Rightarrow \begin{vmatrix} \phi_1 & \phi_2 & c_1\phi_1 + c_2\phi_2 \\ \phi_1' & \phi_2' & c_1\phi_1' + c_2\phi_2' \\ \phi_1'' & \phi_2'' & c_1\phi_1'' + c_2\phi_2'' \end{vmatrix} = \begin{vmatrix} \phi_1 & 0 & c_1\phi_1 \\ \phi_1' & 0 & c_1\phi_1' \\ \phi_1'' & 0 & c_1\phi_1'' \end{vmatrix} + \begin{vmatrix} 0 & \phi_2 & c_2\phi_2 \\ 0 & \phi_2' & c_2\phi_2' \\ 0 & \phi_2'' & c_2\phi_2'' \end{vmatrix}$$

$$= 0 + 0 = 0.$$

2.17: b) $\frac{d^2}{dx^2} : \mathcal{L}^2(0,\infty) \cap \mathcal{C}^2(0,\infty) \longrightarrow \mathcal{L}^2(0,\infty)$.

we show that $m = \pm\sqrt{\lambda}$ in (a), where $\lambda \in \mathbb{C}$.

Now, we want to show that the eigen functions ~~$y = e^{\pm\sqrt{\lambda}x}$~~

$$y = c_1 \underbrace{e^{\sqrt{\lambda}x}}_{y_1} + c_2 \underbrace{e^{-\sqrt{\lambda}x}}_{y_2} \in \mathcal{L}^2(0,\infty) \cap \mathcal{C}^2(0,\infty).$$

i) Let $y_1 = e^{\sqrt{\lambda}x}$; $\lambda \in \mathbb{C}$:

$$\|y_1\|^2 = \int_0^\infty y_1 \overline{y_1}\, dx$$

$$= \int_0^\infty e^{\sqrt{\lambda}x} \overline{e^{\sqrt{\lambda}x}}\, dx$$

$$= \int_0^\infty e^{\sqrt{\lambda}x + \overline{\sqrt{\lambda}x}}\, dx$$

$$= \int_0^\infty e^{2\mathrm{Re}\sqrt{\lambda}x}\, dx$$

$$= \lim_{t\to\infty} \int_0^t e^{2\mathrm{Re}\sqrt{\lambda}x}\, dx$$

$$= \lim_{t\to\infty} \left. \frac{e^{2\mathrm{Re}\sqrt{\lambda}x}}{2\mathrm{Re}\sqrt{\lambda}} \right|_0^t$$

$$= \lim_{t\to\infty} \left(\frac{e^{2\mathrm{Re}\sqrt{\lambda}t}}{2\mathrm{Re}\sqrt{\lambda}} - \frac{1}{2\mathrm{Re}\sqrt{\lambda}} \right)$$

$$= \frac{e^{\infty} - 1}{2\mathrm{Re}\sqrt{\lambda}} = \infty$$

$\Rightarrow y_1 \notin \mathcal{L}^2(0,\infty) \Rightarrow y_1 \notin \mathcal{L}^2(0,\infty) \cap \mathcal{C}^2(0,\infty)$.

ii) Let $y_2 = e^{-\sqrt{\lambda}x}$, $\lambda \in \mathbb{C}$.

$$\|y_2\|^2 = \int_0^\infty e^{-\sqrt{\lambda}\,x}\;\overline{e^{-\sqrt{\lambda}\,x}}\,dx$$

$$= \int_0^\infty e^{-2\mathrm{Re}\sqrt{\lambda}\,x}\,dx$$

$$= \lim_{t\to\infty} \int_0^t e^{-2\mathrm{Re}\sqrt{\lambda}\,x}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{-1}{2\mathrm{Re}\sqrt{\lambda}}\, e^{-2\mathrm{Re}\sqrt{\lambda}\,x}\right]_0^t$$

$$= \lim_{t\to\infty} \left[\frac{-e^{-2\mathrm{Re}\sqrt{\lambda}\,t}}{2\mathrm{Re}\sqrt{\lambda}} + \frac{1}{2\mathrm{Re}\sqrt{\lambda}}\right]$$

$$= \frac{-e^{-2\mathrm{Re}\sqrt{\lambda}\,\infty}}{2\mathrm{Re}\sqrt{\lambda}} + \frac{1}{2\mathrm{Re}\sqrt{\lambda}}$$

$$= \frac{-1}{2\mathrm{Re}\sqrt{\lambda}\; e^{\infty}} + \frac{1}{2\mathrm{Re}\sqrt{\lambda}} = \frac{1}{2\mathrm{Re}\sqrt{\lambda}} < \infty$$

s.t. $\mathrm{Re}\sqrt{\lambda} > 0$.

$\therefore y_2 = e^{-\sqrt{\lambda}\,x} \in \mathcal{L}^2(0,\infty) \cap C^2(0,\infty)$ such that $\mathrm{Re}\sqrt{\lambda} > 0$.

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* ملاحظة هامة:

- في فقرة (a) من هذا السؤال لا نأخذ حالات لـ λ ...

- نأخذ حالات لـ λ عندما تكون المسألة Self-adjoint.

2.18: $\because p=-1,\ q=0,\ r=0 \in \mathbb{R}$

$\Rightarrow p' = 0 = q$

$\therefore \dfrac{-d^2}{dx^2}$ is formally self-adjoint.

Thm 2.14:

i) $p=-1,\ q=0,\ r=0 \in \mathbb{R}$

ii) $q = p'$

$* \ p(f'\bar{g} - f\bar{g}')\Big|_0^{\pi} = p(\pi)\left(f'(\pi)\bar{g}(\pi) - f(\pi)\bar{g}'(\pi)\right)$

$\qquad - p(0)\left(f'(0)\bar{g}(0) - f(0)\bar{g}'(0)\right)$

$\qquad = p(\pi)(0-0) - p(0)(0-0)$

$\qquad = 0$

$\Longrightarrow L$ is self-adjoint:

$\therefore L \equiv \dfrac{-d^2}{dx^2}$ satisfied the conditions of Thm. 2.14.

$$\frac{-d^2}{dx^2}u = \lambda u$$

$$\Longrightarrow \frac{d^2}{dx^2}u + \lambda u = 0$$

-suppose that $u = e^{mx}$, Then $u' = me^{mx} \to u'' = m^2 e^{mx}$

$\therefore m^2 e^{mx} + \lambda e^{mx} = 0$

$m^2 + \lambda = 0 \Longrightarrow m^2 = -\lambda \Rightarrow m = \pm\sqrt{-\lambda}$

1) if $\lambda > 0$:

$\therefore m = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\, i$

$\therefore u = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x.$

$\because u(0) = 0$

$\therefore 0 = c_1 \cos 0 + \underbrace{c_2 \sin 0}_{0} \Rightarrow \boxed{c_1 = 0}$

$\therefore u = c_2 \sin\sqrt{\lambda}\, x$

$\because u'(\pi) = 0$

$u' = c_2\sqrt{\lambda} \cos\sqrt{\lambda}\, x$

$0 = c_2\sqrt{\lambda} \cos\sqrt{\lambda}\, \pi$

$\Rightarrow \cos\sqrt{\lambda}\, \pi = 0$

$\Rightarrow \sqrt{\lambda_n}\, \pi = \left(\frac{2n+1}{2}\right)\pi \,; n \in \mathbb{N} \cup \{0\}.$

$\Rightarrow \lambda_n = \left(\frac{2n+1}{2}\right)^2 ; n \in \mathbb{N} \cup \{0\}$

$\therefore$ The eigen values are given by the sequence $\left\{ \frac{(2n+1)^2}{4} ; n \in \mathbb{N} \cup \{0\} \right\}$.

$\therefore$ The corresponding eigen fuction is given by:

$u_n = c_2 \sin\left(\frac{2n+1}{2}\right)x .$

Now, we wait to prove the (iii) from theorem (2.14).

i.e. we wat to prove that:

$$\langle u_n, u_m \rangle = 0 \text{ s.t. } u_n = c_2 \sin\left(\frac{2n+1}{2}\right)x$$

$$u_m = c_3 \sin\left(\frac{2m+1}{2}\right)x$$

$$\therefore \langle u_n, u_m \rangle = \int_0^{\pi} c_2 \sin\left(\frac{2n+1}{2}\right)x \; c_3 \sin\left(\frac{2m+1}{2}\right)x$$

$$= \frac{c_2 c_3}{2} \int_0^{\pi} \left(\cos\left(\frac{2n+1-2m-1}{2}\right)x - \cos\left(\frac{2n+1+2m+1}{2}\right)x\right) dx$$

$$= \frac{c_2 c_3}{2} \int_0^{\pi} \left(\cos(n-m)x - \cos(n+m+1)x\right) dx$$

$$= \frac{c_2 c_3}{2} \left[\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m+1)x}{n+m+1}\right]_0^{\pi}$$

$$= \frac{c_2 c_3}{2} \left[\left(\overset{\text{zero}}{\cancel{\frac{\sin(n-m)\pi}{n-m}}} - \overset{\text{zero}}{\cancel{\frac{\sin(n+m+1)\pi}{n+m+1}}}\right) - \left(\underset{\text{zero}}{\cancel{\frac{\sin(n-m)(0)}{n-m}}} - \underset{\text{zero}}{\cancel{\frac{\sin 0}{n+m+1}}}\right)\right]$$

$$= 0$$

$$\therefore u_n \perp u_m.$$

2) <u>if $\lambda = 0$:</u> $m = \pm\sqrt{-\lambda} = 0.$

$$\Longrightarrow u = c_1 + c_2 x$$

$$\because u(0) = 0 \Longrightarrow \boxed{c_1 = 0}$$

$\therefore \quad u = c_2 x$

$\because u'(\pi) = 0$

$\therefore u' = c_2$

$\Rightarrow \boxed{0 = c_2}$

$\therefore u = 0 \in \mathcal{L}^2(0,\pi) \cap C^2(0,\pi).$

also, the trivial solutions satisfied (iii) from Thm.(2.14).

3) <u>if $\lambda < 0$:</u>

$\therefore m = \pm\sqrt{-\lambda} \in \mathbb{R}.$

$\therefore u = c_1 \cosh\sqrt{-\lambda}\, x + c_2 \sinh\sqrt{-\lambda}\, x.$

$\because u(0) = 0,$

$\Rightarrow 0 = c_1 + 0 \Rightarrow \boxed{c_1 = 0}$

~~$\therefore u = c_1 \cosh\sqrt{-\lambda}\, x$~~

$\therefore u = c_2 \sinh\sqrt{-\lambda}\, x$

$\because u'(\pi) = 0 \Rightarrow u' = c_2\sqrt{-\lambda} \cosh\sqrt{-\lambda}\, x.$

$\therefore 0 = c_2 \cancel{\sinh} \sqrt{-\lambda} \cosh\sqrt{-\lambda}\, \pi$

$\because \sqrt{-\lambda} \neq 0$ and $\cosh\sqrt{-\lambda}\, \pi \neq 0$ $\left(\begin{array}{l}\underline{\text{since,}} \\ \cos\sqrt{-\lambda}\,\pi \\ = \dfrac{e^{\sqrt{-\lambda}\,\pi} + e^{-\sqrt{-\lambda}\,\pi}}{2}\end{array}\right)$

$\therefore \boxed{c_2 = 0}$