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# Adaptive Control of a Production and Maintenance System with Unknown Deterioration and Obsolescence Rates

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**Abstract**

***This paper deals with a continuous-time model of a production and maintenance system whose performance deteriorates over time. The problem of adaptive control of a production-maintenance system with unknown deterioration rate is presented. Using a feedback the scheduled production rate, preventive maintenance rate, and updating rule of deterioration and obsolescence rates will be derived.***

***Keywords:* Adaptive control, Liapunov technique, Obsolescence rate, Production inventory system, Preventive maintenance level.**

## 1. INTRODUCTION

Applications of optimal control theory to management science, in general, and to production planning, in particular, are proving to be quite fruitful; see [1]. Naturally, with the optimal control theory, optimal control techniques came to be applied to production planning problems.

The production inventory problem for manufacturing systems, which is subject to uncertainties such as demand fluctuations, machine failures, and others, has attracted the attention of numerous researchers. During the past years, a number of methods have been reported for determining economic quantities for different products on a single or multiple machines [2]-[4].

A preventive maintenance model for a production inventory system is developed in [5] using information on the systems conditions (such as finished product demand, inventory position, costs of repair and maintenance, etc.) and a continuous probability distribution characterizing the machine failure process.

A manufacturing system with preventive maintenance that produces a single part type is considered in by [6]. An inventory is maintained according to a machine-age-dependant hedging point policy. In this paper they have conjecture that, for such a system, the failure frequencies can be reduced through preventive maintenance resulting in possible increase in system performance. Traditional preventive maintenance policies, such as age replacement, periodic replacement, are usually studied without finished goods inventories. In the cases where the finished goods inventories have been considered, restrictive assumptions have used, such as not allowing breakdown during the stock build up period and during backlog situations due to the complexity of the mathematical model.

This paper studies a production-inventory model which produces a single item with a certain production rate and seeks an adaptive actual production and inventory rates. This model has a dynamic nature and an adaptive control approach seems particularly well suited to achieve its goal level and production rate. Also, the unknown deterioration rate will be derived from the conditions of asymptotic stabilization about the steady-state.

Besides adjusting its production rate, we also assume that the firm has set an inventory goal level and penalties are incurred for the inventory level to deviate from its goal and for the desired production rate to deviate from the actual production rate.

The rest of the paper is organized as follows. Following this introduction, the model is introduced in *Section 2* in the case of an infinite planning horizon. *Section 3* discusses the problem of adaptive control of a production and maintenance system. The Liapunov technique is used to prove the asymptotic stability of the system about its steady state. The conditions of the asymptotic are used to derive the schedule production rate and preventive maintenance rate. In *Section 4* we present some illustrative examples. *Section 5* concludes the paper.

## 2. MATHEMATICAL MODEL

Consider a manufacturing system that produces a single product, selling some units and adding others to inventory. We start by writing the differential equation models. These can be conveniently formulated in terms of Moreover, the following variables and parameters are used:
$x(t)$: Inventory level at time t,
$p(t)$ : Proportion of good units of end item produced at time t,
$S(t)$ : Demand rate at time t,
$u(t)$: scheduled production rate at time t,
$\theta(t)$ : Natural deterioration rate at time t,
$m(t)$ : Preventive maintenance rate applied at time t to reduce the proportion of defective units,
$\alpha(t)$ : Obsolescence rate of process performance in the absence of maintenance.
We make the following assumptions:

- There is a non-negative initial inventory level: $x(0)=x_0>0$.
- All demand must be satisfied: $x(t)>0$.
- Negative production is not allowed.
- The maintenance level is bounded by a lower limit of zero and an upper limit of $0 \le m(t) \le M$ .
- $0 \le p(t) \le 1$.
- All functions are assumed to be nonnegative, continuous and differentiable.

Assume that at the time instant t, the demand occurs at rate $S(t)$, the scheduled production at rate $u(t)$, the preventive maintenance rate $m(t)$ , and the obsolescence rate $\alpha(t)$. It follows that the inventory level $x(t)$ and the preventive maintenance rate $p(t)$ evolve according to the following first order differential equations:

$$\left.\begin{aligned}\dot{x}(t) &= p(t)u(t) - S(t) - \theta(t)x(t),\\ \dot{p}(t) &= m(t) - \big(\alpha(t)+m(t)\big)p(t),\end{aligned}\right\} \tag{1}$$

Where dot means the differentiation with respect to time and the initial state is $x(0)=x_0$, $p(0)=p_0$.
These balance equations were also used by [6]-[8] without deterioration rate $\theta$. This paper will generalize the above equations which were used by Danny *et al.* to contain deterioration rate $\theta$ for inventory items.
The steady-states of this system can be derive by setting $\dot{x}(t)=0$ and $\dot{p}(t)=0$. That is:

$$\bar{x}=\theta^{-1}(\bar{u}\,\bar{p}-\bar{S}),\ \ \bar{p}=\frac{\bar{m}}{\bar{\alpha}+\bar{m}}, \tag{2}$$

Where $\bar{x}$ , $\bar{p}$ and $\bar{S}$ are the values of inventory level, proportion of good units rate, and demand rate at the steady-state.

In what follows, we will display the numerical solution of the system (1) for the following set of parameters values:

| Par. | $u$ | S | $\alpha$ | $\theta$ | $m$ |
|---|---|---|---|---|---|
| Val. | 50 | 1 + sin (5 t) | 1 + sin (t + 2) | 0.01 | 20 |

With the following values of initial inventory level, proportion of good units: $x(t) = 50$ and $p(t) = 0.6$ respectively.

The numerical results are illustrated in Figure 1a to Figure 1c.

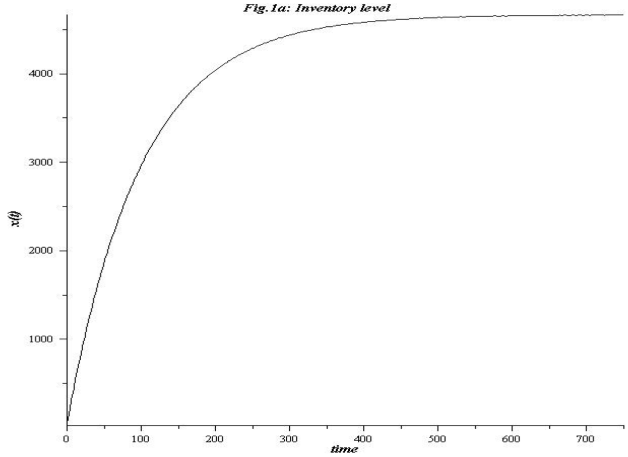



Fig.1a: Inventory level

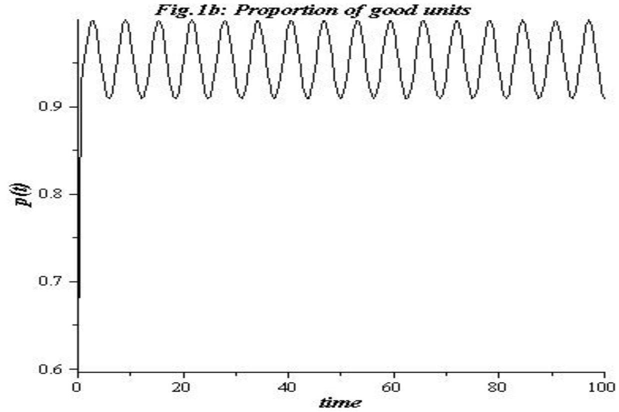



Fig.1b: Proportion of good units

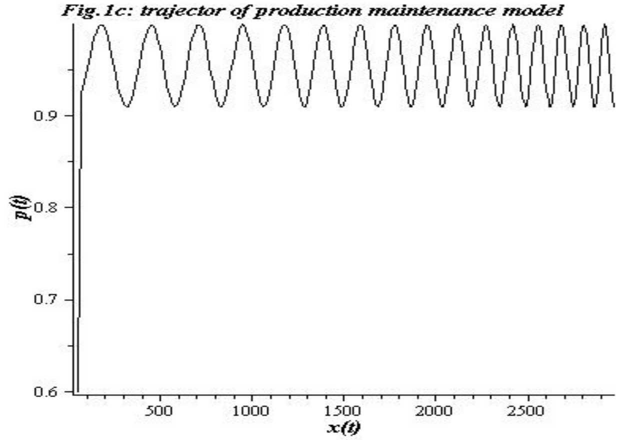



Fig.1c: trajector of production maintenance model

We conclude that the stream of the whole system in the same direction the proportion of good units.

Next, in what follows we will discuss the adaptive control of the production and maintenance model with unknown deterioration rate using Liapunov technique.

## 3. ADAPTIVE CONTROL PROBLEM

In this section, we will discuss in details the adaptive control problem of the production and maintenance model with unknown deterioration rate. Both the scheduled production rate and preventive maintenance rate will be derived. Also the updating rule of deterioration rate will be derived from the conditions of asymptotic stability of the reference model.

To study the adaptive control problem we start by considering the required values of both inventory goal level and, proportion of good units' rate as their values at the steady-state of system (1). Assume that the unknown deterioration rate $\theta$ has the estimator $\hat{\theta}(t)$ and our reference model will take the following form:

$$\left.\begin{aligned} \dot{x}(t) &= p(t)u(t) - S(t) - \hat{\theta}(t)x(t), \\ \dot{p}(t) &= m(t) - \big(\alpha(t) + m(t)\big)p(t), \end{aligned}\right\} \tag{3}$$

This system will be satisfied by the special solution (2) if the estimator of the unknown deterioration rate tends to the real values of deterioration rate. That is: $\hat{\theta}(t) = \theta$.

In order to study the adaptive control problem, we start by obtaining the perturbed system of the production and preventive maintenance model about its steady states $(\bar{x}, \bar{p})$. To obtain this perturbed system we will introduce the following new variables:

$$\left.\begin{aligned} e_1(t) &= x(t) - \bar{x}, \quad e_2(t) = p(t) - \bar{p}, \\ v_1(t) &= u(t) - \bar{u}, \quad v_2(t) = m(t) - \bar{m}, \\ \eta_1(t) &= \alpha(t) - \bar{\alpha}, \quad \eta_2(t) = \hat{\theta}(t) - \theta, \\ d(t) &= S(t) - \bar{S}, \end{aligned}\right\} \tag{4}$$

Substituting from (4) into (3) and taking into account the steady- state solution (2), one gets:

$$\left.\begin{aligned} \dot{e}_1 &= v_1(t)(\bar{p} + e_2(t)) + \bar{u}e_2(t) - (\theta + \eta_2(t))e_1(t) - \bar{x}\eta_2(t) - d(t), \\ \dot{e}_2 &= v_2(t) - (\eta_1(t) + v_2(t))(\bar{p} + e_2(t)) - (\bar{\alpha} + \bar{m})e_2(t), \end{aligned}\right\} \tag{5}$$

The system solution (5) will be used to study the adaptive control problem using the Liapunov technique.

**Theorem 1.** (*Reference [9]*). *Let* $\underline{e}(t) = \underline{0}$, *be an equilibrium point for a general nonlinear system modeled by:* $\dot{e} = f(\underline{e}, t)$ *where* $t \in \Re^+$ *and* $\underline{e}(t) \in \Re^n$ *: Let* $D \in \Re^n$ *be a domain containing the equilibrium point* $\underline{e}(t) = \underline{0}$ *and define:* $\Phi : D \to \Re$ *be a continuously differentiable function such that:* $\forall \Phi(\underline{0}) = 0$, *and* $\Phi(\underline{e}) > 0$, $\forall \underline{e} \in D - 0$:

1. *If* $\Phi(\underline{e}) \le 0$, $\forall \underline{e}(t) \in D$, *then* $\underline{e}(t) = \underline{0}$ *is stable. This means that given* $\forall \varepsilon > 0$*; r >0 ,* $\exists r \in (0, \varepsilon]$ *such that* $B_r = \{\underline{e}(t) \in \Re^n : \| \underline{e}(t) \| < r < \varepsilon\} \subset D$.
2. *If* $\Phi(\underline{e}, t) < 0$, $\forall\ \underline{e}(t) \in D - 0$, *then* $\underline{e}(t) = 0$ *is asymptotically stable, i.e.,* $\forall\ \underline{e}(0) \in D, \| \underline{e}(0) \| < \delta \| \underline{e}(t) \| \to 0$ *as* $t \to \infty$.

*Using this theorem, we will show that our closed loop system is asymptotically stable.*

The following theorem gives the desired production rate and updating rule for deterioration rate ensure the asymptotic stability of the reference reverse logistic inventory model with uncertain different types deterioration and disposal rate.

***Theorem 2.*** *The solution* $(\bar{x}, \bar{p})$ *of the system (1) is asymptotically stable in Liapunov since if and only if:*

1. *there exist a production rate and preventive maintenance rate:*

$$\left.\begin{aligned} u(t) &= \bar{u} + \frac{S(t)-\bar{S}-k_1(x(t)-\bar{x})-\bar{u}(p(t)-\bar{p})}{p(t)}, \\ m(t) &= \bar{m} - \frac{k_2(p(t)-\bar{p})}{1-p(t)}, \end{aligned}\right\} \qquad (6)$$

2. *there exist two continuous functions such that :*

$$\left.\begin{aligned} \alpha(t) &= \bar{\alpha} + \left[c_1 + \int_0^t f_1(p,v)\,dv\right] e^{-l_1 t}, \\ \hat{\theta}(t) &= \theta + \left[c_2 + \frac{\bar{x}^2 e^{l_2 t}}{l_2} + \int_0^t f_2(x,v)\,dv\right] e^{-l_2 t}, \end{aligned}\right\} \qquad (7)$$

*Where,* $k_i,\ l_i\ (i=1,2)$ *are positive real control gains parameters that can be select by the firm, while* $c_1$ *and* $c_2$ *are two real constants, and* $f_1$*, and* $f_2$ *are:*

$$\left.\begin{aligned} f_1(p,t) &= p(t)(p(t)-\bar{p})e^{l_1 t}, \\ f_2(x,t) &= \left(x^2(t)-\bar{x}\,x(t)\right)e^{l_2 t}, \end{aligned}\right\} \qquad (8)$$

***Proof:*** *The proof of this theorem is based on the choosing of the suitable Liapunov function for the system perturbed system which consists of the two systems (5) and (7).*
*Assume this function has the form:*

$$\Phi(e_1,e_2,\eta_2) = \sum_{i=1}^{2}\left(e_i^2+\eta_i^2\right), \qquad (9)$$

*This function is a positive definite function of the var*iables $e_i(t)$, (i=1,2) and $\eta_2(t)$. The total time derivative of the Liapunov function (8) along the trajectory of the systems (5) and (7) gives:

$$\dot{\Phi} = -\left[(\theta+k_1)e_1^2+(\bar{\alpha}+\bar{m}+k_2)e_2^2+l_1\eta_1^2+l_2\eta_2^2\right] \le 0, \qquad (10)$$

Since, $\theta+k_1>0,\ \bar{\alpha}+\bar{m}+k_2>0,\ l_i>0,\ (i=1,2)$, then the total time $\dot{\Phi}$ of the Liapunov function is a negative definite function of the variables $e_i(t),\ \eta_2(t),\ (i=1,2)$, so $e_i(t)=0,$ and $v_i(t)=0,$ where $(i=1,2)$, and $\eta_2(t)=0$ the special solution is asymptotically stable in the Liapunov sense. This completes the proof.

**Lemma:** *There are always schedule production rate u(t) and preventive maintenance rate m(t) for all t>0. The proof of this Lemma is obvious. It follows immediately from the second condition of theorem 1.*

Therefore, using the nonlinear feedback (6) for scheduled production rate, and preventive maintenance rate and updating rule for the unknown deterioration rate, the production and maintenance system with unknown deterioration is asymptotically stable about its steady state.

Now substituting the adaptive controlled the scheduled production rate and preventive maintenance rate (6) into (5) we get the following controlled system:

$$\left.\begin{aligned} \dot{x}(t) &= -k_1(x(t)-\bar{x})-\hat{\theta}(t)x(t)+\theta\bar{x} \\ \dot{p}(t) &= -k_2(p(t)-\bar{p})-(\bar{\alpha}+\bar{m})(p(t)-\bar{p})-(\alpha(t)-\bar{\alpha})p(t) \\ \dot{\alpha}(t) &= p(t)(p(t)-\bar{p})-l_1(\alpha(t)-\bar{\alpha}) \\ \dot{\hat{\theta}}(t) &= x^2(t)-\bar{x}\,x(t)-l_2(\hat{\theta}(t)-\theta) \end{aligned}\right\} \qquad (11)$$

The above nonlinear system of differential equations is used to study the time evolution of inventory level, the proportion of good units' rate and dynamic estimators of deterioration rate. It appears from (10) that the analytical solution of the system is difficult to derive since it is non-linear and therefore we solve it numerically in the next section.

4. Numerical Solution

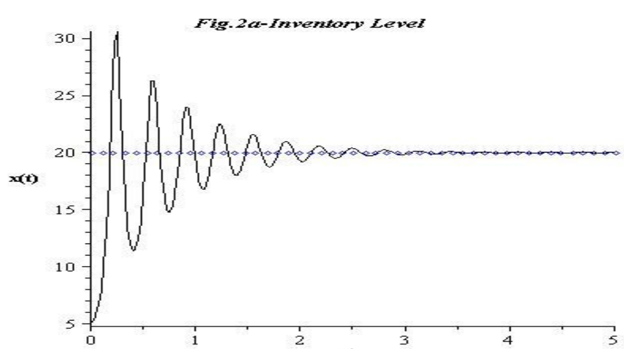
To illustrate the solution procedure described in *section 3,* we carry out the numerical solution of the system (10) using numerical integration approach. The objective of this section is to study and discuss the numerical solution of the problem to determine an adaptive control strategy for a production and preventive maintenance inventory model with unknown deterioration and obsolescence rates. The numerical solution algorithm is based on numerical integration of the system (10) using the Runge-Kutta method. This numerical solution will displayed graphically for different values of the parameters. The sensitively analysis of numerical solution of the system (10) is discussed for different cases of the system parameters and initial states.

### *A. Constant demand rate*

In this example we discuss the numerical solution of the perturbation of constant demand rate.
The following set of parameter values is assumed:

| Par. | $\bar{u}$ | S | $S$ | $\bar{\alpha}$ | $\theta$ | $\bar{m}$ |
|---|---|---|---|---|---|---|
| Val. | 10 | 25 | 15 | 20 | 0.125 | 25 |
| Par. | $\bar{p}$ | $\bar{x}$ | $k_1$ | $k_2$ | $l_1$ | $l_2$ |
| Val. | 10 | 20 | 0.35 | 0.25 | 0.01 | 2.5 |

with the following values of initial inventory level, proportion of good units: $x(0)=5$, $p(0)=0.7$, $\alpha(0)=20.02$, and $\theta(0)=0.2$ respectively.
The numerical results are illustrated in Figure 2a to Figure 2f.



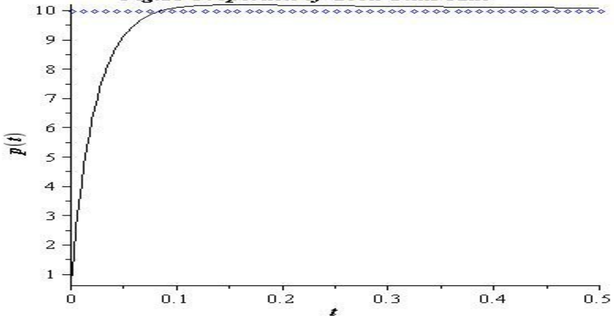



Fig.2b-Proportion of Good Units Rate

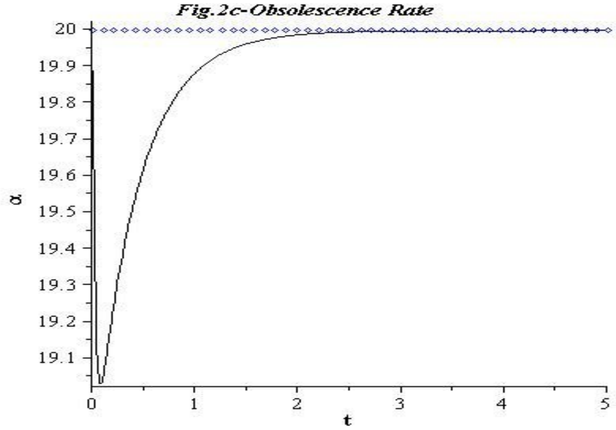



Fig.2c-Obsolescence Rate

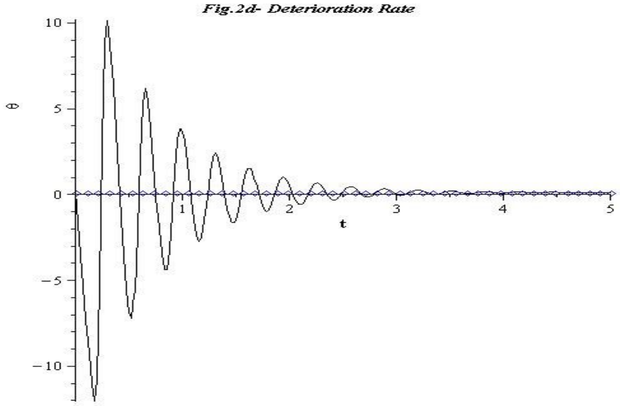



Fig.2d- Deterioration Rate

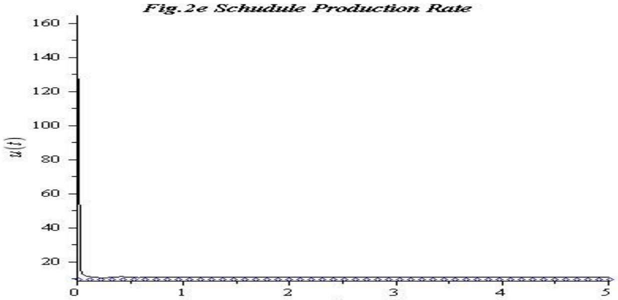
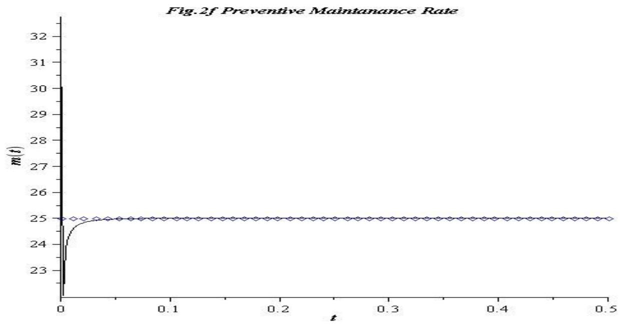



Fig.2e Schudule Production Rate




We can see that the inventory level and the deterioration rate are oscillating about their goal levels and rates, respectively. While the proportion of good units' rate, the obsolescence rate, the schedule production rate, and the preventive maintenance rate are exponentially tend to their goal rates.

*B. Exponential demand rate*

In this example we discuss the numerical solution of the perturbation of exponential demand rate.
The following set of parameter values is assumed:

| Par. | $\bar{u}$ | S | $S$ | $\bar{\alpha}$ | $\theta$ | $\bar{m}$ |
|---|---|---|---|---|---|---|
| Val. | 5 | $3e^{-3t}$ | 3 | 0.5 | 0.25 | 50 |
| Par. | $\bar{p}$ | $\bar{x}$ | $k_1$ | $k_2$ | $l_1$ | $l_2$ |
| Val. | 10 | 5 | 3 | 2 | 1 | 2 |

With the following values of initial inventory level, proportion of good units: $x(0)=5$, $p(0)=0.7$, $\alpha(0)=20.02$, and $\theta(0)=0.2$ respectively.
The numerical results are illustrated in Figure 3a to Figure 3f.

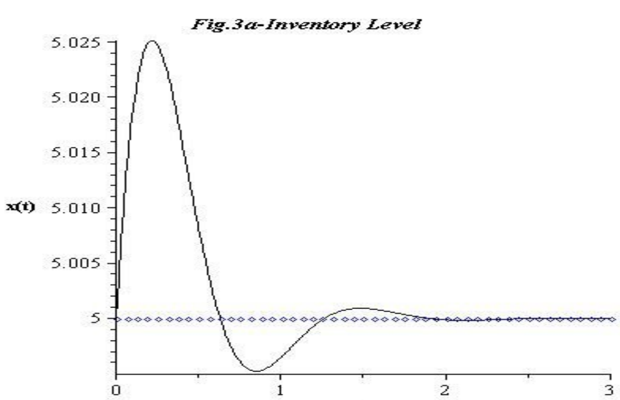


*Fig.3b-Proportion of Good Units Rate*

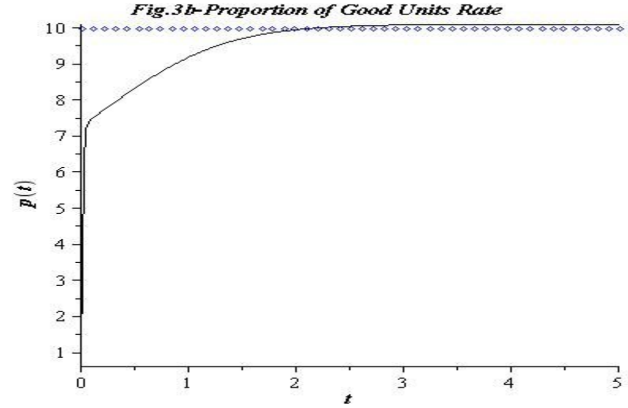



*Fig.3c-Obsolescence Rate*

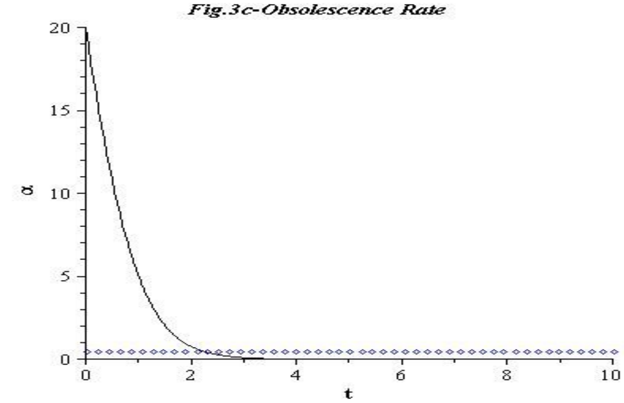



*Fig.3d- Deterioration Rate*

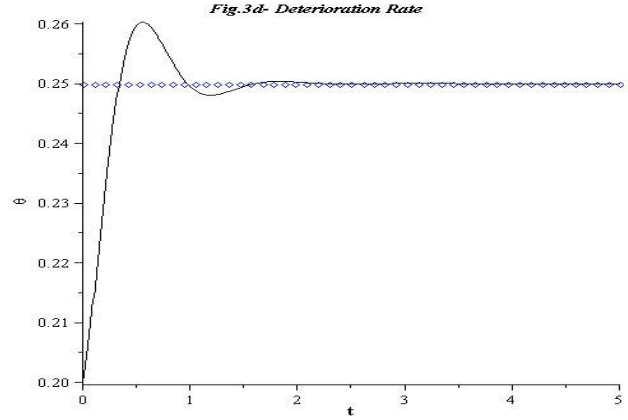
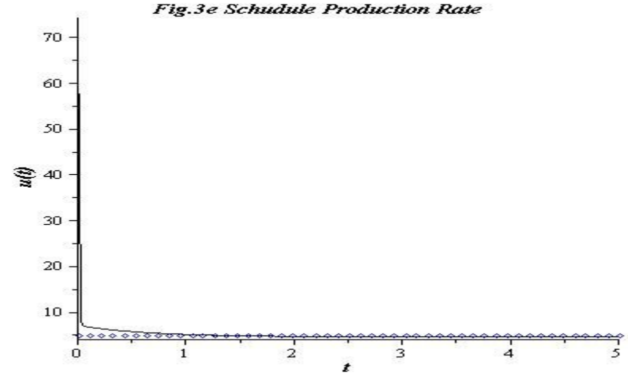
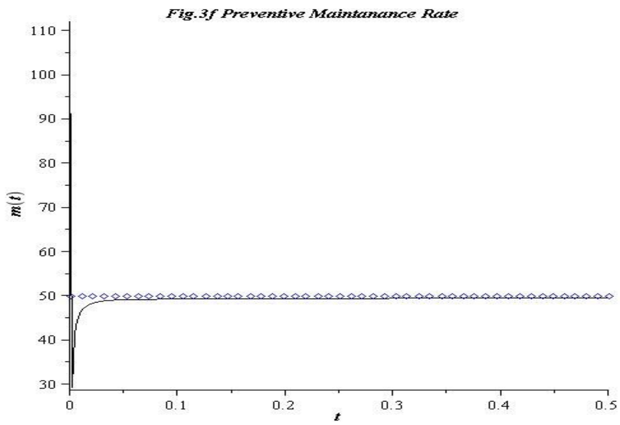



*Fig.3e Schudule Production Rate*






We can see that the inventory level and the deterioration rate are oscillating about their goal levels and rates, respectively. While the proportion of good units' rate, the obsolescence rate, the schedule production rate, and the preventive maintenance rate are exponentially tend to their goal rates.

## 5. CONCLUSION

In this paper we discussed the problem of adaptive control of a production and maintenance system with unknown deterioration and obsolescence rates. The numerical solution showed that the inventory level and the deterioration rate oscillate about their goal levels and rates, respectively. While the proportion of good units rate, the obsolescence rate, the schedule production rate, and the preventive maintenance rate are exponentially tends to their goal rates. The preventive maintenance rate and the obsolescence rate are obtained.

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