M 106

Integral Calculus and Applications

Contents

1 The Indefinite Integrals 4

1.1 Antiderivatives and Indefinite Integrals 4

1.1.1 Antiderivatives 4

1.1.2 Indefinite Integrals 5

1.2 Properties of the Indefinite Integral 7

1.3 Integration By Substitution 9

2 The Definite Integrals 16

2.1 Summation Notation 16

2.2 Riemann Sum and Area 19

2.3 Properties of the Definite Integral 23

2.4 The Fundamental Theorem of Calculus 26

2.5 Numerical Integration 33

2.5.1 Trapezoidal Rule 33

2.5.2 Simpson's Rule 35

3 Logarithmic and Exponential Functions 44

3.1 The Natural Logarithmic Function 44

3.1.1 Properties of the Natural Logarithmic Function 45

3.1.2 Differentiating and Integrating the Natural Logarithmic Function 46

3.2 The Natural Exponential Function 50

3.2.1 Properties of the Natural Exponential Function 50

3.2.2 Differentiating and Integrating the Natural Exponential Function 51

3.3 General Exponential and Logarithmic Functions 53

3.3.1 General Exponential Function 53

3.3.2 General Logarithmic Function 56

4 Inverse Trigonometric and Hyperbolic Functions 62

4.1 Inverse Trigonometric Functions 62

4.2 Hyperbolic Functions 66

4.2.1 Properties of the Hyperbolic Functions 67

4.2.2 Differentiating and Integrating the Hyperbolic Functions 70

4.3 Inverse Hyperbolic Functions 72

4.3.1 Properties the Inverse Hyperbolic Functions 72

4.3.2 Differentiating and Integrating the Inverse Hyperbolic Functions 75

5 Techniques of Integration 81

5.1 Integration by Parts 81

5.2 Trigonometric Functions **84**
5.2.1 Integration of Powers of Trigonometric Functions . . . 84
5.2.2 Integration of Forms sin ux cos vx, sin ux sin vx and cos ux cos vx . . . 88
5.3 Trigonometric Substitutions **89**
5.4 Integrals of Rational Functions **92**
5.5 Integrals Involving Quadratic Forms **95**
5.6 Miscellaneous Substitutions **97**
5.6.1 Fractional Functions in sin x and cos x . . . 97
5.6.2 Integrals of Fractional Powers . . . 98
5.6.3 Integrals of Form $\sqrt[n]{f(x)}$ . . . 99

6 Indeterminate Forms and Improper Integrals . . . 103
6.1 Indeterminate Forms **103**
6.2 Improper Integrals **107**
6.2.1 Infinite Intervals . . . 108
6.2.2 Discontinuous Integrands . . . 109

7 Application of Definite Integrals . . . 113
7.1 Areas **113**
7.2 Solids of Revolution **117**
7.3 Volumes of Solids of Revolution **119**
7.3.1 Disk Method . . . 119
7.3.2 Washer Method . . . 122
7.3.3 Method of Cylindrical Shells . . . 128
7.4 Arc Length and Surfaces of Revolution **131**
7.4.1 Arc Length . . . 132
7.4.2 Surfaces of Revolution . . . 134

8 Parametric Equations and Polar Coordinates . . . 142
8.1 Parametric Equations of Plane Curves **142**
8.1.1 Tangent Lines . . . 144
8.1.2 Arc Length and Surface Area of Revolution . . . 146
8.2 Polar Coordinates System **150**
8.2.1 The Relationship between Rectangular and Polar Coordinates . . . 151
8.2.2 Tangent Line to Polar Curves . . . 153
8.2.3 Graphs in Polar Coordinates . . . 154
8.3 Area in Polar Coordinates **161**
8.3.1 Arc Length and Surface Area of Revolution in Polar Coordinates . . . 166

Appendix . . . 173
Appendix (1): Basic Mathematical Concepts **173**

Appendix . . . 187
Appendix (1): Integration Rules and Integrals Table **187**
Appendix (2): Answers to Exercises **190**

Homework . . . 207

Chapter 1

The Indefinite Integrals

1.1 Antiderivatives and Indefinite Integrals

We begin with the definition of the antiderivatives and indefinite integrals. Then, we provide basic integration rules.

1.1.1 Antiderivatives

Definition 1.1 A function F is called an antiderivative of f on an interval I if

$$F^{'}(x) = f(x) \text{ for every } x \in I.$$

Example 1.1

(1) Let $F(x) = x^2 + 3x + 1$ and $f(x) = 2x + 3$.
Since $F^{'}(x) = f(x)$, then the function $F(x)$ is an antiderivative of $f(x)$.
(2) Let $G(x) = \sin\ x + x$ and $g(x) = \cos\ x + 1$.
Since $G^{'}(x) = \cos\ x + 1$, then the function $G(x)$ is an antiderivative of $g(x)$.

If $F(x)$ is an antiderivative of $f(x)$, then every function $F(x)+c$ is also antiderivative of $f(x)$, where c is a constant. The upcoming theorem states that any antiderivative $G(x)$, which is different from $F(x)$ can be expressed as $F(x)+c$ where c is an arbitrary constant.

Theorem 1.1 If functions F and G are antiderivatives of a function f on an interval I, there exists a constant c such that $G(x) = F(x) + c$.

Proof. Let H be a function defined as follows:

$$H(x) = G(x) - F(x)\ \forall x \in I$$

where F and G are antiderivatives of the function f. Let $a, b \in I$ such that $a < b$. Since F and G are antiderivatives of f, then

$$H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$$

for every $x \in I$. Since the function H is differentiable, it is continuous. From the mean value theorem on $[a,b]$, there is a number $c \in (a,b)$ such that

$$H'(c) = \frac{H(b) - H(a)}{b - a}.$$

[1] Since $H'(x) = 0$ on I, then $H'(c) = 0$. This implies $H(a) = H(b)$ and this means H is a constant function. ■

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[1] If f is continuous on $[a,b]$ and differentiable on (a,b), there exists a number $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Example 1.2 Let $f(x) = 2x$. The functions

$F(x) = x^2 + 2,$

$G(x) = x^2 - \frac{1}{2},$

$H(x) = x^2 - \sqrt[3]{2},$

are antiderivatives of the function f. Therefore, $F(x) = x^2 + c$ is a general form of the antiderivatives of the function $f(x) = 2x$.

Example 1.3 Find the general form of the antiderivatives of $f(x) = 6x^5$.

Solution:

If $F(x) = x^6$, then $F'(x) = 6x^5$. The function $F(x) = x^6 + c$ is the general antiderivative of f.

1.1.2 Indefinite Integrals

From Theorem 1.1, if the function $F(x) + c$ is an antiderivative of $f(x)$, then there exist no antiderivatives in different forms for the function $f(x)$. This leads us to define the indefinite integral.

Definition 1.2 Let f be a continuous function on an interval I. The indefinite integral of f is the general antiderivative of f on I:

$$\int f(x)\, dx = F(x) + c.$$

The function f is called the integrand, the symbol $\int$ is the integral sign, x is called the variable of the integration and c is the constant of the integration.

Now, by using the previous definition, the general antiderivatives in Example 1.1 are

1 $\int (2x+3)\, dx = x^2 + 3x + c.$

2 $\int (\cos\, x + 1)\, dx = \sin\, x + x + c.$

We can now work out how to evaluate some integrals. To do that, we should remember differentiation rules of some functions.

Basic Integration Rules

■ Rule 1: Power of x.

$$\frac{d}{dx}x^{n+1} = (n+1)x^n, \text{ so } \int (n+1)x^n\, dx = x^{n+1} + c$$

Generally, for $n \neq -1$,

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c.$$

In words, to integrate the function x^n, we add 1 to the power and divide the function by $n+1$. If $n = 1$, we have a special case

$$\int 1\, dx = x + c.$$

■ Rule 2: Trigonometric functions.

$$\frac{d}{dx}\sin\, x = \cos\, x, \text{ so } \int \cos\, x\, dx = \sin\, x + c$$

$$\frac{d}{dx}\cos\, x = -\sin\, x, \text{ so } \int -\sin\, x\, dx = \cos\, x + c$$

Therefore, $\int \sin\, x\, dx = -\cos\, x + c.$

The other trigonometric functions with the previous rules are listed in the following table:

Derivative	Indefinite Integral
$\frac{d}{dx}(x) = 1$	$\int 1\ dx = x + c$
$\frac{d}{dx}(\frac{x^{n+1}}{n+1}) = x^n, n \neq -1$	$\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}(\sin\ x) = \cos\ x$	$\int \cos\ x\ dx = \sin\ x + c$
$\frac{d}{dx}(-\cos\ x) = \sin\ x$	$\int \sin\ x\ dx = -\cos\ x + c$
$\frac{d}{dx}(\tan\ x) = \sec^2\ x$	$\int \sec^2\ x\ dx = \tan\ x + c$
$\frac{d}{dx}(-\cot\ x) = \csc^2\ x$	$\int \csc^2\ x\ dx = -\cot\ x + c$
$\frac{d}{dx}(\sec\ x) = \sec\ x\ \tan\ x$	$\int \sec\ x\ \tan\ x\ dx = \sec\ x + c$
$\frac{d}{dx}(-\csc\ x) = \csc\ x\ \cot\ x$	$\int \csc\ x\ \cot\ x\ dx = -\csc\ x + c$

Table 1.1: The list of basic integration rules.

Example 1.4 Evaluate the integral.

(1) $\int x^{-3}\ dx$

(2) $\int \frac{1}{\cos^2\ x}\ dx$

Solution:

(1) $\int x^{-3}\ dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$

(2) $\int \frac{1}{\cos^2\ x}\ dx = \int \sec^2\ x\ dx = \tan\ x + c.$ $\quad(\sec\ x = \frac{1}{\cos\ x} \Rightarrow \sec^2\ x = \frac{1}{\cos^2\ x})$

Note that we sometimes need to express an integrand in a form in which we can recognize its derivative like item 2 in the previous example.

Exercise 1.1

1 - 8 Evaluate the integral.

1 $\int \frac{1}{\sqrt{x}}\ dx$

2 $\int \frac{1}{x^{\frac{5}{4}}}\ dx$

3 $\int \frac{1}{\sin^2\ x}\ dx$

4 $\int -\ \csc^2\ x\ \tan^2\ x\ dx$

5 $\int \frac{1}{\sqrt[5]{x}}\ dx$

6 $\int \frac{\tan x}{\cos\ x}\ dx$

7 $\int \frac{\sqrt{x}}{x^3}\ dx$

8 $\int \sqrt{\sin^4\ x}\ \csc\ x\ dx$

1.2 Properties of the Indefinite Integral

In this section, we list some properties of the indefinite integrals.

Theorem 1.2 Assume f and g have antiderivatives on an interval I, then

1. $\frac{d}{dx}\int f(x)\,dx = f(x)$.
2. $\int \frac{d}{dx}(F(x))\,dx = F(x)+c$.
3. $\int \big(f(x)\pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.
4. $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a constant.

Proof. For items 1 and 2, let F be an antiderivative of f.

1. $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\big(F(x)+c\big) = f(x)$.
2. $\int \frac{d}{dx}(F(x))\,dx = \int f(x)\,dx = F(x)+c$.
3. Let F and G be antiderivatives of f and g, respectively. By differentiating the left side, we have
$$\begin{aligned}\frac{d}{dx}\Big(\int \big(f(x)\pm g(x)\big)\,dx\Big) &= \frac{d}{dx}\big(F(x)\pm G(x)\big)\\ &= f(x)\pm g(x).\end{aligned}$$
Hence, $\int \big(f(x)\pm g(x)\big)\,dx = F(x)\pm G(x)+c_1$.
From the right side, we have
$$\int f(x)\,dx \pm \int g(x)\,dx = F(x)\pm G(x)+c_2$$
For any special case, we can choose the values of the constants such that $c_1=c_2$ and this prove item 3.
4. By differentiating the left side, we have
$$\begin{aligned}\frac{d}{dx}\Big(\int kf(x)\,dx\Big) = k\frac{d}{dx}\Big(\int f(x)\,dx\Big) &= k\frac{d}{dx}(F(x))\\ &= kf(x).\end{aligned}$$
Hence, $\int kf(x)\,dx = kF(x)+c_1$.
From the right side, we have
$$k\int f(x)\,dx = kF(x)+c_2$$
We can choose the values of the constants such that $c_1=c_2$ and this prove item 4. ■

In the following example, we use the previous properties and the table of the basic integration rules to evaluate some indefinite integrals.

Example 1.5 Evaluate the integral.

(1) $\int (4x+3)\,dx$

(2) $\int (2\sin\,x+3\cos\,x)\,dx$

(3) $\int (\sqrt{x}+\sec^2\,x)\,dx$

(4) $\int \frac{d}{dx}(\sin\, x)\, dx$
(5) $\frac{d}{dx}\int \sqrt{x+1}\, dx$

Solution:

(1) $\int (4x+3)\, dx = \frac{4x^2}{2} + 3x + c = 2x^2 + 3x + c.$
(2) $\int (2\sin\, x + 3\cos\, x)\, dx = -2\cos\, x + 3\sin\, x + c.$
(3) $\int (\sqrt{x} + \sec^2\, x)\, dx = \frac{x^{\frac{3}{2}}}{3/2} + \tan\, x + c = \frac{2x^{\frac{3}{2}}}{3} + \tan\, x + c.$
(4) $\int \frac{d}{dx}(\sin\, x)\, dx = \sin\, x + c.$
(5) $\frac{d}{dx}\int \sqrt{x+1}\, dx = \sqrt{x+1}.$

Example 1.6 If $\int f(x)\, dx = x^2 + c_1$ and $\int g(x)\, dx = \tan\, x + c_2$, find $\int \big(3f(x) - 2g(x)\big)\, dx$.

Solution:

From the third and fourth properties, $\int \big(3f(x) - 2g(x)\big)\, dx = 3\int f(x)\, dx - 2\int g(x)\, dx = 3x^2 - 2\tan\, x + c$, where $c = 3c_1 - 2c_2$.

Example 1.7 Solve the differential equation $f'(x) = x^3$ subject to the initial condition $f(0) = 1$.

Solution:

$$\int f'(x)\, dx = \int x^3\, dx$$
$$\Rightarrow f(x) = \frac{1}{4}x^4 + c.$$

If $x = 0$, then $f(0) = \frac{1}{4}(0)^4 + c = 1$ and this implies $c = 1$. Hence, the solution of the differential equation is $f(x) = \frac{1}{4}x^4 + 1$.

Example 1.8 Solve the differential equation $f'(x) = 6x^2 + x - 5$ subject to the initial condition $f(0) = 2$.

Solution:

$$\int f'(x)\, dx = \int (6x^2 + x - 5)\, dx$$
$$\Rightarrow f(x) = 2x^3 + \frac{1}{2}x^2 - 5x + c.$$

Use the condition $f(0) = 2$ by substituting $x = 0$ into the function $f(x)$. We have

$$f(0) = 0 + 0 - 0 + c = 2 \Rightarrow c = 2.$$

Therefore, the solution of the differential equation is $f(x) = 2x^3 + \frac{1}{2}x^2 - 5x + 2$.

Example 1.9 Solve the differential equation $f''(x) = 5\cos\, x + 2\sin\, x$ subject to the initial conditions $f(0) = 3$ and $f'(0) = 4$.

Solution:

$$\int f''(x)\, dx = \int (5\cos\, x + 2\sin\, x)\, dx$$
$$\Rightarrow f'(x) = 5\sin\, x - 2\cos\, x + c$$

Using the condition $f'(0) = 4$ gives

$$f'(0) = 0 - 2 + c = 4 \Rightarrow c = 6.$$ (use values of the trigonometric functions given on page 180)

Hence, $f'(x) = 5\sin\,x - 2\cos\,x + 6$. Now, again

$$\int f'(x)\,dx = \int (5\sin\,x - 2\cos\,x + 6)\,dx$$
$$\Rightarrow f(x) = -5\cos\,x - 2\sin\,x + 6x + c.$$

Use the condition $f(0) = 3$ by substituting $x = 0$ into $f(x)$. We obtain

$$f(0) = -5 - 0 + 0 + c = 3 \Rightarrow c = 8.$$

Hence, the solution of the differential equation is $f(x) = -5\cos\,x - 2\sin\,x + 6x + 8$.

Notes:

■ We can always check our answers by differentiating the results.

■ In the previous examples, we use x as a variable of the integration. However, for this role, we can use any variable such as y, z, t, etc . That is, instead of $f(x)\,dx$, we can integrate $f(y)\,dy$ or $f(t)\,dt$.

■ The properties of the indefinite integral and the table of the basic integrals are elementary for simple functions. Meaning that, for more complex functions, we need some techniques to simplify the integrals. Section 1.3, we shall provide one of these techniques.

Exercise 1.2

1 - 10 ■ Evaluate the integral.

1 $\int \sqrt{x^5}\,dx$

2 $\int (x^{\frac{3}{4}} + x^2 + 1)\,dx$

3 $\int x(x^3 + 2x + 1)\,dx$

4 $\int (x^2 + \sec^2\,x)\,dx$

5 $\int (\csc^2\,x - \sqrt{x})\,dx$

6 $\int \frac{3\sin^2\,x + 4}{\sin^2\,x}\,dx$

7 $\int \frac{x^2 - 1}{x^4}\,dx$

8 $\int \left(4x^{\frac{2}{5}} - 2x^{\frac{2}{3}} + x\right)\,dx$

9 $\int \left(\frac{3}{x^3} + \frac{2}{x^4} + 1\right)\,dx$

10 $\int \frac{x^2 + x + 1}{\sqrt[3]{x}}\,dx$

11 - 12 ■ Evaluate.

11 $\frac{d}{dx}\left(\int \sqrt{\cos^3\,x + 1}\,dx\right)$

12 $\int \frac{d}{dx}\left(\sqrt{\cos^3\,x + 1}\right)\,dx$

13 - 17 ■ Solve the differential equation subject to the given conditions.

13 $f'(x) = 4x^3 + 2x + 1$; $f(0) = 1$.

14 $f''(x) = \sin\,x + 2\cos\,x$; $f(0) = 1$ and $f'(0) = 3$.

15 $f'(x) = \sqrt{x}$; $f(0) = 0$.

16 $f'(x) = \cos x$; $f(\pi) = 1$.

17 $f'(x) = \sec^2 x$; $f(\frac{\pi}{4}) = 0$.

1.3 Integration By Substitution

The integration by substitution (known as u-substitution) is a technique for solving some composite functions. The method is based on changing the variable of the integration to obtain a simple indefinite integral. The following theorem shows how the substitution technique works.

Theorem 1.3 Let g be a differentiable function on an interval I where the derivative is continuous. Let f be continuous on the interval J contains the range of the function g. If F is an antiderivative of the function f on J, then

$$\int f(g(x))g^{'}(x)\,dx = F(g(x)) + c,\ \ x \in I.$$

Proof. Since F is an antiderivative of f, then $\frac{d}{dx}F(g(x)) = F^{'}(g(x))g^{'}(x) = f(g(x))g^{'}(x)$. Hence,

$$\int f(g(x))g^{'}(x)\,dx = \int \frac{d}{dx}F(g(x))\,dx = F(g(x)) + c. \ \blacksquare$$

The task here is to recognize whether an integrand has the form $f(g(x))g^{'}(x)$. The following two examples explain this task.

Example 1.10 Evaluate the integral $\int 2x\,(x^2+1)^3\,dx$.

Solution:

We can use the previous theorem as follows:

let $f(x) = x^3$ and $g(x) = x^2+1$, then $f\big(g(x)\big) = (x^2+1)^3$. Since $g^{'}(x) = 2x$, then from Theorem 1.3, we have

$$\int 2x(x^2+1)^3\,dx = \frac{(x^2+1)^4}{4} + c.$$

We can end with the same solution by using the five steps of the substitution method given below.

Steps of the integration by substitution:

Step 1: Choose a new variable u.

Step 2: Determine the value of du.

Step 3: Make the substitution i.e., eliminate all occurrences of x in the integral by making the entire integral is in terms of u.

Step 4: Evaluate the new integral.

Step 5: Return the evaluation to the initial variable x.

In Example 1.10, let $u = x^2+1$, then $du = 2x\,dx$. By substituting that into the original integral, we have

$$\int u^3\,du = \frac{u^4}{4} + c.$$

Now, by returning the evaluation to the initial variable x, we have $\int 2x(x^2+1)^3\,dx = \frac{(x^2+1)^4}{4} + c$.

Example 1.11 Evaluate the integral $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$.

Solution:

We use Theorem 1.3 for the integral $2\int \frac{\sec^2\sqrt{x}}{2\sqrt{x}}\,dx$. Let $f(x) = \sec^2 x$ and $g(x) = \sqrt{x}$, then $f\big(g(x)\big) = \sec^2\sqrt{x}$. Since $g^{'}(x) = 1/(2\sqrt{x})$, then we have

$$\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx = 2\tan\sqrt{x} + c.$$

By using the steps of the substitution method, let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\,dx$. By substitution, we obtain

$$2\int \sec^2 u\,du = 2\tan u + c = 2\tan\sqrt{x} + c.$$

Example 1.12 Evaluate the integral $\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx$.

Solution:

Let $u = x^3-3x+1$, then $du = 3(x^2-1)\,dx$. By substitution, we have

$$\frac{1}{3}\int u^{-6}\,du = \frac{1}{3}\,\frac{1}{-5u^5}+c = \frac{-1}{15(x^3-3x+1)^5}+c.$$

The upcoming corollary simplifies the process of the substitution method for some functions.

Corollary 1.1 If $\int f(x)\,dx = F(x)+c$, then for any $a \neq 0$,

$$\int f(ax\pm b)\,dx = \frac{1}{a}F(ax\pm b)+c.$$

Proof. To verify the previous result, it is sufficient to choose the variable $u = ax\pm b$, then $du = a\,dx$. This implies $dx = \frac{1}{a}\,du$. By substitution, we have

$$\int f(ax\pm b)\,dx = \int f(u)\,\frac{du}{a} = \frac{1}{a}\int f(u)\,du = \frac{1}{a}F(u) = \frac{1}{a}F(ax\pm b)+c. \blacksquare$$

Example 1.13 Evaluate the integral.

(1) $\int \sqrt{2x-5}\,dx$ **(2)** $\int \cos\,(3x+4)\,dx$

Solution:

From Corollary 1.1, we have

(1) $\int \sqrt{2x-5}\,dx = \frac{1}{2}\,\frac{(2x-5)^{3/2}}{3/2}+c = \frac{(2x-5)^{3/2}}{3}+c.$

(2) $\int \cos\,(3x+4)\,dx = \frac{1}{3}\sin\,(3x+4)+c.$

Notes:

■ The substitution method turns the integral into a simpler integral involving the variable u. The new integral can be evaluated by using either the table of the basic integrals or other techniques of the integration.

■ When using the substitution method, we need to return to the original variable. All examples above expressed in terms of the original variable x.

■ Students should distinguish between integrals that can be evaluated by the substitution method. We must choose u so that du is already sitting in the integrand, regardless of a constant k. For example, the integral $\int \cos\,x^2\,dx$ cannot be evaluated by the substitution method. To see this, let $u = x^2$, this implies $du = 2x\,dx$. However, the term x is not in the integrand. Therefore, the integral cannot be evaluated by the substitution method.

■ The substitution method may be used as a first step in simplifying an integral. It might be followed by other techniques given in Chapter 5.

Exercise 1.3

1 - 16 ■ Evaluate the integral.

1 $\int x\sqrt{1+x^2}\,dx$

2 $\int x\sqrt{x-1}\,dx$

3 $\int x^2\sqrt{x-1}\,dx$

4 $\int \frac{\tan x}{\cos^2 x}\,dx$

5 $\int \sin^5 x \cos x\,dx$

6 $\int \frac{x}{\sqrt{2x^2+1}}\,dx$

7 $\int \cos t\sqrt{1-\sin t}\,dt$

8 $\int \frac{\cos^3 x}{\csc x}\,dx$

9 $\int \cos(3x+4)\,dx$

10 $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx$

11 $\int \sec 4x \tan 4x\,dx$

12 $\int \frac{\sqrt{\cot x}}{\sin^2 x}\,dx$

13 $\int (1+\frac{1}{t})t^{-2}\,dt$

14 $\int \frac{x}{\sqrt{2x-1}}\,dx$

15 $\int x^2(4x^3-6)^7\,dx$

16 $\int \sin^2 3x \cos 3x\,dx$

Review Exercises

1 - 34 Evaluate the integral.

1 $\int 2x\,dx$

2 $\int (3x^2+1)\,dx$

3 $\int (\frac{1}{2}x^3+x)\,dx$

4 $\int (x^4+x^3)\,dx$

5 $\int (x^2+3x-1)\,dx$

6 $\int (1-2x-5x^3)\,dx$

7 $\int \frac{1}{x^2}\,dx$

8 $\int \sqrt{x^5}\,dx$

9 $\int \frac{1}{\sqrt{x^3}}\,dx$

10 $\int (x-1)(x+1)\,dx$

11 $\int (2x^3-3\sqrt{x}+\frac{4}{x^5})\,dx$

12 $\int \sqrt[5]{1+x}\,dx$

13 $\int (x^3+1)(x-1)\,dx$

14 $\int \frac{x^2-1}{x-1}\,dx$

15 $\int \frac{x-3}{\sqrt{x}}\,dx$

16 $\int \frac{x+1}{\sqrt[3]{x}}\,dx$

17 $\int (x^3-1)^2\,dx$

18 $\int \sin\,(x+1)\,dx$

19 $\int (\cos\,x-x)\,dx$

20 $\int (\sec^2\,x-4)\,dx$

21 $\int (\sec\,x\,\tan\,x+x)\,dx$

22 $\int (\csc^2\,x+x^2+1)\,dx$

23 $\int \frac{1}{\cos^2\,x}\,dx$

24 $\int \frac{1}{\sin^2\,x}\,dx$

25 $\int \frac{\tan\,x}{\cos\,x}\,dx$

26 $\int \sec\,x\,(\tan\,x-\sec\,x)\,dx$

27 $\int (2+\tan^2\,x)\,dx$ (Hint: $\tan^2\,x=\sec^2\,x-1$)

28 $\int \frac{\cos\,x}{\sin^2\,x}\,dx$

29 $\int \frac{\tan\,x}{\cos^2\,x}\,dx$

30 $\int \sin\,x\,\sec^2\,x\,dx$

31 $\int \cos\,x\,\csc^2\,x\,dx$

32 $\int \sec\,x\,(\sec\,x+2\tan\,x)\,dx$

33 $\int \csc\,x\,(\csc\,x+3\cot\,x)\,dx$

34 $\int \sin\,x\,\sqrt{\cos^3\,x}\,dx$

35 - 64 ■ Evaluate the integral.

35 $\int x^4(3x^5+1)^{10}\,dx$

36 $\int x\sqrt{x^2+1}\,dx$

37 $\int (2x+1)\sqrt{x^2+x+2}\,dx$

38 $\int (x^2-1)\sqrt[3]{x^3-3x+2}\,dx$

39 $\int (5x+1)(5x^2+2x-5)^3\,dx$

40 $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

41 $\int \frac{\sin^2\sqrt{x}}{\sqrt{x}\cos^2\sqrt{x}}\,dx$

42 $\int \frac{\cos^2\sqrt{x}}{\sqrt{x}\sin^2\sqrt{x}}\,dx$

43 $\int \frac{\sin 2x}{\cos^2 2x}\,dx$

44 $\int \frac{\cos\sqrt{x}}{\sqrt{x}\sin^2\sqrt{x}}\,dx$

45 $\int x\sin x^2\,dx$

46 $\int \frac{x}{\cos^2 x^2}\,dx$

47 $\int \frac{x+1}{\sin^2(x^2+2x-1)}\,dx$

48 $\int \frac{\csc^2\sqrt[3]{x}}{\sqrt[3]{x^2}}\,dx$

49 $\int \frac{\sec^2(\sqrt[5]{x}+1)}{\sqrt[5]{x^4}}\,dx$

50 $\int \frac{x}{\sqrt{x^2+9}}\,dx$

51 $\int \frac{x}{\sqrt[3]{x^2-1}}\,dx$

52 $\int \cos^2 x\sin x\,dx$

53 $\int \frac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}\,dx$

54 $\int \frac{\cos^3\sqrt{x}\sin\sqrt{x}}{\sqrt{x}}\,dx$

55 $\int \frac{2+\cos x}{\sin^2 x}\,dx$

56 $\int \frac{x}{\sqrt[5]{x+1}}\,dx$

57 $\int x\sqrt{x-3}\,dx$

58 $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)^3}\,dx$

59 $\int \frac{2-x}{\sqrt{x}\sqrt{4-x}}\,dx$

60 $\int \sin x\left(\cos^2(x+1)\right)dx$

61 $\int \frac{\sin 2x}{(5+\cos 2x)^3}\,dx$

62 $\int \frac{x^3}{\sqrt{x^4-1}}\,dx$

63 $\int \frac{x^{\frac{3}{4}}}{\sqrt[4]{x}}\,dx$

64 $\int \frac{\sec\sqrt[3]{x}\tan\sqrt[3]{x}}{x^{\frac{2}{3}}}\,dx$

65 - 70 Choose the correct answer.

65 The value of the integral $\int \frac{\sin\ x}{\sqrt{2+\cos\ x}}\,dx$ is equal to

(a) $-2\sqrt{2+\cos\ x}+c$
(b) $\sqrt{2+\cos\ x}+c$
(c) $-\sqrt{2+\cos\ x}+c$
(d) $2\sqrt{2+\cos\ x}+c$

66 The value of the integral $\int \frac{\sin\ (\tan\ x)}{\cos^2\ x}\,dx$ is equal to

(a) $\cos\ (\tan\ x)+c$
(b) $\sin\ (\tan\ x)+c$
(c) $-\cos\ (\tan\ x)+c$
(d) $-\sin\ (\tan\ x)+c$

67 The integral $\int x\sqrt{x^2+1}\,dx$ is equal to

(a) $\frac{1}{2}x^2\sqrt{x^2+1}+c$
(b) $\frac{2}{3}(x^2+1)^{\frac{3}{2}}+c$
(c) $-\frac{2}{3}(x^2+1)^{\frac{3}{2}}+c$
(d) $\frac{1}{3}(x^2+1)^{\frac{3}{2}}+c$

68 The integral $\int \frac{x}{\cos^2\ x^2}\,dx$ is equal to

(a) $\frac{1}{2}\tan\ x^2+c$
(b) $\tan\ x^2+c$
(c) $\frac{1}{2}\tan\ x+c$
(d) $-\frac{1}{\cos\ x^2}+c$

69 The value of the integral $\int \frac{\sec^2\ x}{\cot^2\ x}\,dx$ is equal to

(a) $\frac{1+\cos^2\ x}{3\cos^3\ x}+c$
(b) $\frac{1-3\cos^2\ x}{3\cos^3\ x}+c$
(c) $\frac{\cot^4\ x}{4}+c$
(d) $\frac{\tan^3\ x}{3}+c$

70 The value of the integral $\int \frac{\cos\ x}{\sqrt{4+\sin\ x}}\,dx$

(a) $\frac{1}{2}\sqrt{\sin\ x+4}+c$
(b) $\sqrt{\sin\ x+4}+c$
(c) $2\sqrt{\sin\ x+4}+c$
(d) $-2\sqrt{\sin\ x+4}+c$

Chapter 2

The Definite Integrals

2.1 Summation Notation

Summation (or sigma notation) is a simple form used to give a concise expression for a sum of values.

Definition 2.1 Let $\{a_1, a_2, ..., a_n\}$ be a set of numbers. The symbol $\sum_{k=1}^{n} a_k$ represents their sum:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n.$$

Example 2.1 Evaluate the sum.

(1) $\sum_{i=1}^{3} i^3$

(2) $\sum_{j=1}^{4} (j^2+1)$

(3) $\sum_{k=1}^{3} (k+1)k^2$

Solution:

(1) $\sum_{i=1}^{3} i^3 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36.$

(2) $\sum_{j=1}^{4} (j^2+1) = (1^2+1) + (2^2+1) + (3^2+1) + (4^2+1) = 2 + 5 + 10 + 17 = 34.$

(3) $\sum_{k=1}^{3} (k+1)k^2 = (1+1)(1)^2 + (2+1)(2)^2 + (3+1)(3)^2 = 2 + 12 + 36 = 50.$

Theorem 2.1 Let $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_n\}$ be sets of real numbers. If n is any positive integer, then

1. $\sum_{k=1}^{n} c = \underbrace{c + c + ... + c}_{\text{n-times}} = nc$ for any $c \in \mathbb{R}$.

2. $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k.$

3. $\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k$ for any $c \in \mathbb{R}$.

Example 2.2 Evaluate the sum.

(1) $\sum_{k=1}^{10} 15$

(2) $\sum_{k=1}^{4} (k^2+2k)$

(3) $\sum_{k=1}^{3} 3(k+1)$

Solution:

(1) $\sum_{k=1}^{10} 15 = (10)(15) = 150.$

(2) $\sum_{k=1}^{4} (k^2+2k) = \sum_{k=1}^{4} k^2 + 2\sum_{k=1}^{4} k = (1^2+2^2+3^2+4^2)+2(1+2+3+4) = 30+20 = 50.$

(3) $\sum_{k=1}^{3} 3(k+1) = 3\sum_{k=1}^{3} (k+1) = 3(2+3+4) = 27.$

In the following theorem, we present summations of some polynomial expressions. They will be used later in a Riemann sum.

Theorem 2.2

1. $\sum_{k=1}^{n} k = 1+2+3+...+n = \frac{n(n+1)}{2}$
2. $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+...+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Proof. We prove this theorem by induction.

1. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
   (a) If $n=1$, then both left and right sides equal 1.
   (b) Assume the equality holds for n, that is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. We want to prove that the equality holds for $n+1$. The right side for $n+1$ is $\frac{(n+1)(n+2)}{2}$. The left side is

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}.$$

   Hence, the result follows.

2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
   (a) If $n=1$, then both left and right sides equal 1.
   (b) Assume the equality holds for n i.e., $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. The task is to prove the equality for $n+1$. The right side for $n+1$ is $\frac{(n+1)(n+2)(2n+3)}{6}$. The left side for $n+1$ is

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$
$$= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}.$$

3. $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.
   (a) If $n=1$, then both left and right sides equal 1.

(b) Assume the equality holds for n i.e., $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$. We want to prove the equality for $n+1$. The right side for $n+1$ is $\left[\frac{(n+1)(n+2)}{2}\right]^2$ and the left side is

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 = \frac{(n+1)^2(n^2+4n+4)}{4} = \left[\frac{(n+1)(n+2)}{2}\right]^2.$$

Hence, the formula is proved. ■

Example 2.3 Evaluate the sum.

(1) $\sum_{k=1}^{100} k$ **(2)** $\sum_{k=1}^{10} k^2$ **(3)** $\sum_{k=1}^{10} k^3$

Solution:

(1) $\sum_{k=1}^{100} k = \frac{100(100+1)}{2} = 5050.$

(2) $\sum_{k=1}^{10} k^2 = \frac{10(11)(21)}{6} = 385.$

(3) $\sum_{k=1}^{10} k^3 = \left[\frac{10(11)}{2}\right]^2 = 3025.$

Example 2.4 Express the sum in terms of n.

(1) $\sum_{k=1}^{n} (k+1)$

(2) $\sum_{k=1}^{n} (k^2-k-1)$

Solution:

(1) $\sum_{k=1}^{n} (k+1) = \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}.$

(2) $\sum_{k=1}^{n} (k^2-k-1) = \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = -\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} - n = \frac{n(n^2-4)}{3}.$

Exercise 2.1

1 - 6 Evaluate the sum.

1 $\sum_{i=1}^{3} (i+1)$

2 $\sum_{j=0}^{5} j^2$

3 $\sum_{k=1}^{4} \frac{k}{k+1}$

4 $\sum_{i=1}^{10} 5i$

5 $\sum_{k=1}^{30} 4$

6 $\sum_{j=1}^{3} (3-2j)^2$

7 - 9 Express the sum in terms of n.

7 $\sum_{k=1}^{n} (k-1)$

8 $\sum_{k=1}^{n} (k^2+1)$

9 $\sum_{k=1}^{n} (k^3+2k^2-k+1)$

2.2 Riemann Sum and Area

A Riemann sum is a mathematical form used in this book to approximate the area of a region underneath the graph of a function. Before start-up in this issue, we provide some basic definitions.

Definition 2.2 A set $P = \{x_0, x_1, x_2, ..., x_n\}$ is called a partition of a closed interval $[a,b]$ if for any positive integer n,

$$a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b.$$

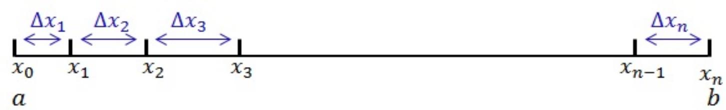


Figure 2.1: A partition of the interval $[a,b]$.

Notes:

- The division of the interval $[a,b]$ by the partition P generates n subintervals: $[x_0,x_1],[x_1,x_2],[x_2,x_3],...,[x_{n-1},x_n]$.
- The length of each subinterval $[x_{k-1},x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
- The union of subintervals gives the whole interval $[a,b]$.

Definition 2.3 The norm of the partition of P is the largest length among $\Delta x_1, \Delta x_2, \Delta x_3, ..., \Delta x_n$ i.e.,

$$\| P \| = max\{\Delta x_1, \Delta x_2, \Delta x_3, ..., \Delta x_n\}.$$

Example 2.5 If $P = \{0, 1.2, 2.3, 3.6, 4\}$ is a partition of the interval $[0,4]$, find the norm of the partition P.

Solution:

We need to find the subintervals and their lengths.

Subinterval $[x_{k-1},x_k]$	Length Δx_k
$[0,1.2]$	$1.2-0=1.2$
$[1.2,2.3]$	$2.3-1.2=1.1$
$[2.3,3.6]$	$3.6-2.3=1.3$
$[3.6,4]$	$4-3.6=0.4$

The norm of P is the largest length among

$$\{\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4\}.$$

Hence, $\| P \| = \Delta x_3 = 1.3$

Remark 2.1
1. The partition P of the interval $[a,b]$ is regular if $\Delta x_0 = \Delta x_1 = \Delta x_2 = ... = \Delta x_n = \Delta x$.
2. For any positive integer n, if the partition P is regular then

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_k = x_0 + k\,\Delta x.$$

Indeed, let P be a regular partition of the interval $[a,b]$. Since $x_0 = a$ and $x_n = b$, then

$$x_1 = x_0 + \Delta x\ ,$$
$$x_2 = x_1 + \Delta x = (x_0 + \Delta x) + \Delta x = x_0 + 2\Delta x\ ,$$
$$x_3 = x_2 + \Delta x = (x_0 + 2\Delta x) + \Delta x = x_0 + 3\Delta x.$$

By continuing doing so, we have $x_k = x_0 + k\ \Delta x$.

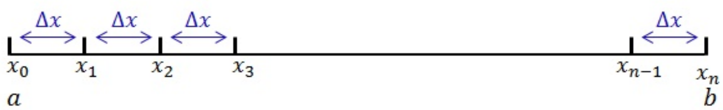


Figure 2.2: A regular partition of the interval $[a,b]$.

Example 2.6 Define a regular partition P that divides the interval $[1,4]$ into 4 subintervals.

Solution:

Since P is a regular partition of $[1,4]$ where $n = 4$, then

$$\Delta x = \frac{4-1}{4} = \frac{3}{4} \ \text{and}\ \ x_k = 1 + k\ \frac{3}{4}.$$

Therefore,

$x_0 = 1$
$x_1 = 1 + \frac{3}{4} = \frac{7}{4}$
$x_2 = 1 + 2(\frac{3}{4}) = \frac{5}{2}$
$x_3 = 1 + 3(\frac{3}{4}) = \frac{13}{4}$
$x_4 = 1 + 4(\frac{3}{4}) = 4$

The regular partition is $P = \{1, \frac{7}{4}, \frac{5}{2}, \frac{13}{4}, 4\}$.

Now, we are ready to define a Riemann sum .

Definition 2.4 Let f be a function defined on a closed interval $[a,b]$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a mark on the partition P where $\omega_k \in [x_{k-1}, x_k]$, $k = 1, 2, 3, \ldots, n$. Then, a Riemann sum of f for P is

$$R_p = \sum_{k=1}^{n} f(\omega_k)\Delta x_k.$$

As shown in Figure 2.3, the amount $f(\omega_1)\Delta x_1$ is the area of the rectangle A_1, $f(\omega_2)\Delta x_2$ is the area of the rectangle A_2 and so on. The sum of these areas approximates the area of the whole region under the graph of the function f from $x = a$ to $x = b$.

This indicates that if f is a defined and positive function on a closed interval $[a,b]$ and P is a partition of that interval where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a mark on the partition P, then the Riemann sum estimates the area of the region under f from $x = a$ to $x = b$. As the number of the subintervals increases $n \to \infty$ (i.e., $||P|| \to 0$), the estimation becomes better. Therefore,

$$A = \lim_{\|P\| \to 0} R_p = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(\omega_k)\Delta x_k$$

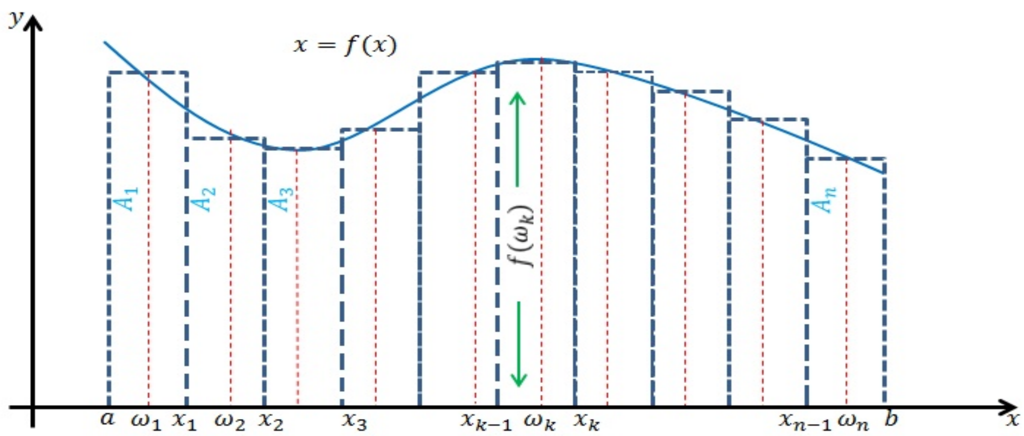


Figure 2.3: A region under a function f from $x = a$ to $x = b$.

Example 2.7 Find a Riemann sum R_p of the function $f(x) = 2x - 1$ for the partition $P = \{-2, 0, 1, 4, 6\}$ of the interval $[-2, 6]$ by choosing the mark,

(1) the left-hand endpoint,
(2) the right-hand endpoint,
(3) the midpoint.

Solution:

(1) Choose the left-hand endpoint of each subinterval.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\,\Delta x_k$
$[-2,0]$	$0-(-2)=2$	-2	-5	-10
$[0,1]$	$1-0=1$	0	-1	-1
$[1,4]$	$4-1=3$	1	1	3
$[4,6]$	$6-4=2$	4	7	14
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				6

(2) Choose the right-hand endpoint of each subinterval.

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\,\Delta x_k$
$[-2,0]$	$0-(-2)=2$	0	-1	-2
$[0,1]$	$1-0=1$	1	1	1
$[1,4]$	$4-1=3$	4	7	21
$[4,6]$	$6-4=2$	6	11	22
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				42

(3) Choose the midpoint of each subinterval.[1]

Subintervals	Length Δx_k	ω_k	$f(\omega_k)$	$f(\omega_k)\,\Delta x_k$
$[-2,0]$	$0-(-2)=2$	-1	-3	-6
$[0,1]$	$1-0=1$	0.5	0	0
$[1,4]$	$4-1=3$	2.5	4	12
$[4,6]$	$6-4=2$	5	9	18
$R_p = \sum_{k=1}^{4} f(\omega_k)\Delta x_k$				24

[1] $\omega_k = \frac{x_{k-1}+x_k}{2}$.

Example 2.8 Let A be the area under the graph of $f(x) = x+1$ from $x = 1$ to $x = 3$. Find the area A by taking the limit of the Riemann sum such that the partition P is regular and the mark ω is the right-hand endpoint of each subinterval.

Solution:

For a regular partition P, we have

1. $\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$, and
2. $x_k = x_0 + k\,\Delta x$ where $x_0 = 1$.

Since the mark ω is the right endpoint of each subinterval, then $\omega_k = x_k = 1 + \frac{2k}{n}$. Therefore,

$$f(\omega_k) = (1+\frac{2k}{n})+1 = \frac{2k}{n}+2 = \frac{2}{n}(n+k).$$

From Definition 2.4,

$$\begin{aligned} R_p = \sum_{k=1}^{n} f(\omega_k)\Delta x_k &= \frac{4}{n^2}\sum_{k=1}^{n}(n+k) \\ &= \frac{4}{n^2}\left[n^2 + \frac{n(n+1)}{2}\right] \\ &= 4 + \frac{2(n+1)}{n}. \end{aligned}$$

(1) $\sum_{k=1}^{n}(n+k) = \sum_{k=1}^{n} n + \sum_{k=1}^{n} k$
(2) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Hence, $\lim_{n\to\infty} R_p = 4+2 = 6$.

The following definition shows that the definite integral of a defined function f on a closed interval $[a,b]$ is a Riemann sum when $\| P \| \to 0$.

Definition 2.5 Let f be a defined function on a closed interval $[a,b]$ and let P be a partition of $[a,b]$. The definite integral of f on $[a,b]$ is

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_k f(\omega_k)\Delta x_k$$

if the limit exists. The numbers a and b are called the limits of the integration.

Example 2.9 Evaluate the integral $\int_2^4 (x+2)\,dx$.

Solution:

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a regular partition of the interval $[2,4]$, then $\Delta x = \frac{4-2}{n} = \frac{2}{n}$ and $x_k = x_0 + \Delta x$. Also, let the mark ω be the right endpoint of each subinterval, so $\omega_k = x_k = 2 + \frac{2k}{n}$ and then $f(\omega_k) = \frac{2}{n}(2n+k)$.

The Riemann sum of f for P is

$$R_p = \sum_k f(\omega_k)\Delta x_k = \frac{4}{n^2}\sum_k (2n+k) = \frac{4}{n^2}(2n^2 + \frac{n(n+1)}{2}) = 8 + \frac{2(n+1)}{n}.$$

From Definition 2.5, $\int_2^4 (x+2)\,dx = \lim_{n\to\infty} R_p = 8 + \lim_{n\to\infty} \frac{2n(n+1)}{n^2} = 8+2 = 10.$

Exercise 2.2

1 - 8 ■ If P is a partition of the interval $[a,b]$, find the norm of the partition P.

1 $P = \{-1, 0, 1.3, 4, 4.1, 5\}, \quad [-1, 5]$

2 $P = \{0, 0.5, 1, 2.5, 3.1, 4\}, \quad [0, 4]$

3 $P = \{-3, 0, 2.3, 4.6, 4.8, 5.5, 6\}, \quad [-3, 6]$

4 $P = \{-2, 0, 2.3, 3, 3.5, 4\}, \quad [-2, 4]$

5 $P = \{3, 3.5, 3.6, 4, 4.9, 7\}, \quad [3, 7]$

6 $P = \{-2, 0, 1.3, 2, 2.5, 3.4, 5.5\}, \quad [-2, 5.5]$

7 $P = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}, \quad [-1, 2]$

8 $P = \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\}, \quad [0, \pi]$

9 - 12 ■ Define a regular partition P that divides the interval $[a,b]$ into n subintervals.

9 $[a,b] = [0,3] \quad n = 5$

10 $[a,b] = [-1,4] \quad n = 6$

11 $[a,b] = [-4,4] \quad n = 8$

12 $[a,b] = [0,1] \quad n = 4$

13 - 15 ■ Find a Riemann sum R_p of the function $f(x) = x^2 + 1$ for the partition $P = \{0, 1, 3, 4\}$ of the interval $[0, 4]$ by choosing the mark,

13 the left-hand endpoint,

14 the right-hand endpoint,

15 the midpoint.

16 - 19 ■ Let A be the area under the graph of f from a to b. Find the area A by taking the limit of a Riemann sum such that the partition P is regular and the mark ω is the right-hand endpoint of each subinterval.

16 $f(x) = x/3 \quad a = 1, \ b = 2$

17 $f(x) = x - 1 \quad a = 0, \ b = 3$

18 $f(x) = 5 - x^2 \quad a = -1, \ b = 1$

19 $f(x) = x^3 - 1 \quad a = 0, \ b = 4$

2.3 Properties of the Definite Integral

In this section, we present some properties of the definite integral.

Theorem 2.3

1. $\displaystyle\int_a^b c\, dx = c(b-a)$,

2. $\displaystyle\int_a^a f(x)\, dx = 0$ if $f(a)$ exists.

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be a mark on P.

1. Let f be a constant function defined by $f(x) = c$. From Definition 2.5,

$$\begin{aligned}
\int_a^b c\, dx &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} c\, \Delta x_k \\
&= \lim_{\|P\|\to 0} c \sum_{k=1}^{n} \Delta x_k && \text{(property 3 on page 16)} \\
&= \lim_{\|P\|\to 0} c(b-a) && \text{($\textstyle\sum_k \Delta x_k$ is the length of the interval $[a,b]$)} \\
&= c(b-a).
\end{aligned}$$

2. From Definition 2.5,

$$\int_a^a f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(\omega_k)\ (0) \qquad (\Delta x_k = 0 \text{ for } k = 1,2,3,\dots,n)$$
$$= \lim_{\|P\|\to 0}\sum_{k=1}^{n} (0)$$
$$= \lim_{\|P\|\to 0} 0 = 0. \blacksquare$$

Theorem 2.4

1. If f and g are integrable on $[a,b]$, then $f+g$ and $f-g$ are integrable on $[a,b]$ and

$$\int_a^b \big(f(x)\pm g(x)\big)\,dx = \int_a^b f(x) \pm \int_a^b g(x)\,dx.$$

2. If f is integrable on $[a,b]$ and $k\in\mathbb{R}$, then $k\,f$ is integrable on $[a,b]$ and

$$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx.$$

Proof. Let $P=\{x_0,x_1,\dots,x_n\}$ be a partition of $[a,b]$ and let $\omega=(\omega_1,\omega_2,\dots,\omega_n)$ be a mark on P.

1. From Definition 2.5,

$$\int_a^b (f\pm g)(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n}(f\pm g)(\omega_k)\Delta x_k$$
$$= \lim_{\|P\|\to 0}\Big(\sum_{k=1}^{n} f(\omega_k)\Delta x_k \pm \sum_{k=1}^{n} g(\omega_k)\Delta x_k\Big)$$
$$= \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(\omega_k)\Delta x_k \pm \lim_{\|P\|\to 0}\sum_{k=1}^{n} g(\omega_k)\Delta x_k$$
$$= \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

2. From Definition 2.5,

$$\int_a^b kf(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} kf(\omega_k)\Delta x_k = \lim_{\|P\|\to 0} k\sum_{k=1}^{n} f(\omega_k)\Delta x_k$$
$$= k\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(\omega_k)\Delta x_k$$
$$= k\int_a^b f(x)\,dx. \blacksquare$$

Theorem 2.5

1. If f and g are integrable on $[a,b]$ and $f(x)\geq g(x)$ for all $x\in[a,b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

2. If f is integrable on $[a,b]$ and $f(x)\geq 0$ for all $x\in[a,b]$, then

$$\int_a^b f(x)\,dx \geq 0.$$

Proof. Let $P=\{x_0,x_1,\dots,x_n\}$ be a partition of $[a,b]$ and let $\omega=(\omega_1,\omega_2,\dots,\omega_n)$ be a mark on P.

1. Since $f(x) \geq g(x)$ for all $x \in [a,b]$, then $f(\omega_k) \geq g(\omega_k)\ \forall\ k = 1,2,\ldots,n$. Hence,

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\omega_k)\Delta x_k \geq \lim_{\|P\|\to 0} \sum_{k=1}^{n} g(\omega_k)\Delta x_k$$
$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

2. Since $f(x) \geq 0$ for all $x \in [a,b]$, then $f(\omega_k) \geq 0\ \forall\ k = 1,2,\ldots,n$. Hence,

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(\omega_k)\Delta x_k \geq 0$$
$$\Rightarrow \int_a^b f(x)\,dx \geq 0. \blacksquare$$

Theorem 2.6 If f is integrable on the intervals $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ contains $c = x_k$ and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be a mark on P. Assume $P_1 = \{x_0, x_1, \ldots, x_k\}$ is a partition of $[a,c]$ with a mark $u = (\omega_1, \omega_2, \ldots, \omega_k)$ and $P_2 = \{x_{k+1}, x_{k+2}, \ldots, x_n\}$ is a partition of $[c,b]$ with a mark $v = (\omega_{k+1}, \omega_{k+2}, \ldots, \omega_n)$.

Now, if $\| P \| \to 0$, then $\| P_1 \| \to 0$ and $\| P_2 \| \to 0$. Also,

$$\lim_{\|P\|\to 0} \sum_{[a,b]} f(\omega_k)\Delta x_k = \lim_{\|P_1\|\to 0} \sum_{[a,c]} f(\omega_k)\Delta x_k + \lim_{\|P_2\|\to 0} \sum_{[c,b]} f(\omega_k)\Delta x_k$$
$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \blacksquare$$

Theorem 2.7 If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Proof. From Theorems 2.6 and 2.3 (item 2), we have

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0.$$

Therefore,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx. \blacksquare$$

Example 2.10 Evaluate the integral.

(1) $\displaystyle\int_0^2 3\,dx$ **(2)** $\displaystyle\int_2^2 (x^2+4)\,dx$

Solution:

(1) $\displaystyle\int_0^2 3\,dx = 3(2-0) = 6.$

(2) $\displaystyle\int_2^2 (x^2+4)\,dx = 0.$

Example 2.11 If $\int_a^b f(x)\,dx = 4$ and $\int_a^b g(x)\,dx = 2$, then find $\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\,dx$.

Solution:

$$\int_a^b \left(3f(x) - \frac{g(x)}{2}\right)\,dx = 3\int_a^b f(x)\,dx - \frac{1}{2}\int_a^b g(x)\,dx = 3(4) - \frac{1}{2}(2) = 11.$$

Example 2.12 Prove that $\int_0^2 (x^3+x^2+2)\,dx \geq \int_0^2 (x^2+1)\,dx$ without evaluating the integrals.

Solution: Let $f(x) = x^3+x^2+2$ and $g(x) = x^2+1$. We can find that $f(x)-g(x) = x^3+1 > 0$ for all $x \in [0,2]$. This implies that $f(x) > g(x)$ and from Theorem 2.5, we have

$$\int_0^2 (x^3+x^2+2)\,dx \geq \int_0^2 (x^2+1)\,dx.$$

Exercise 2.3

1 - 2 ■ Evaluate the integral.

1 $\int_0^5 7\,dx$

2 $\int_1^1 \sqrt{3x^2+1}\,dx$

3 - 6 ■ Verify the inequality without evaluating the integrals.

3 $\int_1^2 (3x^2+4)\,dx \geq \int_1^2 (2x^2+5)\,dx$

4 $\int_1^4 (2x+2)\,dx \leq \int_1^4 (3x+1)\,dx$

5 $\int_2^4 (x^2-6x+8)\,dx \leq 0$

6 $\int_0^{2\pi} (1+\sin\, x)\,dx \geq 0$

7 - 10 ■ If $\int_a^b f(x)\,dx = 2$ and $\int_a^b g(x)\,dx = 3$, then find

7 $\int_a^b \left(6f(x) - \frac{g(x)}{3}\right)\,dx$

8 $\int_b^a \big(f(x)+g(x)\big)\,dx$

9 $\int_a^a \sqrt{(f.g)(x)}\,dx$

10 $\int_c^a f(x)\,dx + \int_b^c f(x)\,dx$ where $c \in (a,b)$

■

2.4 The Fundamental Theorem of Calculus

In this section, we formulate one of the most important results of calculus, the Fundamental Theorem. This theorem links together the notions of integrals and derivatives.

Theorem 2.8 Suppose that f is continuous on the closed interval $[a,b]$.

1. If $F(x) = \int_a^x f(t)\,dt$ for every $x \in [a,b]$, then $F(x)$ is an antiderivative of f on $[a,b]$.
2. If $F(x)$ is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

The proof of this theorem is given on page 29.

The Fundamental Theorem simplifies the process of calculating the definite integrals. The following corollary shows how the definite integral can be evaluated.

Corollary 2.1 If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

Notes:

■ From the previous corollary, a definite integral $\int_a^b f(x)\,dx$ is evaluated by two steps:

Step 1: Find an antiderivative F of the integrand,

Step 2: Evaluate the antiderivative F at upper and lower limits by substituting $x = b$ and $x = a$ (evaluate at lower limit) into F, then subtracting the latter from the former i.e., calculate $F(b) - F(a)$.

■ When using substitution to evaluate the definite integral $\int_a^b f(x)\,dx$, we have two options:

Option 1: Change the limits of integration to the new variable. For example, $\int_0^1 2x\sqrt{x^2+1}\,dx$. Let $u = x^2 + 1$, this implies $du = 2x\,dx$. Change the limits $u(0) = 1$ and $u(1) = 2$. By substitution, we have $\int_1^2 u^{1/2}\,du$. Then, evaluate the integral without returning to the original variable.

Option 2: Leave the limits in terms of the original variable. Evaluate the integral, then return to the original variable. After that, substitute $x = b$ and $x = a$ into the antiderivative as in step 2 above.

Example 2.13 Evaluate the integral.

(1) $\int_{-1}^2 (2x+1)\,dx$

(2) $\int_0^3 (x^2+1)\,dx$

(3) $\int_1^2 \frac{1}{\sqrt{x^3}}\,dx$

(4) $\int_0^{\frac{\pi}{2}} (\sin\, x+1)\,dx$

(5) $\int_{\frac{\pi}{4}}^{\pi} (\sec^2\, x-4)\,dx$

(6) $\int_0^{\frac{\pi}{3}} (\sec\, x\, \tan\, x+x)\,dx$

Solution:

(1) $\int_{-1}^2 (2x+1)\,dx = \Big[x^2+x\Big]_{-1}^2 = \big(4+2\big) - \big((-1)^2+(-1)\big) = 6 - 0 = 6.$

(2) $\int_0^3 (x^2+1)\,dx = \Big[\frac{x^3}{3}+x\Big]_0^3 = (\frac{27}{3}+3) - 0 = 12.$

(3) $\int_1^2 \frac{1}{\sqrt{x^3}}\,dx = \Big[\frac{-2}{\sqrt{x}}\Big]_1^2 = \frac{-2}{\sqrt{2}} - (-2) = \frac{-2+2\sqrt{2}}{\sqrt{2}} = -\sqrt{2}+2.$

(4) $\int_0^{\frac{\pi}{2}} (\sin\, x+1)\,dx = \Big[-\cos\, x+x\Big]_0^{\frac{\pi}{2}} = (-\cos\, \frac{\pi}{2}+\frac{\pi}{2}) - (-\cos\, 0+0) = \frac{\pi}{2}+1.$

(5) $\int_{\frac{\pi}{4}}^{\pi} (\sec^2\, x-4)\,dx = \Big[\tan\, x-4x\Big]_{\frac{\pi}{4}}^{\pi} = (\tan\, \pi-4\pi) - (\tan\, \frac{\pi}{4}-4\frac{\pi}{4}) = -4\pi - (1-\pi) = -3\pi-1.$

(6) $\int_0^{\frac{\pi}{3}} (\sec\, x\, \tan\, x+x)\,dx = \Big[\sec\, x+\frac{x^2}{2}\Big]_0^{\frac{\pi}{3}} = (\sec\, \frac{\pi}{3}+\frac{(\frac{\pi}{3})^2}{2}) - (\sec\, 0+\frac{0}{2}) = 2+\frac{\pi^2}{18}-1 = 1+\frac{\pi^2}{18}.$

Example 2.14 If $f(x) = \begin{cases} x^2 & : x < 0 \\ x^3 & : x \geq 0 \end{cases}$, find $\int_{-1}^2 f(x)\,dx.$

Solution:

The definition of the function f changes at 0. Since $[-1,2]=[-1,0]\cup[0,2]$, then from Theorem 2.6,

$$\begin{aligned}\int_{-1}^{2} f(x)\,dx &= \int_{-1}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx\\ &= \int_{-1}^{0} x^2\,dx + \int_{0}^{2} x^3\,dx\\ &= \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^4}{4}\right]_{0}^{2}\\ &= \frac{1}{3}+\frac{16}{4}=\frac{13}{3}.\end{aligned}$$

Example 2.15 Evaluate the integral $\int_0^2 |\,x-1\,|\;dx$.

Solution:

$$|\,x-1\,| = \begin{cases} -(x-1) & : x<1 \\ x-1 & : x\geq 1\end{cases}$$

Since $[0,2]=[0,1]\cup[1,2]$, then from Theorem 2.6,

$$\begin{aligned}\int_0^2 |\,x-1\,|\;dx &= \int_0^1 (-x+1)\,dx + \int_1^2 (x-1)\,dx\\ &= \left[\frac{-x^2}{2}+x\right]_0^1 + \left[\frac{x^2}{2}-x\right]_1^2\\ &= (\frac{1}{2}-0)+(0+\frac{1}{2}) = 1.\end{aligned}$$

Theorem 2.9 If f is continuous on a closed interval $[a,b]$, then there is at least a number $z\in(a,b)$ such that

$$\int_a^b f(x)\,dx = f(z)(b-a).$$

Proof. If f is constant i.e., $f(x)=k$, then

$$\int_a^b f(x)\,dx = k(b-a) = f(z)(b-a)$$

for any $z\in(a,b)$ and this means the equality is satisfied.

Therefore, assume the function f is not constant. Since the function f is continuous, then from the extreme value theorem, there exist $u,\ v\in[a,b]$ such that $f(u)=m$ is the minimum value and $f(v)=M$ is the maximum value of f.[2] Now, $\forall\, x\in[a,b]$, we have

$$m \leq f(x) \leq M.$$

This implies

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx.$$

Then,

$$\begin{aligned} m(b-a) &\leq \int_a^b f(x)\,dx \leq M(b-a).\\ \Rightarrow m &\leq \frac{\int_a^b f(x)\,dx}{(b-a)} \leq M\\ \Rightarrow f(u) &\leq \frac{\int_a^b f(x)\,dx}{(b-a)} \leq f(v)\end{aligned}$$

[2]If f is a continuous function on a closed interval $[a,b]$, then f takes a minimum value and a maximum value at least once in $[a,b]$.

From the intermediate value theorem,[3] there exists a number $z \in (a,b)$ such that

$$\frac{\int_a^b f(x)\,dx}{(b-a)} = f(z) \Rightarrow \int_a^b f(x)\,dx = (b-a)f(z). \blacksquare$$

Example 2.16 Find a number z that satisfies the conclusion of the Mean Value Theorem for the function f on the given interval.
(1) $f(x) = 1+x^2$, $[0,2]$
(2) $f(x) = \sqrt[3]{x}$, $[0,1]$

Solution:

(1) From Theorem 2.9,

$$\begin{aligned}\int_0^2 (1+x^2)\,dx &= (2-0)f(z)\\ \left[x+\frac{x^3}{3}\right]_0^2 &= 2(1+z^2)\\ \frac{14}{3} &= 2(1+z^2)\\ \frac{7}{3} &= 1+z^2\end{aligned}$$

This implies $z^2 = \frac{4}{3}$, then $z = \pm\frac{2}{\sqrt{3}}$. However, $-\frac{2}{\sqrt{3}} \notin (0,2)$, so $z = \frac{2}{\sqrt{3}} \in (0,2)$.

(2) From Theorem 2.9,

$$\begin{aligned}\int_0^1 \sqrt[3]{x}\,dx &= (1-0)f(z)\\ \frac{3}{4}\left[x^{\frac{4}{3}}\right]_0^1 &= \sqrt[3]{z}\end{aligned}$$

This implies $z = \frac{27}{64} \in (0,1)$.

In the following, we prove the Fundamental Theorem.

Proof. **1.** We want to prove that if $x \in [a,b]$, then $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$. Note that

$$\begin{aligned}F(x+h) = \int_a^{x+h} f(t)\,dt &= \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt\\ &= F(x) + \int_x^{x+h} f(t)\,dt.\end{aligned}$$

Hence, $F(x+h)-F(x) = \int_x^{x+h} f(t)\,dt$. Since the function f is continuous on the interval $[x,x+h]$, then from the Mean Value Theorem for integrals, there is $z \in (x,x+h)$ such that

$$\begin{aligned}\int_x^{x+h} f(t)\,dt &= f(z)h\\ \Rightarrow \frac{F(x+h)-F(x)}{h} &= f(z).\end{aligned}$$

When $h \to 0$, $(x+h) \to x$ and this means $z \to x$. This implies $f(z) \to f(x)$ since f is continuous. Therefore,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x).$$

[3] If f is continuous on a closed interval $[a,b]$ and If u is any number between $f(a)$ and $f(b)$, then there is at least a number $z \in [a,b]$ such that $f(z) = w$.

2. Assume that F and G are antiderivatives of f on the interval $[a,b]$. Then, form Theorem 1.1, there is a constant c such that $F(x) = G(x) + c$. Now, if $x = a$, then $F(a) = \int_a^a f(t)\ dt = 0$. Thus

$$F(a) = G(a) + c \Rightarrow c = -G(a) \Rightarrow F(x) = G(x) - G(a),\ \forall x \in [a,b].$$

If we assume $x = b$, then

$$F(b) = \int_a^b f(x)\ dx = G(b) - G(a). \blacksquare$$

In the following, we define the average value of the function f on the interval $[a,b]$.

Definition 2.6 If f is continuous on the interval $[a,b]$, then the average value f_{av} of f on $[a,b]$ is

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)\ dx.$$

Example 2.17 Find the average value of the function f on the given interval.

(1) $f(x) = x^3 + x - 1, \qquad [0,2]$

(2) $f(x) = \sqrt{x}, \qquad [1,3]$

Solution:

(1) $f_{av} = \frac{1}{2-0}\int_0^2 (x^3 + x - 1)\ dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^2}{2} - x\right]_0^2 = \frac{1}{2}\Big[(4+2-2)-(0)\Big] = 2.$

(2) $f_{av} = \frac{1}{3-1}\int_1^3 \sqrt{x}\, dx = \frac{1}{2}\frac{2}{3}\left[\, x^{\frac{3}{2}}\, \right]_1^3 = \frac{3\sqrt{3}-1}{3}.$

From the Fundamental Theorem, if f is continuous on $[a,b]$ and $F(x) = \int_c^x f(t)\ dt$ where $c \in [a,b]$, then

$$\frac{d}{dx}\int_a^x f(t)\ dt = \frac{d}{dx}\Big[F(x) - F(a)\Big] = f(x)\ \ \forall x \in [a,b].$$

This result can be generalized as follows:

Theorem 2.10 Let f be continuous on $[a,b]$. If g and h are in the domain of f and differentiable, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\ dt = f(h(x))h'(x) - f(g(x))g'(x)\ \ \forall x \in [a,b].$$

Proof. Let $F(x) = \int_{g(x)}^{h(x)} f(t)\ dt$. For any constant a, we can write

$$F(x) = \int_{g(x)}^{a} f(t)\ dt + \int_a^{h(x)} f(t)\ dt.$$

Assume $H(x) = \int_a^{h(x)} f(t)\ dt$ and let $u = h(x)$. Then, from the chain rule, we have

$$H'(x) = \frac{dH}{dx} = \frac{dH}{du}\frac{du}{dx} = f(u)h'(x) = f(h(x))h'(x)$$

Similarly, assume $G(x) = \int_{g(x)}^{a} f(t)\ dt = -\int_a^{g(x)} f(t)\ dt$. This implies, $G'(x) = -f(g(x))g'(x)$. Thus,

$$F'(x) = H'(x) + G'(x) = f(h(x))h'(x) - f(g(x))g'(x). \blacksquare$$

Corollary 2.2 Let f be continuous on $[a,b]$. If g and h are in the domain of f and differentiable, then

1. $\frac{d}{dx}\int_a^{h(x)} f(t)\, dt = f(h(x))h'(x) \ \ \forall x \in [a,b]$,

2. $\frac{d}{dx}\int_{g(x)}^{a} f(t)\, dt = -f(g(x))g'(x) \ \ \forall x \in [a,b]$.

Proof. The proof of this corollary is straightforward from Theorem 2.10 by assuming $g(x) = a$ in item 1 and $h(x) = a$ in item 2. ■

Example 2.18 Find the derivative.

(1) $\frac{d}{dx}\int_1^x \sqrt{\cos\, t}\, dt$

(2) $\frac{d}{dx}\int_1^{x^2} \frac{1}{t^3+1}\, dt$

(3) $\frac{d}{dx}\Big(x\int_x^{x^2} (t^3-1)\, dt\Big)$

(4) $\frac{d}{dx}\int_{x+1}^{3} \sqrt{t+1}\, dt$

(5) $\frac{d}{dx}\int_1^{\sin\, x} \frac{1}{1-t^2}\, dt$

(6) $\frac{d}{dx}\int_{-x}^{x} \cos\,(t^2+1)\, dt$

(7) $\frac{d}{dx}\int_{-x}^{x^2} \frac{1}{t^2+1}\, dt$

(8) $\frac{d}{dx}\int_{\cos\, x}^{\sin\, x} \sqrt{1+t^4}\, dt$

Solution:

(1) $\frac{d}{dx}\int_1^x \sqrt{\cos\, t}\, dt = \sqrt{\cos\, x}\,(1) = \sqrt{\cos\, x}.$

(2) $\frac{d}{dx}\int_1^{x^2} \frac{1}{t^3+1}\, dt = \frac{1}{(x^2)^3+1}(2x) = \frac{2x}{x^6+1}.$

(3) $\frac{d}{dx}\Big(x\int_x^{x^2} (t^3-1)\, dt\Big) = \int_x^{x^2} (t^3-1)\, dt + x\big(2x(x^6-1)-(x^3-1)\big)$ (Let $f(x) = x$ and $g(x) = \int_x^{x^2}(t^3-1)\, dt$. Then, find $\frac{d}{dx}(fg)(x)$)

(4) $\frac{d}{dx}\int_{x+1}^{3} \sqrt{t+1}\, dt = 0 - \sqrt{(x+1)+1} = -\sqrt{x+2}.$

(5) $\frac{d}{dx}\int_1^{\sin\, x} \frac{1}{1-t^2}\, dt = \frac{1}{1-\sin^2\, x}\cos\, x = \frac{\cos\, x}{\cos^2\, x} = \sec\, x.$ (use the identity $\cos^2\, x + \sin^2\, x = 1$)

(6) $\frac{d}{dx}\int_{-x}^{x} \cos\,(t^2+1)\, dt = \cos\,(x^2+1) + \cos\,(x^2+1) = 2\cos\,(x^2+1).$

(7) $\frac{d}{dx}\int_{-x}^{x^2} \frac{1}{t^2+1}\, dt = \frac{2x}{x^4+1} + \frac{1}{x^2+1}.$

(8) $\frac{d}{dx}\int_{\cos\, x}^{\sin\, x} \sqrt{1+t^4}\, dt = \sqrt{1+\sin^4\, x}\,\cos\, x + \sqrt{1+\cos^4\, x}\,\sin\, x.$

Example 2.19 If $F(x) = (x^2-2)\int_2^x \big(t+3F'(t)\big)\, dt$, find $F'(2)$.

Solution:

$$F'(x) = 2x\int_2^x \big(t+3F'(t)\big)\, dt + (x^2-2)\big(x+3F'(x)\big)$$

Letting $x = 2$ gives

$$F'(2) = 4\int_2^2 \big(t+3F'(t)\big)\, dt + (4-2)\big(2+3F'(2)\big)$$
$$\Rightarrow F'(2) = 2\big(2+3F'(2)\big).$$

Hence, $-5F'(2) = 4 \Rightarrow F'(2) = -\frac{4}{5}$.

Exercise 2.4

1 - 10 Evaluate the integral.

1 $\int_0^3 (2 - x + x^2)\, dx$

2 $\int_{-1}^1 (x^2 + 3x + 1)\, dx$

3 $\int_0^{10} (x^{\frac{3}{2}} + 1)\, dx$

4 $\int_1^2 \frac{2}{\sqrt{x}}\, dx$

5 $\int_0^{\pi} \cos\, x\, dx$

6 $\int_0^{\frac{\pi}{4}} (\sin\, x + \cos\, x)\, dx$

7 $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\, x\, (\tan\, x + \sec\, x)\, dx$

8 $\int_0^2 |\, x - 1\, |\; dx$

9 $\int_{-1}^1 |\, 3x + 1\, |\; dx$

10 $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin^2\, x}\, dx$

11 - 16 Verify that the function f satisfies the hypotheses of the Mean Value Theorem on the given interval. Then, find all numbers z that satisfy the conclusion of the Mean Value Theorem.

11 $f(x) = (x+1)^3, \quad [-1, 1]$

12 $f(x) = 1 - x^3, \quad [-2, 0]$

13 $f(x) = \sqrt{x}, \quad [1, 4]$

14 $f(x) = \frac{2}{\sqrt{x}}, \quad [1, 4]$

15 $f(x) = \sin\, x, \quad [0, \pi]$

16 $f(x) = \cos\, x, \quad [0, \frac{\pi}{2}]$

17 - 20 Find the average value of the function f on the given interval.

17 $f(x) = x^3 + x^2 - 1, \quad [0, 2]$

18 $f(x) = \sqrt[3]{x}, \quad [-1, 3]$

19 $f(x) = \frac{1}{x^3}, \quad [1, 5]$

20 $f(x) = \sin\, x, \quad [0, \frac{\pi}{6}]$

21 - 28 Find the derivative.

21 $\frac{d}{dx} \int_{\cos\, x}^{\sin\, x} \sqrt{t+1}\, dt$

22 $\frac{d}{dx} \int_x^{\sqrt{x}} \frac{1}{t^2+1}\, dt$

23 $\frac{d}{dx} \int_1^x (t-1)\, dt$

24 $\frac{d}{dx} \int_{x+1}^{3(x-1)} \frac{1}{t-1}\, dt$

25 $\frac{d}{dx}\left(\sin\, x \int_1^x \sqrt{t}\, dt\right)$

26 $\frac{d}{dx} \int_{-2x}^x \sin\, (t+1)\, dt$

27 $\frac{d}{dx} \int_{x^3}^{\pi} \frac{1}{t^4+1}\, dt$

28 $\frac{d}{dx} \int_{\tan\, x}^{\sec\, x} \sqrt{1+t^4}\, dt$

29 - 32 Find the derivative at the indicated value.

29 $F(x) = \int_2^x \sqrt{3t^2+1}\, dt, \quad F(2), F'(2)$ and $F''(2)$.

30 $G(x) = \int_x^0 \frac{\sin\, t}{t+1}\, dt, \quad G(0), G'(0)$ and $G''(0)$.

31 $H(x) = \int_x^{x^2} \sqrt[5]{t+1}\, dt, \quad H'(2)$.

32 $F(x) = \sin x \int_0^x \left(1 + F'(t)\right)\, dt$, $F(0)$ and $F'(0)$.

2.5 Numerical Integration

Sometimes we face definite integrals that cannot be solved even if the integrands are continuous functions such as $\sqrt{1+x^3}$ and e^{x^2}. In our discussion in this book so far, we are not able to evaluate such integrals. We exploit this to show the reader a new technique to approximate the definite integrals.

2.5.1 Trapezoidal Rule

As discussed in Section 2.2, if f is a defined and positive function on a closed interval $[a,b]$, a Riemann sum approximates the area underneath the graph of f from $x = a$ to $x = b$ as follows. Assume P is a regular partition of $[a,b]$. We divide the interval $[a,b]$ by the partition P into n subintervals : $[x_0,x_1],[x_1,x_2],[x_2,x_3],...,[x_{n-1},x_n]$. Then, we find the length of the subintervals: $\Delta x_k = \frac{b-a}{n}$. Using Riemann sum, we have

$$\int_a^b f(x)\, dx \approx \sum_{k=1}^{n} f(\omega_k)\Delta x_k = \frac{b-a}{n}\sum_{k=1}^{n} f(\omega_k)\ ,$$

where $\omega = (\omega_1,\omega_2,...,\omega_n)$ is a mark on the partition P.

As shown in Figure 2.4, we take the mark as follows:

1. The left-hand endpoint. We choose $\omega_k = x_{k-1}$ in each subinterval. Then,
$$\int_a^b f(x)\, dx \approx \frac{b-a}{n}\sum_{k=1}^{n} f(x_{k-1}).$$
2. The right-hand endpoint. We choose $\omega_k = x_k$ in each subinterval. Then,
$$\int_a^b f(x)\, dx \approx \frac{b-a}{n}\sum_{k=1}^{n} f(x_k).$$
3. The average of the previous two approximations is more accurate,
$$\frac{b-a}{2n}\Big[\sum_{k=1}^{n} f(x_{k-1}) + \sum_{k=1}^{n} f(x_k)\Big].$$

Trapezoidal Rule

Let f be continuous on $[a,b]$. If $P = \{x_0,x_1,...,x_n\}$ is a regular partition of $[a,b]$, then

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2n}\Big[f(x_0)+2f(x_1)+2f(x_2)+...+2f(x_{n-1})+f(x_n)\Big].$$

Error Estimation

Although the numerical methods give an approximated value of a definite integral, there is a possibility that an error occurs. The numerical method and the number of subintervals play a role in determining the error. The way of estimating the error under the trapezoidal rule is given without proof in the following theorem.

Theorem 2.11 Suppose that f'' is continuous on $[a,b]$ and M is the maximum value for f'' over $[a,b]$. If E_T is the error in calculating $\int_a^b f(x)\, dx$ under the trapezoidal rule, then

$$|\, E_T \,| \leq \frac{M(b-a)^3}{12\, n^2}.$$

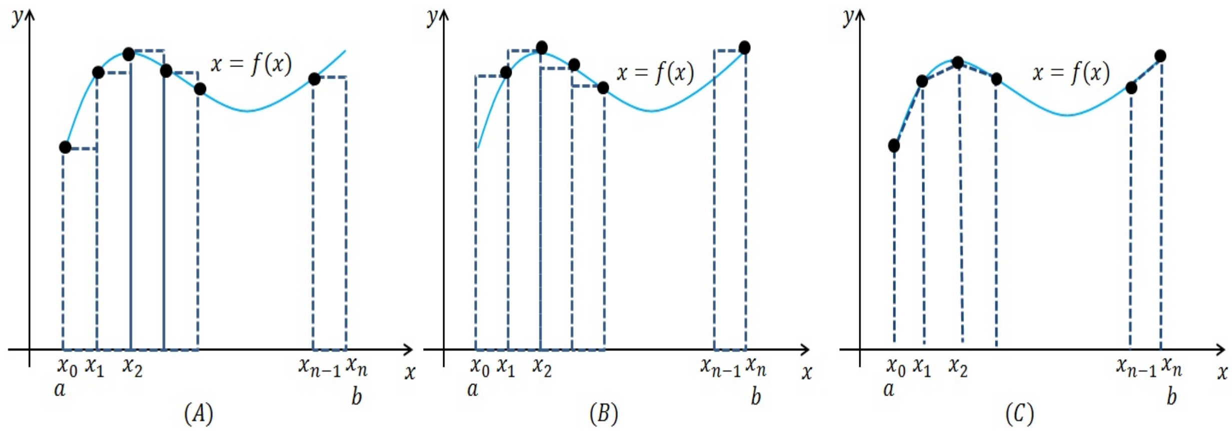


Figure 2.4: Approximation of a definite integral by using the trapezoidal rule.

Example 2.20 By using the trapezoidal rule with $n = 4$, approximate the integral $\int_1^2 \frac{1}{x}\,dx$. Then, estimate the error.

Solution:

(1) We approximate the integral $\int_1^2 \frac{1}{x}\,dx$ by the trapezoidal rule.

(a) Find a regular partition $P = \{x_0, x_1, x_2, ..., x_n\}$ where $\Delta x = \frac{(b-a)}{n}$ and $x_k = x_0 + k\Delta x$.
We divide the interval $[1,3]$ into four subintervals where the length of each subinterval is $\Delta x = \frac{2-1}{4} = \frac{1}{4}$ as follows:

$x_0 = 1$
$x_1 = 1 + \frac{1}{4} = 1\frac{1}{4}$
$x_2 = 1 + 2(\frac{1}{4}) = 1\frac{1}{2}$
$x_3 = 1 + 3(\frac{1}{4}) = 1\frac{3}{4}$
$x_4 = 1 + 4(\frac{1}{4}) = 2$

The partition is $P = \{1, 1.25, 1.5, 1.75, 2\}$.

(b) Approximate the integral by using the following table:

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	1	1	1	1
1	1.25	0.8	2	1.6
2	1.5	0.6667	2	1.3334
3	1.75	0.5714	2	1.1428
4	2	0.5	1	0.5
Sum = $\sum_{k=1}^{4} m_k f(x_n)$				5.5762

Hence,

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{8}\big[5.5762\big] = 0.697.$$

(2) We estimate the error by using Theorem 2.11.

$$f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3} \Rightarrow f'''(x) = -\frac{6}{x^4}.$$

Since $f''(x)$ is a decreasing function on the interval $[1,2]$, then $f''(x)$ is maximized at $x = 1$.
Hence, $M = \mid f''(1) \mid = 2$ and

$$\mid E_T \mid \leq \frac{2(2-1)^3}{12(4)^2} = \frac{1}{96} = 0.0104.$$

Remark 2.2 By knowing the error, we can determine the number of the subintervals n before starting approximating.

Example 2.21 Find the minimum number of subintervals to approximate the integral $\int_1^2 \frac{1}{x}\, dx$ such that the error is less than 10^{-3}.

Solution:

From the previous example, we had $M = 2$. Therefore, $|\, E_T \,| \leq \frac{2(2-1)^3}{12n^2} \leq 10^{-3}$.

This implies that

$$n^2 \geq \frac{2(2-1)^3}{12} 10^3 = \frac{10^3}{6} \Rightarrow n \geq \sqrt{\frac{500}{3}} = 12.91.$$

Therefore, $n = 13$.

2.5.2 Simpson's Rule

Simpson's rule is another numerical method to approximate the definite integrals. The question that can be raised here is that how the trapezoidal method differs from Simpson's method? The trapezoidal method depends on building trapezoids from the subintervals, then taking the average of the left and right endpoints. The Simpson's rule is built on approximating the area of the graph in each subinterval with area of some parabola (Figure 4.1).

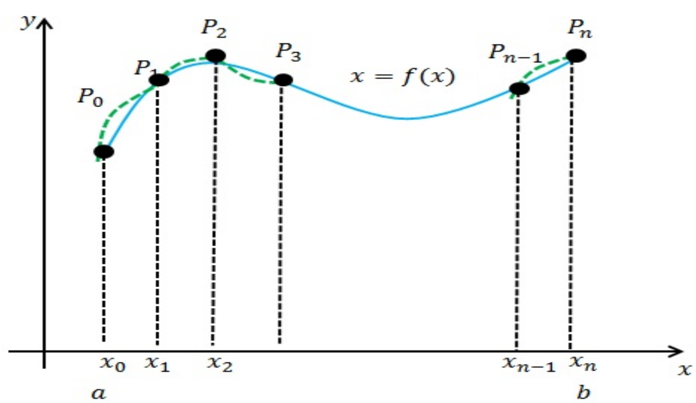


Figure 2.5: Approximation of a definite integral by using Simpson's rule.

First, let P be a regular partition of the interval $[a,b]$ to generate n subintervals such that $|\, P \,| = \frac{(b-a)}{n}$ and n is an even number.

Now, take three points lying on the parabola as shown in the next figure. Assume for simplicity that $x_0 = -h$, $x_1 = 0$ and $x_2 = h$. Since the equation of a parabola is

$$y = ax^2 + bx + c$$

, then from the figure, the area under the graph bounded by $[-h,h]$ is

$$\int_{-h}^{h} (ax^2 + bx + c)\, dx = \frac{h}{3}(2ah^2 + 6c).$$

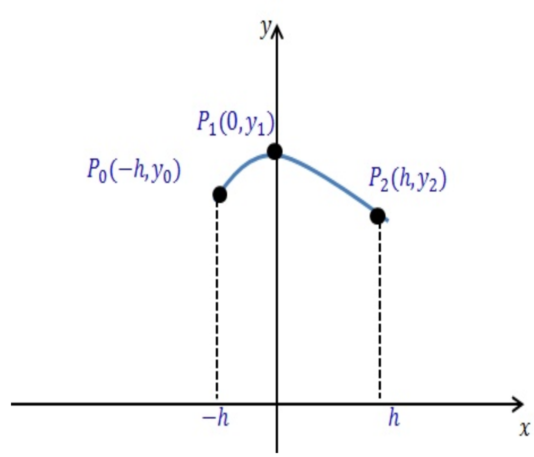


Figure 2.6

Thus, since the points P_0, P_1 and P_2 lie on the parabola, then

$$\begin{aligned} y_0 &= ah^2 - bh + c \\ y_1 &= c \\ y_2 &= ah^2 + bh + c. \end{aligned}$$

Some computations lead to $2ah^2 + 6c = y_0 + 4y_1 + y_2$. Therefore,

$$\int_{-h}^{h} (ax^2 + bx + c)\, dx = \frac{h}{3}(y_0 + 4y_1 + y_2) = \frac{h}{3}\big(f(x_0) + 4f(x_1) + f(x_2)\big).$$

Generally, for any three points P_{k-1}, P_k and P_{k+1}, we have

$$\frac{h}{3}(y_{k-1} + 4y_k + y_{k+1}) = \frac{h}{3}\big(f(x_{k-1}) + 4f(x_k) + f(x_{k+1})\big).$$

By summing the areas of all parabolas, we have

$$\begin{aligned} \int_a^b f(x)\, dx &= \frac{h}{3}\big(f(x_0) + 4f(x_1) + f(x_2)\big) \\ &+ \frac{h}{3}\big(f(x_2) + 4f(x_3) + f(x_4)\big) \\ &\ldots \\ &+ \frac{h}{3}\big(f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big) \\ &= \frac{b-a}{3n}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big] \end{aligned}$$

Simpson's Rule

Let f be continuous on $[a,b]$. If $P = \{x_0, x_1, \ldots, x_n\}$ is a regular partition of $[a,b]$ where n is even, then

$$\int_a^b f(x)\, dx \approx \frac{(b-a)}{3n}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big].$$

Error Estimation

The estimation of the error under Simpson's method is given by the following theorem.

Theorem 2.12 Suppose $f^{(4)}$ is continuous on $[a,b]$ and M is the maximum value for $f^{(4)}$ on $[a,b]$. If E_S is the error in calculating $\int_a^b f(x)\, dx$ under Simpson's rule, then

$$|\, E_S \,| \leq \frac{M(b-a)^5}{180\, n^4}.$$

Example 2.22 By using Simpson's rule with $n = 4$, approximate the integral $\int_1^3 \sqrt{x^2+1}\, dx$. Then, estimate the error.

Solution:

1. We approximate the integral $\int_1^3 \sqrt{x^2+1}\, dx$ under Simpson's rule.
   (a) Find the partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ where $\Delta x = \frac{(b-a)}{n}$ and $x_k = x_0 + k\Delta x$.
   We divide the interval $[1,3]$ into four subintervals where the length of each subinterval is $\Delta x = \frac{3-1}{4} = \frac{1}{2}$ as follows:

$x_0 = 1$
$x_1 = 1 + \frac{1}{2} = 1\frac{1}{2}$
$x_2 = 1 + 2(\frac{1}{2}) = 2$
$x_3 = 1 + 3(\frac{1}{2}) = 2\frac{1}{2}$
$x_4 = 1 + 4(\frac{1}{2}) = 3$

The partition is $P = \{1, 1.5, 2, 2.5, 3\}$.

(b) Approximate the integral by using the following table:

k	x_k	$f(x_k)$	m_k	$m_k f(x_k)$
0	1	1.4142	1	2
1	1.5	1.8028	4	7.2112
2	2	2.2361	2	4.4722
3	2.5	2.6926	4	10.7704
4	3	3.1623	1	10
Sum = $\sum_{k=1}^{4} m_k f(x_k)$				27.0302

Hence, $\int_1^3 \sqrt{x^2+1}\,dx \approx \frac{2}{12}\left[27.0302\right] = 4.5050$.

2. We estimate the error by using Theorem 2.12.
Since $f^{(5)}(x) = -(15x(4x^2-3))/\sqrt{(x^2+1)^9}$, then $f^{(4)}(x)$ is a decreasing function on the interval $[1,3]$. Therefore, $f^{(4)}(x)$ is maximized at $x = 1$. Then, $M = \mid f^{(4)}(1) \mid = 0.7955$ and

$$\mid E_s \mid < \frac{(0.7955)(3-1)^5}{180(4)^4} = 5.5243 \times 10^{-4}.$$

Example 2.23 Find the minimum number of subintervals to approximate the integral $\int_1^3 \sqrt{x^2+1}\,dx$ such that the error is less than 10^{-2}.

Solution:

From the previous example, we know that $M = 0.7955$. Thus, $\mid E_S \mid < \frac{(0.7955)(3-1)^5}{180n^4} < 10^{-2}$. This implies that

$$n^4 > \frac{(0.7955)(32)}{180} 10^2 \Rightarrow n > 14.14.$$

Therefore, $n = 14$.

Exercise 2.5

1 - 4 By using the trapezoidal rule, approximate the definite integral for the given n, then estimate the error.

1 $\int_{-1}^{1} \sqrt{x^2+1}\,dx$, $\quad n = 4$

2 $\int_{2}^{4} \sqrt{x}\,dx$, $\quad n = 5$

3 $\int_{0}^{4} \frac{x}{x+1}\,dx$, $\quad n = 4$

4 $\int_{0}^{\pi} \sin x\,dx$, $\quad n = 4$

5 - 8 ■ By using Simpson's rule, approximate the definite integral for the given n, then estimate the error.

5 $\ln(2) = \int_1^2 \frac{1}{x}\, dx,$ $\quad n = 4$

6 $\int_0^1 \frac{x}{\sqrt{x^4+1}}\, dx,$ $\quad n = 6$

7 $\int_0^2 \sqrt{x^3+1}\, dx,$ $\quad n = 10$

8 $\int_1^3 \sqrt{\ln x}\, dx,$ $\quad n = 4$

9 - 10 ■ Consider the function f, and the integral $I(f)$. What is the minimum number of points to be used to ensure an error $\leq 5 \times 10^{-2}$.

9 $f(x) = e^x$ and $I(f) = \int_0^2 e^x\, dx$ under the trapezoid rule.

10 $f(x) = \cos\, x^2$ and $I(f) = \int_0^2 \cos\, x\, dx$ under Simpson's rule.

Review Exercises

1 - 4 Express the sum in terms of n.

1 $\sum_{k=1}^{n}(k-1)$

2 $\sum_{k=1}^{n}(2k+1)$

3 $\sum_{k=1}^{n}(k^2-k+1)$

4 $\sum_{k=1}^{n}(k^3+2k+1)$

5 - 8 Evaluate the sum.

5 $\sum_{k=1}^{4}(2k+1)$

6 $\sum_{j=1}^{5}\frac{1}{j+1}$

7 $\sum_{k=1}^{3}(k^2+2k)$

8 $\sum_{i=1}^{4}(i-1)^2$

9 - 12 For the partition P, find the norm $\| P \|$.

9 $P=\{0,1.01,1.1,2.5,3.6,4,6\}$

10 $P=\{1,2.5,3,4,5.1,6\}$

11 $P=\{-3,-2.5,-1,0.5,1.2,2\}$

12 $P=\{0,1.04,1.09,2.15,3.7,4,5\}$

13 - 16 Find a Riemann sum R_P for the given function f by choosing the mark ω,

(a) the left-hand endpoint,

(b) the right-hand endpoint,

(c) the midpoint,

13 $f(x)=x+1, \quad \{1,2.5,3,3.5,4,5,6\}$

14 $f(x)=2x-1, \quad \{-1,0,1,1.5,2,3,3.5\}$

15 $f(x)=x^2+1, \quad \{1,1.5,2,2.5,3,3.5,4\}$

16 $f(x)=1-x^3, \quad \{-2,-1,0,1,3,5,6\}$

17 - 28 Find the area under the graph of f from a to b by taking the limit of a Riemann sum.

17 $f(x)=x+3, \quad a=1,\ b=3$

18 $f(x)=3-x, \quad a=0,\ b=1$

19 $f(x)=x^2, \quad a=-1,\ b=1$

20 $f(x)=x^2-x+1, \quad a=-1,\ b=3$

21 $f(x)=\frac{x}{2}, \quad a=2,\ b=4$

22 $f(x)=x^3+x+1, \quad a=0,\ b=2$

23 $f(x)=x, \quad a=1,\ b=3$

24 $f(x)=(1-x)^2, \quad a=0,\ b=1$

25 $f(x)=\frac{x}{3}, \quad a=-1,\ b=1$

26 $f(x)=x(x-1), \quad a=0,\ b=3$

27 $f(x)=5x, \quad a=1,\ b=3$

28 $f(x)=x^2+1, \quad a=0,\ b=2$

29 - 42 Evaluate the integral.

29 $\int_{-2}^{4} 2\,dx$

30 $\int_{0}^{5} (3-x)\,dx$

31 $\int_{-1}^{4} (2x^2+x-1)\,dx$

32 $\int_{2}^{2} (6x^2+3)\,dx$

33 $\int_{0}^{1} (x^3-4x^4)\,dx$

34 $\int_{-1}^{1} x\sqrt{x^2+1}\,dx$

35 $\int_{0}^{5} |\,x-1\,|\;dx$

36 $\int_{0}^{3} |\,2x-3\,|\;dx$

37 $\int_{1}^{3} (x-2)(x+3)\,dx$

38 $\int_{0}^{\pi} \cos\,x\,dx$

39 $\int_{0}^{\frac{\pi}{2}} \sin\,x\,dx$

40 $\int_{0}^{\pi} \sec\,x\,(\tan\,x-\sec\,x)\,dx$

41 $\int_{0}^{\pi} x\,\cos\,x^2\,dx$

42 $\int_{\pi/4}^{\pi} \frac{\csc^2\,\sqrt{x}}{\sqrt{x}}\,dx$

43 - 48 If $\int_{a}^{b} f(x)\,dx=2$, $\int_{b}^{c} f(x)\,dx=2$ and $\int_{a}^{b} g(x)\,dx=3$ where $c\in(a,b)$, evaluate the integral.

43 $\int_{b}^{a} f(x)\,dx$

44 $\int_{a}^{c} f(x)\,dx$

45 $\int_{a}^{b} \big(2f(x)+g(x)\big)\,dx$

46 $\int_{b}^{a} \big(5f(x)-3g(x)\big)\,dx$

47 $\int_{a}^{b} \big(\frac{1}{3}f(x)+7g(x)\big)\,dx$

48 $\int_{a}^{a} \big(4f(x)+g(x)\big)\,dx$

49 - 54 Use the properties of the definite integrals to prove the inequality without evaluating the integrals.

49 $\int_{0}^{1} x\,dx \geq \int_{0}^{1} x^2\,dx$

50 $\int_{0}^{3} \frac{x}{x^3+2}\,dx \geq \int_{0}^{3} x\,dx$

51 $\int_{1}^{4} (2x+2)\,dx \geq \int_{1}^{4} (3x+1)\,dx$

52 $\int_{0}^{3} (x^2-3x+4)\,dx \geq 0$

53 $\int_{1}^{2} \sqrt{5-x}\,dx \geq \int_{1}^{2} \sqrt{x+1}\,dx$

54 $2 < \int_{-1}^{2} \sqrt{1+x^2}\,dx$

55 - 59 Find the average value of the function f on the given interval.

55 $f(x)=x^2,\quad [1,4]$

56 $f(x)=9-x^2,\quad [0,3]$

57 $f(x)=x-x^2,\quad [0,2]$

58 $f(x)=x^3+1,\quad [-1,2]$

59 $f(x)=6x^2-2x+4,\quad [-1,3]$

60 - 63 Find the number z that satisfies the Mean Value Theorem for the function f on the given interval.

60 $f(x)=2+x^2,\quad [0,4]$

61 $f(x)=x^3,\quad [-1,3]$

62 $f(x)=\sqrt{x},\quad [0,9]$

63 $f(x)=4x^3-1,\quad [1,2]$

64 - 71 Find the derivative of the functions.

64 $\int_0^x \sin \sqrt{t}\, dt$

65 $\int_1^x \frac{1}{t}\, dt$

66 $\int_{3x}^{x^3} \sin (t^3+1)^{10}\, dt$

67 $\int_2^{x+1} \frac{1}{t^2+1}\, dt$

68 $\int_{\cos x}^{x} \cos t^2\, dt$

69 $\int_0^x \sqrt{t^2+1}\, dt$

70 $\int_{6x-1}^{0} \sqrt[3]{2t+4}\, dt$

71 $\int_3^{\sqrt{x}} \tan t^2\, dt$

72 - 75 By using the trapezoidal rule, approximate the definite integral for the given n, then estimate the error.

72 $\int_1^5 x^3\, dx, \qquad n=4$

73 $\int_1^2 \frac{1}{x}\, dx, \qquad n=10$

74 $\int_0^1 e^x\, dx, \qquad n=4$

75 $\int_1^3 \sqrt{1+x^3}\, dx, \qquad n=6$

76 - 79 By using Simpson's rule, approximate the definite integral for the given n, then estimate the error.

76 $\int_0^{\pi} \frac{1}{2-\sin x}\, dx, \qquad n=4$

77 $\int_0^1 \ln(1+e^x)\, dx, \qquad n=6$

78 $\int_1^2 e^x\, dx, \qquad n=6$

79 $\int_0^{\pi} \cos x^2\, dx, \qquad n=4$

80 - 81 Find the minimum number of subintervals to approximate the integral $\int_1^3 x^5+1\, dx$ by using the trapezoidal rule such that the error is less than

80 10^{-2}

81 10^{-4}

82 - 83 Find the minimum number of subintervals to approximate the integral $\int_1^3 x^5+1\, dx$ by using Simpson's rule such that the error is less than

82 0.5

83 2.55

84 - 106 Choose the correct answer.

84 The sum $\sum_{k=1}^{n^2} (k-1)$ is equal to

(a) $\frac{n^2(n-1)}{2}$ (b) $\frac{n(n-1)}{2}$ (c) $\frac{n^2(n^2+1)}{2}$ (d) $\frac{n^2(n^2-1)}{2}$

85 The sum $\lim_{n\to\infty} \sum_{k=1}^{n} (\frac{k}{n^2})$ is equal to

(a) 0 (b) ∞ (c) 2 (d) $\frac{1}{2}$

86 If $\sum_{k=1}^{n} (k+\alpha) = \frac{n^2}{2} (n \geq 1)$, then the value of α is equal to

(a) $-\frac{n}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1

87 If $\sum_{k=1}^{4}(k+a)=14$, then the value of a is equal to
(a) 1 (b) 4 (c) -4 (d) -1

88 If $\sum_{k=1}^{5}(\alpha k^2+k-1)=20$, then the value of α is equal to
(a) $\frac{2}{11}$ (b) $\frac{-2}{11}$ (c) $\frac{1}{11}$ (d) $\frac{-1}{11}$

89 If $\sum_{k=1}^{6}(k^2+3k+2\alpha)=130$, then the value of α is equal to
(a) 2 (b) -2 (c) 1 (d) 3

90 The average value of the function $f(x)=\sqrt[3]{x+1}$ on $[-2,0]$ is equal to
(a) 3 (b) 0 (c) -1 (d) -3

91 The average value of the function $f(x)=\sin\, x\cos\, x$ on $[0,\frac{\pi}{4}]$ is equal to
(a) $-\frac{1}{\pi}$ (b) $\frac{1}{4}$ (c) $\frac{1}{\pi}$ (d) $-\frac{1}{4}$

92 The average value of $f(x)=|\, x-1\,|$ on $[0,1]$ is equal to
(a) $-\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 0 (d) $\frac{1}{2}$

93 The average value of $f(x)=\sin\, x\cos\, x$ on $[-\pi,\pi]$ is equal to
(a) $\frac{1}{2\pi}$ (b) $\frac{1}{\pi}$ (c) 1 (d) 0

94 If $F(x)=\int_1^{x^2}\sqrt[3]{t^4+1}\,dt$, the $F'(x)$ is equal to
(a) $\sqrt[3]{x^8+1}$ (b) $x^2\sqrt[3]{x^8+1}$ (c) $2x\sqrt[3]{x^8+1}$ (d) $2x\sqrt[3]{x^4+1}$

95 The value of the integral $\displaystyle\int_0^2 |\, x-1\,|\, dx$ is equal to
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) 2

96 If $f(1)=3$, $f(4)=7$, $f(2)=4$ and $f(14)=23$, the value of the integral $\displaystyle\int_1^2(x^2+1)f'(x^3+3x)\,dx$ is equal to
(a) $\frac{1}{3}$ (b) 16 (c) 1 (d) $\frac{16}{3}$

97 If $F(x)=x\displaystyle\int_{\sqrt{\pi}}^{x}\cos\, t^2\,dt$, then $F'(\sqrt{\pi})$ is equal to
(a) 0 (b) $\sqrt{\pi}$ (c) $-\sqrt{\pi}$ (d) 1

98 If $F(x)=\displaystyle\int_{2x}^{x^2}\sin\, t^3\,dt$, then $F'(x)$ is equal to
(a) $2x\sin\, x^6-\sin\, 8x^3$
(b) $2x\sin\, x^6-2\sin\, 8x^3$
(c) $2x\sin\, x^6-2\sin\, 6x^3$
(d) $2x\sin\, x^6+2\sin\, 8x^3$

99 The number z that satisfies the Mean Value Theorem for $f(x)=x^2$ on $[0,2]$ is
(a) $\sqrt{\frac{8}{3}}$ (b) $\frac{8}{\sqrt{3}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\frac{2}{\sqrt{3}}$

100 The number z that satisfies the Mean Value Theorem for $f(x)=1+x^2$ on $[-3,0]$ is
(a) $-\sqrt{3}$ (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) $-\sqrt{2}$

101 If $F(x)=\displaystyle\int_{x-1}^{x+1}\tan(t^2)\,dt$, then $F'(x)$ is equal to
(a) $\tan\,(x^2+2x+1)+\tan\,(x^2-2x+1)$
(b) $\tan\,(x^2+2x+1)-\tan\,(x^2-2x+1)$
(c) $\tan\,(x^2+1)-\tan\,(x^2-1)$
(d) 0

102 If $F(x)=\displaystyle\int_1^{x^3}\sqrt{5+t^2}\,dt$, then $F'(1)$ is equal to
(a) 0 (b) $3\sqrt{6}$ (c) $\sqrt{6}$ (d) $\frac{2}{\sqrt{6}}$

103 If $\int_0^{x^2} f(\sqrt{t})\, dt = x$, then $f(x)$ is equal to

(a) 1 (b) $\frac{1}{2x}$ (c) $\frac{1}{x^2}$ (d) $\frac{1}{2}$

104 The value of the integral $\int_{-1}^{1} 2\,|\,x\,|^3\ dx$

(a) 2 (b) 1 (c) 0 (d) -1

105 The derivative of the integral $\int_0^x \left(1+\frac{d\tan\ t}{dt}\right) dt$ is equal to

(a) $1+\tan\ x$ (b) $1-\tan\ x$ (c) $1-\sec^2\ x$ (d) $1+\sec^2\ x$

106 If $G(x)=\int_e^{x^2} \frac{\ln t}{4}\, dt$, then $G'(e)$ is equal to

(a) $2e$ (b) 1 (c) e (d) $4e$

Chapter 3

Logarithmic and Exponential Functions

3.1 The Natural Logarithmic Function

In chapter 1, we found that $\int x^r\, dx = \frac{x^{r+1}}{r+1} + c$ (see Table 1.1). If $r = -1$, does the previous rule hold? The answer is no because the denominator will become zero. The task now is to find a general antiderivative of the function $\frac{1}{x}$; meaning that we are looking for a function $F(x)$ such that $F^{'}(x) = \frac{1}{x}$.

Consider the function $f(t) = \frac{1}{t}$. It is continuous on the interval $(0, +\infty)$ and this implies that the function is integrable on the interval $[1, x]$. Figure 3.1 shows the graph of the function $f(t) = \frac{1}{t}$ from $t = 1$ to $t = x$ where $x > 0$. The area of the region under the graph can be expressed as

$$f(x) = \int_1^x \frac{1}{t}\, dx$$

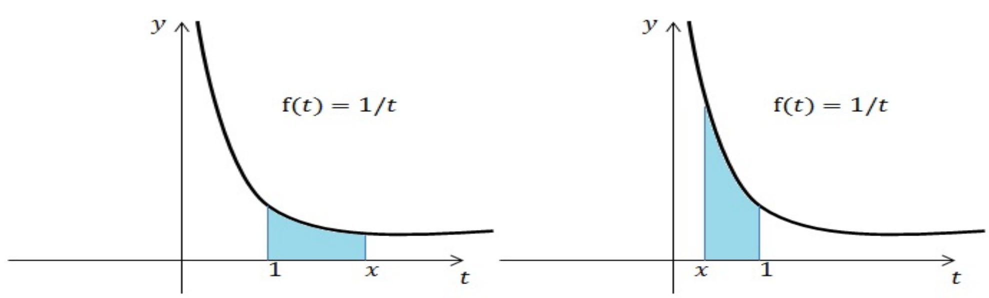


Figure 3.1: The area under the graph of the function $f(t) = \frac{1}{t}$ in the interval $[1, x]$ where $x > 0$.

In the following definition, we introduce the antiderivative of the function $f(t) = \frac{1}{t}$.

Definition 3.1 The natural logarithmic function is defined as follows:

$$\ln : (0, \infty) \to \mathbb{R}\ ,$$

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

for every $x > 0$.

3.1.1 Properties of the Natural Logarithmic Function

1. From the Definition 3.1, the domain of the function $\ln x$ is $(0,\infty)$.
2. The range of the function $\ln x$ is $\mathbb{R}$ as follows:

$$y = \begin{cases} \ln x > 0 & : x > 1 \\ \ln x = 0 & : x = 1 \\ \ln x < 0 & : 0 < x < 1 \end{cases}$$

To see this, let $x = 1$, then $\ln x = \int_1^1 \frac{1}{t}\, dt = 0$. Now, since $\int_1^x \frac{1}{t}\, dt = -\int_x^1 \frac{1}{t}\, dt$, then for $0 < x < 1$, the integral is the negative of the area of the region under $f(t) = \frac{1}{t}$ from $t = x$ to $x = 1$. This means that $\ln x$ is negative for $0 < x < 1$ and positive for $x > 1$.

3. The function $\ln x$ is differentiable and continuous on the domain. From the fundamental theorem of calculus, we have

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\int_1^x \frac{1}{t}\, dt = \frac{1}{x}, \forall x > 0.$$

Therefore, the function $\ln x$ is increasing on the interval $(0,\infty)$.

4. The second derivative $\frac{d^2}{dx^2}(\ln x) = \frac{-1}{x^2} < 0$ for all $x \in (0,\infty)$. Therefore, the function $\ln x$ is concave downward on the interval $(0,\infty)$.
5. Rules of the natural logarithmic function:

Theorem 3.1 If $a, b > 0$ and $r \in \mathbb{Q}$, then

1. $\ln ab = \ln a + \ln b$.
2. $\ln \frac{a}{b} = \ln a - \ln b$.
3. $\ln a^r = r \ln a$.

Proof. 1. Let $f(x) = \ln ax$ and $g(x) = \ln x + \ln a$ for all $x \in (0,\infty)$. Then,

$$f'(x) = \frac{1}{ax} a = \frac{1}{x} \text{ and}$$

$$g'(x) = \frac{1}{x} + 0 = \frac{1}{x}.$$

Since f and g have the same derivative on the interval $(0,\infty)$, they differ by a constant (Theorem 1.1). By taking $x = 1$, $f(1) = \ln a$ and $g(1) = \ln a$. This implies that the constant they differ by is 0, that is $f(x) = g(x)$.

2. From item (1), we have

$$\ln a = \ln(\frac{a}{b} b) = \ln \frac{a}{b} + \ln b.$$

This implies

$$\ln \frac{a}{b} = \ln a - \ln b.$$

3. Let $f(x) = \ln x^r$ for all $x > 0$ and $r \in \mathbb{Q}$. Then,

$$f'(x) = \frac{1}{x^r}\, r\, x^{r-1} = \frac{r}{x}.$$

Since $\frac{d}{dx}(r \ln x) = \frac{r}{x}$, then there is a constant c such that

$$\ln x^r = r \ln x + c,\ \forall x > 0.$$

If $x = 1$, $\ln 1^r = r \ln 1 + c$ and this implies $c = 0$. Hence, $\ln x^r = r \ln x$. Therefore, for any $a > 0$, we have

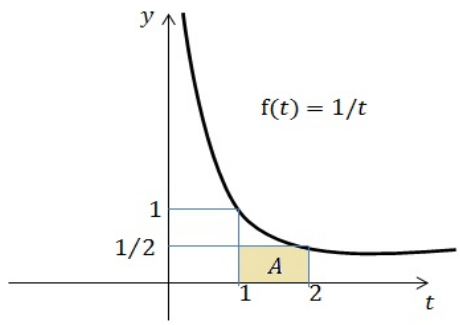
$$\ln a^r = r \ln a. \blacksquare$$

6. $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to 0^+} \ln x = -\infty$.

To see this, the figure on the right shows the region of $f(t)=\frac{1}{t}$ from $t=1$ to $t=x$. The area $A=(1)(\frac{1}{2})=\frac{1}{2}$. From Definition 3.1, $\ln 2=\int_1^2 \frac{1}{t}\,dt>\frac{1}{2}=$ area of A.
Since $\ln x$ is increasing function, then

$$\ln x>\ln 2^m=m\ln 2>\frac{m}{2}\ \forall m\in\mathbb{N}$$

where if m is sufficiently large, $x\geq 2^m$. This implies $\lim_{x\to\infty}\ln x>\frac{m}{2}$, then $\lim_{x\to\infty}\ln x=\infty$.



Now, let $u=\frac{1}{x}$ as $x\to 0^+$, $u\to\infty$. Since $x=\frac{1}{u}\Rightarrow \ln x=\ln\frac{1}{u}=-\ln u$. This implies

$$\lim_{x\to 0^+}\ln x=\lim_{x\to\infty}(-\ln u)=-\lim_{x\to\infty}\ln u=-\infty.$$

From the previous properties, we have the graph of the function $y=\ln x$.

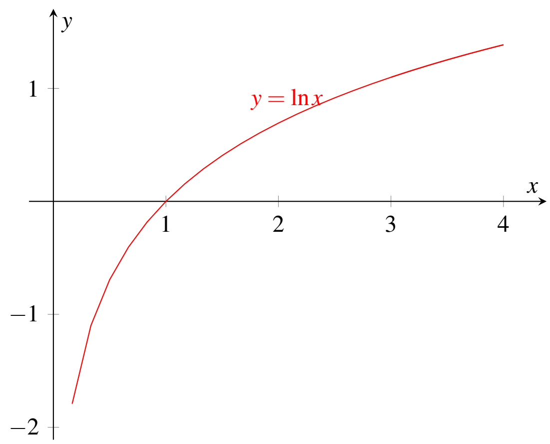


Figure 3.2: The graph of the function $y=\ln x$.

3.1.2 Differentiating and Integrating the Natural Logarithmic Function

From our discussion above, we found that

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Hence,

$$\frac{d}{dx}\ln(-x)=\frac{1}{-x}(-1)=\frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}\ln(|x|)=\frac{1}{x}\ \forall x\neq 0.$$

In the following theorem, we generalize the previous result.

Theorem 3.2 If $u=g(x)$ is differentiable, then

1. $\frac{d}{dx}\ln u = \frac{1}{u}u'$ if $u>0$

2. $\frac{d}{dx}\ln|u|=\frac{1}{u}u'$ if $u\neq 0$

Proof. **1.** If $y=\ln u$ where $u=g(x)$ is differentiable, then from the chain rule and the previous result, we have

$$\frac{d}{dx}\ln u=\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{u}u'.$$

2. If $u > 0$, then $|\, u \,| = u$. From the previous item, we have

$$\frac{d}{dx}\ln|\, u \,| = \frac{d}{dx}\ln u = \frac{1}{u}u'.$$

If $u < 0$, then $|\, u \,| = -u > 0$. This implies

$$\frac{d}{dx}\ln|\, u \,| = \frac{d}{dx}\ln(-u) = \frac{1}{-u}\frac{d}{dx}(-u') = \frac{1}{u}u'. \blacksquare$$

Henceforth we will assume that the domain of the function $u = g(x)$ is restricted to the domain of the natural logarithmic function. Therefore, we sometimes do not put the function $g(x)$ with the absolute value.

Example 3.1 Find the derivative of the function.

(1) $f(x) = \ln(x+1)$

(2) $g(x) = \ln(x^3 + 2x - 1)$

(3) $h(x) = \ln\sqrt{x^2+1}$

(4) $y(x) = \sqrt{\ln x}$

(5) $f(x) = \ln\cos\, x$

(6) $g(x) = \sqrt{x}\,\ln x$

(7) $h(x) = \sin\,(\ln x)$

(8) $y(x) = \ln(x + \ln x)$

Solution:

(1) $f'(x) = \frac{1}{x+1}$.

(2) $g'(x) = \frac{3x^2+2}{x^3+2x-1}$.

(3) $h'(x) = \frac{1}{\sqrt{x^2+1}}\frac{2x}{2\sqrt{x^2+1}} = \frac{x}{x^2+1}$.

(4) $y'(x) = \frac{1}{2\sqrt{\ln x}}\frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$.

(5) $f'(x) = \frac{-\sin\, x}{\cos\, x} = -\tan\, x$.

(6) $g'(x) = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{\ln x+2}{2\sqrt{x}}$.

(7) $h'(x) = \cos\,(\ln x)(\frac{1}{x}) = \frac{\cos\,(\ln x)}{x}$.

(8) $y'(x) = \frac{1}{x+\ln x}(1+\frac{1}{x}) = \frac{x+1}{x(x+\ln x)}$.

In the following, we present a simple application of the natural logarithmic function. We know that the derivative of composite functions takes an effort and time. This problem can be solved by using the differentiation of the natural logarithmic function. Specifically, we use the derivative of the natural logarithmic function and Theorem 3.1 to simplify the differentiation of the composite functions.

Example 3.2 Find the derivative of the function $y = \sqrt[5]{\frac{x-1}{x+1}}$.

Solution:
We can solve this example using the derivative rules. However, for simplicity, we use the natural logarithmic function.
By Taking the logarithm function of each side, we have

$$\ln|\, y \,| = \ln\left|\sqrt[5]{\frac{x-1}{x+1}}\right| = \frac{1}{5}\Big(\ln|x-1| - \ln|x+1|\Big).$$

By differentiating both sides with respect to x, we have

$$\frac{y'}{y} = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big) \qquad \Big(\frac{d}{dx}\ln y = \frac{y'}{y}\Big)$$

By multiplying both sides by y, we obtain

$$y' = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big)y$$

$$\Rightarrow y' = \frac{1}{5}\Big(\frac{1}{x-1} - \frac{1}{x+1}\Big)\sqrt[5]{\frac{x-1}{x+1}}.$$

Example 3.3 Find the derivative of the function $y = \dfrac{\sqrt{x}\cos\, x}{(x+1)\sin\, x}$.

Solution:
Take the natural logarithm of each side. This implies

$$\ln|\,y\,| = \ln\left|\frac{\sqrt{x}\cos\, x}{(x+1)\sin\, x}\right| = \ln\sqrt{x} + \ln|\cos\, x| - \ln|x+1| - \ln|\sin\, x|.$$

By differentiating both sides, we have

$$\frac{y'}{y} = \frac{1}{2x} - \frac{\sin\, x}{\cos\, x} - \frac{1}{x+1} - \frac{\cos\, x}{\sin\, x}.$$

Multiply both sides by y to have

$$y' = \left(\frac{1}{2x} - \tan\, x - \frac{1}{x+1} - \cot\, x\right)\frac{\sqrt{x}\cos\, x}{(x+1)\sin\, x}.$$

Recall, $\frac{d}{dx}\ln|\,u\,| = \frac{u'}{u}$ where $u = g(x)$ is a differentiable function. By integrating both sides, we have

$$\begin{aligned}\int \frac{u'}{u}\,dx &= \int \frac{d}{dx}\ln|\,u\,|\;dx\\ &= \ln|\,u\,| + c.\end{aligned}$$

This can be stated as follows:

$$\boxed{\int \frac{u'}{u}\,dx \;=\; \ln|\,u\,| + c}$$

If $u = x$, we have the following special case

$$\boxed{\int \frac{1}{x}\,dx \;=\; \ln|\,x\,| + c}$$

Example 3.4 Evaluate the integral.

(1) $\displaystyle\int \frac{2x}{x^2+1}\,dx$

(2) $\displaystyle\int \frac{6x^2+1}{4x^3+2x+1}\,dx$

(3) $\displaystyle\int_2^e \frac{dx}{x\,\ln x}$

(4) $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$

(5) $\displaystyle\int \tan\, x\,dx$

(6) $\displaystyle\int \cot\, x\,dx$

(7) $\displaystyle\int \sec\, x\,dx$

(8) $\displaystyle\int \csc\, x\,dx$

Solution:

(1) $\displaystyle\int \frac{2x}{x^2+1}\,dx = \ln(x^2+1) + c.$

(2) $\displaystyle\int \frac{6x^2+1}{4x^3+2x+1}\,dx = \frac{1}{2}\int \frac{12x^2+2}{4x^3+2x+1}\,dx = \frac{1}{2}\ln|\,4x^3+2x+1\,| + c.$

(3) Let $u = \ln x$, then $du = \frac{1}{x}\,dx$. By substitution, we obtain $\displaystyle\int \frac{1}{u}\,du = \ln|\,u\,|$.

By returning the evaluation to the initial variable x, we have $\displaystyle\int \frac{dx}{x\,\ln x} = \ln(\ln x)$. Hence,

$$\int_2^e \frac{dx}{x\,\ln x} = \Big[\ln(\ln x)\Big]_2^e = \ln(\ln e) - \ln(\ln 2) = \ln(1) - \ln(\ln 2) = -\ln(\ln 2).$$

(4) For $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$, let $u = 1+\sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\,dx$. By substitution, we have $\displaystyle 2\int \frac{1}{u}\,du = 2\ln|\,u\,|$.

By returning the evaluation to the initial variable x, we have $\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \ln|1+\sqrt{x}|$. Hence,

$$\int_1^4 \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = 2\Big[\ln|1+\sqrt{x}|\Big]_1^4 = 2(\ln 3 - \ln 2).$$

(5) We know that $\tan x = \frac{\sin x}{\cos x}$. Therefore,

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx \\ &= -\ln|\cos x| + c = \ln|\sec x| + c. \qquad (\sec x = \tfrac{1}{\cos x})\end{aligned}$$

(6) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + c = -\ln|\csc x| + c \qquad (\csc x = \frac{1}{\sin x})$

(7) $\int \sec x\,dx = \int \frac{\sec x\,(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + c.$

(8) $\int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{(\csc x - \cot x)}\,dx = \int \frac{\csc^2 x - \csc x\,\cot x}{\csc x - \cot x}\,dx = \ln|\csc x - \cot x| + c.$

Exercise 3.1

1 - 20 Find the derivative of the function.

1 $y = \ln(x+1)$

2 $y = \ln(x^3 + 2x - 4)$

3 $y = \ln(\sqrt{x})$

4 $y = \ln(\sqrt[3]{x^2})$

5 $y = \ln(\frac{1}{x})$

6 $y = \ln(\sin x + x + 1)$

7 $y = \ln(\sec x + x^2)$

8 $y = \ln(\cos^2 x)$

9 $y = \ln(\sin^2 x)$

10 $y = \ln(\cos^2 x)$

11 $y = \ln(\sin^2 x)$

12 $y = \ln(\sec x \tan x)$

13 $y = \csc x \ln x$

14 $y = \sqrt[3]{x^2}\ln(x^3+1)$

15 $y = \ln\left(\sqrt{\frac{x^2-1}{x+2}}\right)$

16 $y = \ln\big((x^2+1)(x-1)\big)$

17 $y = \ln(\sqrt{x+1} - \sqrt{x})$

18 $y = \frac{x}{\ln x^2}$

19 $y = \ln(x^3+1)$

20 $y = \ln\big(\ln(\sin x)\big)$

21 - 26 Find the derivative of the function.

21 $y = \sqrt[5]{\frac{2x+1}{3x-1}}$

22 $y = \frac{(x-1)(\sqrt{x^3+2x+1})}{x^3+2x^2+x-1}$

23 $y = \frac{x^2\sqrt{7x+3}}{(1+x^2)^3}$

24 $y = \sqrt[3]{\frac{\tan^2 x \sin x \cos x}{\sqrt{x^3}}}$

25 $y = \left(\frac{x \sec x^2}{\sqrt{x}(x+1)}\right)^{\frac{7}{2}}$

26 $y = \frac{\sqrt[3]{x+1}\cos^2 x}{(x+1)^2 \cos 3x}$

27 - 38 Evaluate the integral.

27 $\int \frac{3x}{x^2+1}\,dx$

28 $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\,dx$

29 $\int \frac{1}{x \ln x^2}\,dx$

30 $\int_0^{\frac{\pi}{4}} \sec x\,dx$

31 $\int \frac{\csc^2 x}{1+\cot x}\,dx$

32 $\int_{-1}^{4} \frac{x}{x^2+1}\,dx$

33 $\int \csc x\,dx$

34 $\int \frac{\cos\left(\sqrt{x+1}\right)}{\sqrt{x+1}}\,dx$

35 $\int \frac{\sqrt{\ln x^2}}{x}\,dx$

36 $\int_1^2 \frac{x+3}{x^2}\,dx$

37 $\int \frac{\cos(\ln x)}{x}\,dx$

38 $\int_2^3 \frac{1}{x\,(\ln x)^5}\,dx$

3.2 The Natural Exponential Function

Since the natural logarithmic function $\ln : (0,\infty) \longrightarrow \mathbb{R}$ is a strictly increasing function (see Figure 3.3), it is one-to-one. The function ln is also onto and this implies that the natural logarithmic function has an inverse function. The inverse function is called the natural exponential function.

Definition 3.2 The natural exponential function is defined as follows:

$$exp : \mathbb{R} \longrightarrow (0,\infty) \ ,$$

$$y = exp\ x \Leftrightarrow \ln y = x$$

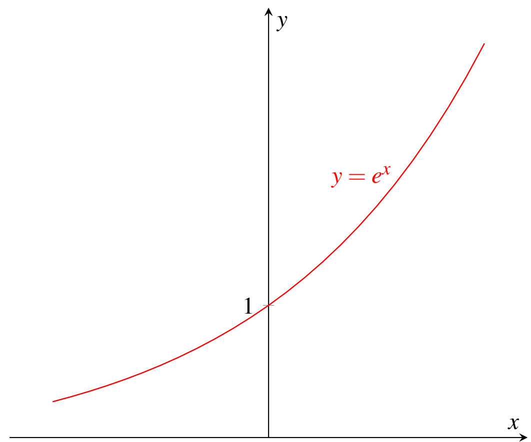


Figure 3.3: The graph of the function $y = e^x$.

3.2.1 Properties of the Natural Exponential Function

1. From the definition, the domain of the function $exp\ x$ is $\mathbb{R}$.

2. The range of the function $exp\ x$ is $(0,\infty)$ as follows:

$$y = \begin{cases} exp\ x > 1 & : x > 0 \\ exp\ x = 1 & : x = 0 \\ exp\ x < 1 & : x < 0 \end{cases}$$

3. Usually, the symbol $exp\ x$ is written as e^x, so $exp\ (1) = e \approx 2.71828$. From Definition 3.2, we have $\ln e = 1$ and $\ln e^r = r \ln e = r\ \forall\ r \in \mathbb{Q}$.

4. The function e^x is continuous and differentiable on the domain. From Definition 3.2, we have

$$y = e^x \Rightarrow \ln y = x.$$

By differentiating both sides, we have

$$\frac{d}{dx} \ln y = \frac{y'}{y} = 1 \Rightarrow y' = y.$$

Hence,

$$\frac{d}{dx} e^x = e^x\ \forall x \in \mathbb{R}.$$

Therefore, the function e^x is increasing on the domain $\mathbb{R}$.

5. The second derivative $\frac{d^2}{dx^2} e^x = e^x > 0$ for all $x \in \mathbb{R}$. Hence, the function e^x is concave upward on the domain $\mathbb{R}$.

6. $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.

7. Since e^x and $\ln x$ are inverse functions, then

$$\ln e^x = x,\ \forall x \in \mathbb{R}\ ,$$

$$e^{\ln x} = x,\ \forall x \in (0,\infty).$$

8. Rules of the natural exponential function:

Theorem 3.3 If $a, b > 0$ and $r \in \mathbb{Q}$, then
(a) $e^a e^b = e^{a+b}$

(b) $\frac{e^a}{e^b} = e^{a-b}$

(c) $(e^a)^r = e^{ar}$

Proof. **(a)** From the properties of the natural logarithmic function, we have

$$\ln(e^a e^b) = \ln e^a + \ln e^b = a \ln e + b \ln e = a + b \text{ , and}$$

$$\ln e^{a+b} = a + b.$$

Since the function ln is injective, then $e^a e^b = e^{a+b}$.

(b) From the properties of the natural logarithmic function, we have

$$\ln\left(\frac{e^a}{e^b}\right) = a \ln e - b \ln e = \ln e^a - \ln e^b = a - b \text{ , and}$$

$$\ln e^{a-b} = a - b.$$

Since the function ln is injective, then $\frac{e^a}{e^b} = e^{a-b}$.

(c) Since $\ln(e^a)^r = r \ln e^a = ra$ and $\ln e^{(ar)} = a\, r$, then $(e^a)^r = e^{ar}$. ■

Example 3.5 Solve for x.
(1) $\ln x = 2$

(2) $\ln(\ln x) = 0$

(3) $(x-1)e^{-\ln \frac{1}{x}} = 2$

(4) $xe^{2 \ln x} = 8$

Solution:
(1) $\ln x = 2 \Rightarrow e^{\ln x} = e^2 \Rightarrow x = e^2$. (take *exp* of both sides)
(2) $\ln(\ln x) = 0 \Rightarrow e^{\ln(\ln x)} = e^0 \Rightarrow \ln x = 1 \Rightarrow e^{\ln x} = e^1 \Rightarrow x = e$. (take *exp* twice)

(3) $(x-1)e^{-\ln \frac{1}{x}} = 2 \Rightarrow (x-1)e^{\ln(x^{-1})^{-1}} = 2 \Rightarrow (x-1)e^{\ln x} = 2$. This implies

$$x(x-1) = 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1 \text{ or } x = 2.$$

We have to ignore $x = -1$ since the domain of the natural logarithmic function is $(0, \infty)$.

(4) $xe^{2 \ln x} = 8 \Rightarrow xe^{\ln x^2} = 8 \Rightarrow x^3 = 8 \Rightarrow x = 2$.

Example 3.6 Simplify the expressions.
(1) $\ln(e^{\sqrt{x}})$

(2) $e^{\frac{1}{3} \ln x}$

(3) $(x+1)\ln(e^{x-1})$

(4) $e^{(\sqrt{x}+2\ln x)}$

Solution:
(1) $\ln(e^{\sqrt{x}}) = \sqrt{x}$.

(2) $e^{\frac{1}{3} \ln x} = e^{\ln \sqrt[3]{x}} = \sqrt[3]{x}$.

(3) $(x+1)\ln(e^{x-1}) = (x+1)(x-1) = x^2 - 1$.

(4) $e^{(\sqrt{x}+2\ln x)} = e^{\sqrt{x}} e^{\ln x^2} = x^2 e^{\sqrt{x}}$.

3.2.2 Differentiating and Integrating the Natural Exponential Function

From the discussion above, we found that

$$\frac{d}{dx} e^x = e^x$$

Generally, assume that $y = e^u$ where $u = g(x)$ is differentiable. By using the chain rule, we have

$$\frac{d}{dx}e^u = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u u'.$$

Theorem 3.4 If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}e^u = e^u u'.$$

Example 3.7 Find the derivative of the function.

(1) $y = e^{\sqrt[3]{x+1}}$

(2) $y = e^{-5x^2}$

(3) $y = e^{3\cos\, x - 4x^2}$

(4) $y = e^{\frac{1}{x}} - \frac{1}{e^x}$

(5) $y = e^{\ln \sin\, x}$

(6) $y = \ln(e^{2x} + \sqrt{1 - e^x})$

Solution:

(1) $y' = e^{\sqrt[3]{x+1}}(\frac{1}{3(x+1)^{2/3}})$.

(2) $y' = e^{-5x^2}(-10x)$.

(3) $y' = e^{3\cos\, x - 4x^2}(-3\sin\, x - 8x)$.

(4) $y' = e^{\frac{1}{x}}(\frac{-1}{x^2}) - (-e^{-x}) = \frac{1}{e^x} - \frac{e^{\frac{1}{x}}}{x^2}$.

(5) $y' = e^{\ln \sin\, x}\,(\frac{\cos\, x}{\sin\, x}) = \cos\, x$.

(6) $y' = \frac{1}{e^{2x} + \sqrt{1-e^x}}(2e^{2x} - \frac{e^x}{2\sqrt{1-e^x}})$.

Recall that $\frac{d}{dx}e^u = e^u u'$ where $u = g(x)$ is a differentiable function. By integrating both sides, we have

$$\int e^u u'\, dx = \int \frac{d}{dx}e^u\, dx = e^u + c.$$

This can be stated as follows:

$$\boxed{\int e^u u'\, dx = e^u + c}$$

If $u = x$, we have the following special case

$$\boxed{\int e^x\, dx = e^x + c}$$

Example 3.8 Evaluate the integral.

(1) $\int xe^{-x^2}\, dx$

(2) $\int_0^{ln5} e^x(3 - 4e^x)\, dx$

(3) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx$

(4) $\int \frac{e^{\tan\, x}}{\cos^2\, x}\, dx$

Solution:

(1) Let $u = -x^2$, then $du = -2x\, dx$. We substitute that into the integral to obtain

$$\frac{-1}{2}\int e^u\, du = \frac{-1}{2}e^u + c = \frac{-1}{2}e^{-x^2} + c.$$

(2) Let $u = 3 - 4e^x$ and this implies $du = -4e^x\ dx$. By substitution, we have

$$\frac{-1}{4}\int u\ du = -\frac{u^2}{8} + c.$$

Return the evaluation to the initial variable x to obtain $\int e^x(3-4e^x)\ dx = -\frac{1}{8}(3-4e^x)^2$. Hence,

$$\int_0^{\ln 5} e^x(3-4e^x)\ dx = -\frac{1}{8}\Big[(3-4e^x)^2\Big]_0^{\ln 5} = -\frac{1}{8}\big[(-17)^2 - (-1)^2\big] = -36.$$

(3) Let $u = e^x - e^{-x}$, then $du = e^x + e^{-x}\ dx$. By substitution, we have

$$\int \frac{1}{u}\ du = \ln|u| + c = \ln|e^x - e^{-x}| + c.$$

(4) Let $u = \tan\ x$, then $du = \sec^2\ x\ dx$. By substitution, we have

$$\int e^u\ du = e^u + c = e^{\tan x} + c \qquad (\sec^2\ x = \frac{1}{\cos^2\ x})$$

Exercise 3.2

1 - 4 Simplify the expressions.

1 $\sin^2\ x + e^{2\ln\cos\ x}$

2 $\ln e^{\sqrt[5]{x}}$

3 $(x+2)e^{\ln(x-2)}$

4 $\ln(e^{3+2\ln x})$

5 - 8 Solve for x.

5 $\ln x^2 = 4$

6 $\ln(\ln x) = 1$

7 $x\ e^{\ln x} = 27$

8 $\ln e^{x(x+2)} = 3$

9 - 18 Find the derivative of the function.

9 $y = e^{\sin\ x - 3x^2}$

10 $y = x\ e^{x\sqrt{x}}$

11 $y = e^x \cos\ (\ln x)$

12 $y = e^{\frac{1}{x}} \ln x$

13 $y = \ln(e^{-x} + \sqrt{x}\ e^{-x})$

14 $y = e^{\sqrt[3]{x}}\ \sin\ x$

15 $y = \ln(\tan\ e^x)$

16 $y = \sqrt{e^x}$

17 $y = (e^x+1)(\sqrt{e^{-x}+1})$

18 $y = \sec^2\ (e^{3x})$

19 - 28 Evaluate the integral.

19 $\int_0^1 e^{2x+1}\ dx$

20 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\ dx$

21 $\int \frac{e^{\sin\ x}}{\sec\ x}\ dx$

22 $\int \frac{(1-2\sqrt{x}\sin\ x)\ e^{\sqrt{x}+\cos\ x}}{\sqrt{x}}\ dx$

23 $\int \frac{e^{\frac{1}{x}}}{x^2}\ dx$

24 $\int_0^{\pi/4} \frac{e^{\sec\ x}\sin\ x}{\cos^2\ x}\ dx$

25 $\int \frac{1}{\sqrt{x}\ e^{\sqrt{x}}}\ dx$

26 $\int \frac{e^x}{(1+e^x)^5}\ dx$

27 $\int e^{\ln\cos\ x}\ dx$

28 $\int_1^2 \frac{e^x}{e^x+1}\ dx$

3.3 General Exponential and Logarithmic Functions

3.3.1 General Exponential Function

In Section 3.2, we defined the natural exponential function a^x when $a = e$. In the following, we define the general exponential function a^x with $a > 0$.

Definition 3.3 The general exponential function is defined as follows:

$$a^x : \mathbb{R} \to (0,\infty)\ ,$$

$$a^x \ = \ e^{x\ln a} \ \text{ for every } \ a>0.$$

Since $\ln a^x = x\ln a\ \forall x \in \mathbb{Q}$, then by taking the natural exponential function of both sides, we can write

$$a^x \ = \ e^{x\ln a}.$$

The function a^x is called the general exponential function with base a.

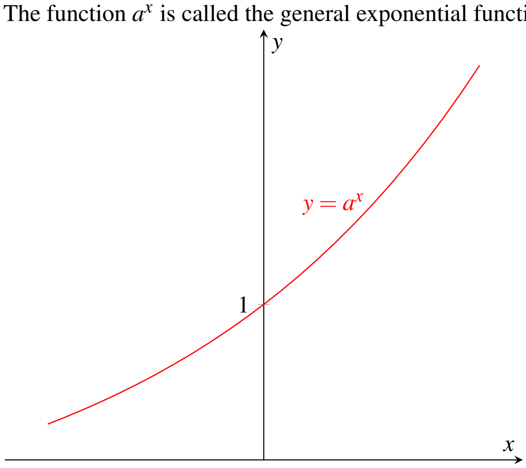


Figure 3.4: The function $y = a^x$ for $a > 1$.

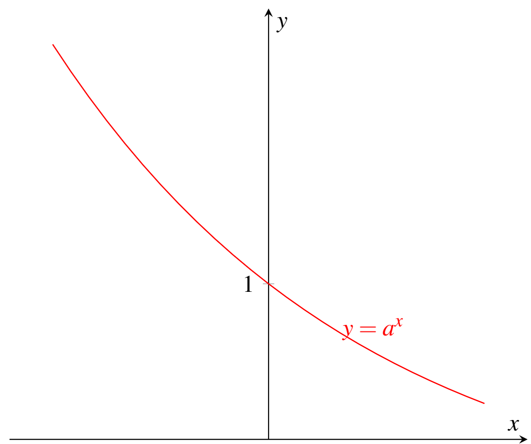


Figure 3.5: The function $y = a^x$ for $a < 1$.

Properties of the General Exponential Function

Let $f(x) = a^x\ \forall x \in \mathbb{R}$.

1. From Definition 3.3, the domain of $f(x)$ is $\mathbb{R}$ and the range is $(0,\infty)$ where

$$y = \begin{cases} a^x > 1 & : x > 0 \\ a^x = 1 & : x = 0 \\ a^x < 1 & : x < 0 \end{cases}$$

2. If $a > 1$, $\ln a > 0$ and this implies that $x\ln a$ and $f(x)$ are increasing functions as shown in Figure 3.4.

3. If $a < 1$, $\ln a < 0$ and this implies that $x\ln a$ and $f(x)$ are decreasing functions (see Figure 3.5).

4. Rules of the general exponential function:

Theorem 3.5 If $a,b > 0$ and $x,y \in \mathbb{R}$, then

a. $a^x a^y = a^{x+y}$

b. $\frac{a^x}{a^y} = a^{x-y}$

c. $(a^x)^y = a^{x\,y}$

d. $(ab)^x = a^x b^x$

Proof. We prove this theorem by using Definition 3.3 and the properties of the functions e^x and $\ln x$.

a. $a^x a^y = e^{x\ln a} e^{y\ln a} = e^{x\ln a + y\ln a} = e^{(x+y)\ln a} = e^{\ln a^{(x+y)}} = a^{x+y}$.

b. $\frac{a^x}{a^y} = \frac{e^{x\ln a}}{e^{y\ln a}} = e^{x\ln a - y\ln a} = e^{(x-y)\ln a} = e^{\ln a^{(x-y)}} = a^{x-y}$.

c. $(a^x)^y = e^{y\ln a^x} = e^{\ln a^{x\,y}} = a^{x\,y}$.

d. $(ab)^x = e^{x\ln ab} = e^{x(\ln a + \ln b)} = e^{x\ln a} e^{x\ln b} = a^x b^x$.

Note that the previous result generalizes Theorem 3.3. ∎

Differentiating and Integrating the General Exponential Function

Since $a^x = e^{x\ln a}$, then

$$\begin{aligned}\frac{d}{dx}a^x &= \frac{d}{dx}e^{x\ln a}\\ &= e^{x\ln a}\ln a\\ &= a^x\ln a.\end{aligned}$$

This can be stated as follows:

$$\boxed{\frac{d}{dx}a^x = a^x \ln a}$$

The following theorem generalizes the previous result.

Theorem 3.6 If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}a^u = a^u \ln a u'.$$

Proof. From Definition 3.3 and the chain rule, we have

$$\begin{aligned}\frac{d}{dx}a^u &= \frac{d}{dx}e^{u\ln a}\\ &= e^{u\ln a}u'\ln a\\ &= a^u u'\ln a.\end{aligned}$$

Note that by applying the previous theorem for $a = e$, we have Theorem 3.4. ■

Example 3.9 Find the derivative of the function.

(1) $y = 2^{\sqrt{x}}$

(2) $y = 3^{x^2 \sin\, x}$

(3) $y = \sin\, 3^x$

(4) $y = x(7^{-3x})$

(5) $y = \ln(\tan\, 5^x)$

(6) $y = (10^x + 10^{-x})^{10}$

Solution:

(1) $y' = 2^{\sqrt{x}}\, \ln 2\, \frac{1}{2\sqrt{x}} = \frac{2^{\sqrt{x}}\ln 2}{2\sqrt{x}}$.

(2) $y' = 3^{x^2 \sin\, x}\, \ln 3\, (2x\, \sin\, x + x^2\, \cos\, x)$.

(3) $y' = \cos\,(3x)\big(3^x \ln 3\big) = \big(3^x\, \ln 3\big)\, \cos\, 3^x$.

(4) $y' = 7^{-3x} + x\,\big((-3\ln 7)\, 7^{-3x}\big) = 7^{-3x}\big(1-(3\ln 7)\, x\big)$.

(5) $y' = \frac{\sec^2\, 5^x\, (5^x\, \ln 5)}{\tan\, 5^x} = \frac{(5^x\, \ln 5)\, \sec^2\, 5^x}{\tan\, 5^x}$.

(6) $y' = 10\,(10^x + 10^{-x})^9\,(10^x\, \ln 10 - 10^{-x}\, \ln 10) = 10\, \ln 10\,(10^x + 10^{-x})^9\,(10^x - 10^{-x})$.

Example 3.10 Find the derivative of the function $y = (\sin\, x)^x$.

Solution:
Take the natural logarithm of both sides to have $\ln y = x\, \ln(\sin\, x)$. By differentiating both sides, we have

$$\begin{aligned}\frac{y'}{y} &= \ln(\sin\, x) + \frac{x\, \cos\, x}{\sin\, x}\\ \Rightarrow y' &= \big(\ln(\sin\, x) + x\cot\, x\big)(\sin\, x)^x.\end{aligned}$$

From Theorem 3.6, we have

$$\boxed{\int a^u\, u'\, dx = \frac{1}{\ln a} a^u + c}$$

Example 3.11 Evaluate the integral.

(1) $\int x 3^{-x^2}\, dx$

(2) $\int 5^x \sqrt{5^x+1}\, dx$

(3) $\int 3^x \sin\, 3^x\, dx$

(4) $\int \frac{2^x}{2^x+1}\, dx$

Solution:

(1) Let $u=-x^2$, then $du=-2x\, dx$. By substitution, we have

$$\frac{-1}{2}\int 3^u\, du = \frac{-1}{2\ln 3}3^u + c = \frac{-1}{2\ln 3}3^{-x^2} + c.$$

(2) Let $u=5^x+1$, then $du=5^x\ln 5\, dx$. By substitution, we obtain

$$\frac{1}{\ln 5}\int u^{\frac{1}{2}}\, du = \frac{1}{\ln 5}\frac{u^{\frac{3}{2}}}{3/2} + c = \frac{2(5^x+1)^{\frac{3}{2}}}{3\ln 5} + c.$$

(3) Let $u=3^x$, then $du=3^x\ln 3\, dx$. By substitution, we have

$$\frac{1}{\ln 3}\int \sin\, u\, du = -\frac{1}{\ln 3}\cos\, u + c = -\frac{1}{\ln 3}\cos\, 3^x + c.$$

(4) Let $u=2^x+1$, then $du=2^x\ln 2\, dx$. By substituting that into the integral, we have

$$\frac{1}{\ln 2}\int \frac{1}{u}\, du = \frac{1}{\ln 2}\ln|\, u\,| + c = \frac{1}{\ln 2}\ln(2^x+1) + c.$$

3.3.2 General Logarithmic Function

We know that if $a \neq 1$, the function a^x is strictly increasing or decreasing, depending on the value of a. In any case, the function a^x is one-to-one and onto and this implies that the function a^x has an inverse function. The inverse function is called the general logarithmic function $\log_a x$ with base a.

Definition 3.4 The general logarithmic function is defined as follows:

$$\log_a : (0,\infty) \to \mathbb{R}\,,$$

$$x = a^y \Leftrightarrow y = \log_a x.$$

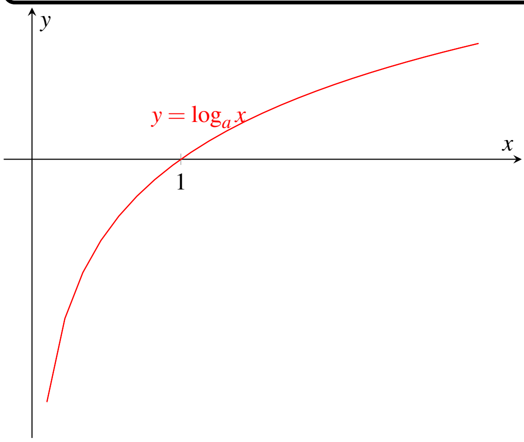


Figure 3.6: The function $y = \log_a x$ for $a > 1$.

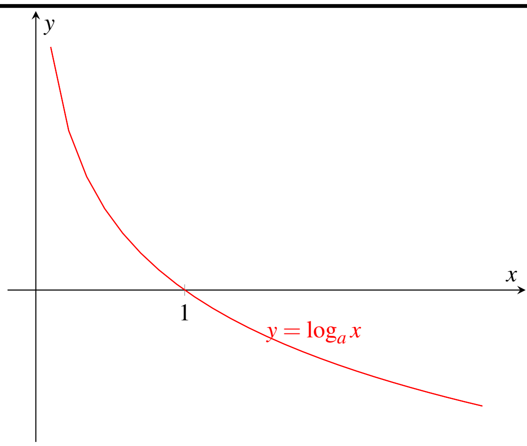


Figure 3.7: The function $y = \log_a x$ for $a < 1$.

Properties of the General Logarithmic Function

1. The general logarithmic function $\log_a x = \frac{\ln x}{\ln a}$.
To verify this, from Definition 3.4, we have $y = \log_a x \Rightarrow x = a^y$.
By taking the natural logarithm of both sides, we have

$$\ln x = \ln a^y = y \ln a \Rightarrow y = \frac{\ln x}{\ln a}.$$

2. If $a > 1$, the function $\log_a x$ is increasing while if $0 < a < 1$, the function $\log_a x$ is decreasing (see Figures 3.6 and 3.7).

3. The natural logarithmic function $\ln x = \log_e x$.

4. The general logarithmic function $\log_{10} x = \log x$.

5. The general logarithmic function $\log_a a = 1$.

6. Rules of the general logarithmic function:

Theorem 3.7 If $x, y > 0$ and $r \in \mathbb{R}$, then

a. $\log_a xy = \log_a x + \log_a y$

b. $\log_a \frac{x}{y} = \log_a x - \log_a y$

c. $\log_a x^r = r \log_a x$

Proof. To prove the theorem, we use the formula $\log_a x = \frac{\ln x}{\ln a}$ and the properties of the natural logarithmic function.

a. $\log_a xy = \frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a} = \log_a x + \log_a y.$ $\quad (\ln a\,b = \ln a + \ln b)$

b. $\log_a \frac{x}{y} = \frac{\ln(\frac{x}{y})}{\ln a} = \frac{\ln x}{\ln a} - \frac{\ln y}{\ln a} = \log_a x - \log_a y.$ $\quad (\ln \frac{a}{b} = \ln a - \ln b)$

c. $\log_a x^r = \frac{\ln x^r}{\ln a} = r\frac{\ln x}{\ln a} = r \log_a x.$ $\quad (\ln a^r = r \ln a)$

The previous result generalizes Theorem 3.1. ■

Differentiating and Integrating the General Logarithmic Function

Since $\log_a x = \frac{\ln x}{\ln a}$, then

$$\frac{d}{dx}\left(\log_a x\right) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x \ln a}.$$

By integrating both sides, we have

$$\int \frac{1}{x \ln a}\, dx = \log_a |x| + c.$$

Theorem 3.8 If $u = g(x)$ is differentiable, then

$$\frac{d}{dx}\left(\log_a |u|\right) = \frac{d}{dx}\left(\frac{\ln |u|}{\ln a}\right) = \frac{1}{u \ln a} u'$$

From the previous theorem, we have

$$\boxed{\int \frac{1}{u \ln a} u'\, dx = \log_a |u| + c}$$

Note that

$$\int \frac{1}{u \ln a} u'\, dx = \frac{1}{\ln a} \int \frac{u'}{u}\, dx = \frac{\ln |u|}{\ln a} = \log_a |u| + c\,.$$

Example 3.12 Find the derivative of the function.

(1) $y = \log_3 \sin\, x$ **(2)** $y = \log \sqrt{x}$

Solution:

(1) $y' = \frac{1}{\ln 3} \frac{\cos\ x}{\sin\ x} = \frac{\cot\ x}{\ln 3}$.

(2) $y' = \frac{1}{(2 \ln 10)\ x}$.

Example 3.13 Evaluate the integral.

(1) $\int \frac{1}{x \log x}\, dx$

(2) $\int \frac{1}{x\ \log_2 \sqrt{x}}\, dx$

Solution:

(1) Let $u = \log x \Rightarrow du = \frac{dx}{x \ln 10}$. By substitution, we have

$$\ln 10 \int \frac{1}{u}\, du = \ln 10\ \ln |\, u \,| + c = \ln 10\ \ln |\, \log x \,| + c.$$

(2) Let $u = \log_2 \sqrt{x} \Rightarrow du = \frac{dx}{2 \ln 2 \sqrt{x}}$. By substitution, we have

$$2 \ln 2 \int \frac{1}{u}\, du = 2 \ln 2\ \ln |\, u \,| + c = 2 \ln 2\ \ \ln |\, \log_2 \sqrt{x} \,| + c.$$

Exercise 3.3

1 - 10 Find the derivative of the function.

1 $y = 3^x$

2 $y = 2^{\sin\ x\ \cos\ x}$

3 $y = \ln 2^x$

4 $y = \log_2 \cos\ x$

5 $y = \log \sqrt[3]{x+1}$

6 $y = 5^{\sqrt{x}\ \tan\ x}$

7 $y = x\, 4^{-2x}$

8 $y = \log(x+1)$

9 $y = \ln(\sec\ 5^{x+1})$

10 $y = \log_5 x^{\frac{3}{2}}$

11 - 14 Find the derivative of the function.

11 $y = (\sin\ x)^x$

12 $y = e^{x^x}$

13 $y = x^{e^x}$

14 $y = (x^2 - x)^{\ln x}$

15 - 20 Evaluate the integral.

15 $\int x^2 5^{x^3}\, dx$

16 $\int 2^x \cos\,(2^x + 1)\, dx$

17 $\int \frac{1}{x \log x^2}\, dx$

18 $\int \frac{3^x}{\sqrt{3^x + 1}}\, dx$

19 $\int 7^{3x} \sqrt{7^{3x} + 1}\, dx$

20 $\int \frac{\log_2 \sin\ x}{\tan\ x}\, dx$

Review Exercises

1 - 6 Solve for x.

1 $x = e^{\ln 2}$

2 $\ln x = 1$

3 $\ln x = \ln 3 - 2\ln 8$

4 $\ln x^2 = \ln 4 + \ln 2$

5 $\ln x = \ln(x+1) + \ln(x-1)$

6 $e^{2x} + 2e^x - 8 = 0$

7 - 12 Find each limit if it exists.

7 $\lim_{x\to 0} \ln \cos\ x$

8 $\lim_{x\to\infty} \frac{1}{1+\ln x}$

9 $\lim_{x\to\infty} e^{-x} + 1$

10 $\lim_{x\to\infty} \ln e^x$

11 $\lim_{x\to\infty} \log_2 x + e^x$

12 $\lim_{x\to 0^+} \ln \sin\ x$

13 - 44 Find the derivative of the function.

13 $f(x) = \ln x^2$

14 $f(x) = \ln(x^2 + 3x + 1)$

15 $f(x) = \ln \cos^3\ x$

16 $f(x) = \ln \sin\ x^2$

17 $f(x) = \ln \sqrt{x^3 + x - 1}$

18 $f(x) = \ln(\sqrt{x} - \sqrt{x-1})$

19 $f(x) = \sin\ x\ \ln \cos\ x$

20 $f(x) = \ln(\frac{x^2 \sin\ x}{\sqrt{x+1}})$

21 $f(x) = \frac{1}{\ln x} + \ln(\frac{1}{x})$

22 $f(x) = (\ln x^3)^2$

23 $f(x) = \sqrt{x}\ \ln(x^2 + x - 2)$

24 $f(x) = e^x \sec\ x$

25 $f(x) = e^{\ln x^2 + x - 1}$

26 $f(x) = e^{x+1} \sin^3\ x$

27 $f(x) = e^{\frac{x}{x+1}}$

28 $f(x) = \ln(\sin\ e^x)$

29 $f(x) = e^{2x+1}$

30 $f(x) = e^{\sin\ x}$

31 $f(x) = e^{\sec^2\ x}$

32 $f(x) = \sin(e^{2x^3 + x - 1})$

33 $f(x) = e^{2x+1}$

34 $f(x) = \frac{e^x}{x+1}$

35 $f(x) = \frac{e^x}{\ln x}$

36 $f(x) = e^{x \tan\ x}$

37 $f(x) = e^x \ln x$

38 $f(x) = x^2 e^{\sqrt{x}}$

39 $f(x) = \pi^{\cos\ x}$

40 $f(x) = 2^{\sin^2\ x}$

41 $f(x) = 10^{3x}$

42 $f(x) = \tan(2^{\sin\ x})$

43 $f(x) = \log_3(\frac{6x+1}{2x-1})$

44 $f(x) = \log(\ln x)$

45 - 50 Find the derivative of the function.

45 $y = (\tan\ x)^{\tan\ x}$

46 $y = x^x$

47 $y = x^{\sqrt{x}}$

48 $y = x^{4x}$

49 $y = x^{\sin\ x}$

50 $y = (\ln x)^{\tan\ x}$

51 - 72 Evaluate the integral.

51 $\int \frac{x^2}{x^3+2}\, dx$

52 $\int \frac{\sin\ x}{\cos\ x}\, dx$

53 $\int \frac{x+1}{x^2+2x}\, dx$

54 $\int \frac{\sqrt{\ln x}}{x}\, dx$

55 $\int_0^1 \frac{x}{x^2+1}\, dx$

56 $\int_{-2}^0 \frac{x}{x^2+3}\, dx$

57 $\int \frac{\cos\ (\ln x)}{x}\, dx$

58 $\int (\sqrt{x}+\frac{1}{\sqrt{x}})^2\, dx$

59 $\int \frac{1}{x(\ln x)^2}\, dx$

60 $\int \frac{\sin\ x - \cos\ x}{\sin\ x + \cos\ x}\, dx$

61 $\int x3^{-x^2}\, dx$

62 $\int 2xe^{x^2}\, dx$

63 $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

64 $\int \frac{\cos\ x\, e^{\ln(\sin\ x)}}{\sin\ x}\, dx$

65 $\int e^{\tan\ x} \sec^2\ x\, dx$

66 $\int \frac{5^{\sqrt{x}}}{\sqrt{x}}\, dx$

67 $\int_0^{\ln 2} e^x(2-3e^x)\, dx$

68 $\int \frac{x^3}{x^4+1}\, dx$

69 $\int 4^{3x}\, dx$

70 $\int_0^3 x3^{-x^2}\, dx$

71 $\int x10^{x^2+1}\, dx$

72 $\int \frac{a^{\sqrt{x+1}}}{\sqrt{x+1}}\, dx$ where $a > 0$

73 - 89 Choose the correct answer.

73 If $f(x) = \log_2 \frac{x}{x-1} = 1$, then x is equal to

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) -1

74 The value of the integral $\int_0^1 5^x\, dx$ is equal to

(a) $\frac{4\ln 5}{5}$ (b) $\frac{\ln 5}{4}$ (c) $\frac{4}{\ln 5}$ (d) $\frac{5\ln 5}{4}$

75 If $f(x) = x^{x+1}$, then $f'(x)$ is equal to

(a) $(1+\frac{1}{x}+\ln x)x^{x+1}$ (b) $(\ln x+\frac{1}{x})x^{x+1}$ (c) $(1+\ln x)x^{x+1}$ (d) $(1+\frac{1}{x}+\ln x)x^x$

76 $\lim_{x\to\infty} \frac{e^x+e^{2x}}{1+e^{2x}}$ is equal to

(a) ∞ (b) 1 (c) 0 (d) None of these

77 The integral $\int \tan\ 2x\, dx$ is equal to

(a) $\frac{-1}{2} \ln|\sec\ 2x| + c$ (b) $\frac{1}{2}\sec^2\ 2x + c$ (c) $\frac{-1}{2}\ln|\cos\ 2x| + c$ (d) $2\sec^2\ 2x + c$

78 The integral $\int \ln(2^{\sin\ x})\, dx$ is equal to

(a) $\frac{1}{2}\ln(2)\sin\ x + c$ (b) $2^{-\sin\ x}\cos\ x + c$ (c) $-\sin\ x + c$ (d) $-\ln 2\cos\ x + c$

79 The integral $\int_0^1 \frac{e^x}{(e^x+1)^2}\, dx$ is equal to

(a) $\frac{e-1}{2(1+e)}$ (b) 0 (c) -1 (d) $\frac{1}{(1+e)^2}$

80 If $f(x) = x^{\ln x}$ then $f'(e)$ is equal to
(a) 2 (b) $2e$ (c) 0 (d) e

81 $\lim_{x\to 0^+} \frac{\sin x}{\ln x}$ is equal to
(a) ∞ (b) 0 (c) 1 (d) $-\infty$

82 If $f(x) = \ln(\ln x)$ then $f'(x)$ is equal to
(a) $\frac{1}{\ln x}$ (b) $\frac{1}{x\ln x}$ (c) $-\frac{1}{(\ln x)^2}$ (d) $-\frac{1}{x\ln x}$

83 The integral $\int 2^{\sin\ x}\cos\ x\, dx$ is equal to
(a) $2^{\sin\ x}+c$ (b) $(\ln 2)2^{\sin\ x}+c$ (c) $\frac{2^{\sin\ x}}{\ln 2}+c$ (d) $-\frac{2^{\sin\ x}}{\ln 2}+c$

84 The integral $\int \frac{\tan^2\ x}{\sec\ x}\, dx$ is equal to
(a) $\ln|\sec\ x+\tan\ x|+\sin\ x+c$ (b) $\ln|\sec\ x+\tan\ x|-\cos\ x+c$
(c) $\ln|\sec\ x+\tan\ x|-\sin\ x+c$ (d) $\ln|\sec\ x|-\sin\ x+c$

85 The value of the integral $\int_0^1 3^x\, dx$ is equal to
(a) $\frac{2}{\ln 3}$ (b) $\frac{3}{\ln 3}$ (c) 3 (d) 2

86 If $f(x) = x^x$, then $f'(1)$ is equal to
(a) 0 (b) e (c) 1 (d) $\frac{1}{e}$

87 The value of the integral $\int_0^1 (7x)7^{x^2}\, dx$ is equal to
(a) $\frac{21}{\ln 7}$ (b) $21\ln 7$ (c) $\frac{49}{\ln 7}$ (d) $\frac{7}{\ln 7}$

88 If $F(x) = x^{\frac{1}{x}}$, the $F'(e)$ is equal to
(a) 0 (b) e (c) $e^{\frac{1}{e}}$ (d) $\frac{e^{\frac{1}{e}}}{e^2}$

89 If $\log_2 \frac{x-1}{x} = 2$, then x is equal to
(a) -1 (b) $\frac{1}{3}$ (c) $-\frac{1}{3}$ (d) 1

Chapter 4

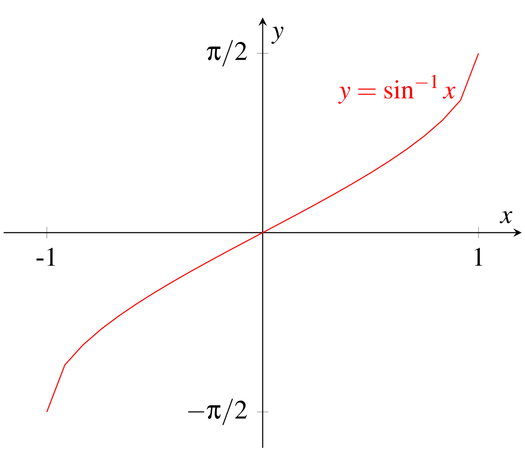
Inverse Trigonometric and Hyperbolic Functions

4.1 Inverse Trigonometric Functions

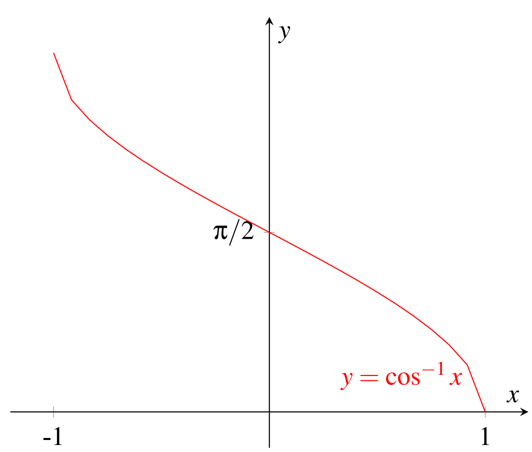
The inverse trigonometric functions are the inverse functions of the trigonometric functions: sine, cosine, tangent, cotangent, secant, and cosecant. While the trigonometric functions give trigonometric ratios, the inverse trigonometric functions give angles from the angle trigonometric ratios. The most common notations to name the inverse trigonometric functions are *arcsinx*, *arccosx*, *arctanx*, etc. However, the notations $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$, etc. are often used as well. In this book, we use the latter notations to denote to the inverse trigonometric functions.[1]

To find the inverse of any function, we need to show that the function is bijective (i.e., is it one-to-one and onto?). From your knowledge, none of the six trigonometric functions are bijective. Therefore, in order to have inverse trigonometric functions, we should consider subsets of their domains. In the following, we show the graph of the inverse trigonometric functions, and their domains and ranges.

■ The inverse sine function
$\sin\ y = x \Leftrightarrow y = \sin^{-1}\ x$
Domain: $[-1, 1]$ Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

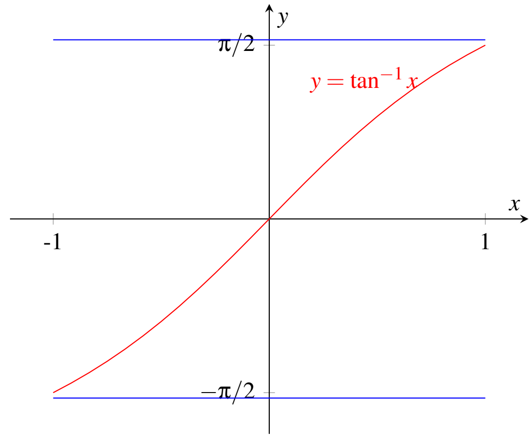


■ The inverse cosine function
$\cos\ y = x \Leftrightarrow y = \cos^{-1}\ x$
Domain: $[-1, 1]$ Range: $[0, \pi]$

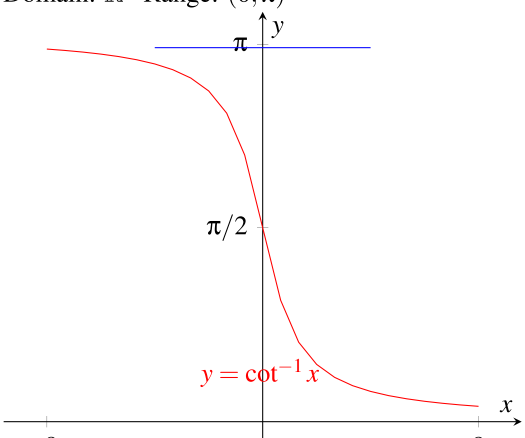


[1] *Common mistake:* some students write $\sin^{-1}\ x = (\sin\ x)^{-1} = \frac{1}{\sin\ x}$ and this is not true.

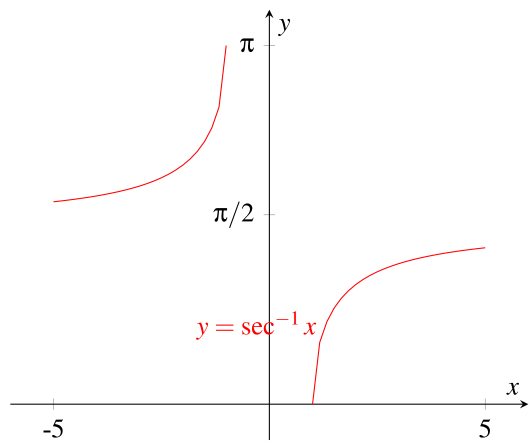
■ The inverse tangent function
$\tan\ y = x \Leftrightarrow y = \tan^{-1}\ x$
Domain: $\mathbb{R}$
Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$



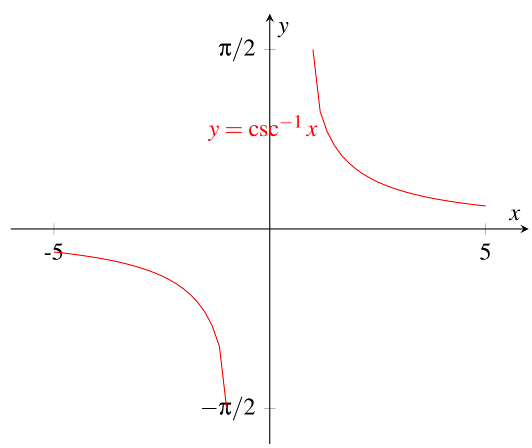
■ The inverse cotangent function
$\cot\ y = x \Leftrightarrow y = \cot^{-1}\ x$
Domain: $\mathbb{R}$ Range: $(0, \pi)$



■ The inverse secant function
$\sec y = x \Leftrightarrow y = \sec^{-1}\ x$
Domain: $\mathbb{R} \setminus (-1, 1)$ Range: $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$



■ The inverse cosecant function
$\csc\ y = x \Leftrightarrow y = \csc^{-1}\ x$
Domain: $\mathbb{R} \setminus (-1, 1)$
Range: $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$



Differentiating and Integrating the Inverse Trigonometric Functions

In the following theorem, we list the derivatives of the inverse trigonometric functions. Then, we list the integration rules.

Theorem 4.1 If $u = g(x)$ is a differentiable function, then

1. $\frac{d}{dx}\sin^{-1}\ u = \frac{1}{\sqrt{1-u^2}}u'$
2. $\frac{d}{dx}\cos^{-1}\ u = \frac{-1}{\sqrt{1-u^2}}u'$
3. $\frac{d}{dx}\tan^{-1}\ u = \frac{1}{u^2+1}u'$
4. $\frac{d}{dx}\cot^{-1}\ u = \frac{-1}{u^2+1}u'$
5. $\frac{d}{dx}\sec^{-1}\ u = \frac{1}{u\sqrt{u^2-1}}u'$
6. $\frac{d}{dx}\csc^{-1}\ u = \frac{-1}{u\sqrt{u^2-1}}u'$

Proof. For simplicity, we assume $u = x$.

1. From the differentiation rule of the inverse functions, $y = sin^{-1}\ x$ is differentiable if $x \in (-1,1)$. By differentiating $\sin\ y = x$ implicitly, we have
$$\cos\ y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos\ y} \Rightarrow \frac{d}{dx}\sin^{-1}\ x = \frac{1}{\sqrt{1-\sin^2\ y}}.$$
This implies
$$\frac{d}{dx}\sin^{-1}\ x = \frac{1}{\sqrt{1-x^2}}.$$
2. The function $\cos^{-1}\ x$ is differentiable if $x \in (-1,1)$. We know that
$$y = \cos^{-1}\ x \Leftrightarrow \cos y = x.$$
By using the implicit differentiation, we obtain
$$-\sin\ y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\sin\ y} \Rightarrow \frac{d}{dx}\cos^{-1}\ x = \frac{-1}{\sqrt{1-\cos^2\ y}}.$$
This implies
$$\frac{d}{dx}\cos^{-1}\ x = \frac{-1}{\sqrt{1-x^2}}.$$
3. The function $\tan^{-1}\ x$ is differentiable if $x \in \mathbb{R}$. Since
$$y = \tan^{-1}\ x \Leftrightarrow \tan y = x\ ,$$
we use the implicit differentiation to have
$$\sec^2\ y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2\ y} \Rightarrow \frac{d}{dx}\tan^{-1}\ x = \frac{1}{1+\tan^2\ y}.$$
Hence,
$$\frac{d}{dx}\tan^{-1}\ x = \frac{1}{1+x^2}.$$
4. This item can be proved in a similar way to item 3.
5. The function $\sec^{-1}\ x$ is differentiable if $x \in (-\infty,-1)\cup(1,\infty)$. Since
$$y = \sec^{-1}\ x \Leftrightarrow \sec y = x\ ,$$
we use the implicit differentiation to have
$$\sec\ y\ \tan\ y\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec\ y\ \tan\ y} \Rightarrow \frac{d}{dx}\sec^{-1}\ x = \frac{1}{x\sqrt{\sec^2\ y-1}}.$$
Hence,
$$\frac{d}{dx}\sec^{-1}\ x = \frac{1}{x\sqrt{x^2-1}}.$$
6. This item can be proved in a similar way to item 5. ■

Example 4.1 Find the derivative of the function.

(1) $y = \sin^{-1}\ 5x$

(2) $y = \tan^{-1}\ e^x$

(3) $y = \sec^{-1}\ 2x$

(4) $y = \sin^{-1}\ (x-1)$

Solution:

(1) $y' = \frac{5}{\sqrt{1-(5x)^2}} = \frac{5}{\sqrt{1-25x^2}}$.

(2) $y' = \frac{e^x}{(e^x)^2+1} = \frac{e^x}{e^{2x}+1}$.

(3) $y' = \frac{2}{2x\sqrt{4x^2-1}} = \frac{1}{x\sqrt{4x^2-1}}$.

(4) $y' = \frac{1}{\sqrt{1-(x-1)^2}} = \frac{1}{\sqrt{2x-x^2}}$.

From the list of the derivatives of the inverse trigonometric functions, we have the following integration rules:

1. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$

2. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$

3. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + c$

The following theorem generalizes the previous integration rules.

Theorem 4.2 For $a > 0$,

1. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1} \frac{x}{a} + c$

2. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1} \frac{x}{a} + c$

3. $\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1} \frac{x}{a} + c$

Proof. We prove item 1 and the others can be done in a similar way.
For simplicity, we assume $u = x$.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{a^2(1-\frac{x^2}{a^2})}}\,dx = \int \frac{1}{a\sqrt{1-(\frac{x}{a})^2}}\,dx.$$

Let $v = \frac{x}{a}$, then $dv = \frac{dx}{a}$. By substitution, we have

$$\frac{1}{a}\int \frac{1}{\sqrt{1-v^2}}\,a\,dv = \int \frac{1}{\sqrt{1-v^2}}\,dv = \sin^{-1}\, v + c = \sin^{-1} \frac{x}{a} + c. \blacksquare$$

Example 4.2 Evaluate the integral.

(1) $\int \frac{1}{\sqrt{4-25x^2}}\,dx$.

(2) $\int \frac{1}{x\sqrt{x^6-4}}\,dx$.

(3) $\int \frac{1}{9x^2+5}\,dx$.

(4) $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$.

Solution:

(1) $\int \frac{1}{\sqrt{4-25x^2}}\,dx = \int \frac{1}{\sqrt{4-(5x)^2}}\,dx$.

Let $u = 5x$, then $du = 5dx \Rightarrow dx = \frac{du}{5}$. By substitution, we have

$$\frac{1}{5}\int \frac{1}{\sqrt{4-u^2}}\,du = \frac{1}{5}\sin^{-1}\frac{u}{2} + c = \frac{1}{5}\sin^{-1}\frac{5x}{2} + c.$$

(2) $\int \frac{1}{x\sqrt{x^6-4}}\,dx = \int \frac{1}{x\sqrt{(x^3)^2-4}}\,dx$.

Let $u = x^3$, then $du = 3x^2dx$. By substitution, we obtain

$$\frac{1}{3}\int \frac{1}{u\sqrt{u^2-4}}\,du = \frac{1}{3}\frac{1}{2}\sec^{-1}\frac{u}{2} + c = \frac{1}{6}\sec^{-1}\frac{x^3}{2} + c.$$

(3) $\displaystyle\int \frac{1}{9x^2+5}\,dx = \int \frac{1}{(3x)^2+5}\,dx.$
Let $u = 3x$, then $du = 3dx$. By substitution, we have

$$\frac{1}{3}\int \frac{1}{u^2+5}\,du = \frac{1}{3}\frac{1}{\sqrt{5}}\tan^{-1}\frac{u}{\sqrt{5}} + c = \frac{1}{3\sqrt{5}}\tan^{-1}\frac{3x}{\sqrt{5}} + c.$$

(4) $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{1}{\sqrt{(e^x)^2-1}}\,dx.$
Let $u = e^x$, $du = e^x dx$. After substitution, we have

$$\int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1} u + c = \sec^{-1} e^x + c.$$

Exercise 4.1

1 - 8 Find the derivative of the function.

1 $y = \sin^{-1} \ln x$

2 $y = \cos^{-1}(4x^2)$

3 $y = \tan^{-1}\sqrt{x}$

4 $y = csc^{-1}\frac{5x}{3}$

5 $y = \sin^{-1}(x^2+x-1)$

6 $y = \tan^{-1}\frac{1}{x}$

7 $y = \cot^{-1} e^{\frac{1}{x}}$

8 $y = \sec^{-1}(\ln\sqrt[3]{x})$

9 - 16 Evaluate the integral.

9 $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

10 $\displaystyle\int \frac{1}{x^2+81}\,dx$

11 $\displaystyle\int \frac{1}{\sqrt{e^{2x}-4}}\,dx$

12 $\displaystyle\int \frac{1}{\sec x\,(\sin^2 x+1)}\,dx$

13 $\displaystyle\int \frac{1}{x\sqrt{x^8-9}}\,dx$

14 $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$

15 $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

16 $\displaystyle\int \frac{\cot x}{\cos^2 x\,\sqrt{\tan^2 x-3}}\,dx$

4.2 Hyperbolic Functions

In this section, we define the hyperbolic functions. They are based on the natural exponential function and this indicates that the properties and the rules of the differentiation of the former functions depend on the latter function.

Definition 4.1 The hyperbolic sine function (sinh) and the hyperbolic cosine function (cosh) are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \forall x \in \mathbb{R},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},\ \forall x \in \mathbb{R}.$$

Other hyperbolic functions can be defined from the hyperbolic sine and the hyperbolic cosine as follows:

$$\begin{aligned}
\tanh\ x &= \frac{\sinh\ x}{\cosh\ x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \forall x \in \mathbb{R} \\
\coth\ x &= \frac{\cosh\ x}{\sinh\ x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad \forall x \in \mathbb{R} \setminus \{0\} \\
\text{sech}\ x &= \frac{1}{\cosh\ x} = \frac{2}{e^x + e^{-x}}, \quad \forall x \in \mathbb{R} \\
\text{csch}\ x &= \frac{1}{\sinh\ x} = \frac{2}{e^x - e^{-x}}, \quad \forall x \in \mathbb{R} \setminus \{0\}
\end{aligned}$$

4.2.1 Properties of the Hyperbolic Functions

1. The graph of the hyperbolic functions depends on the natural exponential functions e^x and e^{-x} (as shown in Figure 4.2).

2. The hyperbolic sine function is an odd function (i.e., $\sinh(-x) = -\sinh x$); whereas the hyperbolic cosine is an even function (i.e., $\cosh(-x) = \cosh x$). Therefore, the functions tanh, coth and csch are odd functions and the function sech is an even function. This in turn indicates that the graphs of the functions sinh, tanh, coth and csch are symmetric with respect to the original point; whereas the graph of the functions cosh and sech are symmetric around the y-axis.

3. $\cosh^2\ x - \sinh^2\ x = 1,\ \forall x \in \mathbb{R}$.
   To verify this item, we have from Definition 4.1 that

$$\cosh\ x - \sinh\ x = e^{-x} \ \text{ and } \ \cosh\ x + \sinh\ x = e^x.$$

   Hence,

$$(\cosh\ x - \sinh\ x)(\cosh\ x + \sinh\ x) = \cosh^2\ x - \sinh^2\ x = e^{-x}e^x = e^0 = 1.$$

4. Since $\cos^2\ t + \sin^2\ t = 1$ for any $t \in \mathbb{R}$, then the point $P(\cos\ t, \sin\ t)$ is located on the unit circle $x^2 + y^2 = 1$. However, for any $t \in \mathbb{R}$, the point $P(\cosh\ t, \sinh\ t)$ is located on the hyperbola $x^2 - y^2 = 1$. Figure 4.1 illustrates this item.

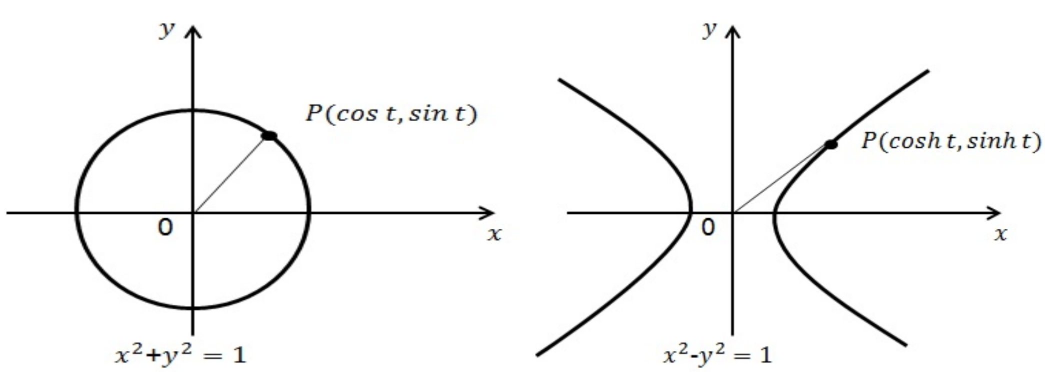


Figure 4.1: sinh x and cosh x versus sin x and cos x.

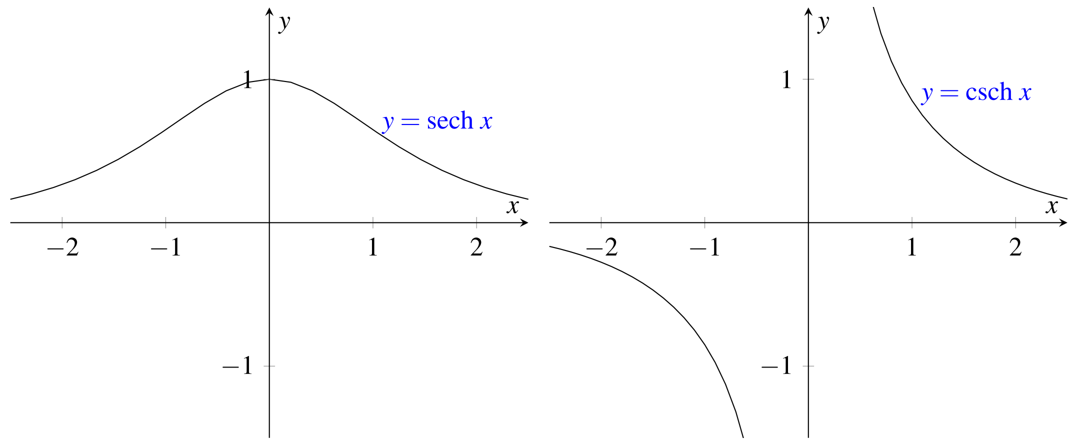
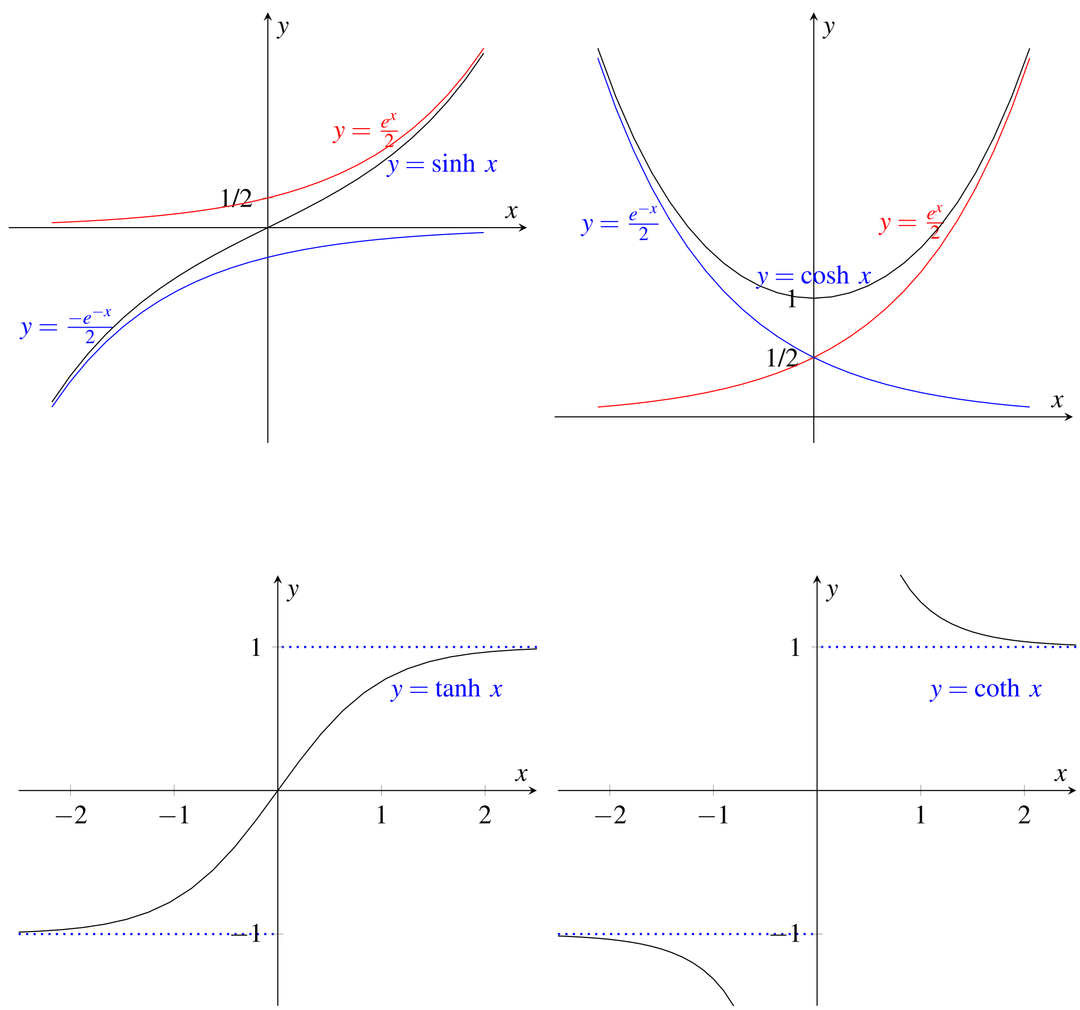


Figure 4.2: The graph of the hyperbolic functions.



Theorem 4.3

1. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
2. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
3. $\sinh 2x = 2\sinh x \cosh x$
4. $\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1 = \cosh^2 x + \sinh^2 x$
5. $1 - \tanh^2 x = \operatorname{sech}^2 x$
6. $\coth^2 x - 1 = \operatorname{csch}^2 x$
7. $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
8. $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Proof. **1.** From the definition of $\cosh x$ and $\sinh x$, we have

$$\begin{aligned}
\sinh x \cosh y + \cosh x \sinh y &= \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2} \\
&= \frac{1}{4}\left(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-(x+y)} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-(x+y)}\right) \\
&= \frac{1}{4}\left(2e^{(x+y)} - 2e^{-(x+y)}\right) \\
&= \frac{e^{(x+y)} - e^{-(x+y)}}{2} = \sinh(x+y).
\end{aligned}$$

$$\begin{aligned}
\sinh x \cosh y - \cosh x \sinh y &= \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} - \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2} \\
&= \frac{1}{4}\left(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-(x+y)} - e^{x+y} + e^{x-y} - e^{-x+y} + e^{-(x+y)}\right) \\
&= \frac{1}{4}\left(2e^{(x-y)} - 2e^{-x+y}\right) \\
&= \frac{e^{(x-y)} - e^{-(x-y)}}{2} = \sinh(x-y).
\end{aligned}$$

2.
$$\begin{aligned}
\cosh x \cosh y + \sinh x \sinh y &= \frac{e^x + e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2}\frac{e^y - e^{-y}}{2} \\
&= \frac{1}{4}\left(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-(x+y)} + e^{x+y} - e^{x-y} - e^{-x+y} + e^{-(x+y)}\right) \\
&= \frac{1}{4}\left(2e^{(x+y)} + 2e^{-(x+y)}\right) \\
&= \frac{e^{(x+y)} + e^{-(x+y)}}{2} = \cosh(x+y).
\end{aligned}$$

$$\begin{aligned}
\cosh x \cosh y - \sinh x \sinh y &= \frac{e^x + e^{-x}}{2}\frac{e^y + e^{-y}}{2} - \frac{e^x - e^{-x}}{2}\frac{e^y - e^{-y}}{2} \\
&= \frac{1}{4}\left(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-(x+y)} - e^{x+y} + e^{x-y} + e^{-x+y} - e^{-(x+y)}\right) \\
&= \frac{1}{4}\left(2e^{(x-y)} + 2e^{-x+y}\right) \\
&= \frac{e^{(x-y)} + e^{-(x-y)}}{2} = \cosh(x-y).
\end{aligned}$$

3.
$$\begin{aligned}
2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{1}{2}\left(e^{2x} + 1 - 1 - e^{-2x}\right) \\
&= \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x.
\end{aligned}$$

4. We prove that $\cosh\ 2x = 2\cosh^2\ x - 1$ and the other equalities can be proven similarly.

$$\begin{aligned} 2\cosh^2\ x - 1 &= 2\Big(\frac{e^x + e^{-x}}{2}\Big)^2 - 1 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{2} - 1 \\ &= \frac{e^{2x} + e^{-2x}}{2} = \cosh\ 2x. \end{aligned}$$

5. From the identity $\cosh^2\ x - \sinh^2\ x = 1$, divide both sides by $\cosh^2\ x$. The result is $1 - \tanh^2\ x = \text{sech}^2\ x$.

6. From the identity $\cosh^2\ x - \sinh^2\ x = 1$, divide both sides by $\sinh^2\ x$. The result is $\coth^2\ x - 1 = \text{csch}^2\ x$.

7. From items 1 and 2 in this theorem, we have

$$\tanh\ (x \pm y) = \frac{\sinh\ (x \pm y)}{\cosh\ (x \pm y)} = \frac{\sinh\ x \cosh\ y \pm \cosh\ x \sinh\ y}{\cosh\ x \cosh\ y \pm \sinh\ x \sinh\ y}.$$

By dividing the numerator and denominator by $\cosh\ x\ \cosh\ y$, we obtain

$$\tanh\ (x \pm y) = \frac{\tanh\ x \pm \tanh\ y}{1 \pm \tanh\ x \tanh\ y}.$$

8. We prove this item by using items 3 and 4.

$$\tanh\ 2x = \frac{\sinh\ 2x}{\cosh\ 2x} = \frac{2\sinh\ x\ \cosh\ x}{\cosh^2\ x + \sinh^2\ x}.$$

By dividing the numerator and denominator by $\cosh^2\ x$, we have

$$\tanh\ 2x = \frac{2\tanh\ x}{1 + \tanh^2\ x}. \blacksquare$$

4.2.2 Differentiating and Integrating the Hyperbolic Functions

Theorem 4.4 lists the differentiation rules of the hyperbolic functions.

Theorem 4.4 If $u = g(x)$ is differentiable function, then

1. $\frac{d}{dx} \sinh\ u = \cosh\ u\ u'$
2. $\frac{d}{dx} \cosh\ u = \sinh\ u\ u'$
3. $\frac{d}{dx} \tanh\ u = \text{sech}^2\ u\ u'$
4. $\frac{d}{dx} \coth\ u = -\text{csch}^2\ u\ u'$
5. $\frac{d}{dx} \text{sech}\ u = -\text{sech}\ u\ \tanh\ u\ u'$
6. $\frac{d}{dx} \text{csch}\ u = -\text{csch}\ u\ \coth\ u\ u'$

Proof. For simplicity, we consider the case $u = x$.

1. $\frac{d}{dx}(\sinh\ x) = \frac{d}{dx}\big(\frac{e^x - e^{-x}}{2}\big) = \frac{e^x + e^{-x}}{2} = \cosh\ x.$

2. $\frac{d}{dx}(\cosh\ x) = \frac{d}{dx}\big(\frac{e^x + e^{-x}}{2}\big) = \frac{e^x - e^{-x}}{2} = \sinh\ x.$

3. $\frac{d}{dx}(\tanh\ x) = \frac{d}{dx}\big(\frac{\sinh\ x}{\cosh\ x}\big) = \frac{\cosh\ x\ \cosh\ x - \sinh\ x\ \sinh\ x}{\cosh^2\ x} = \frac{\cosh^2\ x - \sinh^2\ x}{\cosh^2\ x} = \frac{1}{\cosh^2\ x} = \text{sech}^2\ x.$

4. $\frac{d}{dx}(\coth\ x) = \frac{d}{dx}\big(\frac{\cosh\ x}{\sinh\ x}\big) = \frac{\sinh\ x\ \sinh\ x - \cosh\ x\ \cosh\ x}{\sinh^2\ x} = \frac{\sinh^2\ x - \cosh^2\ x}{\sinh^2\ x} = \frac{-1}{\sinh^2\ x} = -\text{csch}^2\ x.$

5. $\frac{d}{dx}(\text{sech}\ x) = \frac{d}{dx}\big(\frac{1}{\cosh\ x}\big) = \frac{-\sinh\ x}{\cosh^2\ x} = -\text{sech}\ x\ \tanh\ x.$

6. $\frac{d}{dx}(\text{csch}\ x) = \frac{d}{dx}\big(\frac{1}{\sinh\ x}\big) = \frac{-\cosh\ x}{\sinh^2\ x} = -\text{csch}\ x\ \coth\ x. \blacksquare$

Example 4.3 Find the derivative of the functions.

(1) $y = \sinh\ (x^2)$

(2) $y = \sqrt{x}\ \cosh\ x$

(3) $y = e^{\sinh\ x}$

(4) $y = (x+1)\ \tanh^2\ (x^3)$

Solution:

(1) $y' = 2x\ \cosh\ (x^2)$.

(2) $y' = \frac{1}{2\sqrt{x}}\ \cosh\ x + \sqrt{x}\ \sinh\ x$.

(3) $y' = e^{\sinh\ x}\ \cosh\ x$.

(4) $y' = \tanh^2\ (x^3) + 6x^2(x+1)\ \tanh\ (x^3)\ \mathrm{sech}^2\ (x^3)$.

Example 4.4 Find $\frac{dy}{dx}$ if $y = x^{\cosh\ x}$.

Solution: Take the natural logarithm of each side to have

$$\ln y = \cosh\ x\ \ln x.$$

By differentiating both sides, we obtain $\frac{y'}{y} = \sinh\ x\ \ln x + \frac{\cosh\ x}{x}$.Therefore,

$$y' = \left[\sinh\ x\ \ln x + \frac{\cosh\ x}{x}\right] x^{\cosh\ x}.$$

Theorem 4.5

- $\int \sinh\ x\, dx = \cosh\ x + c$
- $\int \cosh\ x\, dx = \sinh\ x + c$
- $\int \mathrm{sech}^2\ x\, dx = \tanh\ x + c$
- $\int \mathrm{csch}^2\ x\, dx = -\coth\ x + c$
- $\int \mathrm{sech}\ x\ \tanh\ x\, dx = -\mathrm{sech}\ x + c$
- $\int \mathrm{csch}\ x\ \coth\ x\, dx = -\mathrm{csch}\ x + c$

Example 4.5 Evaluate the integral.

(1) $\int \sinh^2\ x\ \cosh\ x\, dx$

(2) $\int e^{\cosh\ x}\ \sinh\ x\, dx$

(3) $\int \tanh\ x\, dx$

(4) $\int e^x\ \mathrm{sech}\, x\, dx$

Solution:

(1) Let $u = \sinh\ x$, then $du = \cosh\ x\, dx$. By substitution, we have $\int u^2\ du = u^3/3 + c$. Hence,

$$\int \sinh^2\ x\ \cosh\ x\, dx = \frac{\sinh^3\ x}{3} + c.$$

(2) Let $u = \cosh\ x$, then $du = \sinh\ x\, dx$. By substitution, we have $\int e^u\ du = e^u + c$. Hence,

$$\int e^{\cosh\ x} \sinh\ x\, dx = e^{\cosh\ x} + c.$$

(3) We know that $\tanh\ x = \frac{\sinh\ x}{\cosh\ x}$, so $\int \tanh\ x\, dx = \int \frac{\sinh\ x}{\cosh\ x}\, dx$.

Let $u = \cosh\ x$, then $du = \sinh\ x\, dx$. By substitution, we have $\int \frac{1}{u}\, du = \ln|\,u\,| + c$.

This implies

$$\int \tanh\ x\, dx = \ln \cosh\ x + c.$$

(4) $\int e^x\ \mathrm{sech}\, x\, dx = \int \frac{2e^x}{e^x + e^{-x}}\, dx = \int \frac{2e^{2x}}{e^{2x}+1}\, dx.$

Let $u = e^{2x}$, then $du = 2e^{2x}\, dx$. By substitution, we have $\int \frac{1}{u+1}\, du = \ln|\,u+1\,| + c = \ln\left(e^{2x}+1\right) + c.$

Exercise 4.2

1 - 10 Find the derivative of the functions.

1 $y = \sinh\left(\sqrt{x^3}\right)$

2 $y = \tanh\ 5x$

3 $y = e^{-x}\cosh\ x$

4 $y = e^{\sinh 2x}$

5 $y = \ln(\coth\ x)$

6 $y = \sqrt{\operatorname{csch} x}$

7 $y = \sinh\ (\tan\ x)$

8 $y = \cosh\ (e^{\sqrt{x}})$

9 $y = \tanh\ (\ln x)$

10 $y = \sqrt{x+1}\ \operatorname{csch} x$

11 - 20 Evaluate the integral.

11 $\displaystyle\int \frac{\sinh\ (\sqrt{x})}{\sqrt{x}}\ dx$

12 $\displaystyle\int \frac{\cosh\ (\ln x)}{x}\ dx$

13 $\displaystyle\int e^x \tanh\ e^x\ dx$

14 $\displaystyle\int (1+\tanh\ x)^3 \operatorname{sech}^2\ x\ dx$

15 $\displaystyle\int \frac{e^{\sinh\ x}}{\operatorname{sech} x}\ dx$

16 $\displaystyle\int \frac{\operatorname{sech} x\ \tanh\ x}{1+\operatorname{sech} x}\ dx$

17 $\displaystyle\int \sqrt{3+\cosh\ x}\ \sinh\ x\ dx$

18 $\displaystyle\int \frac{\tanh\ \sqrt{x}\ \left(\operatorname{sech}\ \sqrt{x}+1\right)}{\sqrt{x}}\ dx$

19 $\displaystyle\int \frac{1}{\cosh^2\ x\ \tanh\ x}\ dx$

20 $\displaystyle\int \ln(\coth\ x)\ \operatorname{sech} x\ \operatorname{csch} x\ dx$

4.3 Inverse Hyperbolic Functions

4.3.1 Properties the Inverse Hyperbolic Functions

The function $\sinh : \mathbb{R} \to \mathbb{R}$ is bijective, so it has an inverse function

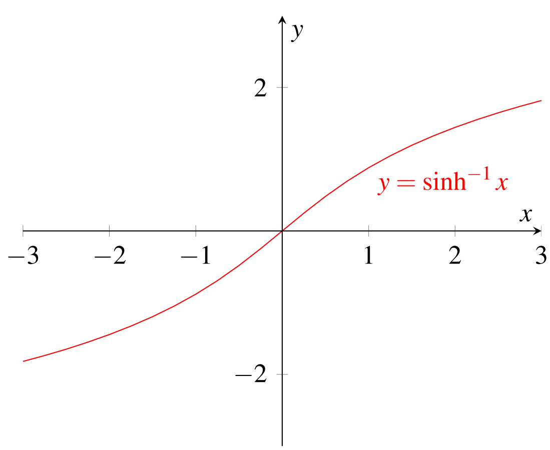
$$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$$

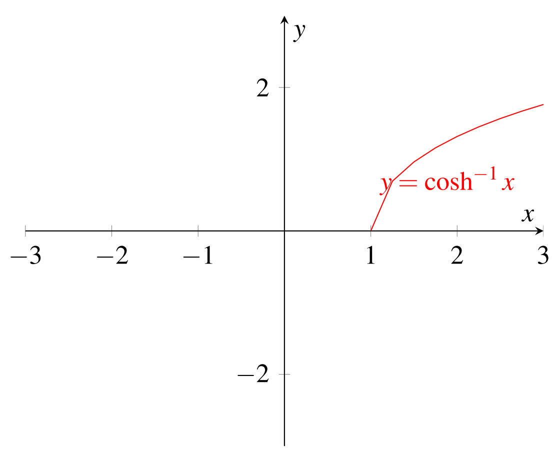
$$\sinh\ y = x \Leftrightarrow y = \sinh^{-1}\ x$$

The function cosh is injective on $[0,\infty)$, so $\cosh : [0,\infty) \to [1,\infty)$ is bijective on $[0,\infty)$. It has an inverse function

$$\cosh^{-1} : [1,\infty) \to [0,\infty)$$

$$\cosh\ y = x \Leftrightarrow y = \cosh^{-1}\ x$$

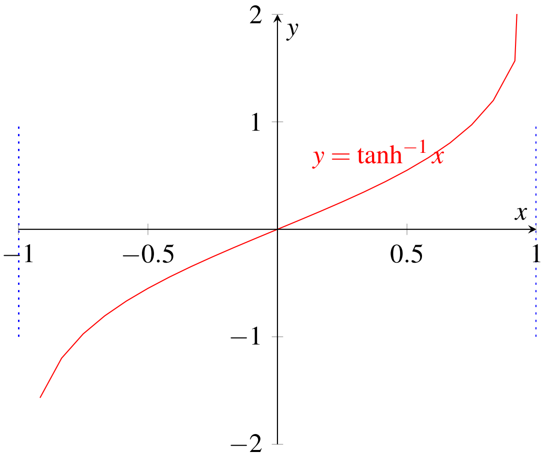




■ The function $\tanh : \mathbb{R} \to (-1,1)$ is bijective, so it has an inverse function

$$\tanh^{-1} : (-1,1) \to \mathbb{R}$$

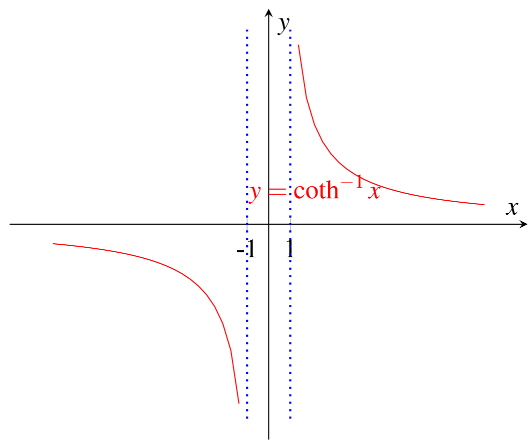
$$\tanh\, y = x \Leftrightarrow y = \tanh^{-1}\, x$$



■ The function $\coth : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus [-1,1]$ is bijective, so it has an inverse function

$$\coth^{-1} : \mathbb{R} \setminus [-1,1] \to \mathbb{R} \setminus \{0\}$$

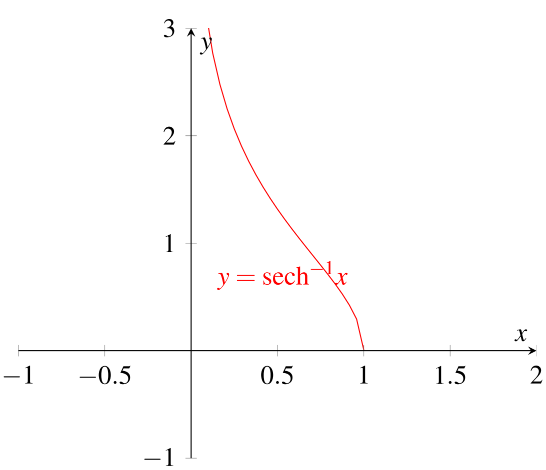
$$\coth\, y = x \Leftrightarrow y = \coth^{-1}\, x$$



■ The function sech is bijective on $[0,\infty)$, so $\operatorname{sech} : [0,\infty) \to (0,1]$ has an inverse function

$$\operatorname{sech}^{-1} : (0,1] \to [0,\infty)$$

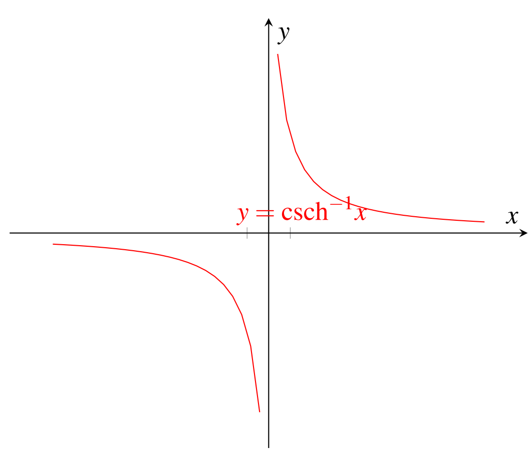
$$\operatorname{sech} y = x \Leftrightarrow y = \operatorname{sech}^{-1} x$$



■ The function $\operatorname{csch} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is bijective. The inverse function is

$$\operatorname{csch}^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$$

$$\operatorname{csch} y = x \Leftrightarrow y = \operatorname{csch}^{-1} x$$



The following theorem shows that the inverse hyperbolic functions can be formalized as functions depend on the natural logarithmic function.

Theorem 4.6

1. $\sinh^{-1}\ x = \ln(x+\sqrt{x^2+1}),\ \forall x \in \mathbb{R}$
2. $\cosh^{-1}\ x = \ln(x+\sqrt{x^2-1}),\ \forall x \in [1,\infty)$
3. $\tanh^{-1}\ x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ \forall x \in (-1,1)$
4. $\coth^{-1}\ x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ \forall x \in \mathbb{R}\setminus[-1,1]$
5. $\operatorname{sech}^{-1}\ x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right),\ \forall x \in (0,1]$
6. $\operatorname{csch}^{-1}\ x = \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right),\ \forall x \in \mathbb{R}\setminus\{0\}$

Proof. **1.** Let $y = \sinh^{-1}\ x$, then

$$x = \sinh\ y = \frac{e^y - e^{-y}}{2} \Rightarrow e^y - 2x - e^{-y} = 0.$$

The last expression can be rewritten as quadratic equation

$$e^{2y} - 2xe^y - 1 = 0\ ,$$

where x represents an unknown variable. By using the discriminant method, we have

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Since $\sqrt{x^2+1} > x$ and $e^y > 0$, then $e^y = x + \sqrt{x^2+1}$. By taking the natural logarithm of both sides, we have

$$y = \sinh^{-1}\ x = \ln(x+\sqrt{x^2+1}).$$

2. If $y = \cosh^{-1}\ x$, we have $x = \cosh\ y = \frac{e^y+e^{-y}}{2}$, then $e^{2y} - 2xe^y + 1 = 0$. By using the discriminant method, we have

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}.$$

Since $\sqrt{x^2-1} > x$ and $e^y > 0$, then $e^y = x + \sqrt{x^2-1}$. Take the natural logarithm of both sides to obtain

$$y = \cosh^{-1}\ x = \ln(x+\sqrt{x^2-1}).$$

3. Let $y = \tanh^{-1}\ x$, then

$$x = \tanh\ y = \frac{e^y - e^{-y}}{e^y + e^{-y}} \Rightarrow e^{2y} - xe^{2y} = 1 + x \Rightarrow e^{2y} = \frac{1+x}{1-x}.$$

By taking the natural logarithm of both sides, we have

$$y = \tanh^{-1}\ x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

4. Let $y = \coth^{-1}\ x$, then

$$x = \coth\ y = \frac{e^y + e^{-y}}{e^y - e^{-y}}.$$

Therefore, $xe^{2y} - x = e^{2y} + 1$ and then $e^{2y} = \frac{x+1}{x-1}$. By taking the natural logarithm of both sides, we obtain

$$y = \coth^{-1}\ x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right).$$

5. Let $y = \operatorname{sech}^{-1} x$, then

$$x = \operatorname{sech} y = \frac{2}{e^y + e^{-y}} \Rightarrow xe^{2y} - 2e^y + x = 0.$$

By using the discriminant method, we have

$$e^y = \frac{2 \pm \sqrt{4-4x^2}}{2x} \Rightarrow e^y = \frac{1+\sqrt{1-x^2}}{x}.$$

By taking the natural logarithm of both sides, we obtain $y = \operatorname{sech}^{-1} x = \ln(\frac{1+\sqrt{1-x^2}}{x})$.

6. Put $y = \operatorname{csch}^{-1} x$, this implies $x = \operatorname{csch} y = \frac{2}{e^y - e^{-y}}$, then $xe^{2y} - 2e^y - x = 0$. By using the discriminant method, we have

$$e^y = \frac{2 \pm \sqrt{4+4x^2}}{2x} \Rightarrow e^y = \frac{1+\sqrt{1+x^2}}{x}.$$

Take the natural logarithm of both sides to obtain $y = \operatorname{csch}^{-1} x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$. ■

4.3.2 Differentiating and Integrating the Inverse Hyperbolic Functions

In this section, we list the derivatives of the inverse hyperbolic functions. To prove the results, we can use either the derivative of the hyperbolic functions or Theorem 4.6.

Theorem 4.7 If $u = g(x)$ is differentiable function, then

1. $\frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{u^2+1}} u'$

2. $\frac{d}{dx} \cosh^{-1} u = \frac{1}{\sqrt{u^2-1}} u'$, $\forall u \in (1,\infty)$

3. $\frac{d}{dx} \tanh^{-1} u = \frac{1}{1-u^2} u'$, $\forall u \in (-1,1)$

4. $\frac{d}{dx} \coth^{-1} u = \frac{1}{1-u^2} u'$, $\forall u \in \mathbb{R} \setminus [-1,1]$

5. $\frac{d}{dx} \operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}} u'$, $\forall u \in (0,1)$

6. $\frac{d}{dx} \operatorname{csch}^{-1} u = \frac{-1}{|u|\sqrt{u^2+1}} u'$, $\forall u \in \mathbb{R} \setminus \{0\}$

Proof. For simplicity, we prove the theorem for the case $u = x$.

1. Let $y = \sinh^{-1} x$, then $x = \sinh y$. By using the implicit differentiation, we have

$$1 = \cosh y\, y' \Rightarrow y' = \frac{1}{\cosh y}.$$

We know that $\cosh^2 y = 1 + \sinh^2 y = 1 + x^2$. This implies $\cosh y = \sqrt{1+x^2}$ since $\cosh y \geq 1$. Hence

$$\frac{d}{dx} \sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}.$$

2. If $y = \cosh^{-1} x$, then $x = \cosh y$. By differentiating both sides, we have

$$1 = \sinh y\, y' \Rightarrow y' = \frac{1}{\sinh y}.$$

We know that $\sinh^2 y = \cosh^2 y - 1 = x^2 - 1$. Since $y \geq 0, \sinh y \geq 0$, then $\sinh y = \sqrt{x^2-1}$. Hence

$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}.$$

3. We can prove this item by using Theorem 4.6.

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2} \ln|1+x| - \frac{1}{2} \ln|1-x|.$$

Hence,

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{2(1+x)} + \frac{1}{2(1-x)} = \frac{1}{1-x^2}.$$

4. From Theorem 4.6,

$$\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1|.$$

Thus,

$$\frac{d}{dx}\coth^{-1} x = \frac{1}{2(x+1)} - \frac{1}{2(x-1)} = \frac{-1}{x^2-1} = \frac{1}{1-x^2}.$$

5. Since $\operatorname{sech}^{-1} x = \ln(\frac{1+\sqrt{1-x^2}}{x}) = \ln(1+\sqrt{1-x^2}) - \ln x$, then

$$\frac{d}{dx}\operatorname{sech}^{-1} x = \frac{-x}{\sqrt{1-x^2}(1+\sqrt{1-x^2})} - \frac{1}{x} = \frac{-(1+\sqrt{1-x^2})}{x\sqrt{1-x^2}(1+\sqrt{1-x^2})} = \frac{-1}{x\sqrt{1-x^2}}.$$

6. Since $\operatorname{csch}^{-1} x = \ln(\frac{1+\sqrt{x^2+1}}{x}) = \ln(1+\sqrt{x^2+1}) - \ln x$, then

$$\frac{d}{dx}\operatorname{csch}^{-1} x = \frac{x}{(1+\sqrt{x^2+1})\sqrt{x^2+1}} - \frac{1}{x} = \frac{-(1+\sqrt{x^2+1})}{x\sqrt{x^2+1}(1+\sqrt{x^2+1})} = \frac{-1}{|x|\sqrt{x^2+1}}. \blacksquare$$

Example 4.6 Find the derivative of the functions.

(1) $y = \sinh^{-1}\sqrt{x}$

(2) $y = \tanh^{-1} e^x$

(3) $y = \cosh^{-1}(4x^2)$

(4) $y = \ln(\sinh^{-1} x)$

(5) $y = \operatorname{csch}^{-1} 4x$

(6) $y = x\tanh^{-1}\frac{1}{x}$

(7) $y = (\tanh^{-1} x)^2$

(8) $y = e^x \operatorname{sech}^{-1} x$

Solution:

(1) $y' = \frac{1}{\sqrt{(\sqrt{x})^2+1}}\,\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(x+1)}}$.

(2) $y' = \frac{e^x}{1-(e^x)^2} = \frac{e^x}{1-e^{2x}}$.

(3) $y' = \frac{8x}{\sqrt{(4x^2)^2-1}} = \frac{8x}{\sqrt{16x^4-1}}$.

(4) $y' = \frac{1}{\sinh^{-1} x}\,\frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}\,\sinh^{-1} x}$.

(5) $y' = \frac{-4}{|4x|\sqrt{16x^2+1}} = \frac{-1}{|x|\sqrt{16x^2+1}}$.

(6) $y' = \tanh^{-1}(\frac{1}{x}) + x\,(\frac{1}{1-(\frac{1}{x})^2})(\frac{-1}{x^2}) = \tanh^{-1}(\frac{1}{x}) - \frac{x}{x^2-1}$.

(7) $y' = 2(\tanh^{-1} x)\,\frac{1}{1-x^2} = \frac{2\tanh^{-1} x}{1-x^2}$

(8) $y' = e^x \operatorname{sech}^{-1} x - \frac{e^x}{x\sqrt{1-x^2}}$

From Theorem 4.7, we have the following list of integrals:

- $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + c$
- $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + c,\ x > 1$
- $\displaystyle\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + c,\ |x| < 1$
- $\displaystyle\int \frac{1}{1-x^2}\,dx = \coth^{-1} x + c,\ |x| > 1$
- $\displaystyle\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1}|x| + c,\ |x| < 1$
- $\displaystyle\int \frac{1}{x\sqrt{x^2+1}}\,dx = -\operatorname{csch}^{-1}|x| + c,\ |x| > 1$

Theorem 4.8 generalizes the previous result.

Theorem 4.8

1. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\tfrac{x}{a}+c$

2. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\tfrac{x}{a}+c,\ x>a$

3. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \tfrac{1}{a}\tanh^{-1}\tfrac{x}{a}+c,\ |x|<a$

4. $\displaystyle\int \frac{1}{a^2-x^2}\,dx = \tfrac{1}{a}\coth^{-1}\tfrac{x}{a}+c,\ |x|>a$

5. $\displaystyle\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\tfrac{1}{a}\operatorname{sech}^{-1}\tfrac{|x|}{a}+c,\ |x|<a$

6. $\displaystyle\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\tfrac{1}{a}\operatorname{csch}^{-1}\tfrac{|x|}{a}+c,\ |x|>a$

Example 4.7 Evaluate the integral.

(1) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$

(2) $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\,dx$

(3) $\displaystyle\int \frac{1}{\sqrt{e^{2x}+9}}\,dx$

(4) $\displaystyle\int \frac{1}{x\sqrt{1-x^6}}\,dx$

(5) $\displaystyle\int_0^1 \frac{1}{16-x^2}\,dx$

(6) $\displaystyle\int_5^7 \frac{1}{16-x^2}\,dx$

Solution:

(1) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx = \cosh^{-1}\tfrac{x}{2}+c.$

(2) $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\,dx = \int \frac{1}{\sqrt{(2x)^2+9}}\,dx.$
Let $u=2x$, then $du=2dx$. By substitution, we have

$$\frac{1}{2}\int \frac{1}{\sqrt{u^2+9}}\,du = \frac{1}{2}\sinh^{-1}\frac{u}{3}+c = \frac{1}{2}\sinh^{-1}\frac{2x}{3}+c.$$

(3) $\displaystyle\int \frac{1}{\sqrt{e^{2x}+9}}\,dx = \int \frac{1}{\sqrt{(e^x)^2+9}}\,dx.$
Let $u=e^x$, then $du=e^x dx$. By substituting that into the integral, we have

$$\int \frac{1}{u\sqrt{u^2+9}}\,du = -\frac{1}{3}\operatorname{csch}^{-1}\frac{|u|}{3}+c = -\frac{1}{3}\operatorname{csch}^{-1}\frac{e^x}{3}+c.$$

(4) $\displaystyle\int \frac{1}{x\sqrt{1-x^6}}\,dx = \int \frac{1}{x\sqrt{1-(x^3)^2}}\,dx.$
Let $u=x^3$, then $du=3x^2dx$. By substitution, we obtain

$$\frac{1}{3}\int \frac{1}{u\sqrt{1-u^2}}\,du = -\frac{1}{3}\operatorname{sech}^{-1}|u|+c = -\frac{1}{3}\operatorname{sech}^{-1}|x^3|+c.$$

(5) Since the interval of the integral is subinterval of $(-4,4)$, then the value of the integral is $\tanh^{-1}$. Hence,

$$\int_0^1 \frac{1}{16-x^2}\,dx = \frac{1}{4}\Big[\tanh^{-1}\frac{x}{4}\Big]_0^1 = \frac{1}{4}\Big[\tanh^{-1}(\frac{1}{4})-\tanh^{-1}(0)\Big] = \frac{1}{4}\Big[\frac{1}{2}\ln(\frac{5}{3})-\frac{1}{2}\ln(1)\Big] = \frac{1}{8}\ln(\frac{5}{3}).$$

(6) The interval of the integral is not subinterval of $(-4,4)$, so the value of the integral is $\coth^{-1}$. This implies

$$\int_5^7 \frac{1}{16-x^2}\,dx = \frac{1}{4}\Big[\coth^{-1}\frac{x}{4}\Big]_5^7 = \frac{1}{4}\Big[\coth^{-1}\frac{7}{4}-\coth^{-1}\frac{5}{4}\Big] = \frac{1}{8}\Big[\ln(11)-3\ln(3)\Big].$$

Exercise 4.3

1 - 6 Find the derivative of the functions.

1 $y = \sinh^{-1}(\tan x)$

2 $y = \cosh^{-1}(e^{\sqrt{x}})$

3 $y = \tanh^{-1}(\ln x)$

4 $y = \sqrt{x+1}\,\operatorname{csch}^{-1} x$

5 $y = \tan x \tanh^{-1} x$

6 $y = (2x-1)^3 \sinh^{-1} \sqrt{x}$

7 - 14 Evaluate the integral.

7 $\displaystyle\int \frac{1}{\sqrt{2x^2-2}}\,dx$

8 $\displaystyle\int \frac{e^x}{1-e^{2x}}\,dx$

9 $\displaystyle\int \frac{1}{x\sqrt{1-x^4}}\,dx$

10 $\displaystyle\int \frac{1}{\sqrt{x^2+25}}\,dx$

11 $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx$

12 $\displaystyle\int \frac{1}{\sec x\,(1-\sin^2 x)}\,dx$

13 $\displaystyle\int \frac{1}{x\sqrt{x^6+2}}\,dx$

14 $\displaystyle\int \frac{1}{\sqrt{4-e^{2x}}}\,dx$

Review Exercises

1 - 18 Find the derivative.

1 $y = \sin^{-1}(3x+1)$

2 $y = \cos^{-1}\sqrt{x}$

3 $y = \tan^{-1}\frac{2}{3}x$

4 $y = \sec^{-1} 3x$

5 $y = \sinh(4x+1)$

6 $y = \cosh e^x$

7 $y = \sqrt{x}\tanh\sqrt{x}$

8 $y = e^{3x}\cosh 2x$

9 $y = \sqrt{\sinh 3x + \cosh 5x}$

10 $y = \tan^{-1}(\sinh x)$

11 $y = e^{\operatorname{sech} x}\cosh(\cosh x)$

12 $y = \frac{\sinh x}{\cosh x}$

13 $y = \operatorname{sech}^{-1} 3x$

14 $y = \coth^{-1}\sqrt{x}$

15 $y = x^4\cosh^{-1} x$

16 $y = e^x\tanh^{-1}\sqrt[3]{x}$

17 $y = \sinh^{-1}(\tanh x)$

18 $y = \tanh^{-1}\left(\frac{1-x}{1+x}\right)$

19 - 22 Find each limit if it exists.

19 $\lim\limits_{x\to\infty}\frac{1}{\sinh x}$

20 $\lim\limits_{x\to-\infty}\cosh x$

21 $\lim\limits_{x\to\infty} e^x\tanh x$

22 $\lim\limits_{x\to\infty} e^{\operatorname{sech} x}$

23 - 42 Evaluate the integral.

23 $\int \sinh^3 x\cosh x\,dx$

24 $\int \tanh^4 x\operatorname{sech}^2 x\,dx$

25 $\int e^{\sinh x}\cosh x\,dx$

26 $\int e^x\operatorname{csch} x\,dx$

27 $\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$

28 $\int x\operatorname{sech} x^2\tanh x^2\,dx$

29 $\int \frac{1}{\operatorname{sech} 3x}\,dx$

30 $\int \tanh x\,dx$

31 $\int \frac{1}{3+x^2}\,dx$

32 $\int \frac{1}{x\sqrt{x^4-4}}\,dx$

33 $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$

34 $\int \frac{x-1}{\sqrt{4-x^2}}\,dx$

35 $\int \frac{x+1}{x\sqrt{25-x^2}}\,dx$

36 $\int \frac{1}{x\sqrt{x^8-16}}\,dx$

37 $\int \frac{1}{\sqrt{1+4x^2}}\,dx$

38 $\int \frac{1}{4-9x^2}\,dx$

39 $\int_4^8 \frac{x}{x^4-16}\,dx$

40 $\int_2^3 \frac{1}{\sqrt{x^2-1}}\,dx$

41 $\int \frac{1}{\sqrt{25+9x^2}}\,dx$

42 $\int \frac{1}{\sqrt{e^{2x}-16}}\,dx$

43 - 55 Choose the correct answer.

43 The derivative of the function $f(x) = \tan^{-1}(\sinh x)$ is equal to

(a) $\operatorname{sech} x$ (b) $\operatorname{csch} x$ (c) $\tanh x$ (d) $-\operatorname{sech} x$

44 The value of the integral $\int_{-1}^{1}\sinh x\,dx$ is equal to

(a) 0 (b) $2e$ (c) $2e^{-1}$ (d) $\frac{1}{2}e$

45 If $f(x) = \cosh^{-1}\sqrt{x}$, then $f'(x)$ is equal to

(a) $\frac{1}{2\sqrt{x^2-x}}$ (b) $\frac{1}{2\sqrt{x-x^2}}$ (c) $\frac{1}{2\sqrt{x^2+x}}$ (d) None of these

46 The integral $\int \frac{x-2}{x\sqrt{x^2-25}}\,dx$, is equal to
(a) $\cosh^{-1}\frac{x}{5} - 2\sec^{-1}\frac{x}{5} + c$ (b) $\cosh^{-1}\frac{x}{5} - \frac{2}{5}\sec^{-1}x + c$
(c) $\cosh^{-1}\frac{x}{5} - \frac{2}{5}\sec^{-1}\frac{x}{5} + c$ (d) None of these

47 If $f(x) = \tanh^{-1}(\cos 3x)$, then $f'(x)$ is equal to
(a) $3\csc 3x$ (b) $-3\csc 3x$ (c) $\frac{-3\sin 3x}{1+\cos^2 3x}$ (d) 0

48 The integral $\int \frac{\cos x}{1+\sin^2 x}\,dx$, is equal to
(a) $\frac{1}{1+\sin x} + c$ (b) $\tan^{-1}(\sin x) + c$ (c) $\frac{1}{1+\cos x} + c$ (d) $\tanh^{-1}(\sin x) + c$

49 The value of the integral $\int \frac{dx}{\sqrt{16-25x^2}}$ is
(a) $-\frac{\cos^{-1}\frac{x}{16}}{25} + c$ (b) $\frac{\cos^{-1}\frac{x}{16}}{25} + c$ (c) $\frac{\sin^{-1}\frac{5x}{4}}{5} + c$ (d) $-\frac{\sin^{-1}\frac{5x}{4}}{5} + c$

50 The value of the integral $\int \frac{1}{\sqrt{x^2+2}}\,dx$ is
(a) $\sin^{-1}x + c$ (b) $\sinh^{-1}x + c$ (c) $\sinh^{-1}\frac{x}{\sqrt{2}} + c$ (d) $\sin^{-1}\frac{x}{\sqrt{2}} + c$

51 The integral $\int \frac{\cosh x}{1-\sinh^2 x}\,dx$ is equal to
(a) $-\tan^{-1}(\sinh x) + c$ (b) $\tan^{-1}(\sinh x) + c$
(c) $\frac{1}{1+\cosh x} + c$ (d) $\tanh^{-1}(\sinh x) + c$

52 If $F(x) = \tan^{-1}x + \tan^{-1}(\frac{1}{x})$ where $x \neq 0$, then $F'(x)$ is equal to
(a) $\frac{2}{1+x^2}$ (b) $\frac{-1}{1+x^2}$ (c) 0 (d) $\frac{x^2}{1+x^2}$

53 The derivative of the function $f(x) = \tan^{-1}(\sinh x)$ is equal to
(a) $\frac{1}{1+\sinh^2 x}$ (b) $\sec^2(\sinh x)$ (c) $\frac{1}{\cosh x}$ (d) $\frac{\cosh x}{1-\sinh^2 x}$

54 The value of the integral $\int \frac{1}{\sqrt{4+x^2}}\,dx$ is equal to
(a) $\sinh^{-1}\frac{x}{2} + c$ (b) $\sin^{-1}\frac{x}{2} + c$ (c) $\frac{1}{2}\sinh^{-1}\frac{x}{2} + c$ (d) $\frac{1}{2}\sin^{-1}\frac{x}{2} + c$

55 The value of the integral $\int_{-1}^{1}\cosh x\,dx$ is equal to
(a) 0 (b) $2e$ (c) $2e^{-1}$ (d) $e - e^{-1}$

Chapter 5

Techniques of Integration

5.1 Integration by Parts

Integration by parts is a method to transfer the original integral to an easier one that can be evaluated. Practically, the integration by parts divides the original integral into two parts u and dv, then we find the du by deriving u and v by integrating dv.

Theorem 5.1 If $u = f(x)$ and $v = g(x)$ such that f' and g' are continuous, then

$$\int u\, dv = uv - \int v\, du.$$

Proof. We know that $\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + f'(x)g(x)$. Thus, $f(x)g'(x) = \frac{d}{dx}\big(f(x)g(x)\big) - f'(x)g(x)$. By integrating both sides, we obtain

$$\begin{aligned}\int f(x)g'(x)\, dx &= \int \frac{d}{dx}\big(f(x)g(x)\big)\, dx - \int f'(x)g(x)\, dx \\ &= f(x)g(x) - \int f'(x)g(x)\, dx.\end{aligned}$$

Since $u = f(x)$ and $v = g(x)$, then $du = f'(x)\, dx$ and $dv = g'(x)\, dx$. Therefore,

$$\int u\, dv = uv - \int v\, du. \blacksquare$$

Theorem 5.1 shows that the integration by parts transfers the integral $\int u\, dv$ into the integral $\int v\, du$ that should be easier than the original integral. The question here is, what we choose as u and what we choose as $dv = v'\, dx$. It is useful to choose u as a function that can be easily differentiated, and to choose dv as a function that can be easily integrated. This statement is clearly explained through the following examples.

Example 5.1 Evaluate the integral $\int x \cos x\, dx$.

Solution:
Let $I = \int x \cos x\, dx$. Let $u = x$ and $dv = \cos x\, dx$. Hence,

$$\begin{aligned} u &= x \Rightarrow du = dx\,, \\ dv = \cos x\, dx &\Rightarrow v = \int \cos x\, dx = \sin x.\end{aligned}$$

Try to choose
$u = \cos x$ and $dv = x\, dx$
Do you have the same result?

From Theorem 5.1, we have

$$I = x \sin x - \int \sin x\, dx = x \sin x + \cos x + c.$$

Example 5.2 Evaluate the integral $\int x\, e^x\, dx$.

Solution:
Let $I = \int x\, e^x\, dx$. Let $u = x$ and $dv = e^x\, dx$. Hence,

$$u = x \Rightarrow du = dx\,,$$
$$dv = e^x\, dx \Rightarrow v = \int e^x\, dx = e^x.$$

From Theorem 5.1, we have

$$I = x\, e^x - \int e^x\, dx = x\, e^x - e^x + c.$$

Try to choose $u = e^x$ and $dv = x\, dx$
We will obtain

$$I = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x\, dx.$$

However, the integral $\int \frac{x^2}{2} e^x\, dx$ is more difficult than the original one $\int x e^x\, dx$.

Remark 5.1

1. Remember that when we consider the integration by parts, we want to obtain an easier integral. As we saw in Example 5.2, if we choose $u = e^x$ and $dv = x\, dx$, we have $\int \frac{x^2}{2}\, e^x\, dx$ which is more difficult than the original one.
2. When considering the integration by parts, we have to choose dv a function that can be integrated (see Examples 5.3 and 5.6).
3. Sometimes we need to use the integration by parts twice as in Examples 5.4 and 5.5.

Example 5.3 Evaluate the integral $\int \ln x\, dx$.

Solution: Let $I = \int \ln x\, dx$. Let $u = \ln x$ and $dv = dx$. Hence,

$$u = \ln x \Rightarrow du = \frac{1}{x}\, dx\,,$$
$$dv = dx \Rightarrow v = \int 1\, dx = x.$$

From Theorem 5.1, we obtain $I = x \ln x - \int x \frac{1}{x}\, dx = x \ln x - \int 1\, dx = x \ln x - x + c$.

Example 5.4 Evaluate the integral $\int e^x \cos\, x\, dx$.

Solution: Let $I = \int e^x \cos\, x\, dx$. Let $u = e^x$ and $dv = \cos\, x\, dx$.

$$u = e^x \Rightarrow du = e^x\, dx\,,$$
$$dv = \cos\, x\, dx \Rightarrow v = \int \cos\, x\, dx = \sin\, x.$$

Hence, $I = e^x \sin\, x - \int e^x \sin\, x\, dx$.

The integral $\int e^x \sin\, x\, dx$ cannot be evaluated. Therefore, we use the integration by parts again where we assume $J = \int e^x \sin\, x\, dx$. Let $u = e^x$ and $dv = \sin\, x\, dx$. Hence,

$$u = e^x \Rightarrow du = e^x\, dx\,,$$
$$dv = \sin\, x\, dx \Rightarrow v = \int \sin\, x\, dx = -\cos\, x.$$

Hence, $J = -e^x \cos\, x + \int e^x \cos\, x\, dx$. By substituting the result of J into I, we have

$$\begin{aligned} I &= e^x \sin\, x - J \\ &= e^x \sin\, x + e^x \cos\, x - \int e^x \cos\, x\, dx \\ \Rightarrow I &= e^x \sin\, x + e^x \cos\, x - I. \end{aligned}$$

This implies

$$2I = e^x \sin\, x + e^x \cos\, x \Rightarrow I = \frac{1}{2}(e^x \sin\, x + e^x \cos\, x) \Rightarrow \int e^x \cos\, x\, dx = \frac{e^x}{2}(\sin\, x + \cos\, x) + c.$$

Example 5.5 Evaluate the integral $\int x^2 e^x\, dx$.

Solution: Let $I = \int x^2 e^x\, dx$. Let $u = x^2$ and $dv = e^x\, dx$. Hence,

$$u = x^2 \Rightarrow du = 2x\, dx\ ,$$
$$dv = e^x dx \Rightarrow v = \int e^x\, dx = e^x.$$

This implies, $I = x^2 e^x - 2\int xe^x\, dx$.
We use the integration by parts again for the integral $\int xe^x\, dx$. Let $J = \int xe^x\, dx$.
Let $u = x$ and $dv = e^x\, dx$. Hence,

$$u = x \Rightarrow du = \ dx\ ,$$
$$dv = e^x dx \Rightarrow v = \int e^x\, dx = e^x.$$

Therefore, $J = xe^x - \int e^x\, dx = xe^x - e^x + c$. By substituting the result into I, we have

$$I = x^2 e^x - 2(xe^x - e^x) + c = e^x(x^2 - 2x + 2) + c.$$

Example 5.6 Evaluate the integral $\int_0^1 \tan^{-1}\, x\, dx$.

Solution:
Let $I = \int \tan^{-1}\, x\, dx$. Let $u = \tan^{-1}\, x$ and $dv = dx$. Hence,

$$u = \tan^{-1}\, x \Rightarrow du = \frac{1}{x^2+1}\, dx\ ,$$
$$dv = dx \Rightarrow v = \int 1\, dx = x.$$

By applying Theorem 5.1, we obtain

$$I = x\, \tan^{-1}\, x - \int \frac{x}{x^2+1}\, dx = x\, \tan^{-1} x - \frac{1}{2}\ln(x^2+1) + c.$$

Therefore,

$$\int_0^1 \tan^{-1}\, x\, dx = \left[x\, \tan^{-1}\, x - \frac{1}{2}\ln(x^2+1)\right]_0^1 = (\tan^{-1}(1) - \frac{1}{2}\ln 2) - (0 - \frac{1}{2}\ln 1) = \frac{\pi}{4} - \ln\sqrt{2}.$$

Exercise 5.1

1 - 16 ■ Evaluate the integral.

1 $\int x^3 \ln x \, dx$

2 $\int_0^{\pi/3} \sin x \ln(\cos x) \, dx$

3 $\int \sin^{-1} x \, dx$

4 $\int x^3 \sqrt{4-x^2} \, dx$

5 $\int x \sin x \, dx$

6 $\int x^2 \cos x \, dx$

7 $\int e^x \sin 2x \, dx$

8 $\int_0^{\frac{1}{\sqrt{3}}} \tan^{-1} x \, dx$

9 $\int e^{2x} \cos x \, dx$

10 $\int (\ln x)^2 \, dx$

11 $\int \frac{\ln x}{x^2} \, dx$

12 $\int x \sin x \cos x \, dx$

13 $\int \frac{1}{x(\ln x)^3} \, dx$

14 $\int_0^1 x^2 e^x \, dx$

15 $\int x \tan^{-1} x \, dx$

16 $\int x e^{-x} \, dx$

5.2 Trigonometric Functions

5.2.1 Integration of Powers of Trigonometric Functions

In this section, we evaluate integrals of forms $\int \sin^n x \cos^m x \, dx$, $\int \tan^n x \sec^m x \, dx$ and $\int \cot^n x \csc^m x \, dx$. Students need the trigonometric relationships that are provided in the beginning of this book on page 179.

Form 1: $\int \sin^n x \cos^m x \, dx$.

This form is treated as follows:

1. If n is an odd integer, write

$$\sin^n x \cos^m x = \sin^{n-1} x \cos^m x \sin x$$

Then, use the identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.

2. If m is an odd integer, write

$$\sin^n x \cos^m x = \sin^n x \cos^{m-1} x \cos x$$

Then, use the identity $\cos^2 x = 1 - \sin^2 x$ and the substitution $u = \sin x$.

3. If m and n are even, use the identities $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$.

Example 5.7 Evaluate the integral.

(1) $\int \sin^3 x \, dx$

(2) $\int \cos^4 x \, dx$

(3) $\int \sin^5 x \cos^4 x \, dx$

(4) $\int \sin^2 x \cos^2 x \, dx$

Solution:

(1) Write $\sin^3 x = \sin^2 x \sin x = (1-\cos^2 x) \sin x$. Hence,

$$\int \sin^3 x \, dx = \int (1-\cos^2 x) \sin x \, dx.$$

Let $u = \cos x$, then $du = -\, \sin x \, dx$. By substitution, we have

$$-\int (1-u^2) \, du = -u + \frac{u^3}{3} + c.$$

This implies

$$\int \sin^3 x\, dx = -\cos x + \frac{1}{3}\cos^3 x + c.$$

(2) Write $\cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$. Hence,

$$\begin{aligned}
\int \cos^4 x\, dx &= \int \Big(\frac{1+\cos 2x}{2}\Big)^2 dx \\
&= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\, dx \\
&= \frac{1}{4}\Big(\int 1\, dx + \int 2\cos 2x\, dx + \int \cos^2 2x\, dx\Big) \\
&= \frac{1}{4}\Big(x+\sin 2x + \frac{1}{2}\int (1+\cos 4x)\, dx\Big) \\
&= \frac{1}{4}\Big(x+\sin 2x + \frac{1}{2}(x+\frac{\sin 4x}{4})\Big) + c.
\end{aligned}$$

(3) Write $\sin^5 x \cos^4 x = \sin^4 x \cos^4 x \sin x = (1-\cos^2 x)^2 \cos^4 x \sin x$.
Let $u = \cos x$, then $du = -\sin x\, dx$. Thus, the integral becomes

$$-\int (1-u^2)^2 u^4\, du = -\int (u^4 - 2u^6 + u^8)\, du = -(\frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9}) + c.$$

This implies $\int \sin^5 x \cos^4 x\, dx = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + c.$

(4) The integrand $\sin^2 x \cos^2 x = (\frac{1-\cos 2x}{2})(\frac{1+\cos 2x}{2}) = \frac{1-\cos^2 2x}{4} = \frac{\sin^2 2x}{4} = \frac{1}{4}(\frac{1-\cos 4x}{2})$. Hence,

$$\int \sin^2 x \cos^2 x\, dx = \frac{1}{8}\int (1-\cos 4x)\, dx = \frac{1}{8}(x - \frac{\sin 4x}{4}) + c.$$

Form 2: $\int \tan^n x \sec^m x\, dx$.
This form is treated as follows:

1. If $n = 0$, write

$$\sec^m x = \sec^{m-2} x \sec^2 x$$

a. If $m > 1$ is odd, use the integration by parts.
b. If m is even, use the identity $\sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$.

2. If $m = 0$ and n is odd or even, write

$$\tan^n x = \tan^{n-2} x \tan^2 x$$

Then, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \tan x$.

3. If n is even and m is odd, use the identity $\tan^2 x = \sec^2 x - 1$ to reduce the power m and then use the integration by parts.

4. If $m \geq 2$ is even, write

$$\tan^n x \sec^m x = \tan^n x \sec^{m-2} x \sec^2 x$$

Then, use the identity $\sec^2 x = 1 + \tan^2 x$ and the substitution $u = \tan x$. Alternatively, write

$$\tan^n x \sec^m x = \tan^{n-1} x \sec^{m-1} x \tan x \sec x$$

Then, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \sec x$.

5. If n is odd and $m \geq 1$, write

$$\tan^n x \sec^m x = \tan^{n-1} x \sec^{m-1} x \tan x \sec x$$

Then, use the identity $\tan^2 x = \sec^2 x - 1$ and the substitution $u = \sec x$.

Example 5.8 Evaluate the integral.

(1) $\int \tan^5 x\, dx$

(2) $\int \tan^6 x\, dx$

(3) $\int \sec^3 x\, dx$

(4) $\int \tan^5 x \sec^4 x\, dx$

(5) $\int \tan^4 x \sec^4 x\, dx$

Solution:

(1) Write $\tan^5 x = \tan^3 x \tan^2 x = \tan^3 x\,(\sec^2 x - 1)$. Thus,

$$\begin{aligned}\int \tan^5 x\, dx &= \int \tan^3 x\,(\sec^2 x - 1)\, dx\\ &= \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx\\ &= \frac{\tan^4 x}{4} - \int \tan x\,(\sec^2 x - 1)\, dx\\ &= \frac{\tan^4 x}{4} - \int \tan x \sec^2 x\, dx + \int \tan x\, dx\\ &= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + c.\end{aligned}$$

(2) Write $\tan^6 x = \tan^4 x \tan^2 x = \tan^4 x\,(\sec^2 x - 1)$. The integral becomes

$$\begin{aligned}\int \tan^6 x\, dx &= \int \tan^4 x\,(\sec^2 x - 1)\, dx\\ &= \int \tan^4 x \sec^2 x\, dx - \int \tan^4 x\, dx\\ &= \frac{\tan^5 x}{5} - \int \tan^2 x\,(\sec^2 x - 1)\, dx\\ &= \frac{\tan^5 x}{5} - \int \tan^2 x \sec^2 x\, dx + \int \tan^2 x\, dx\\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int (\sec^2 x - 1)\, dx\\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c.\end{aligned}$$

(3) Write $\sec^3 x = \sec x \sec^2 x$ and let $I = \int \sec x \sec^2 x\, dx$.
We use the integration by parts to evaluate the integral as follows:

$$u = \sec x \Rightarrow du = \sec x \tan x\, dx\,,$$

$$dv = \sec^2 x\, dx \Rightarrow v = \int \sec^2 x\, dx = \tan x.$$

Hence,

$$\begin{aligned}I &= \sec x \tan x - \int \sec x \tan^2 x\, dx\\ &= \sec x \tan x - \int (\sec^3 x - \sec x)\, dx\\ &= \sec x \tan x - I + \ln|\sec x + \tan x|\\ I &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + c.\end{aligned}$$

(4) Express the integrand $\tan^5 x \sec^4 x$ as follows

$$\tan^5 x \sec^4 x = \tan^5 x \sec^2 x \sec^2 x = \tan^5 x\,(\tan^2 x + 1) \sec^2 x.$$

This implies

$$\begin{aligned}\int \tan^5 x\ \sec^4 x\,dx &= \int \tan^5 x\ (\tan^2 x+1)\ \sec^2 x\,dx\\ &= \int (\tan^7 x+\tan^5 x)\ \sec^2 x\,dx\\ &= \frac{\tan^8 x}{8}+\frac{\tan^6 x}{6}+c.\end{aligned}$$

(5) Write $\tan^4 x\ \sec^4 x = \tan^4 x\ (\tan^2 x+1)\ \sec^2 x$. The integral becomes

$$\begin{aligned}\int \tan^4 x\ \sec^4 x\,dx &= \int \tan^4 x\ (\tan^2 x+1)\ \sec^2 x\,dx\\ &= \int (\tan^6 x+\tan^4 x)\ \sec^2 x\,dx\\ &= \frac{\tan^7 x}{7}+\frac{\tan^5 x}{5}+c.\end{aligned}$$

Form 3: $\int \cot^n x\ \csc^m x\,dx$.

The treatment of this form is similar to the integral $\int \tan^n x\ \sec^m x\,dx$, except we use the identity

$$\cot^2 x+1=\csc^2 x.$$

Example 5.9 Evaluate the integral.

(1) $\int \cot^3 x\,dx$

(2) $\int \cot^4 x\,dx$

(3) $\int \cot^5 x\ \csc^4 x\,dx$

Solution:

(1) Write $\cot^3 x = \cot x\ (\csc^2 x-1)$. Then,

$$\begin{aligned}\int \cot^3 x\,dx &= \int \cot x\ (\csc^2 x-1)\ dx\\ &= \int (\cot x\ \csc^2 x-\cot x)\ dx\\ &= \int \cot x\ \csc^2 x\,dx-\int \cot x\,dx\\ &= -\frac{1}{2}\cot^2 x-\ln|\sin x|+c.\end{aligned}$$

(2) The integrand can be expressed as $\cot^4 x=\cot^2 x\ (\csc^2 x-1)$. Thus,

$$\begin{aligned}\int \cot^4 x\,dx &= \int \cot^2 x\ (\csc^2 x-1)\ dx\\ &= \int \cot^2 x\ \csc^2 x\,dx-\int \cot^2 x\,dx\\ &= -\frac{\cot^3 x}{3}-\int (\csc^2 x-1)\ dx\\ &= -\frac{\cot^3 x}{3}+\cot x+x+c.\end{aligned}$$

(3) Write $\cot^5 x \csc^4 x = \csc^3 x \cot^4 x \csc x \cot x$. This implies

$$\begin{aligned}\int \cot^5 x \csc^4 x\, dx &= \int \csc^3 x \cot^4 x \csc x \cot x\, dx \\ &= \int \csc^3 x\,(\csc^2 x - 1)^2 \csc x \cot x\, dx \\ &= \int (\csc^7 x - 2\csc^5 x + \csc^3 x)\, \csc x \cot x\, dx \\ &= -\frac{\csc^8 x}{8} + \frac{\csc^6 x}{3} - \frac{\csc^4 x}{4} + c.\end{aligned}$$

5.2.2 Integration of Forms $\sin ux\ \cos vx$, $\sin ux\ \sin vx$ and $\cos ux\ \cos vx$

We deal with these integrals by using the following formulas:

$$\begin{aligned}\sin ux \cos vx &= \frac{1}{2}\left(\sin (u-v)\, x + \sin (u+v)\, x\right) \\ \sin ux \sin vx &= \frac{1}{2}\left(\cos (u-v)\, x - \cos (u+v)\, x\right) \\ \cos ux \cos vx &= \frac{1}{2}\left(\cos (u-v)\, x + \cos (u+v)\, x\right)\end{aligned}$$

Example 5.10 Evaluate the integral.

(1) $\displaystyle\int \sin 5x \sin 3x\, dx$

(2) $\displaystyle\int \sin 7x \cos 2x\, dx$

(3) $\displaystyle\int \cos 5x \sin 2x\, dx$

(4) $\displaystyle\int \cos 6x \cos 4x\, dx$

Solution:

(1) From the previous formulas, we have $\sin 5x \sin 3x = \frac{1}{2}\left(\cos 2x - \cos 8x\right)$. Hence,

$$\begin{aligned}\int \sin 5x \sin 3x\, dx &= \frac{1}{2}\int (\cos 2x - \cos 8x)\, dx \\ &= \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + c.\end{aligned}$$

(2) Since $\sin 7x \cos 2x = \frac{1}{2}\left(\sin 5x + \sin 9x\right)$, then

$$\begin{aligned}\int \sin 7x \cos 2x\, dx &= \frac{1}{2}\int (\sin 5x + \sin 9x)\, dx \\ &= -\frac{1}{10}\cos 5x - \frac{1}{18}\cos 9x + c.\end{aligned}$$

(3) Since $\cos 5x \sin 2x = \frac{1}{2}\left(\sin 3x + \sin 7x\right)$, then

$$\begin{aligned}\int \cos 5x \sin 2x\, dx &= \frac{1}{2}\int (\sin 3x + \sin 7x)\, dx \\ &= -\frac{1}{6}\cos 3x - \frac{1}{14}\cos 7x + c.\end{aligned}$$

(4) Since $\cos 6x \cos 4x = \frac{1}{2}\left(\cos 2x + \cos 10x\right)$, then

$$\begin{aligned}\int \cos 6x \cos 4x\, dx &= \frac{1}{2}\int (\cos 2x + \cos 10x)\, dx \\ &= \frac{1}{4}\sin 2x + \frac{1}{20}\sin 10x + c.\end{aligned}$$

Exercise 5.2

1 - 18 Evaluate the integral.

1 $\int \sin^2 x \cos^6 x \, dx$

2 $\int \sin^5 x \cos^2 x \, dx$

3 $\int \sin^3 x \cos^3 x \, dx$

4 $\int \cos^6 4x \, dx$

5 $\int \tan^4 x \, dx$

6 $\int \cot^5 x \, dx$

7 $\int \frac{\sin^2 \sqrt{x}}{\sqrt{x}} \, dx$

8 $\int \cot^2 x \csc^3 x \, dx$

9 $\int \cot^4 x \csc^2 x \, dx$

10 $\int \tan^3 x \sec^3 x \, dx$

11 $\int \tan^2 x \sec^2 x \, dx$

12 $\int \tan^2 x \sec^3 x \, dx$

13 $\int \sec^5 x \, dx$

14 $\int \tan^6 x \, dx$

15 $\int \sin 7x \cos 3x \, dx$

16 $\int \cos 4x \cos 3x \, dx$

17 $\int \sin 5x \sin 3x \, dx$

18 $\int \sin 3x \cos 5x \, dx$

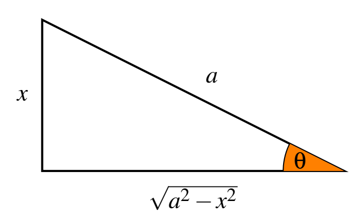
5.3 Trigonometric Substitutions

In this section, we are going to study integrals containing the following expressions $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$ where $a>0$. To get rid of the square roots, we convert them using substitutions involving trigonometric functions. In the following, we explain the conversion of the square roots:

- $\sqrt{a^2-x^2} = a \cos\theta$ if $x = a \sin\theta$.

If $x = a \sin\theta$ where $\theta \in [-\pi/2, \pi/2]$, then

$$\begin{aligned}\sqrt{a^2-x^2} &= \sqrt{a^2-a^2\sin^2\theta}\\ &= \sqrt{a^2(1-\sin^2\theta)}\\ &= \sqrt{a^2\cos^2\theta}\\ &= a\cos\theta.\end{aligned}$$

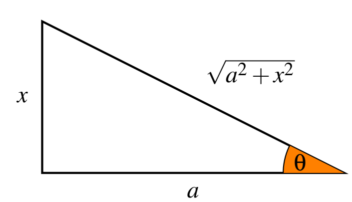


If the expression $\sqrt{a^2-x^2}$ is in a denominator, then we assume $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

- $\sqrt{a^2+x^2} = a \sec\theta$ if $x = a \tan\theta$.

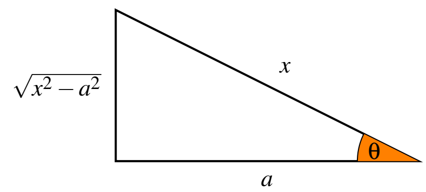
If $x = a \tan\theta$ where $\theta \in (-\pi/2, \pi/2)$, then

$$\begin{aligned}\sqrt{a^2+x^2} &= \sqrt{a^2+a^2\tan^2\theta}\\ &= \sqrt{a^2(1+\tan^2\theta)}\\ &= \sqrt{a^2\sec^2\theta}\\ &= a\sec\theta.\end{aligned}$$



■ $\sqrt{x^2-a^2} = a\ \tan\ \theta$ if $x = a\sec\ \theta$.
If $x = a\ \sec\ \theta$ where $\theta \in [0,\pi/2) \cup [\pi, 3\pi/2)$, then

$$\begin{aligned}\sqrt{x^2-a^2} &= \sqrt{a^2\sec^2\ \theta - a^2} \\ &= \sqrt{a^2(\sec^2\ \theta - 1)} \\ &= \sqrt{a^2\tan^2\ \theta} \\ &= a\ \tan\ \theta.\end{aligned}$$



The previous discussion can be summarized in the following table:

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\ \sin\ \theta,\ \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\ \theta = \cos^2\ \theta$
$\sqrt{a^2+x^2}$	$x = a\ \tan\ \theta,\ \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\ \theta = \sec^2\ \theta$
$\sqrt{x^2-a^2}$	$x = a\ \sec\ \theta,\ \ 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sin^2\ \theta - 1 = \tan^2\ \theta$

Table 5.1: Table of the trigonometric substitutions.

Example 5.11 Evaluate the integral.

(1) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

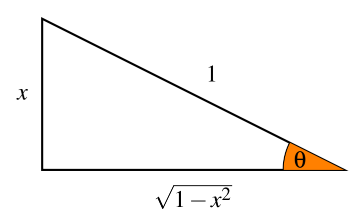
(2) $\displaystyle\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\,dx$

(3) $\displaystyle\int \sqrt{x^2+9}\,dx$

Solution:

(1) Let $x = \sin\ \theta$ where $\theta \in (-\pi/2, \pi/2)$, thus $dx = \cos\ \theta\ d\theta$. By substitution, we have

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^2\ \theta}{\sqrt{1-\sin^2\ \theta}}\ \cos\ \theta\ d\theta \\ &= \int \frac{\sin^2\ \theta\cos\ \theta}{\cos\ \theta}\ d\theta \\ &= \int \sin^2\ \theta\ d\theta \\ &= \frac{1}{2}\int (1-\cos\ 2\theta)\ d\theta \\ &= \frac{1}{2}\Big(\theta - \frac{1}{2}\sin\ 2\theta\Big) + c \\ &= \frac{1}{2}\big(\theta - \sin\ \theta\ \cos\ \theta\big) + c.\end{aligned}$$

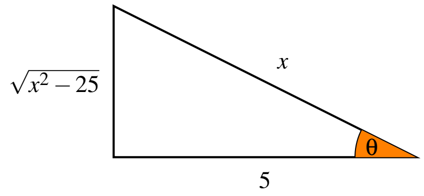


Now, we must return to the original variable x:

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{2}(\sin^{-1}\ x - x\sqrt{1-x^2}) + c.$$

(2) Let $x = 5\ \sec\ \theta$ where $\theta \in [0,\pi/2) \cup [\pi, 3\pi/2)$, thus $dx = 5\ \sec\ \theta\ \tan\ \theta\ d\theta$. After substitution, the integral becomes

$$\int \frac{\sqrt{25\sec^2\theta-25}}{625\sec^4\theta}\,5\sec\theta\tan\theta\,d\theta = \frac{1}{25}\int\frac{\tan^2\theta}{\sec^3\theta}\,d\theta$$
$$= \frac{1}{25}\int \sin^2\theta\cos\theta\,d\theta$$
$$= \frac{1}{75}\sin^3\theta + c.$$



We must return to the original variable x:

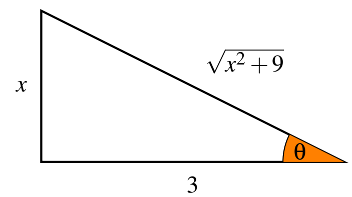
$$\int\frac{\sqrt{x^2-25}}{x^4}\,dx = \frac{(x^2-25)^{3/2}}{75x^3}$$

Hence,

$$\int_5^6 \frac{\sqrt{x^2-25}}{x^4}\,dx = \frac{1}{75}\left[\frac{(x^2-25)^{3/2}}{x^3}\right]_5^6 = \frac{1}{600}.$$

(3) Let $x = 3\tan\theta$ where $\theta\in(-\pi/2,\pi/2)$. This implies $dx = 3\sec^2\theta\,d\theta$. By substitution, we have

$$\int\sqrt{x^2+9}\,dx = \int\sqrt{9\tan^2\theta+9}\,(3\sec^2\theta)\,d\theta$$
$$= 9\int\sec^3\theta\,d\theta$$
$$= \frac{9}{2}\big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\big).$$



This implies

$$\int\sqrt{x^2+9}\,dx = \frac{9}{2}\left(\frac{x\sqrt{x^2+9}}{9}+\ln\left|\frac{\sqrt{x^2+9}+x}{3}\right|\right)+c.$$

Exercise 5.3

1 - 16 Evaluate the integral.

1 $\displaystyle\int\frac{1}{x^2\sqrt{x^2-16}}\,dx$

2 $\displaystyle\int\sqrt{9-x^2}\,dx$

3 $\displaystyle\int\frac{1}{(9x^2-1)^{\frac{3}{2}}}\,dx$

4 $\displaystyle\int\frac{1}{\sqrt{9+x^2}}\,dx$

5 $\displaystyle\int\frac{1}{x^2\sqrt{x^2+4}}\,dx$

6 $\displaystyle\int\frac{x^2}{(16-x^2)^2}\,dx$

7 $\displaystyle\int\frac{x^3}{\sqrt{1-x^8}}\,dx$

8 $\displaystyle\int\frac{\sec^2 x}{\sqrt{9+\tan^2 x}}\,dx$

9 $\displaystyle\int\frac{1}{x^4+2x^2+1}\,dx$

10 $\displaystyle\int\sqrt{x^2-16}\,dx$

11 $\displaystyle\int\sqrt{e^{2x}-25}\,dx$

12 $\displaystyle\int\frac{\cos x}{\sqrt{2-\sin^2 x}}\,dx$

13 $\displaystyle\int\frac{1}{\sqrt{1+x^2}}\,dx$

14 $\displaystyle\int\frac{1}{(1-x^2)^{\frac{5}{2}}}\,dx$

15 $\displaystyle\int e^x\sqrt{1-e^{2x}}\,dx$

16 $\displaystyle\int\frac{\sqrt{9-x^2}}{x^2}\,dx$

5.4 Integrals of Rational Functions

In this section, we study rational functions of form $q(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials. The previous techniques are not suitable to evaluate some integrals that consist of rational functions. Therefore, we need to introduce a new technique to integrate the rational functions. This technique is called decomposition of rational functions into a sum of partial fractions.
The practical steps to evaluate integrals of the rational functions can be summarized as follows:

- Step 1: If the degree of $g(x)$ is less than the degree of $f(x)$, we do polynomial long-division; otherwise we move to step 2.

  From the long division shown on the right side, we have
  $$q(x) = \frac{f(x)}{g(x)} = h(x) + \frac{r(x)}{g(x)} \,,$$
  where $h(x)$ is the quotient and $r(x)$ is the remainder.

  $$\begin{array}{r|l} & \text{h(x)} \\ \hline g(x) \,\big) & \text{f(x)} \\ - & \;\;\dots \\ & \;\;\dots \\ \hline & \text{r(x)} \end{array}$$

- Step 2: Factor the denominator $g(x)$ into irreducible polynomials where the factors are either linear or irreducible quadratic polynomials.[1]

- Step 3: Find the partial fraction decomposition. This step depends on the result of step 2 where the fraction $\frac{f(x)}{g(x)}$ or $\frac{r(x)}{g(x)}$ can be written as a sum of partial fractions:
  $$q(x) = P_1(x) + P_2(x) + P_3(x) + \dots + P_n(x) \,,$$
  each $P_k(x) = \frac{A_k}{(ax+b)^n}, n \in \mathbb{N}$ or $P_k(x) = \frac{A_k x + B_k}{(ax^2+bx+c)^n}$ if $b^2 - 4ac < 0$. The constants A_k and B_k are real numbers and computed later.

- Step 4: Integrate the result of step 3.

Example 5.12 Evaluate the integral $\displaystyle\int \frac{x+1}{x^2-2x-8}\, dx$.

Solution:
Step 1: This step can be skipped since the degree of $f(x) = x+1$ is less than the degree of $g(x) = x^2 - 2x - 8$.
Step 2: Factor the denominator $g(x)$ into irreducible polynomials
$$g(x) = x^2 - 2x - 8 = (x+2)(x-4).$$
Step 3: Find the partial fraction decomposition.
$$\frac{x+1}{x^2-2x-8} = \frac{A}{x+2} + \frac{B}{x-4} = \frac{Ax - 4A + Bx + 2B}{(x+2)(x-4)}.$$
We need to find the constants A and B.

Coefficients of the numerators:
$$A + B = 1 \to (1)$$
$$-4A + 2B = 1 \to (2)$$
By doing some calculation, we obtain $A = \frac{1}{6}$ and $B = \frac{5}{6}$.

Illustration

Multiply equation (1) by 4 and add the result to equation (2)

$$\begin{array}{r} 4A + 4B = 4 \\ -4A + 2B = 1 \\ \hline 6B = 5 \end{array}$$

Step 4: Integrate the result of step 3.
$$\int \frac{x+1}{x^2-2x-8}\, dx = \int \frac{1/6}{x+2}\, dx + \int \frac{5/6}{x-4}\, dx = \frac{1}{6} \ln|x+2| + \frac{5}{6} \ln|x-4| + c.$$

Example 5.13 Evaluate the integral $\displaystyle\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2+3x+2}\, dx$.

[1] For this step, see quadratic equations on page 177.

Solution:
Step 1: Do the polynomial long-division.
Since the degree of the denominator $g(x)$ is less than the degree of the numerator $f(x)$, we do the polynomial long-division given on the right side. Then, we have

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2}.$$

$$\begin{array}{r|rrrr} & 2x & -10 & & \\ \hline x^2+3x+2) & 2x^3 & -4x^2 & -15x & +5 \\ & -(2x^3 & +6x^2 & +4x) & \\ \hline & & (-10x^2 & -19x & +5 \\ & & -(-10x^2 & -30x & -20) \\ \hline & & & 11x & +25 \end{array}$$

Step 2: Factor the denominator $g(x)$ into irreducible polynomials

$$g(x) = x^2+3x+2 = (x+1)(x+2).$$

Step 3: Find the partial fraction decomposition.

$$q(x) = (2x-10) + \frac{11x+25}{x^2+3x+2} = (2x-10) + \frac{A}{x+1} + \frac{B}{x+2} = (2x-10) + \frac{Ax+2A+Bx+B}{(x+1)(x+2)}.$$

We need to find the constants A and B.
Coefficients of the numerators:

$$A+B = 11 \rightarrow 1$$
$$2A+B = 25 \rightarrow 2$$

Illustration

$-2 \times 1 + 2$

$$-2A-2B = -22$$
$$2A+B = 25$$
$$-B = 3$$

By doing some calculation, we have $A = 14$ and $B = -3$.

Step 4: Integrate the result of step 3.

$$\begin{aligned} \int q(x)\,dx &= \int (2x-10)\,dx + \int \frac{14}{x+1}\,dx + \int \frac{-3}{x+2}\,dx \\ &= x^2 - 10x + 14\ln|x+1| - 3\ln|x+2| + c. \end{aligned}$$

Remark 5.2

1. The number of constants $A, B, C, etc.$ is equal to the degree of the denominator $g(x)$. Therefore, in the case of repeated factors of the denominator, we have to check the number of the constants and the degree of $g(x)$.
2. If the denominator $g(x)$ contains irreducible quadratic factors, the numerators of the partial fractions should be polynomials of degree one (see step 3 on page 92).

Example 5.14 Evaluate the integral $\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}dx$.

Solution:
Steps 1 and 2 can be skipped in this example.
Step 3: Find the partial fraction decomposition.
Since the denominator $g(x)$ has repeated factors, then

$$\frac{2x^2-25x-33}{(x+1)^2(x-5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-5} = \frac{A(x^2-4x-5)+B(x-5)+C(x^2+2x+1)}{(x+1)^2(x-5)}.$$

Coefficients of the numerators:

$$A+C = 2 \rightarrow 1$$
$$-4A+B+2C = -25 \rightarrow 2$$
$$-5A-5B+C = -33 \rightarrow 3$$

Illustration

$5 \times 2 + 3 =$

$-25A + 11C = -158 \rightarrow 4$

$25 \times 1 + 4 =$

$36C = -108 \Rightarrow C = -3$

By solving the system of equations, we have $A = 5$, $B = 1$ and $C = -3$.
Step 4: Integrate the result of step 3.

$$\int \frac{2x^2-25x-33}{(x+1)^2(x-5)}\,dx = \int \frac{5}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx + \int \frac{-3}{x-5}\,dx$$
$$= 5\ln|x+1| + \int (x+1)^{-2}\,dx - 3\ln|x-5|$$
$$= 5\ln|x+1| - \frac{1}{(x+1)} - 3\ln|x-5| + c.$$

Example 5.15 Evaluate the integral $\int \frac{x+1}{x(x^2+1)}dx$.

Solution:
Steps 1 and 2 can be skipped in this example.
Step 3: Find the partial fraction decomposition.

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}.$$

Coefficients of the numerators:

$$A+B=0 \rightarrow 1$$
$$C=1 \rightarrow 2$$
$$A=1 \rightarrow 3$$

We have $A=1$, $B=-1$ and $C=1$.
Step 4: Integrate the result of step 3.

$$\int \frac{x+1}{x(x^2+1)}\,dx = \int \frac{1}{x}\,dx + \int \frac{-x+1}{x^2+1}\,dx$$
$$= \ln|x| - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$
$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + c.$$

Exercise 5.4

1 - 20 ■ Evaluate the integral.

1 $\displaystyle\int \frac{1}{x(x-1)}\,dx$

2 $\displaystyle\int_0^2 \frac{1}{x^2+4x+3}\,dx$

3 $\displaystyle\int \frac{1}{x^2-4}\,dx$

4 $\displaystyle\int \frac{1}{x^2-x-2}\,dx$

5 $\displaystyle\int \frac{x+1}{x^2+8x+12}\,dx$

6 $\displaystyle\int \frac{x}{x^2+7x+12}\,dx$

7 $\displaystyle\int_1^5 \frac{x^2-1}{x^2+3x-4}\,dx$

8 $\displaystyle\int \frac{x^3}{x^2-25}\,dx$

9 $\displaystyle\int \frac{x}{x^2+7x+6}\,dx$

10 $\displaystyle\int \frac{1}{x^2+3x+9}\,dx$

11 $\displaystyle\int \frac{1}{(x-1)(x^2+1)}\,dx$

12 $\displaystyle\int \frac{x+2}{(x+1)(x^2-4)}\,dx$

13 $\displaystyle\int \frac{x^3+2x+1}{x^2-3x-10}\,dx$

14 $\displaystyle\int \frac{1}{x^2+1}\,dx$

15 $\displaystyle\int \frac{3x^2+3x-1}{x^3+x^2-x}\,dx$

16 $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\cos^2 x-\cos x-2}\,dx$

17 $\displaystyle\int \frac{2-x}{x^3+x^2}\,dx$

18 $\displaystyle\int_0^1 \frac{1}{1+e^x}\,dx$

19 $\displaystyle\int \frac{e^x}{e^{2x}-2e^x-15}\,dx$

20 $\displaystyle\int \frac{1}{x^4-x^2}\,dx$

■

5.5 Integrals Involving Quadratic Forms

In this section, we provide a new technique for integrals that contain irreducible quadratic expressions ax^2+bx+c where $b \neq 0$. This technique is completing square method: $a^2 \pm 2ab + b^2 = (a \pm b)^2$. Before presenting this method, we explain the word irreducible and show the reader how to complete the square.

Notes:

■ If a quadratic polynomial has real roots, it is called reducible; otherwise it is called irreducible.
For the expression ax^2+bx+c, if $b^2-4ac<0$, then the quadratic expression is irreducible.

■ To complete the square, we need to find $\left(\frac{b}{2a}\right)^2$, then add and subtract it.

Example 5.16 For the quadratic expression $x^2-6x+13$, we have $a=1, b=-6$ and $c=13$. Since $b^2-4ac=-16<0$, then the quadratic expression is irreducible. To complete the square, we find $(\frac{b}{2a})^2=9$, then we add and substrate it as follows:

$$x^2-6x+13 = \underbrace{x^2-6x+9}_{=(x-3)^2} \underbrace{-9+13}_{=4}$$

Hence, $x^2-6x+13=(x-3)^2+4$.

In the following, we use the previous idea to evaluate some integrals.

Example 5.17 Evaluate the integral $\displaystyle\int \frac{1}{x^2-6x+13}\,dx$.

Solution:
The quadratic expression $x^2-6x+13$ is irreducible. By completing the square, we have from the previous example

$$\int \frac{1}{x^2-6x+13}\,dx = \int \frac{1}{(x-3)^2+4}\,dx.$$

Let $u = x - 3$, then $du = dx$. By substitution,

$$\int \frac{1}{u^2+4}\,du = \frac{1}{2}\tan^{-1}\frac{u}{2} + c = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + c.$$

Example 5.18 Evaluate the integral $\displaystyle\int \frac{x}{x^2-4x+8}\,dx$.

Solution:
For the quadratic expression x^2-4x+8, we have $b^2-4ac<0$. Therefore, the quadratic expression x^2-4x+8 is irreducible. By completing the square, we obtain

$$\begin{aligned} x^2-4x+8 &= (x^2-4x+4)+8-4 \\ &= (x-2)^2+4. \end{aligned}$$

Hence

$$\int \frac{x}{x^2-4x+8}\,dx = \int \frac{x}{(x-2)^2+4}\,dx.$$

Let $u = x-2$, then $du = dx$. By substitution,

$$\begin{aligned} \int \frac{u+2}{u^2+4}\,du &= \int \frac{u}{u^2+4}\,du + \int \frac{2}{u^2+4}\,du \\ &= \frac{1}{2}\ln|u^2+4| + \tan^{-1}\frac{u}{2} \\ &= \frac{1}{2}\ln\left((x-2)^2+4\right) + \tan^{-1}\left(\frac{x-2}{2}\right) + c \\ &= \frac{1}{2}\ln\left(x^2-4x+8\right) + \tan^{-1}\left(\frac{x-2}{2}\right) + c. \end{aligned}$$

Example 5.19 Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$.

Solution:
By completing the square, we have $2x-x^2 = -(x^2-2x) = -(x^2-2x+1-1) = 1-(x-1)^2$. Hence

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx.$$

Let $u = x-1$, then $du = dx$. By substitution, the integral becomes

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + c = \sin^{-1}(x-1) + c.$$

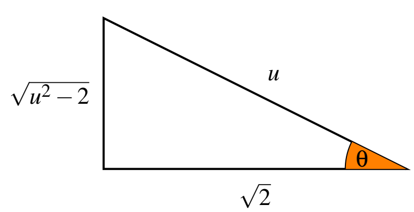
Example 5.20 Evaluate the integral $\displaystyle\int \sqrt{x^2+2x-1}\,dx$.

Solution:
By completing the square, we have $x^2+2x-1 = (x^2+2x+1)-1-1 = (x+1)^2-2$. Hence,

$$\int \sqrt{x^2+2x-1}\,dx = \int \sqrt{(x+1)^2-2}\,dx.$$

Let $u = x+1$, then $du = dx$. The integral becomes $\displaystyle\int \sqrt{u^2-2}\,du$.
Use the trigonometric substitutions, in particular let

$$u = \sqrt{2}\sec\theta \Rightarrow du = \sqrt{2}\sec\theta\tan\theta\,d\theta$$



where $\theta \in [0,\pi/2) \cup [\pi, 3\pi/2)$. By substitution, we have

$$2\int \tan^2\theta\sec\theta\,d\theta = 2\int (\sec^3\theta - \sec\theta)\,d\theta.$$

From Example 5.8, we have

$$2\int (\sec^3\theta - \sec\theta)\,d\theta = \sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right| + c.$$

By returning to the variable u and then to x,

$$\int \sqrt{u^2-2}\,du = \frac{u\sqrt{u^2-2}}{2} - \ln\left|\frac{u+\sqrt{u^2-2}}{\sqrt{2}}\right| + c = \frac{(x+1)\sqrt{(x+1)^2-2}}{2} - \ln\left|\frac{x+1+\sqrt{(x+1)^2-2}}{\sqrt{2}}\right| + c.$$

Exercise 5.5

1 - 12 Evaluate the integral.

1 $\displaystyle\int_0^1 \frac{1}{x^2+4x+5}\,dx$

2 $\displaystyle\int \frac{1}{x^2-6x+1}\,dx$

3 $\displaystyle\int \frac{2x+3}{x^2+2x-3}\,dx$

4 $\displaystyle\int \frac{x^2-2x+5}{2x-x^2}\,dx$

5 $\displaystyle\int_{-1}^0 \frac{1}{\sqrt{8+2x-x^2}}\,dx$

6 $\displaystyle\int \frac{1}{x^2+8x-9}\,dx$

7 $\displaystyle\int \frac{5}{\sqrt{1-4x-x^2}}\,dx$

8 $\displaystyle\int \frac{e^x}{e^{2x}+2e^x-1}\,dx$

9 $\displaystyle\int \frac{1}{\sqrt{6-6x-2x^2}}\,dx$

10 $\displaystyle\int \sqrt{x(2-x)}\,dx$

11 $\displaystyle\int \frac{\sec^2 x}{\tan^2 x-6\tan x+12}\,dx$

12 $\displaystyle\int \sqrt{8-2x-x^2}\,dx$

5.6 Miscellaneous Substitutions

In this section, we study three more important substitutions used in some cases. The first substitution is applied for integrals consisting of rational expressions in $\sin x$ and $\cos x$. The second and third substitutions are applied to integrals of fractional powers.

5.6.1 Fractional Functions in sin x and cos x

The integrals that consist of rational expressions in $\sin x$ and $\cos x$ are treated by using the substitution $u = \tan(x/2)$, $-\pi < x < \pi$. This implies that $du = \frac{\sec^2(x/2)}{2}\,dx$ and since $\sec^2 x = \tan^2 x + 1$, then $du = \frac{u^2+1}{2}\,dx$. Also,

$$\begin{aligned}
\sin x = \sin 2\left(\frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} &= 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos\frac{x}{2}\cos\frac{x}{2} && \text{(multiply and divide by } \cos\tfrac{x}{2}) \\
&= 2\tan\frac{x}{2}\cos^2\frac{x}{2} \\
&= \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} && (\cos x = \tfrac{1}{\sec x}) \\
&= \frac{2u}{u^2+1}.
\end{aligned}$$

For $\cos x$, we have

$$\cos x = \cos 2\left(\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

We can find that

$$\cos\frac{x}{2} = \frac{1}{\sqrt{u^2+1}} \text{ and } \sin\frac{x}{2} = \frac{u}{\sqrt{u^2+1}}.$$

(use the identities $\sec^2\frac{x}{2} = \tan^2\frac{x}{2}+1$ and $\cos^2\frac{x}{2}+\sin^2\frac{x}{2}=1$)

This implies

$$\cos x = \frac{1-u^2}{1+u^2}.$$

The previous discussion can be summarized in the following theorem:

Theorem 5.2 For an integral that contains a rational expression in $\sin x$ and $\cos x$, we assume

$$\sin\, x = \frac{2u}{1+u^2}, \ \text{ and } \ \cos\, x = \frac{1-u^2}{1+u^2}.$$

to produce a rational expression in u where $u = \tan\,(x/2)$, and $du = \frac{1+u^2}{2}\, dx$.

Example 5.21 Evaluate the integral.

(1) $\displaystyle\int \frac{1}{1+\sin\, x}\, dx$

(2) $\displaystyle\int \frac{1}{2+\cos\, x}\, dx$

(3) $\displaystyle\int \frac{1}{1+\sin\, x+\cos\, x}\, dx$

Solution:

(1) Let $u = \tan\, \frac{x}{2}$, then $du = \frac{1+u^2}{2}\, dx$ and $\sin x = \frac{2u}{1+u^2}$. By substituting that into the integral, we have

$$\begin{aligned}\int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}\, du &= 2\int \frac{1}{u^2+2u+1}\, du \\ &= 2\int (u+1)^{-2}\, du \\ &= \frac{-2}{u+1} + c \\ &= \frac{-2}{\tan\, x/2+1} + c.\end{aligned}$$

(2) Let $u = \tan\, \frac{x}{2}$, then $du = \frac{1+u^2}{2}\, dx$ and $\cos\, x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned}\int \frac{1}{2+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\, du &= 2\int \frac{1}{u^2+3}\, du \\ &= \frac{2}{\sqrt{3}}\, \tan^{-1}\, \frac{u}{\sqrt{3}} + c \\ &= \frac{2}{\sqrt{3}}\, \tan^{-1}\, \Big(\frac{\tan\, x/2}{\sqrt{3}}\Big) + c.\end{aligned}$$

(3) Let $u = \tan\, \frac{x}{2}$, this implies $du = \frac{1+u^2}{2}\, dx$, $\sin\, x = \frac{2u}{1+u^2}$ and $\cos\, x = \frac{1-u^2}{1+u^2}$. By substitution, we have

$$\begin{aligned}\int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\, du &= \int \frac{2}{2+2u}\, du \\ &= \int \frac{1}{1+u}\, du \\ &= \ln|\, 1+u\,| + c \\ &= \ln\Big|1+\tan\, \frac{x}{2}\Big| + c.\end{aligned}$$

5.6.2 Integrals of Fractional Powers

In the case of an integrand that consists of fractional powers, it is better to use the substitution $u = x^{\frac{1}{n}}$ where n is the least common multiple of the denominators of the powers. In the following, we provide an example.

Example 5.22 Evaluate the integral $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\, dx.$

Solution:
Let $u = x^{\frac{1}{4}}$, we find $x = u^4$ and $dx = 4u^3 du$. Therefore, $x^{\frac{1}{2}} = (x^{\frac{1}{4}})^2 = u^2$.
By substitution, we have

$$\begin{aligned}\int \frac{1}{u^2+u} 4u^3\, du &= 4\int \frac{u^2}{u+1}\, du \\ &= 4\int (u-1)\, du + 4\int \frac{1}{1+u}\, du \\ &= 2u^2 - 4u + 4\ln\mid u+1\mid + c \\ &= 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln\mid \sqrt[4]{x}+1\mid + c.\end{aligned}$$

5.6.3 Integrals of Form $\sqrt[n]{f(x)}$

If the integrand is of from $\sqrt[n]{f(x)}$, it is useful to assume $u = \sqrt[n]{f(x)}$. This case differs from that given in the substitution method in Chapter 1 i.e., $\sqrt[n]{f(x)}\, f'(x)$ and the difference lies on the existence of the derivative of $f(x)$.

Example 5.23 Evaluate the integral $\int \sqrt{e^x+1}\, dx$.

Solution:
Let $u = \sqrt{e^x+1}$, we obtain $du = \frac{e^x}{2\sqrt{e^x+1}}\, dx$ and $u^2 = e^x + 1$. By substitution, we have

$$\begin{aligned}\int \frac{2u^2}{u^2-1}\, du &= \int 2\, du + 2\int \frac{1}{u^2-1}\, du \\ &= 2u + \int \frac{1}{u-1}\, du + \int \frac{1}{u+1}\, du \\ &= 2u + \ln\mid u-1\mid - \ln\mid u+1\mid + c \\ &= 2\sqrt{e^x+1} + \ln(\sqrt{e^x+1}-1) - \ln(\sqrt{e^x+1}+1) + c.\end{aligned}$$

Exercise 5.6

1 - 12 Evaluate the integral.

1. $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\, dx$
2. $\int \frac{x^{1/2}}{1+x^{3/5}}\, dx$
3. $\int \frac{1}{\sqrt{\cos\, x+1}}\, dx$
4. $\int \frac{\sqrt{x}}{\sqrt{x}+4}\, dx$
5. $\int \frac{1}{1+3\sin\, x}\, dx$
6. $\int \frac{1}{3-\cos\, x}\, dx$
7. $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\, dx$
8. $\int \frac{x^{1/2}}{1+x^{1/4}}\, dx$
9. $\int \frac{1}{\sqrt{e^{2x}+1}}\, dx$
10. $\int \frac{1}{x^{1/2}-x^{3/5}}\, dx$
11. $\int \frac{1}{1-2\cos\, x}\, dx$
12. $\int \frac{1}{\sin\, x+\cos\, x}\, dx$

Review Exercises

1 - 46 Evaluate the integral.

1 $\int x\, e^{2x}\, dx$

2 $\int x\, e^{x^2}\, dx$

3 $\int x \sin x\, dx$

4 $\int x \cos 4x\, dx$

5 $\int \sqrt{x} \ln x\, dx$

6 $\int \cos^{-1} x\, dx$

7 $\int x \sec^2 x\, dx$

8 $\int x\, e^{-4x}\, dx$

9 $\int_1^2 \frac{x}{\sqrt{x^2+1}}\, dx$

10 $\int (\ln x)^3\, dx$

11 $\int \sin^2 x \cos^5 x\, dx$

12 $\int \sin^4 x \cos^4 x\, dx$

13 $\int \tan x \sec^3 x\, dx$

14 $\int \tan^3 x \sec^3 x\, dx$

15 $\int \cot^2 x \csc^3 x\, dx$

16 $\int \cot^4 x \csc^4 x\, dx$

17 $\int \sin 3x \sin x\, dx$

18 $\int \cos 7x \sin 3x\, dx$

19 $\int \cos 4x \cos 2x\, dx$

20 $\int \sqrt{25-x^2}\, dx$

21 $\int \frac{1}{\sqrt{25-x^2}}\, dx$

22 $\int \frac{\sqrt{x^2-16}}{x}\, dx$

23 $\int \frac{x}{(16-x^2)^2}\, dx$

24 $\int \frac{x^3}{(3+x^2)^{\frac{5}{2}}}\, dx$

25 $\int_0^3 x^2\sqrt{9-x^2}\, dx$

26 $\int \frac{1}{x^2-2x}\, dx$

27 $\int \frac{x}{x^2-4x+8}\, dx$

28 $\int \frac{3x+1}{x^2-6x+13}\, dx$

29 $\int \frac{1}{x^2+3x-4}\, dx$

30 $\int \frac{1}{x^3+x^2-x}\, dx$

31 $\int \frac{2x-1}{x^2+x-2}\, dx$

32 $\int_3^7 \frac{x^2}{x^2-x-2}\, dx$

33 $\int \frac{3x^2-10}{x^2-4x+4}\, dx$

34 $\int \frac{x^2-9}{x-1}\, dx$

35 $\int \frac{2x^4-3x^3-10x^2+2x+11}{x^3-x^2-5x-3}\, dx$

36 $\int \frac{1}{1+e^x}\, dx$

37 $\int \frac{x^2}{(x-3)(x+2)^2}\, dx$

38 $\int \frac{x+1}{(x^2+x+2)^2}\, dx$

39 $\int \frac{2x^3-18x^2+29x-4}{(x+1)(x-2)^3}\, dx$

40 $\int \frac{x^5}{\sqrt{x^3+1}}\, dx$

41 $\int \frac{1}{x\sqrt{x^3-1}}\, dx$

42 $\int \frac{\sqrt{x}}{\sqrt{x}+1}\, dx$

43 $\int \frac{1}{3+\cos x}\, dx$

44 $\int \frac{1}{1-\sin x}\, dx$

45 $\int \frac{\sec x}{4-3\tan x}\, dx$

46 $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1+\sin x-\cos x}\, dx$

47 - 72 Choose the correct answer.

47 The partial fraction decomposition of $\frac{1}{x^4-1}$ takes the form

(a) $\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x^2+1}$ (b) $\frac{A}{x^2-1}+\frac{Bx+C}{x^2+1}$ (c) $\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$ (d) None of these

48 The integral $\int \frac{1}{\sqrt{x^2+2x+5}}\, dx$ is equal to

(a) $\sinh^{-1}\left(\frac{x+1}{4}\right)+c$ (b) $\sinh^{-1}\left(\frac{x+1}{2}\right)+c$ (c) $\frac{1}{2}\sinh^{-1}\left(\frac{x+1}{4}\right)+c$ (d) None of these

49 The integral $\int x \sin 2x\, dx$ is equal to

(a) $\frac{x^2}{2}\cos 2x + c$
(b) $\frac{x}{2}\cos 2x - \frac{1}{2}\sin 2x + c$
(c) $-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + c$
(d) None of these

50 The integral $\int \sqrt{1+\sqrt{x}}\, dx$ is equal to

(a) $\frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}} - \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + c$
(b) $\frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}} + c$
(c) $\frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + c$
(d) None of these

51 To evaluate the integral $\int \frac{1}{\sqrt[3]{x}-\sqrt{x}}\, dx$

(a) $u = \sqrt{x}$ (b) $u = \sqrt[4]{x}$ (c) $x = \sqrt[6]{u}$ (d) $u = \sqrt[6]{x}$

52 The integral $\int \frac{2}{x^2-4x+3}\, dx$ is equal to

(a) $\ln(x^2-4x+3)+c$ (b) $\ln\left|\frac{x+1}{x+3}\right|+c$ (c) $\ln\left|\frac{x-3}{x-1}\right|+c$ (d) $\ln\left|\frac{x-1}{x-3}\right|+c$

53 The integral $\int \frac{1}{1+e^x}\, dx$ is equal to

(a) $x - \ln(x+1) + c$
(b) $x - \ln(e^x+1) + c$
(c) $\frac{x^2}{2} - \ln(e^x+1) + c$
(d) $\ln(\frac{x^2}{2}) - \ln(x+1) + c$

54 The integral $\int \frac{1}{\sqrt{4x-x^2}}\, dx$ is equal to

(a) $\sinh^{-1}(\frac{x-2}{2})+c$ (b) $\sin^{-1}(\frac{x-2}{2})+c$ (c) $\frac{1}{2}\sin^{-1}(\frac{x-2}{2})+c$ (d) $\sin^{-1}(\frac{x+2}{2})+c$

55 The integral $\int \frac{1}{\sqrt{x^2+2x+2}}\, dx$ is equal to

(a) $\sinh^{-1}(x+1)+c$ (b) $\sinh^{-1}(\frac{x+1}{2})+c$ (c) $\frac{1}{2}\sinh^{-1}(\frac{x+1}{2})+c$ (d) None of these

56 If $\int \frac{x^{\frac{1}{2}}}{6(x^{\frac{1}{3}}-1)}\, dx = \int \frac{u^8}{u^2-1}\, du$, then

(a) $x = u^2$ (b) $x = u^3$ (c) $x = u^6$ (d) $x = u^8$

57 The substitution used to evaluate the integral $\int \tan^5 x \sec^5 x\, dx$ is

(a) $u = \tan^2 x$ (b) $u = \tan x$ (c) $u = \sec x$ (d) $u = \sin x$

58 To evaluate the integral $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\, dx$, we use the substitution

(a) $u = \sqrt{x}$ (b) $u = \sqrt[4]{x}$ (c) $x = \sqrt[6]{u}$ (d) $u = \sqrt[6]{x}$

59 To evaluate the integral $\int \frac{\sqrt{x^2-25}}{x}\, dx$, we use the substitution

(a) $x = 5\sec\theta$ (b) $x = 25\sec\theta$ (c) $x = 5\tan\theta$ (d) $x = 25\tan\theta$

60 The value of the integral $\int_0^{\frac{\pi}{3}} \sec^2 x\, dx$ is equal to

(a) $\frac{\pi^2}{18}$ (b) $\sqrt{3}$ (c) $\frac{3}{2}$ (d) $\frac{-\pi^2}{18}$

61 To evaluate the integral $\int x^3\sqrt{2x^2+8}\, dx$, we use the substitution

(a) $x = 2\sec\theta$ (b) $x = 2\sqrt{2}\tan\theta$ (c) $x = 2\sqrt{2}\sec\theta$ (d) $x = 2\tan\theta$

62 To evaluate the integral $\int \frac{1}{x^{\frac{2}{3}}+x^{\frac{4}{3}}}\,dx$, we use the substitution

(a) $x=u^2$ (b) $x=u^3$ (c) $x=u^4$ (d) $x=u^{\frac{2}{3}}$

63 To evaluate the integral $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$, we use the substitution

(a) $x=2\tan\,\theta$ (b) $\theta=2\tan\,x$ (c) $x=2\sec\,\theta$ (d) $x=2\sin\,\theta$

64 The value of the integral $\int x\,\sin\,x\,dx$ is equal to

(a) $\sin\,x+x\cos\,x+c$
(b) $-\sin\,x+x\cos\,x+c$
(c) $\sin\,x-x\cos\,x+c$
(d) $-\sin\,x-x\cos\,x+c$

65 The value of the integral $\int \frac{1}{x^2+2x+5}\,dx$ is equal to

(a) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+c$
(b) $\tanh^{-1}\left(\frac{x+1}{2}\right)+c$
(c) $\tan^{-1}\left(\frac{x+1}{2}\right)+c$
(d) $\frac{1}{2}\tanh^{-1}\left(\frac{x+1}{2}\right)+c$

66 The value of the integral $\int_0^{\frac{\pi}{2}} \sin^4\,x\,\cos\,x\,dx$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{5}$ (c) $(\frac{\pi}{2})^5$ (d) $\frac{1}{5}(\frac{\pi}{2})^5$

67 The value of the integral $\int \frac{1}{\sqrt{x^2-8x+25}}\,dx$ is equal to

(a) $\sinh^{-1}\left(\frac{x-4}{3}\right)+c$
(b) $\sinh^{-1}(x-4)+c$
(c) $\sin^{-1}\left(\frac{x-4}{3}\right)+c$
(d) $\frac{1}{3}\sinh^{-1}\left(\frac{x-4}{3}\right)+c$

68 The value of the integral $\int \frac{\sinh\,x}{9+\cosh^2\,x}\,dx$ is equal to

(a) $\tan\left(\frac{\cosh\,x}{3}\right)+c$
(b) $\ln(9+\cosh^2\,x)+c$
(c) $\frac{1}{3}\tan^{-1}\left(\frac{\cosh\,x}{3}\right)+c$
(d) $\tan^{-1}\left(\frac{\cosh\,x}{3}\right)+c$

69 The value of the integral $\int \sin^5\,x\,\cos^3\,x\,dx$ is equal to

(a) $\frac{1}{6}\sin^6\,x-\frac{1}{8}\sin^8\,x+c$
(b) $\frac{1}{5}\sin^5\,x-\frac{1}{3}\sin^3\,x+c$
(c) $\frac{1}{3}\sin^5\,x-\frac{1}{2}\sin^2\,x+c$
(d) $\frac{1}{3}\sin^5\,x-\frac{1}{8}\sin^8\,x+c$

70 The value of the integral $\int_0^{\frac{\pi}{2}} \cos^5\,x\,\sin\,x\,dx$ is equal to

(a) 0 (b) $\frac{1}{3}$ (c) 3 (d) $\frac{1}{6}$

71 The value of the integral $\int \tan^3\,x\,\sec\,x\,dx$ is equal to

(a) $\frac{1}{3}\sec^3\,x+\sec\,x+c$
(b) $-\frac{1}{3}\sec^3\,x-\sec\,x+c$
(c) $-\frac{1}{3}\sec^3\,x+\sec\,x+c$
(d) $\frac{1}{3}\sec^3\,x-\sec\,x+c$

72 The value of the integral $\int \frac{x-2}{x\sqrt{x^2-25}}\,dx$ is equal to

(a) $\cosh^{-1}\,\frac{x}{5}-2\sec^{-1}\,\frac{x}{5}+c$
(b) $\cosh^{-1}\,\frac{x}{5}-\frac{2}{5}\sec^{-1}\,x+c$
(c) $\cosh^{-1}\,\frac{x}{5}-\frac{2}{5}\sec^{-1}\,\frac{x}{5}+c$
(d) $\cosh^{-1}\,\frac{x}{5}+2\sec^{-1}\,\frac{x}{5}+c$

Chapter 6

Indeterminate Forms and Improper Integrals

6.1 Indeterminate Forms

In the beginning of this section, we define the limit of functions and list the rules of the limits.

Definition 6.1 Let f be a defined function on an open interval I and $c \in I$ where f may not be defined at c. Then,

$$\lim_{x \to c} f(x) = L,\ L \in \mathbb{R}$$

means for every $\varepsilon > 0$, there is $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

The following theorem presents the general rules of the limits.

Theorem 6.1 If $\lim\limits_{x \longrightarrow c} f(x)$ and $\lim\limits_{x \longrightarrow c} g(x)$ both exist, then

(1) Sum Rule: $\lim\limits_{x \longrightarrow c} \big(f(x) + g(x)\big) = \lim\limits_{x \longrightarrow c} f(x) + \lim\limits_{x \longrightarrow c} g(x)$.

(2) Difference Rule: $\lim\limits_{x \longrightarrow c} \big(f(x) - g(x)\big) = \lim\limits_{x \longrightarrow c} f(x) - \lim\limits_{x \longrightarrow c} g(x)$.

(3) Product Rule: $\lim\limits_{x \longrightarrow c} \big(f(x).g(x)\big) = \lim\limits_{x \longrightarrow c} f(x) \times \lim\limits_{x \longrightarrow c} g(x)$.

(4) Constant Multiple Rule: $\lim\limits_{x \longrightarrow c} \big(k\, f(x)\big) = k \lim\limits_{x \longrightarrow c} f(x)$.

(5) Quotient Rule: $\lim\limits_{x \longrightarrow c} \big(\frac{f(x)}{g(x)}\big) = \frac{\lim\limits_{x \longrightarrow c} f(x)}{\lim\limits_{x \longrightarrow c} g(x)}$.

(6) Power Rule: $\lim\limits_{x \longrightarrow c} \big(f(x)\big)^{m/n} = \big(\lim\limits_{x \longrightarrow c} f(x)\big)^{m/n}$.

Example 6.1 Find each limit if it exists.

(1) $\lim_{x\to 1} x$

(2) $\lim_{x\to 8} \sqrt{x}$

(3) $\lim_{x\to 0} (x^2 - 2x + 1)$

(4) $\lim_{x\to \pi} \sin\ x\ \cos\ x$

(5) $\lim_{x\to 3^+} \frac{1}{(x-3)}$

(6) $\lim_{x\to 1} \frac{x}{(x^2+1)}$

Solution:

(1) $\lim_{x\to 1} x = 1$

(2) $\lim_{x\to 8} \sqrt{x} = 2\sqrt{2}$

(3) $\lim_{x\to 0} (x^2 - 2x + 1) = \lim_{x\to 0} x^2 - 2\lim_{x\to 0} x + \lim_{x\to 0} 1 = 1.$

(4) $\lim_{x\to \pi} \sin\ x\ \cos\ x = \lim_{x\to \pi} \sin\ x\ \lim_{x\to \pi} \cos\ x = 0$

(5) $\lim_{x\to 3^+} \frac{1}{(x-3)} = \frac{\lim_{x\to 3^+} 1}{\lim_{x\to 3^+}(x-3)} = \infty$

(6) $\lim_{x\to 1} \frac{x}{(x^2+1)} = \frac{\lim_{x\to 1} x}{\lim_{x\to 1}(x^2+1)} = \frac{1}{2}$

In the following, we examine several situations where a function is built up from other functions, but the limits of these functions are not sufficient to determine the overall limit. These situations are called indeterminate forms. The following example shows these forms without finding the final result.

Example 6.2

(1) $\lim_{x\to 0} \frac{\sin\ x}{x} = \frac{0}{0}$

(2) $\lim_{x\to \infty} \frac{e^x}{x} = \frac{\infty}{\infty}$

(3) $\lim_{x\to 0^+} x^2 \ln x = 0.\infty$

(4) $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \infty - \infty$

In the following table, we categorize the indeterminate forms:

Case	Indeterminate Form
Quotient	$\frac{0}{0}$ and $\frac{\infty}{\infty}$
Product	$0.\infty$ and $0.(-\infty)$
Sum & Difference	$(-\infty)+\infty$ and $\infty-\infty$
Exponent	0^0, 1^∞, $1^{-\infty}$ and ∞^0

Table 6.2: List of the indeterminate forms.

The following theorem examines the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Theorem 6.2 Suppose f and g are differentiable on an interval I and $c \in I$ where f and g may not be differentiable at c. If $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = c$ and $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

if $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists or equals to ∞.

Proof. The theorem is proved for the indeterminate form $\frac{0}{0}$ at $x = c$. Assume $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$ and we want to prove that $\lim_{x\to c} \frac{f(x)}{g(x)} = L$. Define two functions F and G on the interval I as follows:

$$F(x) = \begin{cases} f(x) & : x \neq c \\ 0 & : x = c \end{cases} \quad \text{and} \quad g(x) = \begin{cases} g(x) & : x \neq c \\ 0 & : x = c \end{cases}$$

Since $\lim_{x\to c} F(x) = \lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} G(x) = \lim_{x\to c} g(x) = 0$, then F and G are continuous on the interval I. Also, we have $F'(x) = f'(x)$

and $G'(x) = g'(x)$ for $x \neq c$. From Cauchy's formula for the two functions F and G on the interval between x and c,[1] there exists a number z belong to the open interval between c and x such that

$$\frac{F'(z)}{G'(z)} = \frac{F'(x) - F(c)}{G'(z) - G(c)} = \frac{F(x)}{G(x)}.$$

Since $z \to c$ when $x \to c$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{F(x)}{G(x)} = \lim_{z\to c} \frac{F'(z)}{G'(z)} = \lim_{z\to c} \frac{f'(z)}{g'(z)} = L. \blacksquare$$

Remark 6.1

1. L'Hôpital's rule works if $c = \pm\infty$ or when $x \to c^+$ or $x \to c^-$.
2. When applying L'Hôpital's rule, we should calculate the derivatives of $f(x)$ and $g(x)$ separately.
3. Sometimes, we need to apply L'Hôpital's rule twice.

Example 6.3 Use L'Hôpital's rule to find each limit if it exists.

(1) $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25}$

(2) $\lim_{x\to 0} \frac{\sin x}{x}$

(3) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

(4) $\lim_{x\to\infty} \frac{e^x}{x}$

Solution:

(1) Since $\lim_{x\to 5} \sqrt{x-1} - 2 = 0$ and $\lim_{x\to 5} x^2 - 2 = 0$, we have the indeterminate form $\frac{0}{0}$. By applying L'Hôpital's rule, we have

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \lim_{x\to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40}.$$

(2) The quotient has the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule to have

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

(3) The indeterminate form is $\frac{\infty}{\infty}$. Apply L'Hôpital's rule to obtain

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

(4) The indeterminate form is $\frac{\infty}{\infty}$. By applying L'Hôpital's rule, we have

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

Before considering examples of other indeterminate forms, we provide techniques to find the limits.

Techniques for finding the limits of other indeterminate forms:

■ Indeterminate form $0.\infty$.

1. Write $f(x)\,g(x)$ as $\frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$.
2. Apply L'Hôpital's rule to the resulting indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

■ Indeterminate form $\infty - \infty$.

1. Write the form as a quotient or product.

[1]Let f and g be continuous on $[a,b]$ and differentiable on (a,b). If $g'(x) \neq 0$ for every x in (a,b), then exists number $z \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(z)}{g'(z)}$.

2. Apply L'Hôpital's rule to the resulting indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate forms 0^0, 1^∞, $1^{-\infty}$ or ∞^0.

1. Let $y = f(x)^{g(x)}$
2. Take the natural logarithm $\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x)$.
3. Apply L'Hôpital's rule to the resulting indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 6.4 Find each limit if it exists.

(1) $\lim\limits_{x\to 0^+} x^2 \ln x$

(2) $\lim\limits_{x\to \frac{\pi}{4}} (1-\tan\, x)\ \sec\ 2x$

(3) $\lim\limits_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

(4) $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}}$

Solution:

(1) The indeterminate form is $0.(-\infty)$, so we cannot apply L'Hôpital's rule. We need to rearrange the expression in a way that enables us to apply L'Hôpital's rule. By using the previous techniques, we obtain

$$x^2 \ln x = \frac{\ln x}{\frac{1}{x^2}}.$$

The limit of the new expression is of the form $\frac{\infty}{\infty}$. Therefore, we can apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

Hence, $\lim\limits_{x\to 0^+} x^2 \ln x = 0$.

(2) The indeterminate form is $0.\infty$, so we try to rewrite the function to apply L'Hôpital's rule. We know that $\sec\ x = 1/\cos\ x$, thus

$$(1-\tan\ x)\sec\ 2x = \frac{(1-\tan\ x)}{\cos\ 2x}.$$

Now, the limit of the new expression is of the form $\frac{0}{0}$. From L'Hôpital's rule, we have

$$\begin{aligned}\lim_{x\to\frac{\pi}{4}} \frac{(1-\tan\ x)}{\cos\ 2x} &= \lim_{x\to\frac{\pi}{4}} \frac{\sec^2\ x}{2\sin\ 2x} && \text{(L'Hôpital's rule)}\\ &= \frac{(\sqrt{2})^2}{2} = 1.\end{aligned}$$

Hence, $\lim\limits_{x\to\frac{\pi}{4}} (1-\tan\ x)\ \sec\ 2x = 1$.

(3) The indeterminate form is $\infty - \infty$. To treat this form, we write the function as a single fraction

$$\frac{1}{x-1} - \frac{1}{\ln x} = \frac{\ln x - x + 1}{(x-1)\ln x}.$$

The new expression takes the indeterminate form $\frac{0}{0}$. From L'Hôpital's rule,

$$\lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{1-x}{x\ \ln x + x - 1}.$$

We have the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule again to have

$$\lim_{x\to 1^+} \frac{1-x}{x\ \ln x + x - 1} = \lim_{x\to 1^+} \frac{-1}{\ln x + 2} = \frac{-1}{2}.$$

Hence, $\lim\limits_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = -\frac{1}{2}$.

(4) The limit is of the form 1^∞. To treat this form, let $y=(1+x)^{\frac{1}{x}}$. By taking the natural logarithm of both sides, we have

$$\begin{aligned}\ln y &= \frac{1}{x}\ln(1+x)\\ \Rightarrow \lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{1}{x}\ln(1+x)\\ &= \lim_{x\to 0}\frac{\ln(1+x)}{x}.\end{aligned}$$

The indeterminate form is $\frac{0}{0}$. By applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = 1.$$

Hence,

$$\begin{aligned}\lim_{x\to 0}\ln y = 1 &\Rightarrow e^{\lim_{x\to 0}\ln y} = e^1 && \text{(take the natural exponential function of both sides)}\\ &\Rightarrow \lim_{x\to 0} e^{(\ln y)} = e\\ &\Rightarrow \lim_{x\to 0} y = e\\ &\Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e.\end{aligned}$$

Exercise 6.1

1 - 14 Find the limit if it exists.

1 $\lim_{x\to 2}\frac{x^2-4x+4}{x-2}$

2 $\lim_{x\to 3}\frac{x^2-9}{x-3}$

3 $\lim_{x\to \pi^+}\frac{\cos\ x+\sin\ x}{\tan\ x}$

4 $\lim_{x\to 0}\frac{1-e^x}{x}$

5 $\lim_{x\to \pi/2^+}\tan\ x$

6 $\lim_{x\to 0^-}\frac{e^x-1}{x^2}$

7 $\lim_{x\to 0}(e^x+x)^{\frac{1}{x}}$

8 $\lim_{x\to \infty}\frac{x+2}{x-2}$

9 $\lim_{x\to 0^+}\frac{e^x-\ln(e^x)}{\ln x}$

10 $\lim_{x\to \pi/2}\frac{1-\sin\ x}{\cos\ x}$

11 $\lim_{x\to 1}\frac{\ln x}{\tan\ \pi x}$

12 $\lim_{x\to 0}\frac{\tan\ x}{x}$

13 $\lim_{x\to \infty}\frac{\ln(\ln x)}{\sqrt{x}}$

14 $\lim_{x\to 0}(\frac{1}{x^2})^x$

6.2 Improper Integrals

In this section, we deal with integrals over infinite intervals or with integrals that involve discontinuous integrands. In such cases, the integrals are called improper.

Definition 6.2 The integral $\int_a^b f(x)\,dx$ is called a proper integral if

1. the interval $[a,b]$ is finite and closed, and

2. $f(x)$ is defined on $[a,b]$.

If condition 1 or 2 is not satisfied, the integral is improper. In the following, we discuss the improper integrals.

6.2.1 Infinite Intervals

In this section, we study integrals of forms $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{b} f(x)\,dx$, $\int_{\infty}^{-\infty} f(x)\,dx$ where f is a continuous function.

Definition 6.3

1. Let f be a continuous function on $[a,\infty)$. The improper integral $\int_a^{\infty} f(x)\,dx$ is defined as follows:

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \text{ if the limit exists.}$$

2. Let f be a continuous function on $(-\infty,b]$. The improper integral $\int_{-\infty}^{b} f(x)\,dx$ is defined as follows:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx \text{ if the limit exists.}$$

The previous integrals are convergent (or to converge) if the limit exists as a finite number. However, if the limit does not exist or equals $\pm\infty$, the integral is called divergent (or to diverge).

3. Let f be a continuous function on $\mathbb{R}$ and $a\in\mathbb{R}$. The improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as follows:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Note:

1. If an improper integral is convergent, the value of the integral is the value of the limit.

2. If both integrals in item 3 converge, then the value of the improper integral is the sum of values of the two integrals.

Example 6.5 Determine whether the integral converges or diverges.

(1) $\int_0^{\infty}\frac{1}{(x+2)^2}\,dx$ **(2)** $\int_0^{\infty}\frac{x}{1+x^2}\,dx$ **(3)** $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$

Solution:

(1) $\int_0^{\infty}\frac{1}{(x+2)^2}\,dx = \lim_{t\to\infty}\int_0^t\frac{1}{(x+2)^2}\,dx.$

The integral

$$\int_0^t \frac{1}{(x+2)^2}\,dx = \int_0^t (x+2)^{-2}\,dx = \left[\frac{-1}{x+2}\right]_0^t = -\Big(\frac{1}{t+2}-\frac{1}{2}\Big).$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^2}\,dx = -\lim_{t\to\infty}\Big(\frac{1}{t+2}-\frac{1}{2}\Big) = -\Big(0-\frac{1}{2}\Big) = \frac{1}{2}.$$

This implies that the integral converges and has the value $\frac{1}{2}$.

(2) $\int_0^{\infty}\frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t\frac{x}{1+x^2}\,dx.$

The integral

$$\int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2}\Big[\ln(1+x^2)\Big]_0^t = \frac{1}{2}\ln(1+t^2)-\frac{1}{2}\ln(1) = \frac{1}{2}\ln(1+t^2).$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2}\lim_{t\to\infty}\ln(1+t^2) = \infty.$$

The improper integral diverges.

(3) $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0\frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t\frac{1}{1+x^2}\,dx.$

We know that $\int \frac{1}{1+x^2}\,dx = \tan^{-1}\,x + c$, so

$$\begin{aligned}\lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2} &= \lim_{t\to-\infty}\left[0-\tan^{-1}(t)\right] + \lim_{t\to\infty}\left[\tan^{-1}\,t - 0\right]\\ &= -\lim_{t\to-\infty}\tan^{-1}\,t + \lim_{t\to\infty}\tan^{-1}\,t\\ &= -(-\frac{\pi}{2}) + \frac{\pi}{2} = \pi.\end{aligned}$$

The integral is convergent and has the value π.

6.2.2 Discontinuous Integrands

Definition 6.4

1. If f is continuous on $[a,b)$ and has an infinite discontinuity at b i.e., $\lim_{x\to b^-} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx \text{ if the limit exists.}$$

2. If f is continuous on $(a,b]$ and has an infinite discontinuity at a i.e., $\lim_{x\to a^+} f(x) = \pm\infty$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^a f(x)\,dx \text{ if the limit exists.}$$

In items 1 and 2, the integral is convergent if the limit exists as a finite number; otherwise the integral is divergent.

3. If f is continuous on $[a,b]$ except at $c \in (a,b)$ such that $\lim_{x\to c^\pm} f(x) = \pm\infty$, the improper integral $\int_a^b f(x)\,dx$ is defined as follows:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Example 6.6 Determine whether the integral converges or diverges.

(1) $\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx$ **(2)** $\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\,dx$ **(3)** $\int_{-3}^1 \frac{1}{x^2}\,dx$

Solution:

(1) Since $\lim_{x\to4^-}\frac{1}{(4-x)^{\frac{3}{2}}} = \infty$ and the integrand is continuous on $[0,4)$, then from Definition 6.4,

$$\begin{aligned}\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx &= \lim_{t\to4^-}\int_0^t (4-x)^{-\frac{3}{2}}\,dx\\ &= \lim_{t\to4^-}\left[\frac{2}{\sqrt{4-x}}\right]_0^t\\ &= \lim_{t\to4^-}\left(\frac{2}{\sqrt{4-t}}-1\right)\\ &= \infty.\end{aligned}$$

Illustration

$$\begin{aligned}\int (4-x)^{-3/2}\,dx &= -\int -(4-x)^{-3/2}\,dx\\ &= 2(4-x)^{-1/2}+c\\ &= \frac{2}{\sqrt{4-x}}+c\end{aligned}$$

Thus, the integral diverges.

(2) The limit $\lim_{x\to0^+}\frac{\cos\,x}{\sqrt{\sin\,x}} = \infty$ and the integrand is continuous on $(0,\frac{\pi}{4}]$, thus

$$\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\, dx = \lim_{t\to 0^+} \int_t^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\, dx$$
$$= 2 \lim_{t\to 0^+} \left[\sqrt{\sin x}\right]_t^{\frac{\pi}{4}}$$
$$= 2 \lim_{t\to 0^+} \left(\frac{1}{\sqrt[4]{2}} - \sqrt{\sin t}\right)$$
$$= \frac{2}{\sqrt[4]{2}}.$$

The integral converges and has the value $\frac{2}{\sqrt[4]{2}}$.

Illustration

$$\int \frac{\cos x}{\sqrt{\sin x}}\, dx = \int \cos x \sin^{-1/2} x\, dx$$
$$= 2\sin^{1/2} x + c$$

(3) Since $\lim\limits_{x\to 0^-} \frac{1}{x^2} = \lim\limits_{x\to 0^+} \frac{1}{x^2} = \infty$ and the integrand is continuous on $[-3,0)\cup(0,1]$, then

$$\int_{-3}^{1} \frac{1}{x^2}\, dx = \int_{-3}^{0} \frac{1}{x^2}\, dx + \int_0^1 \frac{1}{x^2}\, dx$$
$$= \lim_{t\to 0^-} \int_{-3}^{t} \frac{1}{x^2} + \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}$$
$$= -\lim_{t\to 0^-} \left[\frac{1}{x}\right]_{-3}^{t} - \lim_{t\to 0^+} \left[\frac{1}{x}\right]_t^1$$
$$= -\lim_{t\to 0^-} \left[\frac{1}{t} + \frac{1}{3}\right] - \lim_{t\to 0^+} \left[1 - \frac{1}{t}\right]$$
$$= \infty.$$

The integral diverges.

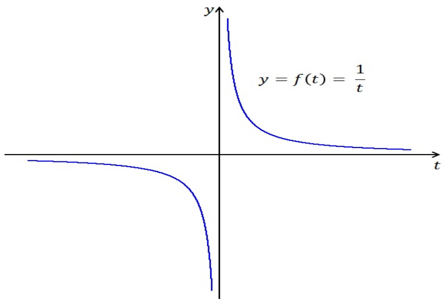


Figure 6.1

Exercise 6.2

1 - 16 Determine whether the integral converges or diverges.

1 $\int_1^{\infty} \frac{1}{x}\, dx$

2 $\int_1^{\infty} \frac{1}{x^2}\, dx$

3 $\int_4^{\infty} \frac{1}{\sqrt{x}}\, dx$

4 $\int_{-\infty}^{0} e^x\, dx$

5 $\int_0^{\infty} e^x\, dx$

6 $\int_2^{\infty} \frac{1}{x-1}\, dx$

7 $\int_1^{2} \frac{1}{1-x}\, dx$

8 $\int_{-1}^{1} \frac{1}{x}\, dx$

9 $\int_0^{3} \frac{dx}{\sqrt{9-x^2}}$

10 $\int_0^{\infty} (1-x)e^{-x}\, dx$

11 $\int_0^{\infty} \frac{dx}{x^2+4}$

12 $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\, dx$

13 $\int_0^{\infty} \frac{1}{x-1}\, dx$

14 $\int_0^{\pi} \sec^2 x\, dx$

15 $\int_0^{2} \frac{1}{x^2+1}\, dx$

16 $\int_0^{\pi/2} \tan x\, dx$

Review Exercises

1 - 10 Find the limit if it exists.

1 $\lim\limits_{x\to\infty} \frac{2^x-1}{x}$

2 $\lim\limits_{x\to 0^-} \frac{\tan x}{x^2}$

3 $\lim\limits_{x\to 0} \frac{e^x-x-1}{\sin x}$

4 $\lim\limits_{x\to\pi^-} \frac{\sin x}{1-\cos x}$

5 $\lim\limits_{x\to 0} \frac{\tan^{-1} x}{x}$

6 $\lim\limits_{x\to 0} \frac{6^x-2^x}{x}$

7 $\lim\limits_{x\to 1} \frac{\ln x}{x-1}$

8 $\lim\limits_{x\to\infty} x \ln(\frac{x+1}{x-1})$

9 $\lim\limits_{x\to 1} x^{1/(1-x)}$

10 $\lim\limits_{x\to 0} (e^x+x)^{\frac{1}{x}}$

11 - 18 Determine whether the integral converges or diverges.

11 $\displaystyle\int_0^\infty \frac{1}{2x^2+3x+1}\, dx$

12 $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\, dx$

13 $\displaystyle\int_{-\infty}^2 \frac{1}{5-2x}\, dx$

14 $\displaystyle\int_{-\infty}^\infty \frac{x}{(x^2+3)^2}\, dx$

15 $\displaystyle\int_0^3 \frac{1}{(x-2)^2}\, dx$

16 $\displaystyle\int_0^4 \frac{1}{x^2+x-6}\, dx$

17 $\displaystyle\int_2^4 \frac{x-2}{x^2-5x+4}\, dx$

18 $\displaystyle\int_0^9 \frac{1}{\sqrt{x}\,(x+9)}\, dx$

19 - 35 Choose the correct answer.

19 $\lim\limits_{x\to 1}(\frac{1}{x-1} - \frac{1}{\ln x})$ is equal to

(a) ∞ (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 0

20 $\lim\limits_{x\to 0} \frac{2^x-3^x}{x}$ is equal to

(a) ∞ (b) $\ln\frac{2}{3}$ (c) $\ln\frac{3}{2}$ (d) -1

21 The improper integral $\displaystyle\int_0^\infty \frac{1}{4+x^2}\, dx$

(a) converges to $\frac{\pi}{4}$ (b) converges to $\frac{\pi}{2}$ (c) diverges (d) None of these

22 The improper integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\, dx$

(a) converges to -2 (b) converges to 1 (c) converges to 2 (d) diverges

23 The improper integral $\displaystyle\int_0^\infty \frac{1}{1+x^2}\, dx$

(a) converges to π (b) converges to $\frac{\pi}{2}$ (c) converges to ∞ (d) diverges

24 The improper integral $\displaystyle\int_0^\infty \frac{1}{x^2+4}\, dx$

(a) converges to 0 (b) converges to $\frac{\pi}{4}$ (c) converges to $\frac{\pi}{2}$ (d) diverges

25 The limit $\lim\limits_{x\to 0}(\frac{\sin x-x}{x^3})$ is equal to

(a) ∞ (b) $-\frac{1}{6}$ (c) $\frac{1}{6}$ (d) 0

26 The improper integral $\displaystyle\int_0^2 \frac{2x}{16-x^4}\,dx$

(a) converges to $\frac{\pi}{2}$ (b) converges to $\frac{\pi}{4}$ (c) converges to π (d) diverges

27 The limit $\lim\limits_{x\to\infty}\left(\frac{x^2}{x-1}-\frac{x^2}{x+1}\right)$ is equal to

(a) ∞ (b) 1 (c) 2 (d) 0

28 The improper integral $\displaystyle\int_0^1 x^{\alpha}\,dx$ converges if and only if

(a) $\alpha > 1$ (b) $-2 < \alpha < -1$ (c) $\alpha > -1$ (d) $\alpha < -2$

29 $\lim\limits_{x\to\infty}\left(\frac{x}{\sqrt{x}-1}-\frac{x}{\sqrt{x}+1}\right)$ is equal to

(a) ∞ (b) 1 (c) 0 (d) 2

30 $\lim\limits_{x\to 0}(1+2x)^{\frac{1}{x}}$ is equal to

(a) 1 (b) e (c) e^2 (d) ∞

31 $\lim\limits_{x\to 0}\dfrac{\int_0^x e^{t^2}\,dt}{x}$ is equal to

(a) ∞ (b) 1 (c) 0 (d) -1

32 The limit $\lim\limits_{x\to 0}(1+3x)^{\frac{1}{x}}$ is equal to

(a) 1 (b) e (c) e^3 (d) ∞

33 The improper integral $\displaystyle\int_0^1 \frac{1}{x^{\frac{3}{2}}}\,dx$

(a) converges to 3 (b) converges to $\frac{2}{3}$ (c) converges to $\frac{3}{2}$ (d) diverges

34 The improper integral $\displaystyle\int_0^{-1} \frac{1}{\sqrt[3]{x+1}}\,dx$

(a) converges to $\frac{1}{2}$ (b) converges to $\frac{3}{2}$ (c) converges to $-\frac{3}{2}$ (d) diverges

35 The improper integral $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^2}\,dx$

(a) converges to 0 (b) converges to 1 (c) converges to -1 (d) diverges

Chapter 7

Application of Definite Integrals

7.1 Areas

The definite integral can be used to calculate areas under graphs. In Chapter 2, we mentioned that if f is continuous and $f \geq 0$ on $[a,b]$, the definite integral $\int_a^b f(x)\,dx$ is exactly the area of the region under the graph of f from a to b.

■ If $y = f(x)$ is a continuous function on $[a,b]$ and $f(x) \geq 0$ for every $x \in [a,b]$, the area of the region under the graph of $f(x)$ from $x = a$ to $x = b$ is given by the integral:

$$\boxed{A = \int_a^b f(x)\,dx}$$

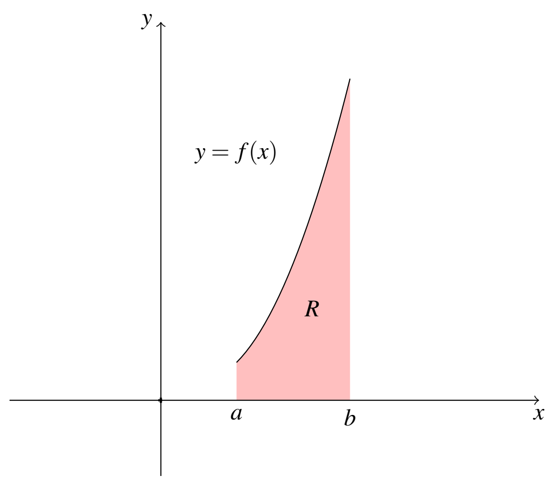


Figure 7.1: The area of the region under the graph of f over $[a,b]$.

■ If f and g are continuous functions and $f(x) \geq g(x)\ \forall x \in [a,b]$, then the area A of the region bounded by the graphs of f (the upper boundary of R) and g (the lower boundary of R) from $x = a$ to $x = b$ is subtracting the area of the region under $g(x)$ from the area of the region under $f(x)$. This can be stated as follows:

$$\boxed{A = \int_a^b \big(f(x) - g(x)\big)\,dx}$$

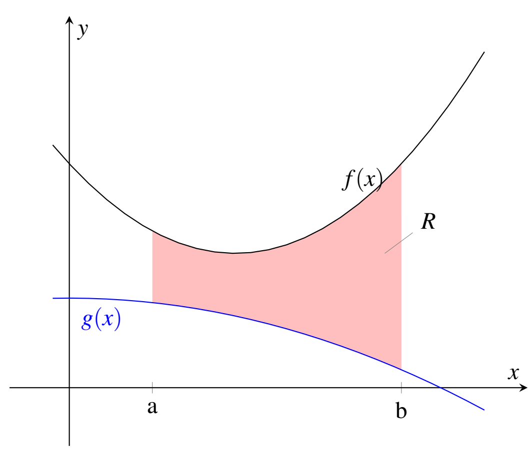


Figure 7.2: The area of the region bounded by the graphs of f and g over $[a,b]$.

■ If $x = f(y)$ is a continuous function on $[c,d]$ and $f(y) \geq 0\ \forall y \in [c,d]$, the area of the region bounded by the graph of $f(y)$ from $y = c$ to $y = d$ is given by the integral:

$$\boxed{A = \int_c^d f(y)\,dy}$$

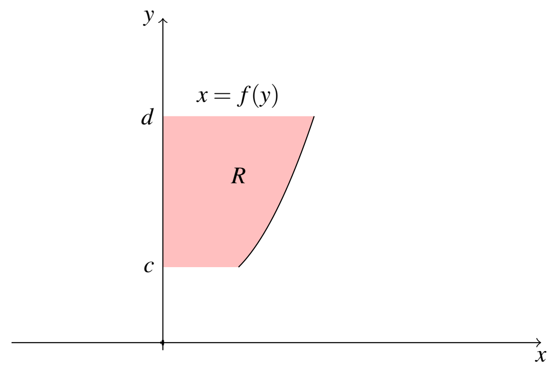


Figure 7.3: The area of the region bounded by the graph of f over $[c,d]$.

■ If f and g are continuous functions and $f(y) \geq g(y)\ \forall y \in [c,d]$, then the area A of the region bounded by the graphs of f (the right boundary of R) and g (the left boundary of R) from $y = c$ to $y = d$ is subtracting the area of the region bounded by $g(x)$ from the area of the region bounded by $f(x)$. This can be stated as follows:

$$\boxed{A = \int_c^d \big(f(y) - g(y)\big)\,dy}$$

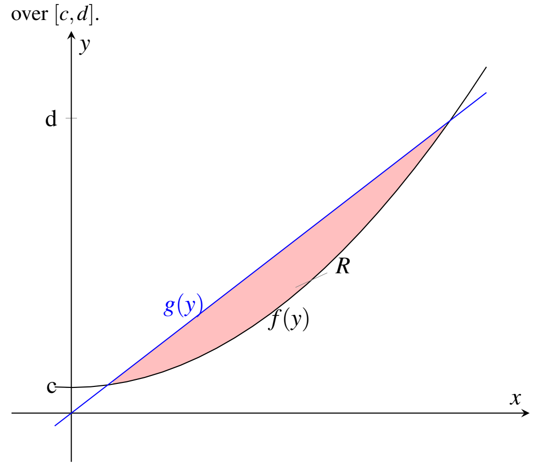


Figure 7.4: The area of the region bounded by the graphs of f and g over $[c,d]$.

Example 7.1 Express the area of the shaded region as a definite integral then find the area.

(1)

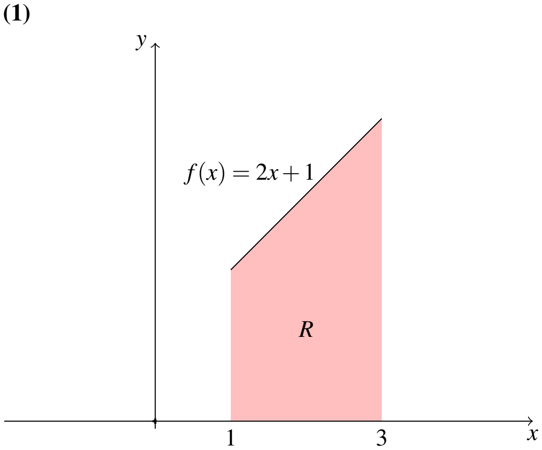


Figure 7.5

(2)

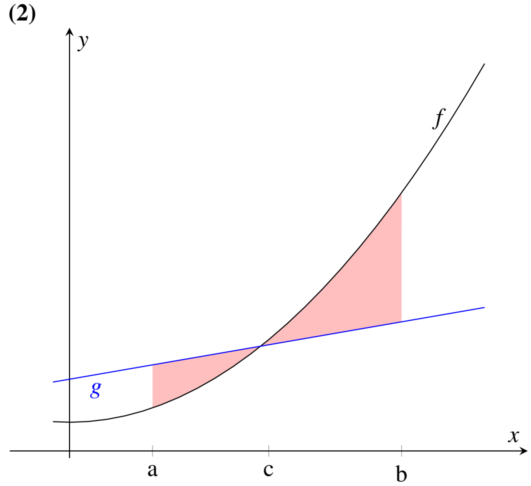


Figure 7.6

Solution:

(1) Area : $A = \int_1^3 (2x+1)\,dx = \Big[x^2 + x\Big]_1^3 = \Big[(3^2+3) - (1^2+1)\Big] = 12 - 2 = 10.$

(2) We have two regions:
Region (1) : in the interval $[a,c]$.
Upper graph: $y = g(x)$
Lower graph: $y = f(x)$
Area: $A_1 = \int_a^c \big(g(x) - f(x)\big)\, dx$.
The total area is $A = A_1 + A_2$.

Region (2) : in the interval $[c,b]$.
Upper graph: $y = f(x)$
Lower graph: $y = g(x)$
Area: $A_2 = \int_c^b \big(f(x) - g(x)\big)\, dx$.

Example 7.2 Sketch the region bounded by the graphs of $y = x^3$ and $y = x$, then find its area.

Solution: The figure on the right shows the region bounded by the two functions.

The region is divided into two regions as follows:

Region (1): in the interval $[-1,0]$
Upper graph: $y = x^3$
Lower graph: $y = x$

$$A_1 = \int_{-1}^{0} (x^3 - x)\, dx = \Big[\frac{x^4}{4} - \frac{x^2}{2}\Big]_{-1}^{0} = \Big[0 - (\frac{1}{4} - \frac{1}{2})\Big] = \frac{1}{4}.$$

Region (2): in the interval $[0,1]$
Upper graph: $y = x$
Lower graph: $y = x^3$

$$A_2 = \int_{0}^{1} (x - x^3)\, dx = \Big[\frac{x^2}{2} - \frac{x^4}{4}\Big]_{0}^{1} = \Big[(\frac{1}{2} - \frac{1}{4}) - 0\Big] = \frac{1}{4}.$$

The total area is $A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

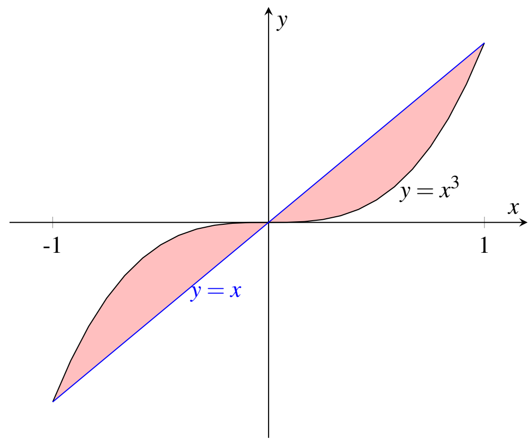


Figure 7.7

Example 7.3 Sketch the region determined by the graphs of $y = \sin\, x$, $y = \cos\, x$, $x = 0$ and $x = \frac{\pi}{4}$, then find its area.

Solution: The figure on the right shows the region bounded by the two functions. Note that over the period $[0, \frac{\pi}{4}]$, the two curves intersect at $\frac{\pi}{4}$.

Hence,

$$\begin{aligned}
\text{Area: } A &= \int_0^{\frac{\pi}{4}} \big(\cos\, x - \sin\, x\big)\, dx \\
&= \Big[\sin\, x + \cos\, x\Big]_0^{\frac{\pi}{4}} \\
&= \Big[\big(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\big) - (1)\Big] \\
&= \sqrt{2} - 1.
\end{aligned}$$

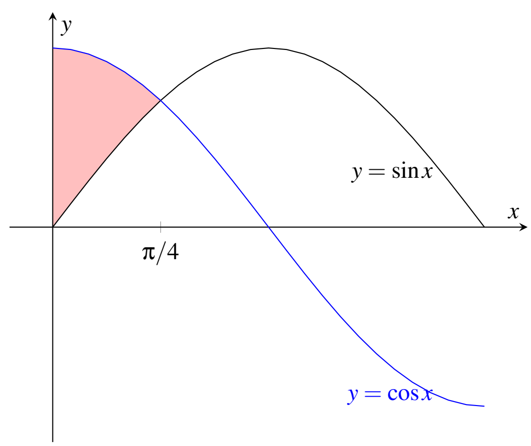


Figure 7.8

Example 7.4 Sketch the region bounded by the graph of $x = \sqrt{y}$ from $y = 0$ to $y = 1$, then find its area.

Solution: The region bounded by the function $x = \sqrt{y}$ from $y = 0$ to $y = 1$ is shown in the figure.

The area of the region is

$$\begin{aligned} A &= \int_0^1 \sqrt{y}\, dy \\ &= \frac{2}{3}\left[\, y^{3/2}\, \right]_0^1 \\ &= \frac{2}{3}. \end{aligned}$$

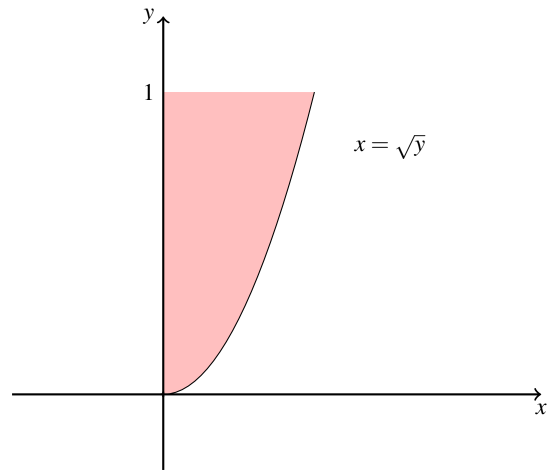


Figure 7.9

Example 7.5 Sketch the region bounded by the graphs of $x = 2y$ and $x = \frac{y}{2} + 3$, then find its area.

Solution:
First, we find the intersection points:

$$2y = \frac{y}{2} + 3 \Rightarrow 4y = y + 6 \Rightarrow y = 2.$$

The two curves intersect at $(4, 2)$.

$$\begin{aligned} \text{Area: } A &= \int_0^2 \left(\frac{y}{2} + 3 - 2y\right)\, dy \\ &= \int_0^2 \left(-\frac{3}{2}y + 3\right)\, dy = \left[-\frac{3}{4}y^2 + 3y\right]_0^2 = -3 + 6 = 3. \end{aligned}$$

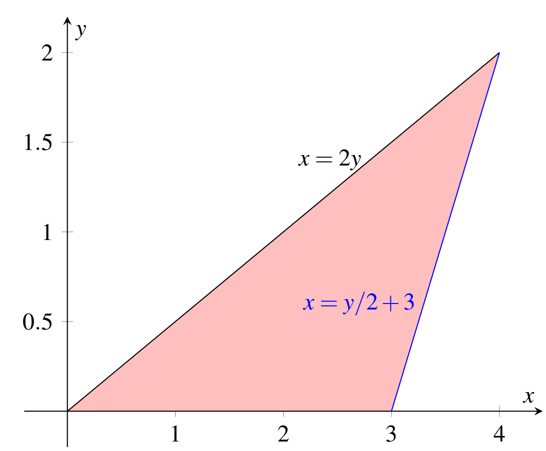


Figure 7.10

Exercise 7.1

1 - 27 Sketch the region bounded by the graphs of the equations, then find its area.

1 $y = \frac{x^2}{2},\ y = 0,\ x = 1,\ x = 3$

2 $y = x^3,\ x = 0,\ x = 2$

3 $y = x + 2,\ x = 1,\ x = 4$

4 $y = x^2 + 1,\ y = 0,\ x = 0,\ x = 2$

5 $y = x^3 + 1,\ y = 0,\ x = 0,\ x = 1$

6 $y = \sin\ x,\ x = 0,\ x = \pi$

7 $y = \tan\ x,\ x = \pi/4,\ x = \pi/3$

8 $y = -x,\ y = x + 1,\ x = 0$

9 $y = \sqrt{x},\ x + y = 2,\ y = 0$

10 $y = x^2,\ x = y - 2,\ y = 0$

11 $x = y^3,\ y = 0,\ y = 2,\ x = 0$

12 $x = \frac{y}{3},\ y = 1,\ y = 3,\ x = 0$

13 $x = (y+1)^2,\ y = 2,\ y = 5,\ x = 0$

14 $y = x^3 - 4x,\ y = 0,\ x = -2,\ x = 0$

15 $y = x^3,\ y = 2$

16 $y = x,\ y = 2x,\ y = -x + 2$

17 $y = \sqrt{x+1},\ x = 1,\ y = 0$

18 $x = y,\ x = y - 5,\ x = 0,\ x = 2$

19 $y = \sqrt{x-1},\ y = x,\ x = 1,\ x = 2$

20 $y = e^x,\ x = -2,\ x = 3$

21 $y = e^{x+1},\ x = 0,\ x = 1$

22 $y = \ln x,\ x = 1,\ x = 5$

23 $x = \sin\ y,\ y = 0,\ y = \pi/4$

24 $x = \sin\ y,\ x = \cos\ y,\ y = 0,\ y = \pi/4$

25 $y = \sin\ x,\ y = \cos\ x,\ x = -\pi/4,\ x = \pi/4$

26 $y = (x+1)^2 + 2,\ x = -2,\ x = 0$

27 $x = \ln y,\ x = 0,\ y = 1,\ y = e$

7.2 Solids of Revolution

In this section, we introduce the solids of revolution.

Definition 7.1 If R is a plane region, the solid of revolution S is a solid generated from revolving R about a line in the same plane where the line is called the axis of revolution.

In the following examples, we show some simple solids of revolution.

Example 7.6 Let $y = f(x) \geq 0$ be continuous for every $x \in [a,b]$. Let R be a region bounded by the graph of f and the x-axis from $x = a$ to $x = b$. Revolution of the region R about the x-axis generates a solid given in Figure 7.11 (right).

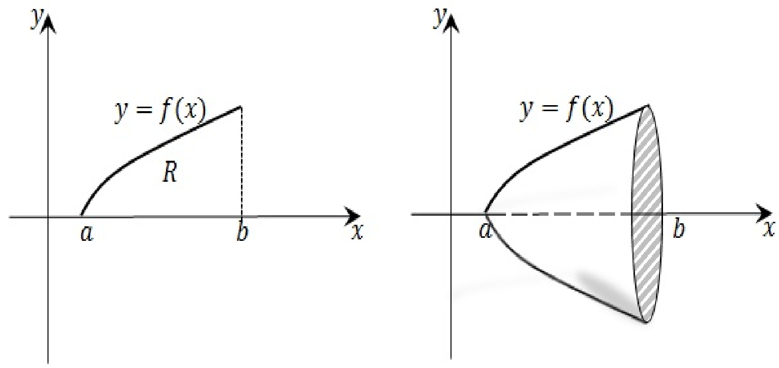


Figure 7.11: Revolution of a region about the x-axis. The figure on the left shows the region under the continuous function $y = f(x)$ on the interval $[a,b]$. The figure on the right shows the solid S generated by revolving the region about the x-axis.

Example 7.7 Let $y = f(x)$ be a constant function from $x = a$ to $x = b$, as in Figure 7.12. The region R is a rectangle and by revolving it about the x-axis, we obtain a circular cylinder.

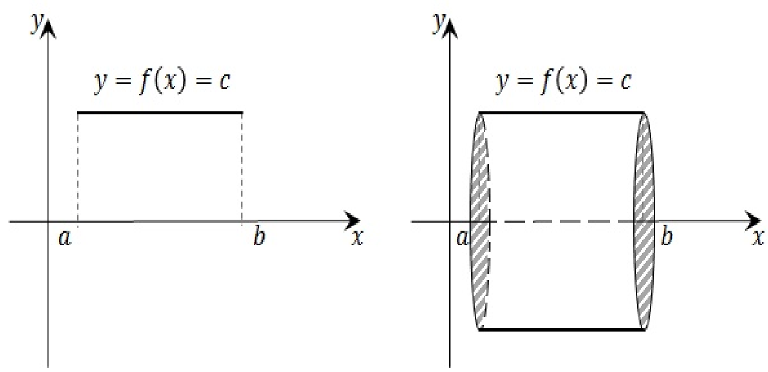


Figure 7.12: Revolution of a rectangular region about the x-axis. The figure on the left shows the region under the constant function $f(x) = c$ on the interval $[a,b]$. The figure on the right shows the circular cylinder generated by revolving the region about the x-axis.

Example 7.8 Consider the region R bounded by the graph of $x = f(y)$ from $y = c$ to $y = d$. Revolution of R about the y-axis generates a solid given in Figure 7.13.

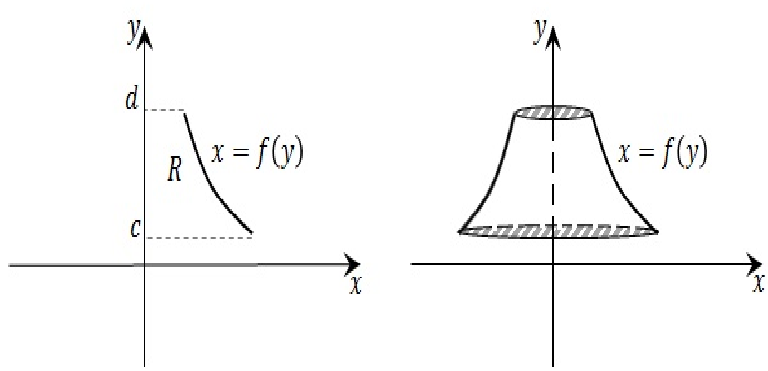


Figure 7.13: Revolution of a region about the y-axis. The figure on the left displays the region under the function $x = f(y)$ on the interval $[c,d]$. The figure on the right displays the solid S generated by revolving the region about the y-axis.

Exercise 7.2

1 - 10 ■ Sketch the region R bounded by the graphs of the given equations, then sketch the solid generated if R is revolved about the specified axis.

1	$y = x^2,\ x = 1,\ y = 4$	about the x-axis
2	$y = \sqrt{x},\ x = 0,\ x = 9$	about the x-axis
3	$y = \ln x,\ x = 0.5,\ x = e^3$	about the x-axis
4	$y = e^x,\ x = -1,\ x = 5$	about the x-axis
5	$y = \sin x,\ x = 0,\ x = \pi$	about the x-axis
6	$y = \cos y,\ y = 0,\ y = \pi/2$	about the y-axis
7	$y = e^{2x},\ y = 0,\ y = 3$	about the y-axis
8	$x = y + 1,\ y = -1,\ y = 5$	about the y-axis
9	$y = x^2,\ y = x$	about the x-axis
10	$y = \sqrt{x},\ y = x$	about the y-axis

7.3 Volumes of Solids of Revolution

One of the interesting applications of the definite integrals is to determine volumes of the revolution solids. In this section, we study three methods to evaluate the volumes of the revolution solids known as disk method, washer method and method of cylindrical shells.

7.3.1 Disk Method

Let f be continuous on $[a,b]$ and let R be the region bounded by the graph of f and the x-axis form $x = a$ to $x = b$. Let S be the solid generated by revolving R about the x-axis. Assume that P is a partition of $[a,b]$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is a mark where $\omega_k \in [x_{k-1}, x_k]$. From each subinterval $[x_{k-1}, x_k]$, we form a rectangle, its high and width are $f(\omega_k)$ and Δx_k, respectively.

The revolution of the vertical rectangle about the x-axis generates a circular disk as shown in Figure 7.15. Its radius and high are

$$r = f(\omega_k)\ , \quad h = \Delta x_k.$$

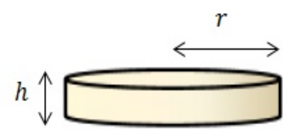


Figure 7.14

$$V = \pi r^2 h$$

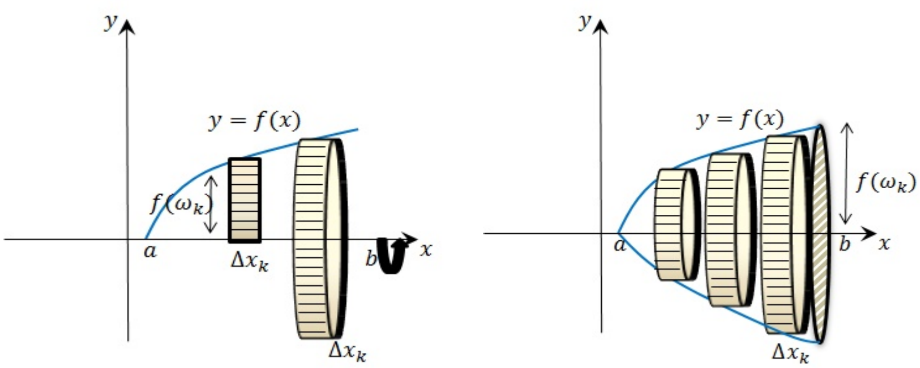


Figure 7.15: The volume by the disk method for a solid generated by revolving the region about the x-axis. The figure on the left shows the region R bounded by a function f on an interval $[a,b]$ and the figure on the right shows the solid S generated by revolving R about the x-axis.

From Figure 7.15, the volume of each circular disk is

$$V_k = \pi(f(\omega_k))^2 \Delta x_k, \ \ k = 1, 2, \ldots, n$$

The sum of volumes of the circular disks approximates the volume of the solid of revolution:

$$V = \sum_{k=1}^{n} V_k = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \pi \left(f(\omega_k) \right)^2 \Delta x_k = \pi \int_a^b \left[f(x) \right]^2 dx.$$

Similarly, we can find the volume of the solid of revolution generated by revolving the region about the y-axis. Let f be continuous on $[c,d]$ and let R be the region bounded by the graph of f and the y-axis from $y=c$ to $y=d$. Let S be the solid generated by revolving R about the y-axis. Assume that P is a partition of $[c,d]$ and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)$ is a mark where $\omega_k\in[y_{k-1},y_k]$. From each $[y_{k-1},y_k]$, we form a rectangle, its high and width are $f(\omega_k)$ and Δy_k, respectively.
The revolution of each horizontal rectangle about the y-axis generates a circular disk as shown in Figure 7.16. Its radius and high are

$$r=f(\omega_k)\ ,\quad h=\Delta y_k.$$

Therefore, the volume of each circular disk is

$$V_k=\pi(f(\omega_k))^2\Delta y_k,\ \ k=1,2,\ldots,n$$

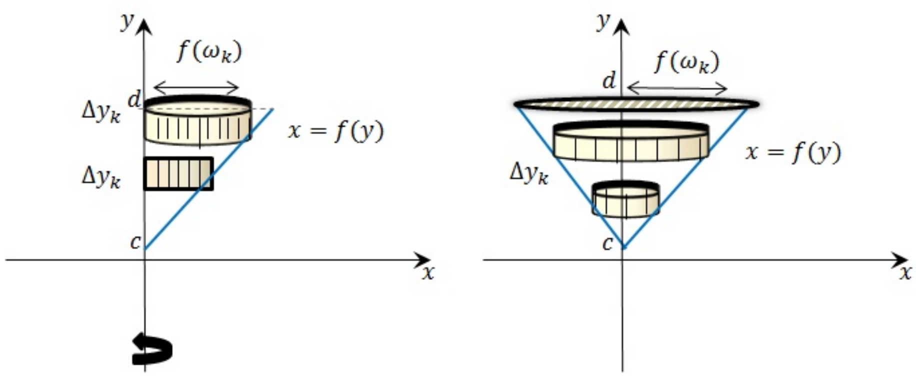


Figure 7.16: The volume by the disk method for a solid generated by revolving the region about the y-axis. The figure on the left shows the region R bounded by a function f on an interval $[c,d]$ and the figure on the right shows the solid S generated by revolving R about the y-axis.

The volume of the solid of revolution given in Figure 7.16 (right) is approximately the sum of the volumes of circular disks:

$$\begin{aligned}V=\sum_{k=1}^{n}V_k&=\lim_{\|P\|\to 0}\sum_{k=1}^{n}\pi(f(\omega_k))^2\Delta y_k\\&=\pi\int_c^d\Big[f(y)\Big]^2\,dy.\end{aligned}$$

These considerations can be summarized in the following theorem:

Theorem 7.1

1. If R is a region bounded by the graph of f on the interval $[a,b]$, the volume of the solid of revolution determined by revolving R about the x-axis is
$$V=\pi\int_a^b\Big[\,f(x)\,\Big]^2\,dx.$$
2. If R is a region bounded by the graph of f on the interval $[c,d]$, the volume of the solid of revolution determined by revolving R about the y-axis is
$$V=\pi\int_c^d\Big[\,f(y)\,\Big]^2\,dy.$$

Example 7.9 Sketch the region R bounded by the graphs of equations $y=\sqrt{x}$, $x=4$ and $y=0$. Then, find the volume of the solid generated by revolving R about the x-axis.

Solution:
The figure shows the solid generated by revolving the region R about the x-axis.

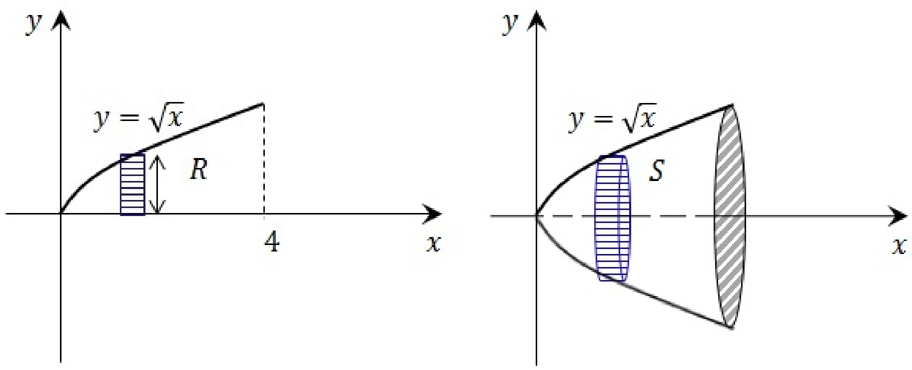


Figure 7.17

Since the revolution is about the x-axis, we have a vertical disk with radius $y=\sqrt{x}$ and thickness dx. Thus, the volume of the solid S is

$$V=\pi\int_0^4(\sqrt{x})^2\,dx=\pi\int_0^4 x\,dx=\frac{\pi}{2}\Big[x^2\Big]_0^4=\frac{\pi}{2}\Big[16-0\Big]=8\pi.$$

Example 7.10 Sketch the region R bounded by the graphs of equations $y=e^x$, $y=e$ and $x=0$. Then, find the volume of the solid generated by revolving R about the y-axis.

Solution:

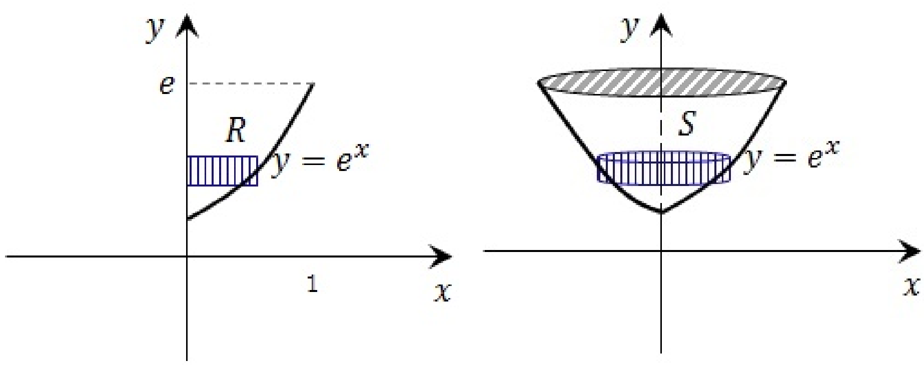


Figure 7.18

The figure shows the region R and the solid S generated by revolving the region about the y-axis. Since the revolution is about the y-axis, then we need to rewrite the function to become $x=f(y)$.

$$y=e^x\Rightarrow\ln y=\ln e^x\Rightarrow x=\ln y=f(y).$$

Now, we have a horizontal disk with radius $x=\ln y$ and thickness dy. Thus, the volume of the solid S is

$$V=\pi\int_1^e(\ln y)^2\,dy=\Big[2y+y\,(\ln y)^2-2y\,\ln y\Big]_1^e=e-2.$$

(use the integration by parts to evaluate the integral $\int(\ln y)^2\,dy$)

Example 7.11 Sketch the region R bounded by the graph of the equation $x=y^2$ on the interval $[0,1]$. Then, find the volume of the solid generated by revolving R about the y-axis.

Solution:

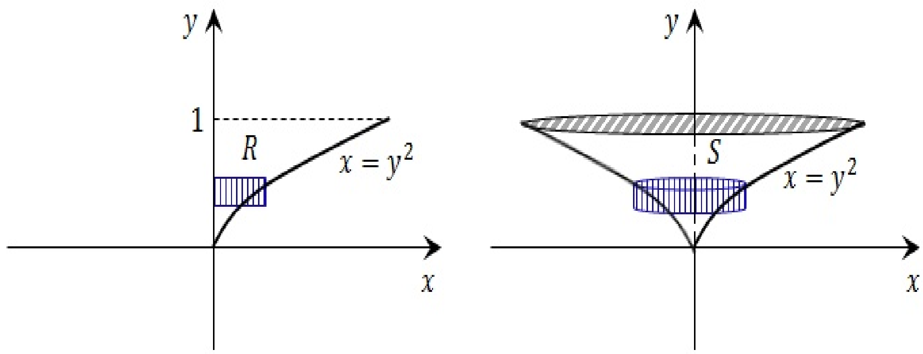


Figure 7.19

Since the revolution of R is about the y-axis, we have a horizontal disk with radius $x = y^2$ and thickness dy. Thus, the volume of the solid S is

$$V = \pi \int_0^1 (y^2)^2 \, dy = \frac{\pi}{5}\Big[\, y^5 \,\Big]_0^1 = \frac{\pi}{5}\Big[1 - 0\Big] = \frac{\pi}{5}.$$

Example 7.12 Sketch the region R bounded by the graph of the equation $y = \cos\ x$ from $x = 0$ to $x = \frac{\pi}{2}$. Then, find the volume of the solid generated by revolving R about the x-axis.

Solution:

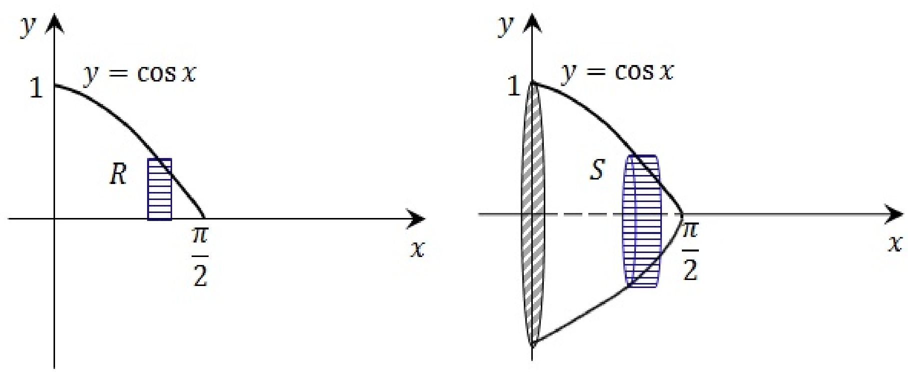


Figure 7.20

The figure shows the region R and the solid S generated by revolving the region about the x-axis. Thus, the disk to evaluate the volume of the generated solid S is vertical where the radius is $y = \cos\ x$ and the thickness is dx. Hence,

$$V = \pi \int_0^{\frac{\pi}{2}} \cos^2\ x \, dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 + \cos\ 2x)\ dx = \frac{\pi}{2}\left[x + \frac{\sin\ 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{4}.$$

7.3.2 Washer Method

The washer method is a generalization of the disk method for a region between two functions f and g. Let R be a region bounded by the graphs of f and g from $x = a$ to $x = b$ such that $f \geq g$ on $[a,b]$ as shown in Figure 7.21). The volume of the solid S generated by revolving the region R about the x-axis can be found by calculating the difference between the volumes of the two solids generated by revolving the regions under f and g about the x-axis as follows:
the outer radius: $y_1 = f(x)$
the inner radius: $y_2 = g(x)$
the thickness: dx
The volume of a washer is $dV = \pi\Big[(\text{the outer radius})^2 - (\text{the inner radius})^2\Big].$ thickness.
This implies $dV = \pi\Big[(f(x))^2 - (g(x))^2\Big]dx.$

Hence, the volume of the solid over the period $[a,b]$ is

$$V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx.$$

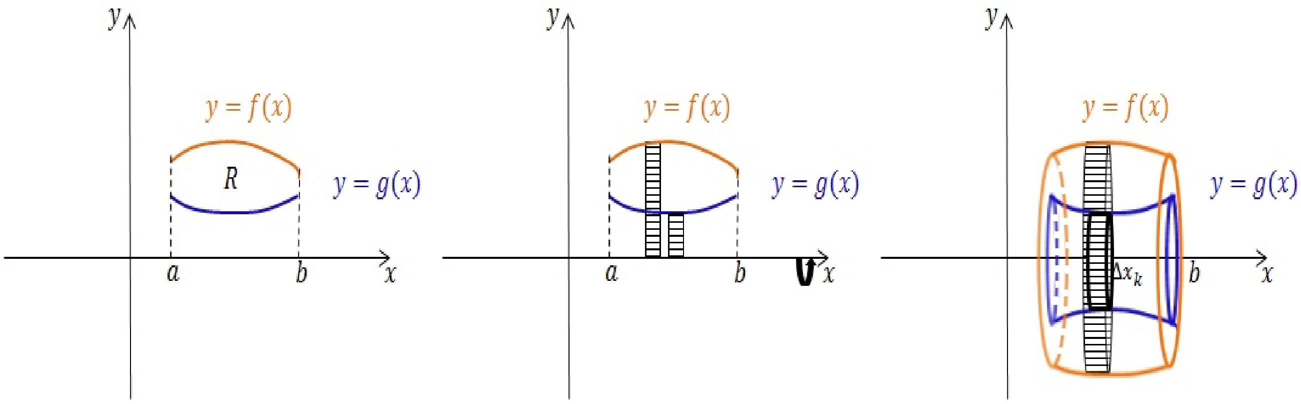


Figure 7.21: The volume by the washer method for a solid generated by revolving the region R about the x-axis.

Similarly, let R be a region bounded by the graphs of f and g such that $f \geq g$ on $[c,d]$ as shown in Figure 7.22. The volume of the solid S generated by revolving R about the y-axis is

$$V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2 \right] dy.$$

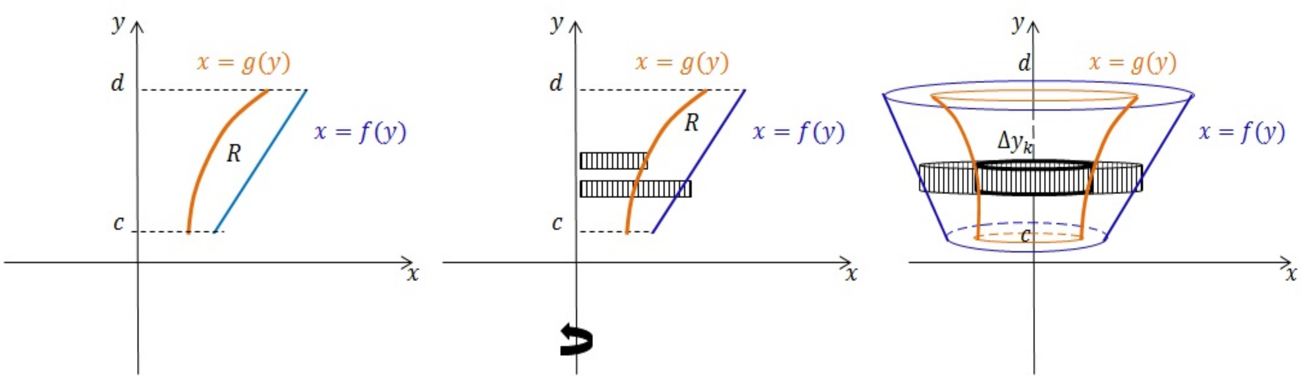


Figure 7.22: The volume by the washer method for a solid generated by revolving the region R about the y-axis.

Theorem 7.2 summarizes the washer method.

Theorem 7.2

1. If R is a region bounded by the graphs of f and g on the interval $[a,b]$ such that $f \geq g$, the volume of the solid of revolution determined by revolving R about the x-axis is

$$V = \pi \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx.$$

2. If R is a region bounded by the graphs of f and g on the interval $[c,d]$ such that $f \geq g$, the volume of the solid of revolution determined by revolving R about the y-axis is

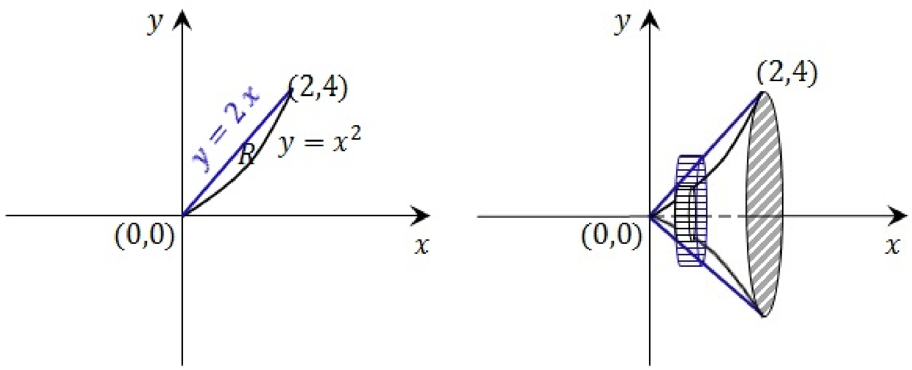
$$V = \pi \int_c^d \left[(f(y))^2 - (g(y))^2 \right] dy.$$

Example 7.13 Let R be a region bounded by the graphs of the functions $y = x^2$ and $y = 2x$. Evaluate the volume of the solid generated by revolving R about the x-axis.

Solution:
Let $f(x) = x^2$ and $g(x) = 2x$. First, we find the intersection points:

$$\begin{aligned} f(x) = g(x) &\Rightarrow x^2 = 2x \\ &\Rightarrow x^2 - 2x = 0 \\ &\Rightarrow x(x-2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 2. \end{aligned}$$

Substituting $x = 0$ into $f(x)$ or $g(x)$ gives $y = 0$. Similarly, if we substitute $x = 2$ into the two functions, we have $y = 2$. Thus, the two curves intersect in two points $(0,0)$ and $(2,4)$.



The figure shows the region R and the solid generated by revolving R about the x-axis. A vertical rectangle generates a washer where
the outer radius: $y_1 = 2x$,
the inner radius: $y_2 = x^2$ and
the thickness: dx.
The volume of the washer is

$$dV = \pi\left[2x - x^2\right]\, dx.$$

Thus, the volume of the solid over the interval $[0,2]$ is

$$\begin{aligned} V = \pi \int_0^2 \left((2x)^2 - (x^2)^2\right)\, dx = \pi \int_0^2 (4x^2 - x^4)\, dx \\ &= \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 \\ &= \pi\left[\frac{32}{3} - \frac{32}{5}\right] \\ &= \frac{64}{15}\pi. \end{aligned}$$

Example 7.14 Consider a region R bounded by the graphs of the functions $y = \sqrt{x}$, $y = 6 - x$ and the x-axis. Revolve this region about the y-axis and find the volume of the generated solid.

Solution:
Since the revolution is about the y-axis, we need to rewrite the functions in terms of y i.e., $x = f(y)$ and $x = g(y)$.

$$y = \sqrt{x} \Rightarrow x = y^2 = f(y)$$

$$y = 6 - x \Rightarrow x = 6 - y = g(y).$$

Now, we find the intersection points:

$$f(y) = g(y) \Rightarrow y^2 = 6 - y \Rightarrow y^2 + y - 6 = 0 \Rightarrow y = -3 \text{ or } y = 2.$$

Since $y = \sqrt{x}$, we ignore the value $y = -3$. By substituting $y = 2$ into the two functions, we have $x = 4$. Thus, the two curves intersect in one point $(4,2)$. The solid S generated by revolving R about the y-axis is shown in the figure.
Since the revolution is about the y-axis, then we have a horizontal rectangle that generates a washer where
the outer radius: $x_1 = 6 - y$,
the inner radius: $x_2 = y^2$ and
the thickness: dy.
The volume of the washer is

$$dV = \pi\big[(6-y)^2 - (y^2)^2\big]\ dy.$$

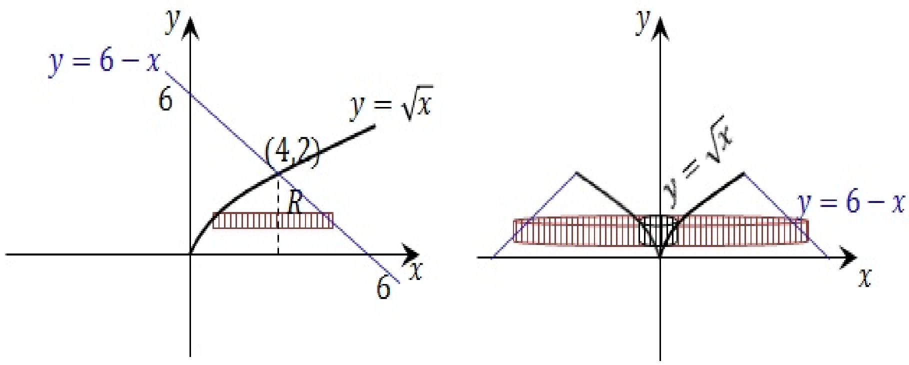


Figure 7.23

The volume of the solid over the interval $[0,2]$ is

$$\begin{aligned} V = \pi\int_0^2 \big[(6-y)^2 - (y^2)^2\big]\ dy &= \pi\Big[-\frac{(6-y)^3}{3} - \frac{y^5}{5}\Big]_0^2 \\ &= \pi\Big[\big(-\frac{64}{3} - \frac{32}{5}\big) - \big(-\frac{216}{3} - 0\big)\Big] \\ &= \frac{664}{15}\pi. \end{aligned}$$

Example 7.15 Consider the same region as in Example 7.14 enclosed by the graphs of $y = \sqrt{x}$, $y = 6 - x$ and the x-axis. Revolve this region about the x-axis instead and find the volume of the generated solid.

Solution:
From the figure, we find that the solid is made up of two separate regions and each requires its own integral. Meaning that, we use the disk method to evaluate the volume of the solid generated by revolving each curve.

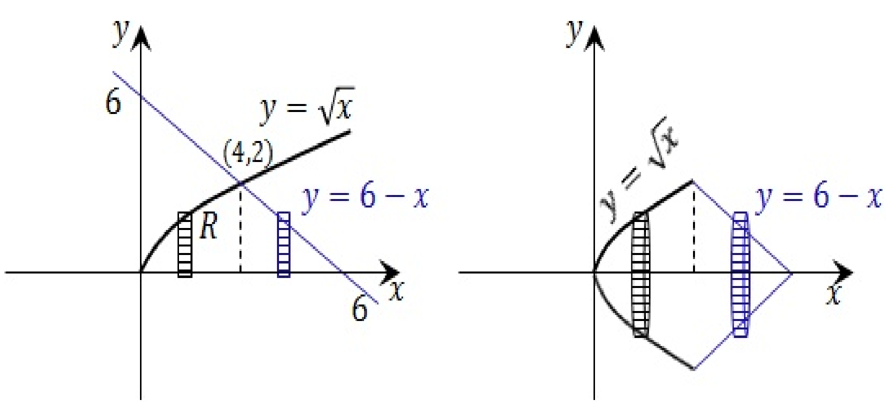


Figure 7.24

$$
\begin{aligned}
V &= \pi\int_0^4 (\sqrt{x})^2\,dx + \pi\int_4^6 (6-x)^2\,dx \\
&= \pi\int_0^4 x\,dx + \pi\int_4^6 (6-x)^2\,dx \\
&= \frac{\pi}{2}\Big[x^2\Big]_0^4 - \frac{\pi}{3}\Big[(6-x)^3\Big]_4^6 \\
&= \frac{\pi}{2}(16-0) - \frac{\pi}{3}(0-8) \\
&= \frac{32}{3}\pi.
\end{aligned}
$$

(we used the substitution method to do the second integral with $u = 6-x$ and $du = dx$)

The revolution of a region is not always about the x-axis or the y-axis. It could be about a line paralleled to the x-axis or the y-axis. If the axis of revolution is a line $y = y_0$, evaluating the volume of the generated solid is similar to the case when the region revolves about the x-axis. Whereas, if the axis of revolution is a line $x = x_0$, evaluating the volume of the generated solid is similar to the case when the region revolves about the y-axis.

Example 7.16 Let R is a region bounded by graphs of the functions $y = x^2$ and $y = 4$. Evaluate the volume of the solid generated by revolving R about the given line.
(a) $y = 4$ **(b)** $x = 2$

Solution:
(a) We have a vertical circular disk:
the radius of the disk: $4 - y = 4 - x^2$, and
the thickness: dx.

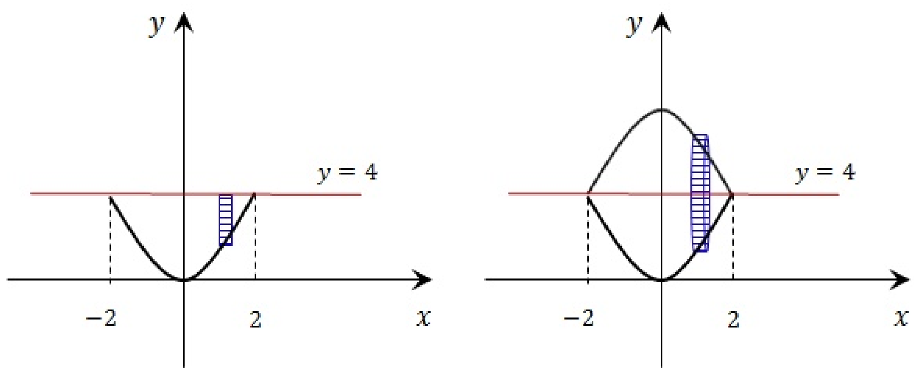


Figure 7.25

The volume of the disk is

$$dV = \pi(4-x^2)^2\,dx.$$

The volume of the solid over the interval $[-2,2]$ is

$$
\begin{aligned}
V = \pi\int_{-2}^2 (4-x^2)^2\,dx &= \pi\int_{-2}^2 (16-8x^2+x^4)\,dx \\
&= \pi\Big[16x - \frac{8x^3}{3} + \frac{x^5}{5}\Big]_{-2}^2 \\
&= \frac{512}{15}\pi.
\end{aligned}
$$

(b) In this case, a horizontal rectangle will generate a washer where
the outer radius: $2+\sqrt{y}$,
the inner radius: $2-\sqrt{y}$ and
the thickness: dy.

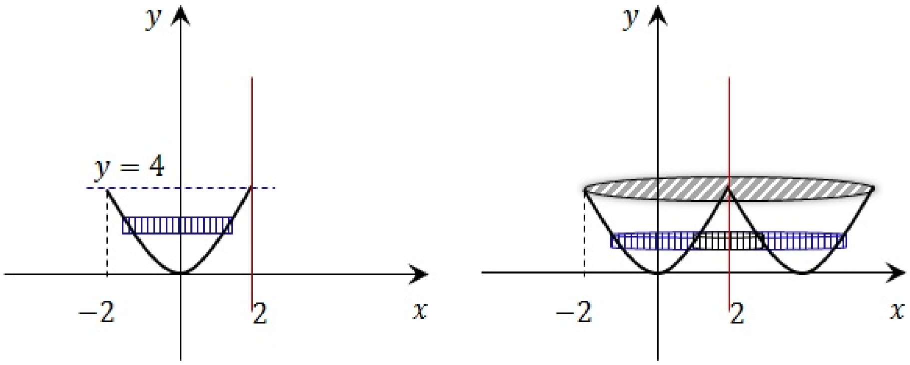


Figure 7.26

The volume of the washer is

$$dV = \pi\left[(2+\sqrt{y})^2 - (2-\sqrt{y})^2\right]\, dy = 8\pi\sqrt{y}\, dy.$$

The volume of the solid over the interval $[0,4]$ is

$$V = 8\pi \int_0^4 \sqrt{y}\, dx = \frac{16\pi}{3}\left[y^{\frac{3}{2}}\right]_0^4 = \frac{128}{3}\pi.$$

Example 7.17 Sketch the region R bounded by graphs of the equations $x=(y-1)^2$ and $x=y+1$. Then, find the volume of the solid generated by revolving R about $x=4$.

Solution:
First, we find the intersection points:

$$\begin{aligned}(y-1)^2 = y+1 &\Rightarrow y^2-2y+1 = y+1\\ &\Rightarrow y^2-3y=0\\ &\Rightarrow y=0 \text{ or } y=3.\end{aligned}$$

Thus, the two curves intersect in two points $(1,0)$ and $(4,3)$.

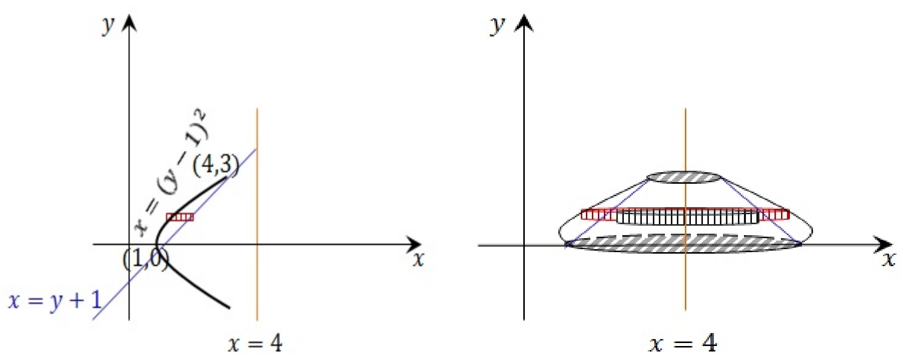


Figure 7.27

The figure shows the region R and the solid S. A horizontal rectangle generates a washer where
the outer radius: $4-(y-1)^2$,
the inner radius: $4-(y+1) = 3-y$ and
the thickness: dy.
The volume of the washer is

$$dV = \pi\left[(4-(y-1)^2)^2 - (3-y)^2\right]\, dy = \pi\left[16-8(y-1)^2+(y-1)^4-(3-y)^2\right]\, dy.$$

Thus, the volume of the solid over the interval $[0,3]$ is

$$\begin{aligned} V &= \pi\Big(\int_0^3 16\,dy - 8\int_0^3 (y-1)^2\,dy + \int_0^3 (y-1)^4\,dy - \int_0^3 (3-y)^2\,dy\Big) \\ &= \pi\Big[16y - \frac{8(y-1)^3}{3} + \frac{(y-1)^5}{5} + \frac{(3-y)^3}{3}\Big]_0^3 \\ &= \frac{108}{5}\pi. \end{aligned}$$

7.3.3 Method of Cylindrical Shells

In the washer method, we assume that the rectangle from each subinterval is vertical to the axis of the revolution while in the method of cylindrical shells, the rectangle is parallel to the axis of the revolution.

As shown in figure, let
r_1 be the inner radius of the shell,
r_2 be the outer radius of the shell,
h be high of the shell,
$\Delta r = r_2 - r_1$ be the thickness of the shell,
$r = \frac{r_1+r_2}{2}$ be the average radius of the shell.

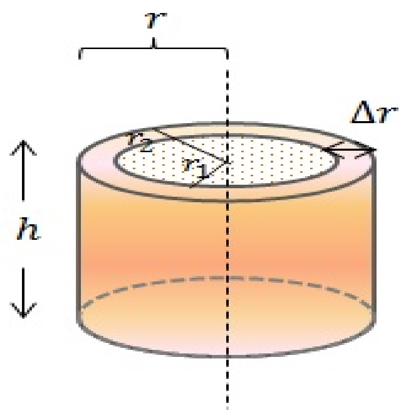


Figure 7.28

The volume of the cylindrical shell is

$$\begin{aligned} V &= \pi r_2^2 h - \pi r_1^2 h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi\Big(\frac{r_2 + r_1}{2}\Big)h(r_2 - r_1) \\ &= 2\pi r h \Delta r. \end{aligned}$$

$$V = \underbrace{V_2}_{\text{the outer cylinder}} - \underbrace{V_1}_{\text{the inner cylinder}}$$

Now, consider the graph shown in Figure 7.29 (A). The revolution of the region R about the y-axis generates a solid given in (B) of the same figure. Let P be a partition of the interval $[a,b]$ and let $\omega = (\omega_1, \omega_2, ..., \omega_n)$ be a mark on P where ω_k is the midpoint of $[x_{k-1}, x_k]$.

The revolution of the rectangle about the y-axis generates a cylindrical shell where
the high = $f(\omega_k)$,
the average radius = ω_k and
the thickness = Δx_k.
Hence, the volume of the cylindrical shell is $V_k = 2\pi\omega_k f(\omega_k)\Delta x_k$. To evaluate the volume of the whole solid, we sum the volumes of all cylindrical shells. This implies

$$V = \sum_{k=1}^{n} V_k = 2\pi \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta x_k.$$

From the Riemann sum

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \omega_k f(\omega_k)\Delta x_k = \int_a^b x f(x)\,dx$$

and this implies

$$V = 2\pi \int_a^b x f(x)\,dx.$$

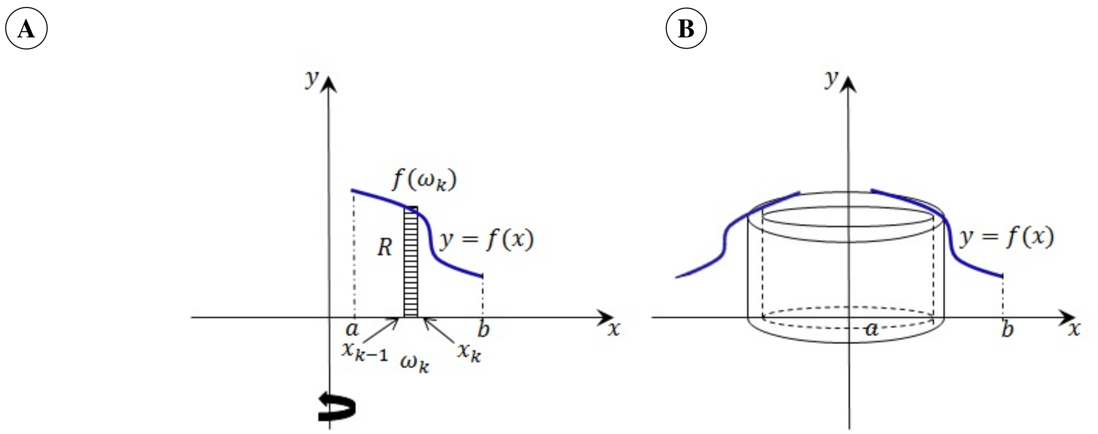


Figure 7.29: The volume by the method of cylindrical shells for a solid generated by revolving a region about the y-axis.

Similarly, if the revolution of the region is about the x-axis, the volume of the solid of revolution is

$$V = 2\pi \int_c^d y f(y)\, dy.$$

Theorem 7.3

1. If R is a region bounded by the graph of f on the interval $[a,b]$, the volume of the solid of revolution determined by revolving R about the y-axis is
$$V = 2\pi \int_a^b x f(x)\, dx.$$
2. If R is a region bounded by the graph of f on the interval $[a,b]$, the volume of the solid of revolution determined by revolving R about the x-axis is
$$V = 2\pi \int_c^d y f(y)\, dy.$$

The method of cylindrical shells is sometimes easier than the washer method. This is because solving equations for one variable in terms of another is not always simple (i.e., solving x in terms of y). For example, for the volume of the solid obtained by revolving the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis, by the washer method, we would have to solve the cubic equation for x in terms of y, but this is not simple.

Example 7.18 Sketch the region R bounded by graphs of the equations $y = 2x - x^2$ and $x = 0$. Then, by the method of the cylindrical shells, find the volume of the solid generated by revolving R about the y-axis.

Solution: The figure shows the region R and the solid S generated by revolving R about the y-axis.

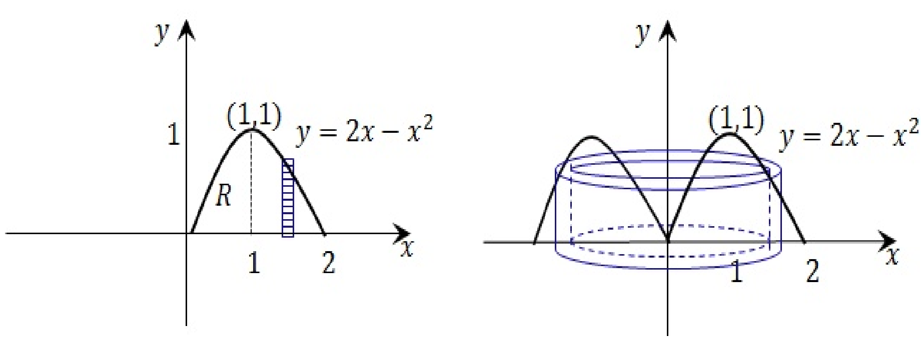


Figure 7.30

Since the revolution is about the y-axis, the rectangle is vertical and by revolving it, we obtain a cylindrical shell where the high: $y = 2x - x^2$,

the average radius: x,
the thickness: dx.
The volume of the cylindrical shell is

$$dV = 2\pi x(2x - x^2)\ dx = 2\pi(2x^2 - x^3)\ dx.$$

Thus, the volume of the solid over the interval $[0,2]$ is

$$\begin{aligned} V &= 2\pi \int_0^2 (2x^2 - x^3)\ dx \\ &= 2\pi \Big[\frac{2x^3}{3} - \frac{x^4}{4}\Big]_0^2 \\ &= 2\pi\Big(\frac{16}{3} - \frac{16}{4}\Big) = \frac{8\pi}{3}. \end{aligned}$$

Example 7.19 Sketch the region R bounded by graphs of the equations $x = \sqrt{y}$ and $x = 2$, and the y-axis. Then, find the volume of the solid generated by revolving R about the x-axis.

Solution:

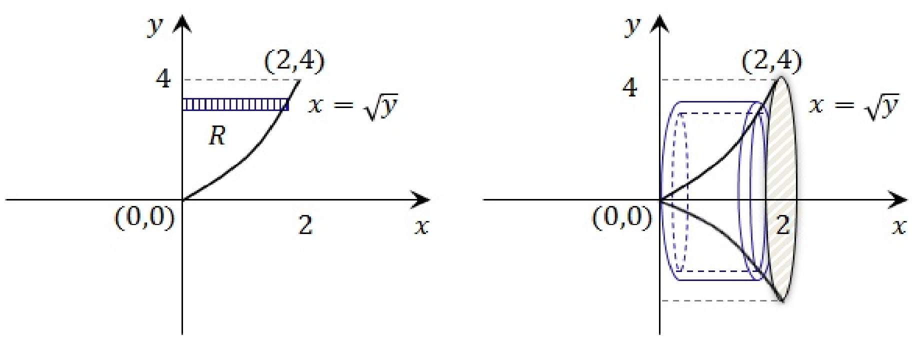


Figure 7.31

Since the revolution is about the x-axis, the rectangle is horizontal and by revolving it, we have a cylindrical shell where
the high: $x = \sqrt{y}$,
the average radius: y
the thickness: dy.
The volume of the cylindrical shell is $dV = 2\pi\, y\, \sqrt{y}\, dy$.
Thus, the volume of the solid over the interval $[0,4]$ is

$$\begin{aligned} V = 2\pi \int_0^4 y\sqrt{y}\ dy &= 2\pi \int_0^4 y^{\frac{3}{2}}\ dy \\ &= \frac{4\pi}{5}\Big[y^{\frac{5}{2}}\Big]_0^4 \\ &= \frac{4\pi}{5}\Big[32 - 0\Big] = \frac{128\pi}{5}. \end{aligned}$$

Exercise 7.3

1 - 8 Sketch the region R bounded by the graphs of the given equations and find the volume of the solid generated by revolving R about the x-axis.

1 $y = x+1, x = 0, x = 1$
2 $y = x^2+1,\ x = 0,\ x = 2$
3 $y = x^3,\ x = 0,\ x = 2$
4 $y = \sqrt{x},\ x = 0,\ x = 4$
5 $y = \sqrt{x},\ x = y$
6 $y = \sin x,\ x = 0,\ x = \pi/2$
7 $y = 1-x^2,\ y = x^2$
8 $y = x^3+1,\ y = x+1$

9 - 16 Sketch the region R bounded by the graphs of the given equations and find the volume of the solid generated by revolving R about the y-axis.

9 $y = x^2,\ y = 1,\ y = 4$
10 $y = \sqrt{x},\ y = 0,\ y = 3$
11 $x = \cos y,\ y = 0,\ y = \pi/2$
12 $x = \ln y,\ y = 1,\ y = e$
13 $y = x,\ y = (x-1)^2+1$
14 $y = e^x,\ x = 1,\ x = 2,\ y = 0$
15 $xy = 4,\ x+y = 5$
16 $y = x^2,\ y^2 = 8x$

17 - 26 Set up and evaluate an integral for the volume of the solid obtained by revolving the region bounded by the given curves about the specified axis or line.

17 $y = x^2,\ y = 1,$ line $x = 1$
18 $y = x^2,\ y = 1,$ x-axis
19 $y = x^2,\ x = y^2$ line $y = -1$
20 $y = \sqrt{x-1},\ y = 0,\ x = 5$ line $x = 5$
21 $y = x^2,\ x = 0,\ y = 1,\ y = 4$ line $y = 1$
22 $y = x-x^2,\ y = 0$ line $x = 2$
23 $y = x^2,\ y = 0,\ x = 1,\ x = 2$ line $x = 1$
24 $y = x^2,\ y = 0,\ x = 1,\ x = 2$ line $x = 4$
25 $y = \sqrt{x-1},\ y = 0,\ x = 5$ line $y = 3$
26 $y = x^4,\ y = \sin\frac{\pi x}{2}$ line $x = -1$

27 - 35 Sketch the region R bounded by graphs of the given equations. Then, by method of the cylindrical shells, find the volume of the solid generated by revolving R about the specified axis or line.

27 $x = 1+y^2,\ x = 0,\ y = 1,\ y = 2$ x-axis
28 $x = \sqrt{y},\ x = 0,\ y = 1$ x-axis
29 $y = x^3,\ y = 8,\ x = 0$ x-axis
30 $y = \frac{1}{x},\ x = 1,\ x = 2$ y-axis
31 $y = x^2,\ y = 0,\ x = 1$ y-axis
32 $y = x^2,\ y = x$ x-axis
33 $y = \sin x,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{4}$ y-axis
34 $y = x^2+x,\ y = 0$ y-axis
35 $y = x+\frac{4}{x},\ y = 5$ line $x = -1$

7.4 Arc Length and Surfaces of Revolution

In this section, we present two other applications of the definite integrals. We use the definite integrals to evaluate the lengths of arcs of functions and areas of surfaces of revolution. We restrict our attention to smooth functions (they have derivatives of all orders in their domains).

7.4.1 Arc Length

Let $y=f(x)$ be a smooth function on $[a,b]$. Assume that $P=\{x_0,x_1,...,x_n\}$ is a regular partition of the interval $[a,b]$ and let $P_0,P_1,...,P_n$ be points on the curve as shown in Figure 7.32.
The distance between any two points of the curve is

$$\begin{aligned}
d(P_{k-1},P_k) &= \sqrt{(\Delta x_k)^2+(\Delta y_k)^2}\\
&= \sqrt{(\Delta x_k)^2+\big(f(x_k)-f(x_{k-1})\big)^2}\\
&= \Delta x_k\sqrt{1+\frac{\big(f(x_k)-f(x_{k-1})\big)^2}{(\Delta x_k)^2}}\\
&= \frac{b-a}{n}\sqrt{1+\Big[\frac{f(x_k)-f(x_{k-1})}{\Delta x_k}\Big]^2}
\end{aligned}$$

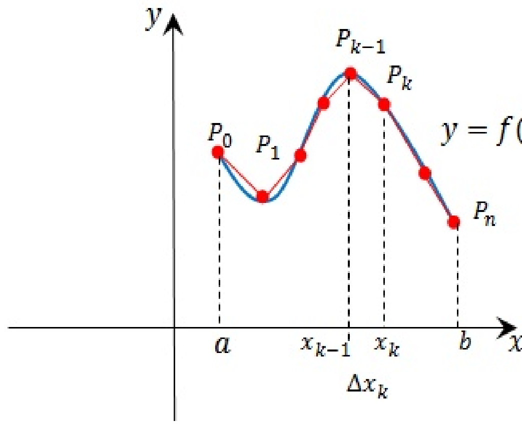


Figure 7.32: The length of the arc of $y=f(x)$ from $(a,f(a))$ to $(b,f(b))$.

From the mean value theorem of differential calculus for the function f on $[x_{k-1},x_k]$, we have

$$f'(c_i)=\frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$$

for some $c_i\in(x_{k-1},x_k)$. Thus, the distance between P_{k-1} and P_k is

$$d(P_{k-1},P_k)=\frac{b-a}{n}\sqrt{1+[f'(c_i)]^2}.$$

The sum of all these distances is

$$\frac{b-a}{n}\Big[\sqrt{1+[f'(c_1)]^2}+\sqrt{1+[f'(c_2)]^2}+...+\sqrt{1+[f'(c_n)]^2}\Big].$$

The previous sum is a Riemann sum for the function $\sqrt{1+[f'(x_k)]^2}$ from a to b where for a better approximation, we let n be large enough. Thus, the arc length of the function f is

$$L(f)=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$$

Similarly, let $x=g(y)$ be a smooth function on $[c,d]$. The length of the arc of the function g from $(g(c),c)$ to $(g(d),d)$ is

$$L(g)=\int_c^d\sqrt{1+[g'(y)]^2}\,dy.$$

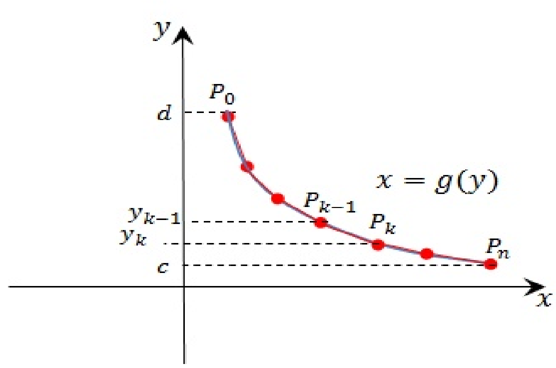


Figure 7.33: The length of the arc of $x=g(y)$ from $(g(c),c)$ to $(g(d),d)$.

Theorem 7.4

1. Let $y = f(x)$ be a smooth function on $[a,b]$. The length of the arc of f from $(a, f(a))$ to $(b, f(b))$ is
$$L(f) = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\, dx.$$
2. Let $x = g(y)$ be a smooth function on $[c,d]$. The length of the arc of g from $(g(c), c)$ to $(g(d), d)$ is
$$L(g) = \int_c^d \sqrt{1+\left[g'(y)\right]^2}\, dy.$$

Example 7.20 Find the arc length of the graph of the given equation from A to B.

(1) $y = 5-\sqrt{x^3}$; $\quad A(0,5),\ B(4,-3)$

(2) $x = 4y$; $\quad A(0,0),\ B(4,1)$

Solution:

(1)
$$\begin{aligned} y = f(x) = 5-\sqrt{x^3} &\Rightarrow f'(x) = -\frac{3}{2}x^{\frac{1}{2}} \\ &\Rightarrow (f'(x))^2 = \frac{9}{4}x \\ &\Rightarrow 1+(f'(x))^2 = \frac{4+9x}{4} \\ &\Rightarrow \sqrt{1+(f'(x))^2} = \frac{\sqrt{4+9x}}{2}. \end{aligned}$$

The length of the curve is
$$\begin{aligned} L(f) = \frac{1}{2}\int_0^4 \sqrt{4+9x}\, dx &= \frac{1}{27}\left[(4+9x)^{\frac{3}{2}}\right]_0^4 \\ &= \frac{1}{27}\left[40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] \\ &= \frac{8}{27}\left[10\sqrt{10}-1\right]. \end{aligned}$$

(2)
$$\begin{aligned} x = g(y) = 4y &\Rightarrow g'(y) = 4 \\ &\Rightarrow (g'(y))^2 = 16 \\ &\Rightarrow 1+(g'(y))^2 = 17 \\ &\Rightarrow \sqrt{1+(g'(y))^2} = \sqrt{17}. \end{aligned}$$

The length of the curve is
$$\begin{aligned} L(g) = \sqrt{17}\int_0^1 dy &= \sqrt{17}\Big[\, y \,\Big]_0^1 \\ &= \sqrt{17}\Big[1-0\Big] = \sqrt{17}. \end{aligned}$$

Example 7.21 Find the arc length of the graph of the given equation over the indicated interval.

(1) $y = \cosh\, x$; $\quad 0 \le x \le 2$

(2) $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$; $\quad -2 \le y \le -1$

Solution:

(1)
$$\begin{aligned} y = f(x) = \cosh x &\Rightarrow f'(x) = \sinh\, x \\ &\Rightarrow (f'(x))^2 = \sinh^2\, x \\ &\Rightarrow 1+(f'(x))^2 = 1+\sinh^2\, x = \cosh^2\, x \\ &\Rightarrow \sqrt{1+(f'(x))^2} = \cosh\, x. \end{aligned}$$

The length of the curve is

$$L(f) = \int_0^2 \cosh x \, dx = \Big[\sinh x\Big]_0^2 = \sinh 2 - \sinh 0 = \sinh 2.$$

$(\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0)$

(2) $x = g(y) = \frac{1}{8}y^4 + \frac{1}{4}y^{-2} \Rightarrow g'(y) = \frac{1}{2}(y^3 - \frac{1}{y^3})$

$$\Rightarrow (g'(y))^2 = \frac{(y^6-1)^2}{4y^6}$$

$$\Rightarrow 1 + (g'(y))^2 = \frac{4y^6 + y^{12} - 2y^6 + 1}{4y^6}$$

$$\Rightarrow 1 + (g'(y))^2 = \frac{y^{12} + 2y^6 + 1}{4y^6}$$

$$\Rightarrow \sqrt{1 + (g'(y))^2} = \sqrt{\frac{(y^6+1)^2}{4y^6}} = \frac{y^6+1}{2y^3}.$$

Since $y < 0$ over $[-2,-1]$, the length of the curve is

$$L(g) = -\frac{1}{2}\int_{-2}^{-1} (y^3 + y^{-3}) \, dy = -\frac{1}{2}\Big[\frac{y^4}{4} - \frac{1}{2y^2}\Big]_{-2}^{-1} = \frac{33}{16}.$$

7.4.2 Surfaces of Revolution

In Section 7.2, we assume that the bounded region revolves about an axis or a line and this process generates a solid. In this section, we assume that only the curve revolves about an axis. This generates a surface called surface of revolution (see Figure 7.34). We show how the definite integral is applied to calculate the area of that surface.

Definition 7.2 Let f is a continuous function on $[a,b]$. The surface of revolution is generated by revolving the graph of the function f about an axis.

Let $y = f(x) \geq 0$ be a smooth function on the interval $[a,b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[a,b]$ and $P_0, P_1, \ldots, P_n$ be the points on the curve as shown in Figure 7.34. Let D_k be a frustum of a cone generated by revolving the line segment $P_{k-1}P_k$ about the x-axis with radii $f(x_{k-1})$ and $f(x_k)$. Since area of the frustum of a cone with radii r_1 and r_2 and slant length ℓ is $S.A = \pi(r_1 + r_2)\ell$, then

$$S.A(D_k) = \pi[f(x_k) + f(x_{k-1})]\Delta\ell_k$$

where $\Delta\ell_k$ is the distance between P_{k-1} and P_k i.e., $\Delta\ell_k = \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2}$.
From the intermediate value theorem, there exists $\omega_k \in (x_{k-1}, x_k)$ such that

$$f(x_k) - f(x_{k-1}) = f'(\omega_k)\Delta x_k.$$

This implies $\Delta\ell_k = \Delta x_k\sqrt{1 + [f'(\omega_k)]^2}$.
For n large, $f(x_k) \approx f(x_{k-1}) \approx f(\omega_k)$ and this implies

$$S.A = \sum_{k=1}^{n} 2\pi f(\omega_k)\sqrt{1 + [f'(\omega_k)]^2}\Delta x_k.$$

From the Riemann sum,

$$S.A = \lim_{\|P\|\to 0}\sum_{k=1}^{n} 2\pi f(\omega_k)\sqrt{1 + [f'(\omega_k)]^2}\Delta x_k = 2\pi\int_a^b |\, f(x)\,| \sqrt{1 + [f'(x)]^2} \, dx = 2\pi\int_a^b |\, y \,| \sqrt{1 + (\frac{dy}{dx})^2} \, dx.$$

If the revolution is about the y-axis, then

$$S.A = 2\pi\int_a^b |\, x\,| \sqrt{1 + [f'(x)]^2} \, dx = 2\pi\int_a^b |\, x\,| \sqrt{1 + (\frac{dy}{dx})^2} \, dx.$$

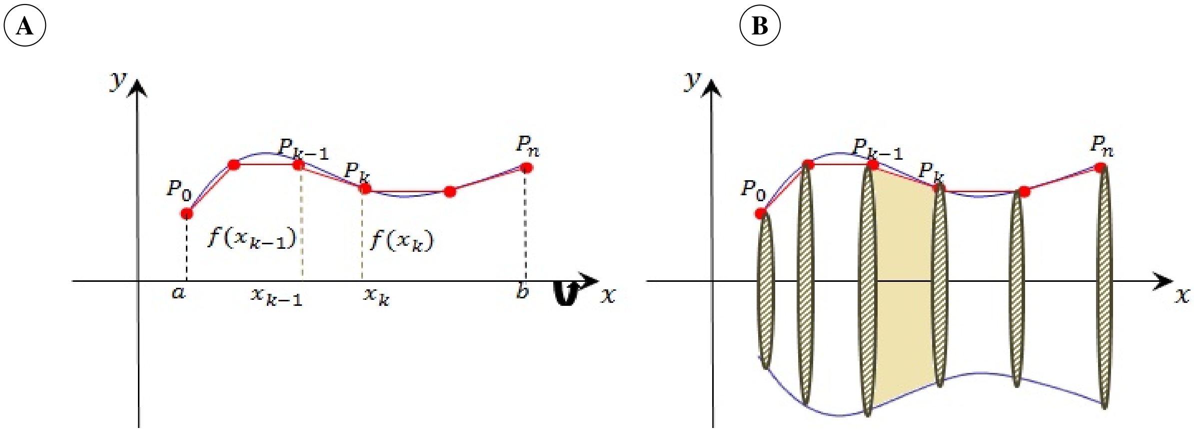


Figure 7.34: The revolution surface generated by revolving the graph of a continuous function about the x-axis.

Similarly, if $x = g(y)$ is a smooth function on $[c,d]$, then the surface area $S.A$ generated by revolving the graph of g about the y-axis from $y = c$ to $y = d$ is

$$S.A = 2\pi \int_c^d |\, g(y) \,| \sqrt{1 + [g'(y)]^2}\, dy = 2\pi \int_c^d |\, x \,| \sqrt{1 + (\frac{dx}{dy})^2}\, dy.$$

If the revolution is about the x-axis, then

$$S.A = 2\pi \int_c^d |\, y \,| \sqrt{1 + [g'(y)]^2}\, dy = 2\pi \int_c^d |\, y \,| \sqrt{1 + (\frac{dx}{dy})^2}\, dy.$$

Theorem 7.5

1. Let $y = f(x)$ be a smooth function on $[a,b]$.

- If the revolution is about the x-axis, the surface area of revolution is

$$S.A = 2\pi \int_a^b |\, y \,| \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

- If the revolution is about the y-axis, the surface are of revolution is

$$S.A = 2\pi \int_a^b |\, x \,| \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

2. Let $x = g(y)$ be a smooth function on $[c,d]$.

- If the revolution is about the y-axis, the surface area of revolution is

$$S.A = 2\pi \int_c^d |\, x \,| \sqrt{1 + \left(g'(y)\right)^2}\, dy.$$

- If the revolution is about the x-axis, the surface area of revolution is

$$S.A = 2\pi \int_c^d |\, y \,| \sqrt{1 + \left(g'(y)\right)^2}\, dy.$$

Note that the absolute value is for the case when the function is negative for some values in the closed interval.

Example 7.22 Find the surface area generated by revolving the graph of the function $\sqrt{4 - x^2}$, $-2 \le x \le 2$ about the x-axis.

Solution:

We apply the formula $S.A = 2\pi \int_a^b |\,y\,| \sqrt{1+(f'(x))^2}\, dx$.

$$\begin{aligned} y = \sqrt{4-x^2} &\Rightarrow f'(x) = \frac{-2x}{2\sqrt{4-x^2}} \\ &\Rightarrow (f'(x))^2 = \frac{x^2}{4-x^2} \\ &\Rightarrow 1+(f'(x))^2 = \frac{4}{4-x^2} \\ &\Rightarrow \sqrt{1+(f'(x))^2} = \frac{2}{\sqrt{4-x^2}}. \end{aligned}$$

The area of the revolution surface is $S.A = 2\pi \int_{-2}^{2} \sqrt{4-x^2}\, \frac{2}{\sqrt{4-x^2}}\, dx = 4\pi\big[2+2\big] = 16\pi$.

Example 7.23 Find the surface area generated by revolving the graph of the function $y = 2x$, $0 \le x \le 3$ about the y-axis.

Solution:
We apply the formula $S.A = 2\pi \int_a^b |\,x\,| \sqrt{1+(f'(x))^2}\, dx$.

$$\begin{aligned} y = 2x &\Rightarrow f'(x) = 2 \\ &\Rightarrow (f'(x))^2 = 4 \\ &\Rightarrow 1+(f'(x))^2 = 5 \\ &\Rightarrow \sqrt{1+(f'(x))^2} = \sqrt{5}. \end{aligned}$$

The area of the revolution surface is $S.A = 2\pi \int_0^3 |\,x\,|\, \sqrt{5}\, dx = \sqrt{5}\pi\Big[\, x^2 \,\Big]_0^3 = 9\sqrt{5}\pi$.

Example 7.24 Find the surface area generated by revolving the graph of the function $x = y^3$ on the interval $[0,1]$ about the y-axis.

Solution:
We apply the formula $S.A = 2\pi \int_c^d |\,x\,| \sqrt{1+(g'(y))^2}\, dy$.

$$\begin{aligned} x = y^3 &\Rightarrow g'(y) = 3y^2 \\ &\Rightarrow (g'(y))^2 = 9y^4 \\ &\Rightarrow 1+(g'(y))^2 = 1+9y^4 \\ &\Rightarrow \sqrt{1+(g'(y))^2} = \sqrt{1+9y^4}. \end{aligned}$$

The area of the revolution surface is $S.A = 2\pi \int_0^1 y^3 \sqrt{1+9y^4}\, dy = \frac{\pi}{27}\Big[(1+9y^4)^{\frac{3}{2}}\Big]_0^1 = \frac{\pi}{27}\Big[10\sqrt{10}-1\Big]$.

Exercise 7.4

1 - 13 Find the arc length of the graph of the given equation over the indicated interval.

1 $y = \ln x,\ 1 \le x \le 3$

2 $y = e^x,\ 0 \le x \le 1$

3 $y = x^2 + 1,\ 1 \le x \le 3$

4 $y = \sqrt{x},\ 1 \le x \le 4$

5 $y = \frac{1}{2}x^2,\ 0 \le x \le 1$

6 $y = \ln(\cos\ x),\ \pi/4 \le x \le \pi/3$

7 $x = \frac{2}{3}(y-1)^{\frac{3}{2}},\ 1 \le y \le 2$

8 $x = \sqrt{4 - y^2},\ 0 \le y \le 1$

9 $x = 4 - 2y,\ 0 \le y \le 2$

10 $x = \cosh\ y,\ 1 \le y \le 3$

11 $x = \frac{y^2}{3},\ 1 \le y \le 4$

12 $x = y^2,\ 0 \le y \le 1$

13 $x = \ln(\sec\ y),\ 0 \le y \le \frac{\pi}{4}$

14 - 24 Find the area of the surface generated by revolving the curve about the specified axis.

14 $y = \sqrt{4 - x^2},\ -1 \le x \le 1$ x-axis

15 $y = x^2,\ 1 \le x \le 2$ y-axis

16 $y = e^x,\ 0 \le x \le 1$ x-axis

17 $y = \ln x,\ 1 \le x \le 3$ y-axis

18 $y = \sin\ x,\ 0 \le x \le \pi/2$ x-axis

19 $x = e^y,\ 1 \le y \le 2$ y-axis

20 $9x = y + 18,\ 0 \le x \le 2$ x-axis

21 $y = x^3,\ 0 \le x \le 2$ x-axis

22 $y = \cos\ 2x,\ 0 \le x \le \pi/6$ x-axis

23 $y = \sqrt[3]{x},\ 1 \le y \le 2$ y-axis

24 $y = 1 - x^2,\ 0 \le x \le 1$ y-axis

Review Exercises

1 - 23 ■ Sketch the region bounded by the graphs of the given equations, then find its area.

1 $y=x^2,\ y=5$

2 $y=x^3,\ x=0,\ y=0,\ x=2$

3 $x=y^3,\ x=0,\ y=1$

4 $x=y^2,\ x=0,\ y=-1,\ y=2$

5 $x^2+y^2=4,$

6 $x=2y,\ y+6=2x,\ y=0$

7 $y=x^2,\ y=\sqrt{x}$

8 $y=x^3,\ y=-x,\ y=8$

9 $x=y^3-y,\ x=0$

10 $y=x,\ x=2-y,\ x=0$

11 $y=x,\ y=x-5,\ x=0,\ y=2$

12 $x=y^2,\ y=x+1,\ y=1,\ y=2$

13 $y=\sin\ x,\ y=\cos\ x,\ x=0,\ x=\frac{\pi}{4}$

14 $y=e^x,\ x=0,\ x=\ln 4$

15 $y=x,\ y=4x,\ y=-x+2$

16 $y=e^{-x}, x=-1,\ x=2$

17 $y=\sin\ x,\ y=\cos\ x,\ x=0,\ x=\frac{\pi}{2}$

18 $y=\cos\ 2x,\ y=0,\ x=\frac{\pi}{4},\ x=\frac{\pi}{2}$

19 $y=\sin\ x,\ x=\frac{-\pi}{4},\ x=\frac{\pi}{2}$

20 $y=\sec\ x,\ y=0,\ x=\frac{-\pi}{4},\ x=\frac{\pi}{4}$

21 $y=\ln x,\ y=0,\ x=\ln 3$

22 $y^2-x^2=1,\ x=-1,\ x=1$

23 $y=\tan\ x,\ y=0,\ x=0,\ x=\frac{\pi}{4}$

24 - 26 ■ Sketch the region bounded by the graphs of the given equations.

24 $x=y^2,\ y-x=2,\ y=-2,\ y=1.$

25 $y=x^2-4,\ y=x+2.$

26 $y=x^2,\ y=-x^2,\ y=-2,\ y=2.$

27 - 39 ■ Sketch the region R bounded by the graphs of the given equations and find the volume of the solid generated by revolving R about the x-axis.

27 $y=\frac{1}{x},\ x=1,\ x=3,\ y=0$

28 $y=x^2,\ y=4-x^2$

29 $y=x,\ x+y=4,\ x=0$

30 $y=x^2,\ y=1-x^2$

31 $y=x^2,\ y=9$

32 $y=x^2,\ y=x$

33 $y=x^2,\ y=x^3$

34 $y=1+x^3,\ x=1,\ x=2,\ y=0$

35 $y=x^2-4x,\ y=0$

36 $y=e^x,\ x=0,\ x=2$

37 $y=\ln x,\ x=1,\ x=4$

38 $y=\sin\ x,\ x=0,\ x=\frac{\pi}{2},\ y=0$

39 $y=\sin\ x,\ y=\cos\ x,\ x=0,\ x=\frac{\pi}{8}$

40 - 52 ■ Sketch the region R bounded by the graphs of the given equations and find the volume of the solid generated by revolving R about the y-axis.

40 $y=x^2,\ x=0,\ y=4$

41 $y=x^3,\ x=0,\ x=1$

42 $x=y^2,\ x=2y$

43 $x=y^2,\ y=x-2$

44 $y^2=1-x,\ x=0$

45 $y=x^2-1,\ x=0,\ y=3$

46 $y=\cos\ x,\ x=0,\ x=\frac{\pi}{2}$

47 $y=\cos\ x,\ y=\sin\ x,\ x=0,\ x=\frac{\pi}{4}$

48 $(x-2)^2+y=1,\ y=0$

49 $y=1-x^2,\ y=1-x$

50 $y=x^2+1,\ x=0,\ x=1$

51 $y=6-3x,\ x=0,\ y=0$

52 $y=9-x^2,\ x=2,\ x=3,\ y=0$

53 - 65 ■ Find the arc length of the graph of the given equation over the indicated interval.

53 $y=x^2,\ 0\le x\le 2$

54 $y=x+1,\ 0\le x\le 4$

55 $y=x^{\frac{3}{2}},\ 1\le x\le 2$

56 $y=2(x-1)^{\frac{3}{2}},\ 1\le x\le 5$

57 $y=x^{\frac{2}{3}},\ 2\le x\le 4$

58 $y=\frac{2}{3}(x^2+1)^{\frac{3}{2}},\ 1\le x\le 4$

59 $y=\frac{2}{3}x^{\frac{3}{2}},\ 0\le x\le 1$

60 $y=\frac{x^3}{6}+\frac{1}{2x},\ 1\le x\le 3$

61 $y=e^{-x},\ 1\le x\le \ln 4$

62 $y=5-2x^{\frac{3}{2}},\ 0\le x\le 11$

63 $y=\ln x,\ 2\le x\le 4$

64 $y=\ln\sec\ x,\ 0\le x\le \frac{\pi}{4}$

65 $y=\sqrt{9-x^2},\ 0\le x\le 4$

66 - 71 Find the area of the surface generated by revolving the given curve about the x-axis.

66 $y=2x,\ 1\le x\le 2$

67 $y=\sqrt{4-x^2},\ 0\le x\le 4$

68 $y=\frac{1}{3}x^3,\ 0\le x\le 3$

69 $y=\sqrt{x},\ 0\le x\le 1$

70 $y=e^x,\ 0\le x\le 1$

71 $y=\cos\ x,\ 0\le x\le \frac{\pi}{2}$

72 - 77 Find the area of the surface generated by revolving the given curve about the y-axis.

72 $x=y^2,\ 0\le y\le 3$

73 $x=\sqrt{1-y^2},\ 0\le y\le 4$

74 $y=\frac{1}{2}x^2,\ 0\le x\le 3$

75 $x=\sqrt{a-y^2},\ 0\le y\le \frac{a}{2}$

76 $y=1-x^2,\ 0\le x\le 1$

77 $x=\sin\ y,\ 0\le y\le \pi$

78 - 84 Choose the correct answer.

78 The area of the region bounded by the graphs of the functions $y=x^2$ and $y=2-x^2$ is equal to
(a) 2 (b) 4 (c) $\frac{3}{8}$ (d) $\frac{8}{3}$

79 The area of the region bounded by the graphs of the functions $y=x$ and $y=-x$ and $y=1$ is equal to
(a) 1 (b) 0 (c) 2 (d) $\frac{1}{2}$

80 The area of the region bounded by the graphs of the functions $y=2x$ and $y=x$ and $0\le x\le 1$ is equal to
(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) 2 (d) $\frac{1}{3}$

81 The arc length of the graph of $y=4x$ from $A(0,0)$ to $B(1,4)$ is equal to
(a) $\sqrt{17}$ (b) $\sqrt{5}$ (c) $4\sqrt{17}$ (d) $4\sqrt{5}$

82 The area of the region bounded by the graphs of the functions $x=-y^2$ and $x=-1$ is equal to

(a) $\frac{4}{3}$ (b) $\frac{1}{9}$ (c) $\frac{1}{6}$ (d) $\frac{8}{3}$

83 The area of the region bounded by the graphs of the functions $y = \cos\, x,\ y = \sin\, x,\ x = 0$ and $x = \frac{\pi}{4}$ is equal to
(a) $\sqrt{2} - 1$ (b) 0 (c) $\sqrt{2} + 1$ (d) $1 - \sqrt{2}$

84 The area of the region bounded by the graphs of the functions $x = y^2$ and $x = 2 - y^2$ is equal to
(a) $\frac{1}{3}$ (b) 8 (c) 1 (d) $\frac{8}{3}$

Chapter 8

Parametric Equations and Polar Coordinates

8.1 Parametric Equations of Plane Curves

In this section, rather than considering only function $y = f(x)$, it is sometimes convenient to view both x and y as functions of a third variable t (called a parameter).

Definition 8.1 A plane curve is a set of ordered pairs $(f(t), g(t))$, where f and g are continuous on an interval I.

If we are given a curve C, we can express it in a parametric form $x(t) = f(t)$ and $y(t) = g(t)$. The resulting equations are called parametric equations. Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a parametric curve.

Definition 8.2 Let C be a curve consists of all ordered pairs $\big(f(t), g(t)\big)$, where f and g are continuous on an interval I. The equations

$$x = f(t),\ y = g(t)\ \text{ for } t \in I$$

are parametric equations for C with parameter t.

Example 8.1 Consider the plane curve C given by $y = x^2$.

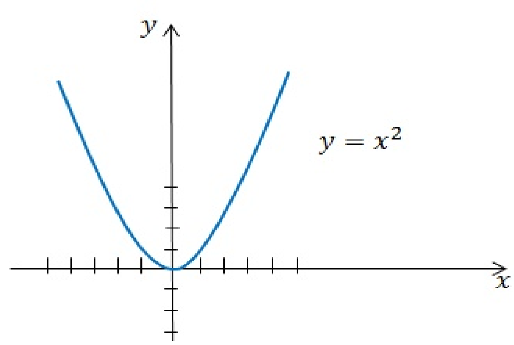


Figure 8.1

If we consider the interval $-1 \leq x \leq 2$, then we have

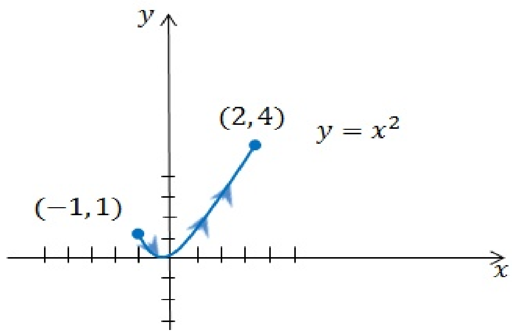


Figure 8.2

Now, let $x = t$ and $y = t^2$ for $-1 \le t \le 2$. We have the same graph where the last equations are called parametric equations for the curve C.

Remark 8.1

1. The parametric equations give the same graph of $y = f(x)$.
2. To find the parametric equations, we introduce a third variable t. Then, we rewrite x and y as functions of t.
3. The parametric equations give the orientation of the curve C indicated by arrows and determined by increasing values of the parameter as shown in Figure 8.2.

Example 8.2 Write the curve given by $x(t) = 2t+1$ and $y(t) = 4t^2 - 9$ as $y = f(x)$.

Solution:
Since $x = 2t+1$, then $t = (x-1)/2$. This implies

$$y = 4t^2 - 9 = 4\left(\frac{x-1}{2}\right)^2 - 9 \Rightarrow y = x^2 - 2x - 8.$$

Example 8.3 Sketch and identify the curve defined by the parametric equations

$$x = 5\cos\, t, \quad y = 2\sin\, t, \quad 0 \le t \le 2\pi.$$

Solution:
First, find the equation in x and y. Since $x = 5\cos\, t$ and $y = 2\sin\, t$, then $\cos\, t = x/5$ and $\sin\, t = y/2$.

By using the identity $\cos^2\, t + \sin^2\, t = 1$, we have

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Thus, the curve is an ellipse.

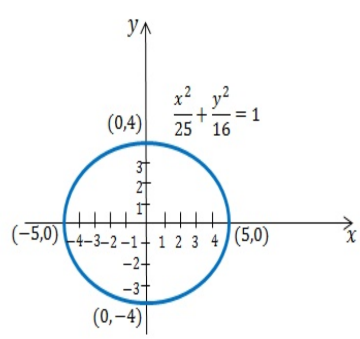


Figure 8.3

Example 8.4 The curve C is given parametrically. Find an equation in x and y, then sketch the graph and indicate the orientation.

(1) $x = \sin\, t,\ y = \cos\, t,\quad 0 \le t \le 2\pi.$

(2) $x = t^2,\ y = 2\ln t,\quad t \ge 1.$

Solution:

(1) By using the identity $\cos^2 t + \sin^2 t = 1$, we obtain

$$x^2 + y^2 = 1.$$

Therefore, the curve is a circle.

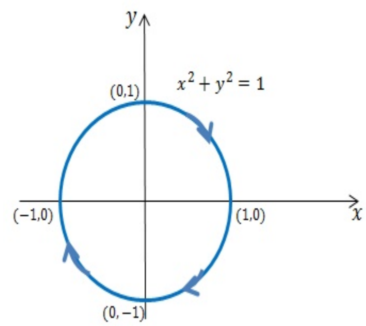


Figure 8.4

The orientation can be indicated as follows:

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	0	1	0	-1	0
y	1	0	-1	0	1
(x,y)	$(0,1)$	$(1,0)$	$(0,-1)$	$(-1,0)$	$(0,1)$

As shown in Figure 8.5, the orientation is indicated by arrows.

(2) Since $y = 2\ln t = \ln t^2$, then $y = \ln x$.

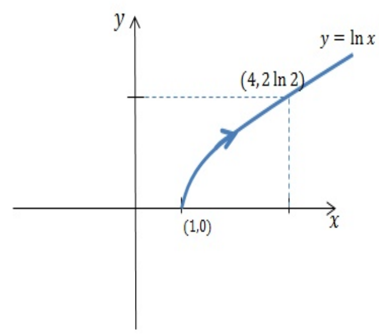


Figure 8.5

The orientation of the curve C for $t \geq 1$:

t	1	2	3
x	1	4	9
y	0	$2\ln 2$	$2\ln 3$
(x,y)	$(1,0)$	$(4,2\ln 2)$	$(9,2\ln 3)$

The orientation of the curve C is determined by increasing values of the parameter t.

8.1.1 Tangent Lines

Suppose that f and g are differentiable functions. We want to find the tangent line to a smooth curve C given by the parametric equations $x = f(t)$ and $y = g(t)$ where y is a differentiable function of x. From the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}.$$

If $dx/dt \neq 0$, we can solve for dy/dx to have the tangent line to the curve C:

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ if } \frac{dx}{dt} \neq 0$$

Remark 8.2
- If $dy/dt = 0$ such that $dx/dt \neq 0$, the curve has a horizontal tangent line.
- If $dx/dt = 0$ such that $dy/dt \neq 0$, the curve has a vertical tangent line.

Example 8.5 Find the slope of the tangent line to the curve at the indicated value.
(1) $x = t+1, y = t^2+3t$; at $t = -1$
(2) $x = t^3-3t, y = t^2-5t-1$; at $t = 2$
(3) $x = \sin t, y = \cos t$; at $t = \frac{\pi}{4}$

Solution:
(1) The slope of the tangent line at $P(x,y)$ is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+3}{1} = 2t+3.$$

The slope of the tangent line at $t = -1$ is 1.

(2) The slope of the tangent line is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-5}{3t^2-3}.$$

The slope of the tangent line at $t = 2$ is $\frac{-1}{9}$.

(3) The slope of the tangent line is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t} = -\tan t.$$

The slope of the tangent line at $t = \frac{\pi}{4}$ is -1.

Example 8.6 Find the equations of the tangent line and the vertical tangent line at $t = 2$ to the curve C given parametrically $x = 2t, \ y = t^2-1$.

Solution:
The slope of the tangent line at $P(x,y)$ is

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t.$$

The slope of the tangent line at $t = 2$ is $m = 2$. Thus, the slope of the vertical tangent line is $\frac{-1}{m} = \frac{-1}{2}$.
At $t = 2$, we have $(x_0, y_0) = (4,3)$. Therefore, the tangent line is

$$y-3 = 2(x-4)$$

Point-Slope form: $y - y_0 = m(x-x_0)$

and the vertical tangent line is

$$y-3 = -\frac{1}{2}(x-4).$$

Example 8.7 Find the points on the curve C at which the tangent line is either horizontal or vertical.
(1) $x = 1-t, y = t^2$.
(2) $x = t^3-4t, y = t^2-4$.

Solution:
(1) The slope of the tangent line is $m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{-1} = -2t$.

For the horizontal tangent line, the slope $m = 0$. This implies $-2t = 0$ and then, $t = 0$. At this value, we have $x = 1$ and $y = 0$. Thus, the graph of C has a horizontal tangent line at the point $(1,0)$.

For the vertical tangent line, the slope $\frac{-1}{m}=0$. This implies $\frac{1}{2t}=0$, but this equation cannot be solved i.e., we cannot find values for t to satisfy $\frac{1}{2t}=0$. Therefore, there are no vertical tangent lines.

(2) The slope of the tangent line is $m=\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2t}{3t^2-4}$.

For the horizontal tangent line, the slope $m=0$. This implies $\frac{2t}{3t^2-4}=0$ and this is acquired if $t=0$. At $t=0$, we have $x=0$ and $y=-4$. Thus, the graph of C has a horizontal tangent line at the point $(0,-4)$.

For the vertical tangent line, the slope $\frac{-1}{m}=0$. This implies $\frac{-3t^2+4}{2t}=0$ and this is acquired if $t=\pm\frac{2}{\sqrt{3}}$. At $t=\frac{2}{\sqrt{3}}$, we obtain $x=-\frac{16}{3\sqrt{3}}$ and $y=-\frac{8}{3}$. At $t=-\frac{2}{\sqrt{3}}$, we obtain $x=\frac{16}{3\sqrt{3}}$ and $y=-\frac{8}{3}$. Thus, the graph of C has vertical tangent lines at the points $(-\frac{16}{3\sqrt{3}},-\frac{8}{3})$ and $(\frac{16}{3\sqrt{3}},-\frac{8}{3})$.

Let the curve C has the parametric equations $x=f(t)$, $y=g(t)$ where f and g are differentiable functions. To find the second derivative $\frac{d^2y}{dx^2}$, we use the formula:

$$\boxed{\frac{d^2y}{dx^2}=\frac{d(y')}{dx}=\frac{dy'/dt}{dx/dt}}$$

Note that $\frac{d^2y}{dx^2}\neq=\frac{d^2y/dt^2}{d^2x/dt^2}$.

Example 8.8 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the indicated value.

(1) $x=t$, $y=t^2-1$ at $t=1$.

(2) $x=\sin t$, $y=\cos t$ at $t=\frac{\pi}{3}$.

Solution:

(1) $\frac{dy}{dt}=2t$ and $\frac{dx}{dt}=1$. Hence, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=2t$, then at $t=1$, we have $\frac{dy}{dx}=2(1)=2$.

The second derivative is $\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=2$.

(2) $\frac{dy}{dt}=-\sin t$ and $\frac{dx}{dt}=\cos t$. Thus, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=-\tan t$, then at $t=\frac{\pi}{3}$, we have $\frac{dy}{dx}=-\sqrt{3}$.

The second derivative is $\frac{d^2y}{dx^2}=\frac{dy'/dt}{dx/dt}=\frac{-\sec^2 t}{\cos t}=-\sec^3 t$. At $t=\frac{\pi}{3}$, we have $\frac{d^2y}{dx^2}=-8$

8.1.2 Arc Length and Surface Area of Revolution

Let C be a smooth curve has the parametric equations $x=f(t)$, $y=g(t)$ where $a\le t\le b$. Assume that the curve C does not intersect itself and f' and g' are continuous.

Let $P=\{t_0,t_1,t_2,...,t_n\}$ is a partition of the interval $[a,b]$. Let $P_k=(x(t_k),y(t_k))$ be a point on C corresponding to t_k. If $d(P_{k-1},P_k)$ is the length of the line segment $P_{k-1}P_k$, then the length of the line given in Figure 8.6 is

$$L_p=\sum_{k=1}^{n} d(P_{k-1},P_k)$$

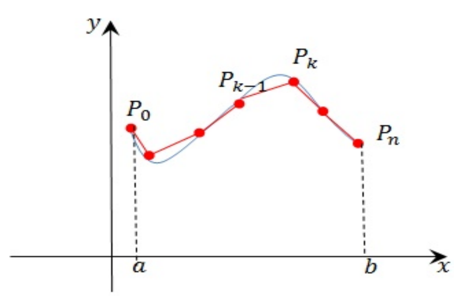


Figure 8.6

In the previous chapter, we found that $L=\lim\limits_{||P||\to 0} L_p$. From the distance formula,

$$d(P_{k-1},P_k)=\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}$$

Therefore, the length of the arc from $t=a$ to $t=b$ is approximately

$$L\approx\lim_{||P||\to 0}\sum_{k=1}^{n}\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}=\lim_{||P||\to 0}\sum_{k=1}^{n}\sqrt{(\Delta x_k/\Delta t_k)^2+(\Delta y_k/\Delta t_k)^2}\Delta t_k$$

From the mean value theorem, there exists numbers $w_k, z_k \in (t_{k-1}, t_k)$ such that

$$\frac{\Delta x_k}{\Delta t_k} = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} = f'(w_k), \quad \frac{\Delta y_k}{\Delta t_k} = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}} = g'(z_k)$$

By substitution, we obtain

$$L \approx \lim_{||P|| \to 0} \sum_{k=1}^{n} \sqrt{\left[f'(w_k)\right]^2 + \left[g'(w_k)\right]^2}$$

If $w_k = z_k$ for every k, then we have Riemann sums for $\sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}$. The limit of these sums is

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\,.$$

In the following, we determine a formula to evaluate the surface area of revolution of parametric curves. Let the curve C has the parametric equations $x = f(t)$, $y = g(t)$ where $a \leq t \leq b$ and f' and g' are continuous. Let the curve C does not intersect itself, except possibly at the point corresponding to $t = a$ and $t = b$. If $g(t) \geq 0$ throughout $[a,b]$, then the area of the revolution surface generated by revolving C about the x-axis is

$$S.A = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_a^b g(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$$

Similarly, if the revolution is about the y-axis such that $f(t) \geq 0$ over $[a,b]$, the area of the revolution surface is

$$S.A = 2\pi \int_a^b f(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$$

Theorem 8.1 Let C be a smooth curve has the parametric equations $x = f(t)$, $y = g(t)$ where $a \leq t \leq b$, and f' and g' are continuous. Assume that the curve C does not intersect itself, except possibly at the point corresponding to $t = a$ and $t = b$.

1. The arc length of the curve is
$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$$
2. If $y \geq 0$ over $[a,b]$, the surface area of revolution generated by revolving C about the x-axis is
$$S.A = 2\pi \int_a^b y\, \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt\ ,$$
3. If $x \geq 0$ over $[a,b]$, the surface area of revolution generated by revolving C about the y-axis is
$$S.A = 2\pi \int_a^b x\, \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt.$$

Example 8.9 Find the arc length of the curve $x = e^t \cos\, t,\ \ y = e^t \sin\, t,\ \ 0 \leq t \leq \frac{\pi}{2}$.

Solution:
First, we find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = e^t \cos\, t - e^t \sin\, t \Rightarrow (\frac{dx}{dt})^2 = (e^t \cos\, t - e^t \sin\, t)^2\ ,$$
$$\frac{dy}{dt} = e^t \sin\, t + e^t \cos\, t \Rightarrow (\frac{dy}{dt})^2 = (e^t \sin\, t + e^t \cos\, t)^2.$$

Thus,

$$\begin{aligned}(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 &= e^{2t}\cos^2\, t - 2e^{2t}\cos\, t\, \sin\, t + e^{2t}\sin^2\, t + e^{2t}\sin^2\, t + 2e^{2t}\sin\, t\, \cos\, t + e^{2t}\sin^2\, t \\ &= e^{2t} + e^{2t} = 2e^{2t}.\end{aligned}$$

Therefore, the arc length of the curve is $L = \sqrt{2}\int_0^{\frac{\pi}{2}} e^t\ dt = \sqrt{2}\left[\ e^t\ \right]_0^{\frac{\pi}{2}} = \sqrt{2}\,\left(e^{\frac{\pi}{2}} - 1\right)$.

Example 8.10 Find the surface area of the solid obtained by revolving the curve $x = 3\cos\, t,\ y = 3\sin\, t,\ 0 \le t \le \frac{\pi}{3}$ about the x-axis.

Solution: Since the revolution is about the x-axis, we apply the formula

$$S.A = 2\pi \int_a^b y\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as follows:

$$\frac{dx}{dt} = -3\sin\, t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9\sin^2\, t \quad \text{and} \quad \frac{dy}{dt} = 3\cos\, t \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9\cos^2\, t.$$

Thus,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9(\sin^2\, t + \cos^2\, t) = 9.$$

This implies

$$S.A = 18\pi \int_0^{\frac{\pi}{3}} \sin\, t\; dt = -18\pi\left[\cos\, t\right]_0^{\frac{\pi}{3}} = -18\pi\left[\frac{1}{2} - 1\right] = 9\pi.$$

Example 8.11 Find the surface area of the solid obtained by revolving the curve $x = t^3,\ y = t,\ 0 \le t \le 1$ about the y-axis.

Solution: Since the revolution is about the y-axis, we apply the formula

$$S.A = 2\pi \int_a^b x\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as follows:

$$\frac{dx}{dt} = 3t^2 \Rightarrow \left(\frac{dx}{dt}\right)^2 = 9t^4 \quad \text{and} \quad \frac{dy}{dt} = 1 \Rightarrow \left(\frac{dx}{dt}\right)^2 = 1.$$

Thus,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9t^4 + 1.$$

This implies

$$S.A = 2\pi \int_0^1 t^3 \sqrt{9t^4 + 1}\, dt = \frac{\pi}{18}\left[(9t^4 + 1)^{\frac{3}{2}}\right]_0^1 = \frac{\pi}{18}\left[10\sqrt{10} - 1\right].$$

Exercise 8.1

1 - 8 The curve C is given parametrically. Find an equation in x and y, then sketch the graph and indicate the orientation.

1 $x = t, y = 2t+1, 1 \le t \le 3$

2 $x = \cos\, 2t, y = \sin\, t, 0 \le t \le \pi/2$

3 $x = 2t, y = (2t)^2, -1 \le t \le 1$

4 $x = 1+\cos\, t, y = 1+\sin\, t, 0 \le t \le 2\pi$

5 $x = \ln t, y = e^t, 1 \le t \le 4$

6 $x = 3\cos\, t, y = 3\sin\, t, 0 \le t \le 2\pi$

7 $x = 3t+2, y = t-1, -1 \le t \le 5$

8 $x = t, y = t^3, 1 \le t \le 3$

9 - 16 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the indicated value.

9 $x = t^2, y = t^3+1$ at $t = 1$

10 $x = t/3, y = t^3/2$ at $t = 2$

11 $x = \sqrt{t^3}, y = 2t+1$ at $t = 1$

12 $x = t^2+1, y = 1-t^3$ at $t = 3$

13 $x = e^t, y = e^{-t}+1$ at $t = 0$

14 $x = t+\cos\, t, y = \sin\, t$ at $t = \pi/4$

15 $x = t\cos\, t, y = t\sin\, t$ at $t = 0$

16 $x = \sqrt[3]{t}, y = t^2$ at $t = 1$

17 - 24 Find the slope of the tangent line to the curve at the indicated value.

17 $x = 2t, y = (2t)^2$ at $t = 1$

18 $x = \sqrt{t^3}, y = 2t+1$ at $t = 2$

19 $x = t^2+1, y = 1-t^3$ at $t = 3$

20 $x = \cos\, 2t, y = \sin\, t$ at $t = \pi/3$

21 $x = 3t+2, y = t-1$ at $t = 1$

22 $x = t+\cos\, t, y = \sin\, t$ at $t = \pi/6$

23 $x = t, y = t^3$, at $t = 1$

24 $x = \sqrt[3]{t}, y = t^2$ at $t = 5$

25 - 30 Find the points on the curve C at which the tangent line is either horizontal or vertical.

25 $x = t, y = t^3, t \in \mathbb{R}$

26 $x = 4t, y = t^2, t \in \mathbb{R}$

27 $x = \ln t, y = e^t, t > 0$

28 $x = t^2, y = t^3-3t, t \in \mathbb{R}$

29 $x = 3t^2-6t, y = \sqrt{t}, t \ge 0$

30 $x = 1-\sin\, t, y = 2\cos\, t, t \in \mathbb{R}$

31 - 38 Find the length of the curve.

31 $x = 3t+2, y = t-1, -1 \le t \le 3$

32 $x = 3t^2, y = 2t^3, 0 \le t \le 2$

33 $x = t, y = t^2, 1 \le t \le 4$

34 $x = \sin\, t, y = \cos\, t, \pi/6 \le t \le \pi/4$

35 $x = \ln t, y = t, 1 \le t \le 4$

36 $x = 1+\cos\, t, y = 1+\sin\, t, 0 \le t \le \pi$

37 $x = 3\cos\, t, y = 3\sin\, t, 0 \le t \le \pi/4$

38 $x = t^2, y = t^3, 0 \le t \le 1/2$

39 - 46 Find the area of the surface generated by revolving the curve about the specified axis.

39 $x = t^2,\ y = t,\ 0 \le t \le 1$ x-axis

40 $x = e^t\cos\, t,\ y = e^t\sin\, t,\ 0 \le t \le \frac{\pi}{2}$ x-axis

41 $x = t,\ y = t^2,\ 1 \le t \le 4$ y-axis

42 $x = t,\ y = \sqrt{t},\ 0 \le t \le 2$ x-axis

43 $x = t^2,\ y = t,\ 0 \le t \le 2$ x-axis

44 $x = 1+\cos\, t,\ y = 1+\sin\, t,\ 0 \le t \le \pi$ y-axis

45 $x = \sin^2\, t,\ y = \cos^2\, t,\ 0 \le t \le \pi/2$ y-axis

46 $x = 3t^2,\ y = t,\ 0 \le t \le 2$ x-axis

8.2 Polar Coordinates System

Previously, we used Cartesian (or Rectangular) coordinates to determine points (x,y). In this section, we are going to study a new coordinate system called polar coordinate system. Figure 8.7 shows the Cartesian and polar coordinates system.

Definition 8.3 The polar coordinate system is a two-dimensional system consisted of a pole and a polar axis (half line). Each point P on a plane is determined by a distance r from a fixed point O called the pole (or origin) and an angle θ from a fixed direction.

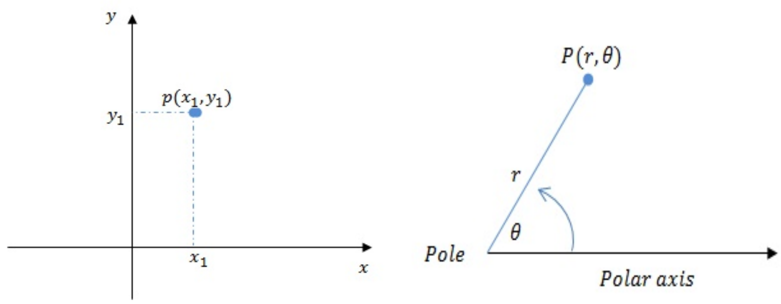


Figure 8.7: The Cartesian and polar coordinates. The Cartesian coordinate system is on the left and the polar coordinate system is on the right.

Remark 8.3

1. From the definition, the point P in the polar coordinate system is represented by the ordered pair (r,θ) where r, θ are called polar coordinates.

2. The angle θ is positive if it is measured counterclockwise from the axis, but if it is measured clockwise the angle is negative.

3. In the polar coordinates, if $r>0$, the point $P(r,\theta)$ will be in the same quadrant as θ; if $r<0$, it will be in the quadrant on the opposite side of the pole with the half line. That is, the points $P(r,\theta)$ and $P(-r,\theta)$ lie in the same line through the pole O, but on opposite sides of O. The point $P(r,\theta)$ with the distance $|r|$ from O and the point $P(-r,\theta)$ with the half distance from O.

4. In the Cartesian coordinate system, every point has only one representation while in a polar coordinate system each point has many representations. The following formula gives all representations of a point $P(r,\theta)$ in the polar coordinate system
$$P(r,\theta+2n\pi)=P(r,\theta)=P(-r,\theta+(2n+1)\pi), \quad n\in\mathbb{Z}.$$

Example 8.12 Plot the points whose polar coordinates are given.

(1) $(1,5\pi/4)$

(2) $(1,-3\pi/4)$

(3) $(1,13\pi/4)$

(4) $(-1,\pi/4)$

Solution:

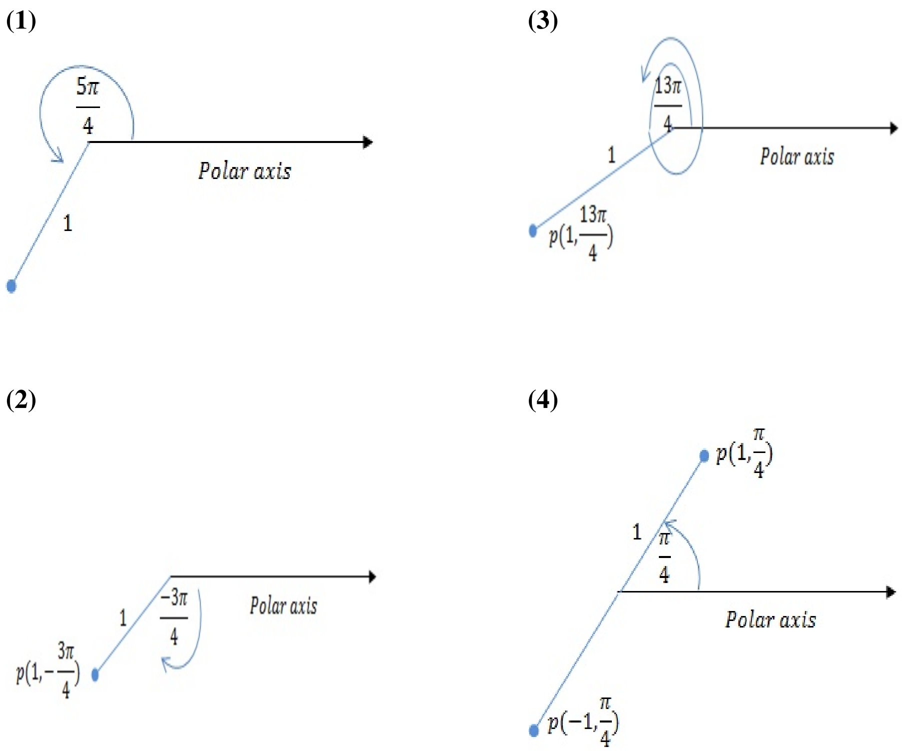


Figure 8.8

8.2.1 The Relationship between Rectangular and Polar Coordinates

Let (x,y) be the rectangular coordinates and (r,θ) be the polar coordinates of the same point P. Let the pole be at the origin of the Cartesian coordinates system, and let the polar axis be the positive x-axis and the line $\theta = \frac{\pi}{2}$ be the positive y-axis as shown in Figure 8.9.

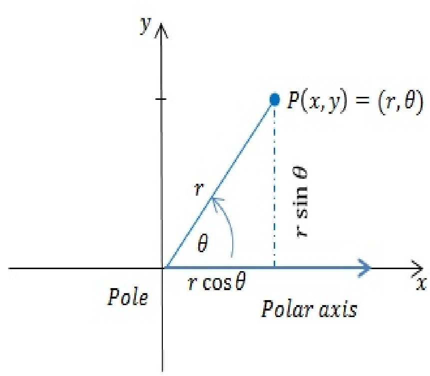


Figure 8.9: The relationship between the rectangular and polar coordinates.

In the triangle, we have

$$\cos\ \theta = \frac{x}{r} \Rightarrow x = r\cos\ \theta\ ,$$
$$\sin\ \theta = \frac{y}{r} \Rightarrow y = r\sin\ \theta.$$

Hence,

$$\begin{aligned} x^2+y^2 &= (r\cos\theta)^2+(r\sin\theta)^2, \\ &= r^2(\cos^2\theta+\sin^2\theta). \end{aligned}$$

This implies, $x^2+y^2=r^2$ and $\tan\ \theta = \frac{y}{x}$ for $x \neq 0$.

The previous relationships can be summarized as follows:

$$x = r\cos\,\theta, \;\; y = r\sin\,\theta$$
$$\tan\,\theta = \frac{y}{x} \;\text{ for }\; x \neq 0$$
$$x^2 + y^2 = r^2$$

Example 8.13 Convert from polar coordinates to rectangular coordinates.

(1) $(1, \pi/4)$

(2) $(2, \pi)$

(3) $(2, -2\pi/3)$

(4) $(4, 3\pi/4)$

Solution:

(1) $r = 1$ and $\theta = \frac{\pi}{4}$.

$$x = r\cos\,\theta = (1)\cos\,\frac{\pi}{4} = \frac{1}{\sqrt{2}}\,,$$
$$y = r\sin\,\theta = (1)\sin\,\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Hence, $(x, y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(2) $r = 2$ and $\theta = \pi$.

$$x = r\cos\,\theta = 2\cos\,\pi = -2\,,$$
$$y = r\sin\,\theta = 2\sin\,\pi = 0.$$

Hence, $(x, y) = (-2, 0)$.

(3) $r = 2$ and $\theta = \frac{-2\pi}{3}$.

$$x = r\cos\,\theta = 2\cos\,\frac{-2\pi}{3} = -1\,,$$
$$y = r\sin\theta = 2\sin\,\frac{-2\pi}{3} = -\sqrt{3}.$$

Hence, $(x, y) = (-1, -\sqrt{3})$.

(4) $r = 4$ and $\theta = \frac{3\pi}{4}$.

$$x = r\cos\,\theta = 4\cos\,\frac{3\pi}{4} = -2\sqrt{2}\,,$$
$$y = r\sin\,\theta = 4\sin\,\frac{3\pi}{4} = 2\sqrt{2}.$$

This implies $(x, y) = (-2\sqrt{2}, 2\sqrt{2})$.

Example 8.14 Convert from rectangular coordinates to polar coordinates for $r \geq 0$ and $0 \leq \theta \leq \pi$.

(1) $(5, 0)$

(2) $(2\sqrt{3}, -2)$

(3) $(-2, 2)$

(4) $(1, 1)$

Solution:

(1) We have $x = 5$ and $y = 0$. By using $x^2 + y^2 = r^2$, we obtain $r = 5$. Also, we have $\tan\theta = \frac{y}{x} = \frac{0}{5} = 0$, then $\theta = 0$. This implies $(r, \theta) = (5, 0)$.

(2) We have $x = 2\sqrt{3}$ and $y = -2$. Use $x^2 + y^2 = r^2$ to have $r = 4$. Also, since $\tan\,\theta = \frac{y}{x} = \frac{-2}{2\sqrt{3}} = \frac{-1}{\sqrt{3}}$, then $\theta = \frac{5\pi}{6}$. Hence, $(r, \theta) = (4, \frac{5\pi}{6})$.

(3) We have $x = -2$ and $y = 2$. Then, $r^2 = x^2 + y^2 = (-2)^2 + 2^2$ and this implies $r = 2\sqrt{2}$. Also, $\tan\,\theta = \frac{y}{x} = \frac{2}{-2} = -1$, then $\theta = \frac{3\pi}{4}$. This implies $(r, \theta) = (2\sqrt{2}, \frac{3\pi}{4})$.

(4) We have $x = 1$ and $y = 1$. By using $x^2 + y^2 = r^2$, we have $r = \sqrt{2}$. Also, by using $\tan\,\theta = \frac{y}{x} = 1$, we obtain $\theta = \frac{\pi}{4}$. This implies, $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$.

A polar equation is an equation in r and θ, $r = f(\theta)$. A solution of the polar equation is an ordered pair (r_0, θ_0) satisfies the equation i.e., $r_0 = f(\theta_0)$. For example, $r = 2\cos\,\theta$ is a polar equation and $(1, \frac{\pi}{3})$, and $(\sqrt{2}, \frac{\pi}{4})$ are solutions of that equation.

Example 8.15 Find a polar equation that has the same graph as the equation in x and y.

(1) $x = 7$

(2) $y = -3$

(3) $x^2 + y^2 = 4$

(4) $y^2 = 9x$

Solution:

(1) $x = 7 \Rightarrow r\cos\,\theta = 7 \Rightarrow r = 7\sec\theta$.

(2) $y = -3 \Rightarrow r\sin\,\theta = -3 \Rightarrow r = -3\csc\theta$.

(3)
$$\begin{aligned} x^2 + y^2 = 4 &\Rightarrow r^2\cos^2\,\theta + r^2\sin^2\,\theta = 4 \\ &\Rightarrow r^2(\cos^2\,\theta + \sin^2\,\theta) = 4 \\ &\Rightarrow r^2 = 4\ . \end{aligned}$$

(4)
$$\begin{aligned} y^2 = 9x &\Rightarrow r^2\sin^2\,\theta = 9r\cos\,\theta \\ &\Rightarrow r\sin^2\,\theta = 9\cos\,\theta \\ &\Rightarrow r = 9\cot\,\theta\csc\,\theta. \end{aligned}$$

Example 8.16 Find an equation in x and y that has the same graph as the polar equation.

(1) $r = 3$

(2) $r = \sin\,\theta$

(3) $r = 6\cos\,\theta$

(4) $r = \sec\,\theta$

Solution:

(1) $r = 3 \Rightarrow \sqrt{x^2 + y^2} = 3 \Rightarrow x^2 + y^2 = 9$.

(2) $r = \sin\theta \Rightarrow r = \frac{y}{r} \Rightarrow r^2 = y \Rightarrow x^2 + y^2 = y \Rightarrow x^2 + y^2 - y = 0$.

(3) $r = 6\cos\,\theta \Rightarrow r = 6\,\frac{x}{r} \Rightarrow r^2 = 6x \Rightarrow x^2 + y^2 - 6x = 0$.

(4) $r = \sec\,\theta \Rightarrow r = \frac{1}{\cos\,\theta} \Rightarrow r\cos\,\theta = 1 \Rightarrow x = 1$.

8.2.2 Tangent Line to Polar Curves

Theorem 8.2 Let $r = f(\theta)$ be a polar curve where f' is continuous. The slope of the tangent line to the graph of $r = f(\theta)$ is
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\,\theta + \sin\,\theta(dr/d\theta)}{-r\sin\,\theta + \cos\,\theta(dr/d\theta)}.$$

Proof. Since $r = f(\theta)$ is a polar curve, then
$$x = f(\theta)\cos\,\theta\ ,\ y = f(\theta)\sin\,\theta.$$
From the chain rule, we have
$$\begin{aligned} \frac{dx}{d\theta} &= -f(\theta)\sin\,\theta + f'(\theta)\cos\,\theta = -r\sin\,\theta + \frac{dr}{d\theta}\cos\,\theta\ , \\ \frac{dy}{d\theta} &= f(\theta)\cos\,\theta + f'(\theta)\sin\,\theta = r\cos\,\theta + \frac{dr}{d\theta}\sin\,\theta. \end{aligned}$$
If $\frac{dx}{d\theta} \neq 0$, the slope of the tangent line to the graph of $r = f(\theta)$ is
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\,\theta + \sin\,\theta(dr/d\theta)}{-r\sin\,\theta + \cos\,\theta(dr/d\theta)}.\ \blacksquare$$

Remark 8.4

1. If $\frac{dy}{d\theta}=0$ such that $\frac{dx}{d\theta}\neq 0$, the curve has a horizontal tangent line.

2. If $\frac{dx}{d\theta}=0$ such that $\frac{dy}{d\theta}\neq 0$, the curve has a vertical tangent line.

3. If $\frac{dx}{d\theta}\neq 0$ at $\theta=\theta_0$, the slope of the tangent line to the graph of $r=f(\theta)$ is

$$\frac{r_0\cos\,\theta_0+\sin\,\theta_0(dr/d\theta)_{\theta=\theta_0}}{-r_0\sin\,\theta_0+\cos\,\theta_0(dr/d\theta)_{\theta=\theta_0}},\quad \text{where}\quad r_0=f(\theta_0)$$

Example 8.17 Find the slope of the tangent line to the graph of $r=\sin\,\theta$ at $\theta=\frac{\pi}{4}$.

Solution:

$$x=r\cos\,\theta\Rightarrow x=\sin\,\theta\cos\,\theta\Rightarrow\frac{dx}{d\theta}=\cos^2\,\theta-\sin^2\,\theta\,,$$

$$y=r\sin\,\theta\Rightarrow y=\sin^2\,\theta\Rightarrow\frac{dy}{d\theta}=2\sin\,\theta\cos\,\theta.$$

Hence,

$$\frac{dy}{dx}=\frac{2\sin\,\theta\cos\,\theta}{\cos^2\,\theta-\sin^2\,\theta}.$$

At $\theta=\frac{\pi}{4}$, $\frac{dy}{d\theta}=1$ and $\frac{dx}{d\theta}=0$. Thus, the slope is undefined. In this case, the curve has a vertical tangent line.

Example 8.18 Find the points on the curve $r=2+2\cos\,\theta$ for $0\leq\theta\leq 2\pi$ at which tangent lines are either horizontal or vertical.

Solution:

$$x=r\cos\,\theta=2\cos\,\theta+2\cos^2\,\theta\Rightarrow\frac{dx}{d\theta}=-2\sin\,\theta-4\cos\,\theta\sin\,\theta\,,$$

$$y=r\sin\,\theta=2\sin\,\theta+2\cos\,\theta\sin\,\theta\Rightarrow\frac{dy}{d\theta}=2\cos\,\theta-2\sin^2\,\theta+2\cos^2\,\theta.$$

For a horizontal tangent line,

$$\frac{dy}{d\theta}=0\Rightarrow 2\cos\,\theta-2\sin^2\,\theta+2\cos^2\,\theta=0\Rightarrow 2\cos^2\,\theta+\cos\,\theta-1=0\Rightarrow(2\cos\,\theta-1)(\cos\,\theta+1)=0.$$

This implies $\theta=\pi$, $\theta=\pi/3$, or $\theta=5\pi/3$. Therefore, the tangent line is horizontal at $(0,\pi)$, $(3,\pi/3)$ or $(3,5\pi/3)$.

For a vertical tangent line,

$$\frac{dx}{d\theta}=0\Rightarrow\sin\,\theta(2\cos\,\theta+1)=0.$$

This implies $\theta=0$, $\theta=\pi$, $\theta=2\pi/3$, or $\theta=4\pi/3$. However, we have to ignore $\theta=\pi$ since at this value $dy/d\theta=0$. Therefore, the tangent line is vertical at $(4,0)$, $(1,2\pi/3)$, or $(1,4\pi/3)$.

8.2.3 Graphs in Polar Coordinates

Before starting sketching polar curves, we study symmetry about the polar axis, or the vertical line $\theta=\frac{\pi}{2}$ or about the pole.

Symmetry in Polar Coordinates

Theorem 8.3

1. **Symmetry about the polar axis.**
   The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if replacing (r,θ) with $(r,-\theta)$ or with $(-r,\pi-\theta)$ does not change the equation.

2. **Symmetry about the vertical line $\theta = \frac{\pi}{2}$.**
   The graph of $r = f(\theta)$ is symmetric with respect to the vertical line if replacing (r,θ) with $(r,\pi-\theta)$ or with $(-r,-\theta)$ does not change the equation.

3. **Symmetry about the pole $\theta = 0$.**
   The graph of $r = f(\theta)$ is symmetric with respect to the pole if replacing (r,θ) with $(-r,\theta)$ or with $(r,\theta+\pi)$ does not change the equation.

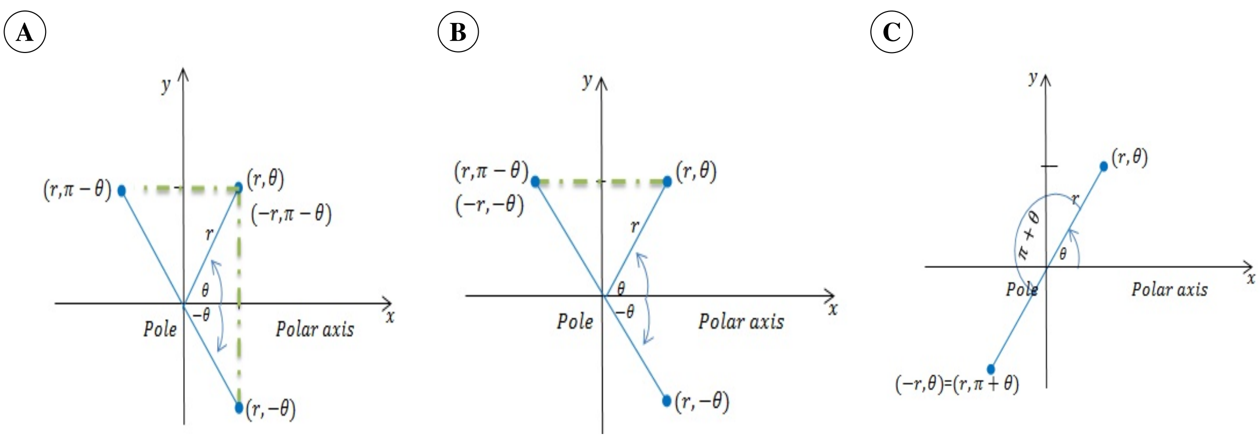


Figure 8.10: Symmetry of the curves in the polar coordinate system. (A) symmetry about the polar axis, (B) symmetry about the vertical line $\theta = \frac{\pi}{2}$, and (C) symmetry about the pole $\theta = 0$.

Example 8.19 **(1)** The graph of $r = 4\cos\,\theta$ is symmetric about the polar axis since

$$4\cos\,(-\theta) = 4\cos\,\theta \;\text{ and }\; -4\cos(\pi-\theta) = 4\cos\,\theta.$$

(2) The graph of $r = 2\sin\,\theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since

$$2\sin\,(\pi-\theta) = 2\sin\,\theta \;\text{ and }\; -2\sin\,(-\theta) = 2\sin\,\theta.$$

(3) The graph of $r^2 = a^2\sin\,2\theta$ is symmetric about the pole since

$$(-r)^2 = a^2\sin\,2\theta,$$
$$\Rightarrow r^2 = a^2\sin\,2\theta.$$

and

$$r^2 = a^2\sin\,\big(2(\pi+\theta)\big),$$
$$= a^2\sin\,(2\pi+2\theta),$$
$$r^2 = a^2\sin\,2\theta.$$

Some Special Polar Graphs

Lines in polar coordinates

1. The polar equation of a straight line $ax+by=c$ is $r = \frac{c}{a\cos\,\theta+b\sin\,\theta}$.
   Since $x = r\cos\,\theta$ and $y = r\sin\,\theta$, then

$$ax+by = c \Rightarrow r(\cos\,\theta+b\sin\,\theta) = c \Rightarrow r = \frac{c}{(\cos\,\theta+b\sin\,\theta)}$$

2. The polar equation of a vertical line $x = k$ is $r = k\sec\,\theta$.
Let $x = k$, then $r\cos\,\theta = k$. This implies $r = \frac{k}{\cos\,\theta} = k\sec\,\theta$.

3. The polar equation of a horizontal line $y = k$ is $r = k\csc\,\theta$.
Let $y = k$, then $r\sin\,\theta = k$. This implies $r = \frac{k}{\sin\,\theta} = r\csc\,\theta$.

4. The polar equation of a line that passes the origin point and makes an angle θ_0 with the positive x-axis is $\theta = \theta_0$.

Example 8.20 Sketch the graph of $\theta = \frac{\pi}{4}$.

Solution:

We are looking for a graph of the set of polar points

$$\{(r,\theta)\ |, r \in \mathbb{R}\}.$$

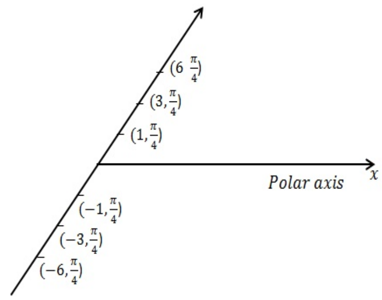


Figure 8.11

Circles in polar coordinates

1. The circle equation with center at the pole O and radius $|a|$ is $r = a$.

2. The circle equation with center at $(a,0)$ and radius $|a|$ is $r = 2a\cos\,\theta$.

3. The circle equation with center at $(0,a)$ and radius $|a|$ is $r = 2a\sin\,\theta$.

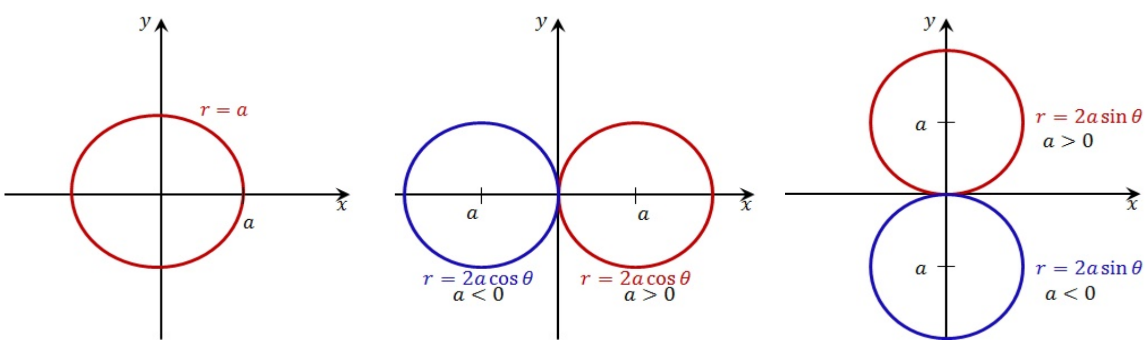


Figure 8.12: Circles in polar coordinates.

Example 8.21 Sketch the graph of $r = 4\sin\,\theta$.

Solution:
Note that the graph of $r = 4\sin\,\theta$ is symmetric about the vertical line $\theta = \frac{\pi}{2}$ since $4\sin\,(\pi - \theta) = 4\sin\,\theta$. Therefore, we restrict our attention to the interval $[0,\pi/2]$ and by the symmetry, we complete the graph. The following table displays polar coordinates of some points on the curve:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	0	2	$4/\sqrt{2}$	$2\sqrt{3}$	4

Cardioid curves

1. $r = a(1 \pm \cos\,\theta)$ **2.** $r = a(1 \pm \sin\,\theta)$

$r = a(1+\cos\,\theta)$ $r = a(1-\cos\,\theta)$ $r = a(1+\sin\,\theta)$ $r = a(1-\sin\,\theta)$

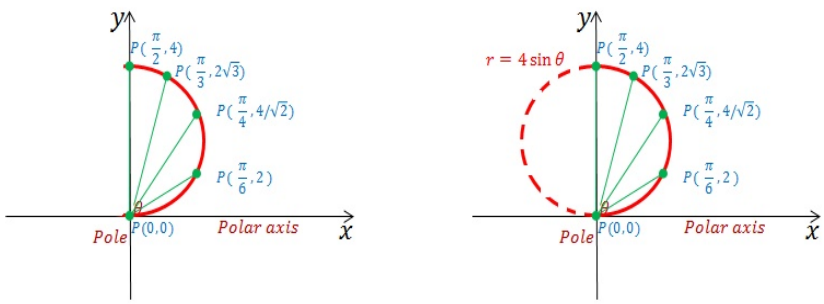


Figure 8.13

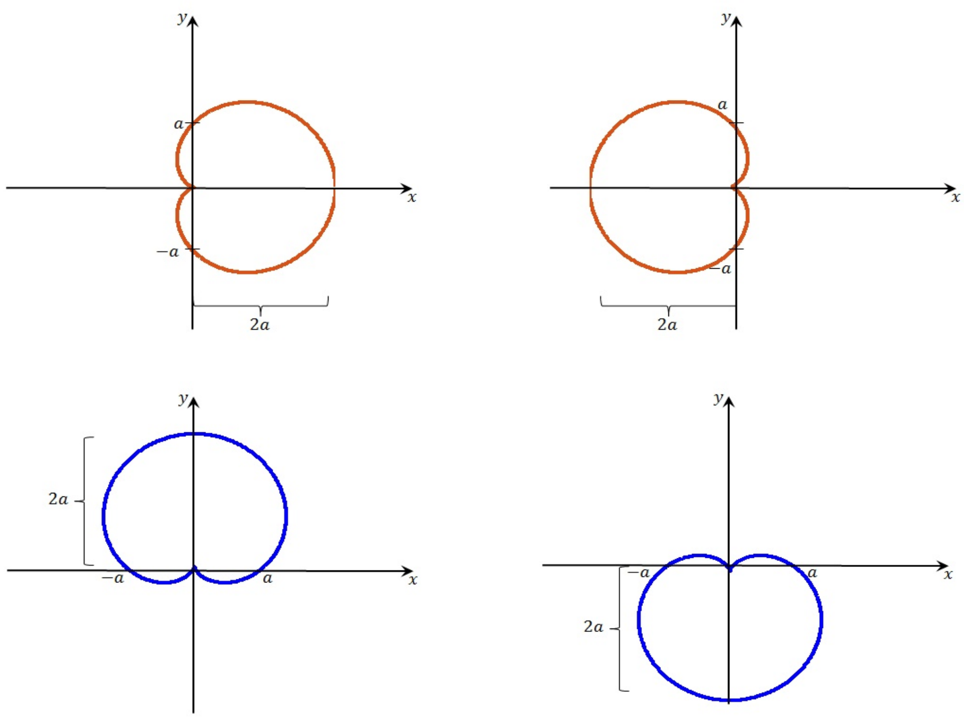


Figure 8.14: Cardioid curves.

Example 8.22 Sketch the graph of $r = a(1-\cos\,\theta)$ where $a > 0$.

Solution:
The curve is symmetric about the polar axis since $\cos\,(-\theta) = \cos\,\theta$. Therefore, we restrict our attention to the interval $[0,\pi]$ and by the symmetry, we complete the graph. The following table displays some solutions of the equation $r = a(1-\cos\,\theta)$:

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$a/2$	a	$3a/2$	$2a$

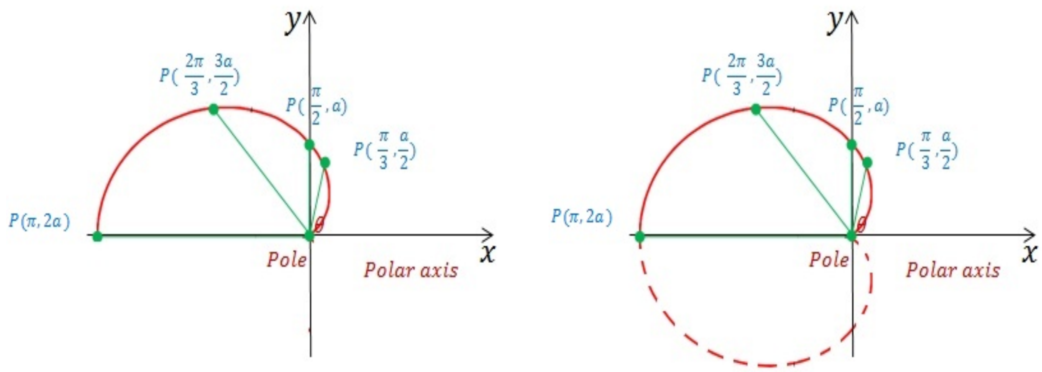


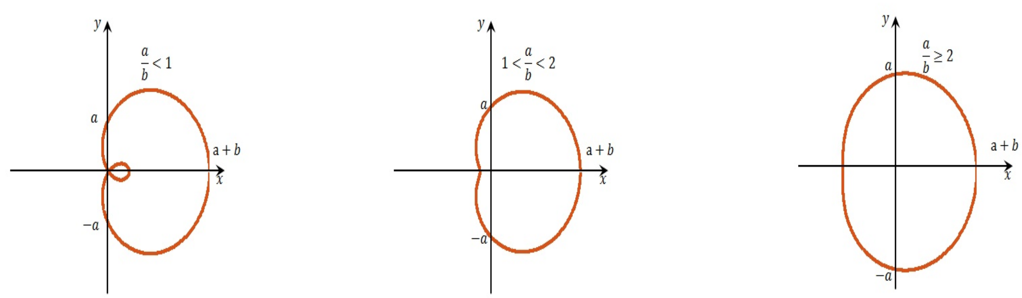
Figure 8.15

Limaçons curves

1. $r = a \pm b\cos\ \theta$ **2.** $r = a \pm b\sin\ \theta$

1. $r = a \pm b\cos\ \theta$

(a) $r = a + b\cos\ \theta$



(b) $r = a - b\cos\ \theta$

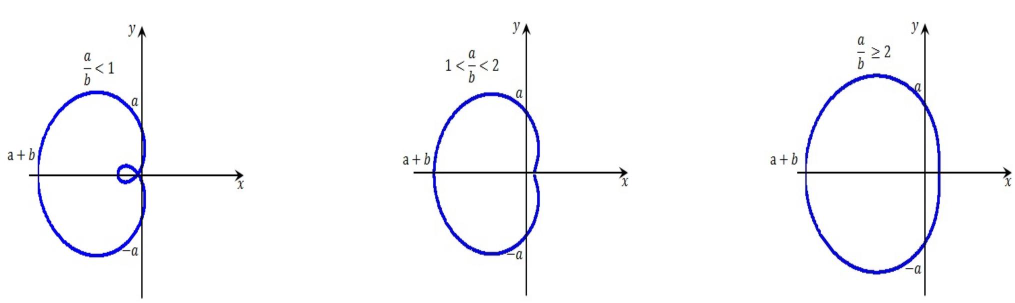
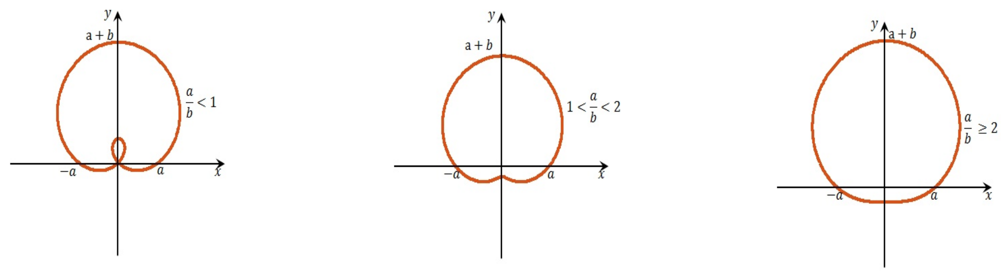


Figure 8.16: Limaçons curves $r = a \pm b\cos\ \theta$.

2. $r = a \pm b\sin\ \theta$

(a) $r = a + b\sin\ \theta$



(b) $r = a - b\sin\,\theta$

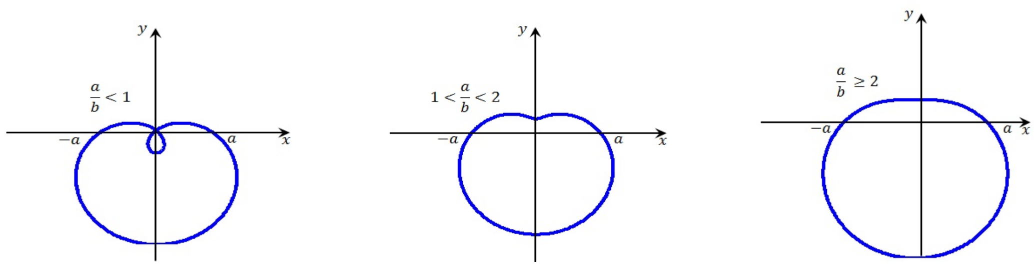
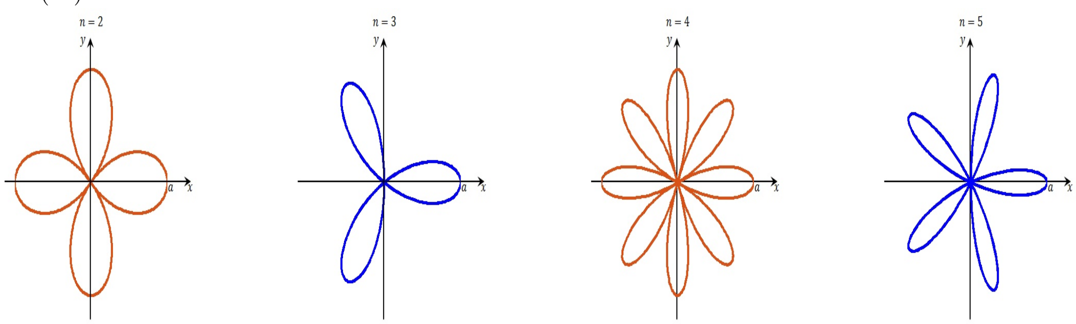


Figure 8.17: Limaçons curves $r = a \pm b\sin\,\theta$.

Roses

1. $r = a\,\cos\,(n\theta)$ **2.** $r = a\,\sin\,(n\theta)$ where $n \in \mathbb{N}$.

1. $r = a\,\cos\,(n\theta)$



2. $r = a\,\sin\,(n\theta)$

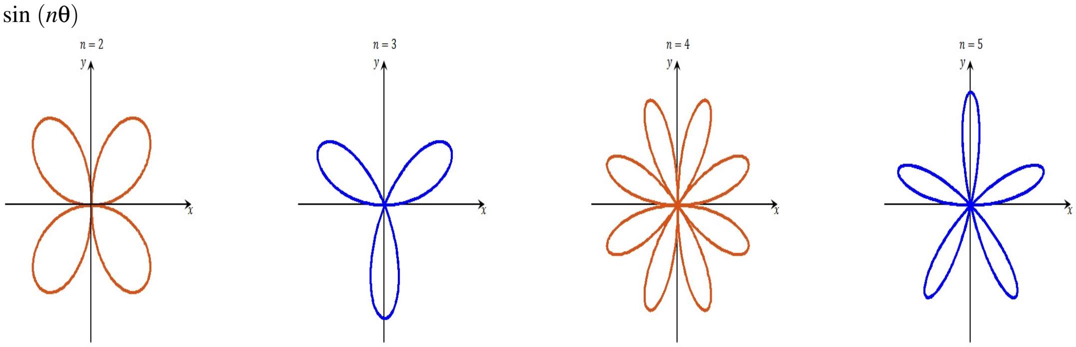


Figure 8.18: Roses in polar coordinates.

Note that if n is odd, there are n petals; however, if n is even, there are $2n$ petals.

Spiral of Archimedes
$r = a\,\theta$

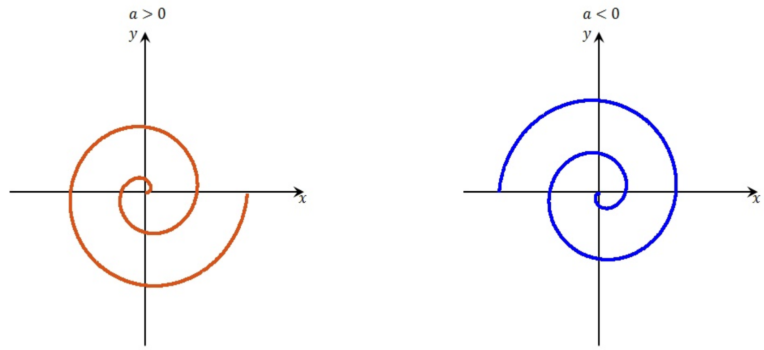


Figure 8.19: Spiral of Archimedes.

Exercise 8.2

1 - 8 Find the corresponding rectangular coordinates for the given polar coordinates.

1 $(1, \frac{\pi}{2})$

2 $(-1, \frac{\pi}{2})$

3 $(2, \frac{\pi}{4})$

4 $(3, \pi)$

5 $(\frac{1}{2}, \frac{3\pi}{2})$

6 $(-3, 2\pi)$

7 $(7, \frac{3\pi}{4})$

8 $(3, \frac{\pi}{6})$

9 - 16 Find the corresponding polar coordinates for the given rectangular coordinates for $r \geq 0$ and $0 \leq \theta \leq \pi$.

9 $(1,1)$

10 $(1,\sqrt{3})$

11 $(-1,1)$

12 $(\sqrt{3},3)$

13 $(2,\sqrt{2})$

14 $(3,0)$

15 $(4,2)$

16 $(-3,-3)$

17 - 24 Find a polar equation that has the same graph as the equation in x and y and vice versa.

17 $x = 9$

18 $x^2 + y^2 = 1$

19 $r = \csc\,\theta$

20 $r = 2\cos\,\theta$

21 $x^2 = 3y$

22 $x^2 - y^2 = 16$

23 $r = \frac{3}{1-\sin\,\theta}$

24 $r = 3 - 2\sin\,\theta$

25 - 28 Sketch the curve of the polar equations.

25 $r = \sec\,\theta$

26 $r = 2\cos\,\theta$

27 $r = 2 + 2\sin\,\theta$

28 $r = 3 + 2\cos\,\theta$

29 - 33 Find the slope of the tangent line to the graph at θ. Then find the points on the curve at which the tangent lines are either horizontal or vertical.

29 $r = 2\sin\,\theta$ at $\theta = \frac{\pi}{3}$

30 $r = 3 + 2\cos\,\theta$ at $\theta = \frac{\pi}{4}$

31 $r = \cos\,7\theta$ at $\theta = \frac{\pi}{2}$

32 $r = 1 + \sin\,\theta$ at $\theta = \frac{\pi}{4}$

33 $r = 1 - \cos\,\theta$ at $\theta = \frac{\pi}{6}$

8.3 Area in Polar Coordinates

Let $r = f(\theta)$ be a continuous function on the interval $[\alpha, \beta]$ such that $0 \leq \alpha \leq \beta \leq 2\pi$. Let $f(\theta) \geq 0$ over that interval and R be a polar region bounded by the polar equations $r = f(\theta)$, $\theta = \alpha$ and $\theta = \beta$ as shown in Figure 8.20.
To find the area of R, we assume $P = \{\theta_1, \theta_2, \ldots, \theta_n\}$ is a regular partition of the interval $[\alpha, \beta]$. Consider the interval $[\theta_{k-1}, \theta_k]$ where $\Delta\theta_k = \theta_k - \theta_{k-1}$. By choosing $\omega_k \in [\theta_{k-1}, \theta_k]$, we have a circular sector where its angle and radius are $\Delta\theta_k$ and $f(\omega_k)$, respectively. The area between θ_{k-1} and θ_k can be approximated by the area of a circular sector.

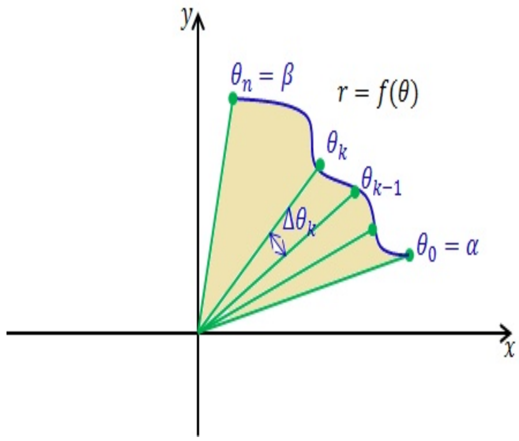


Figure 8.20: Areas in polar coordinates.

Let $f(u_k)$ and $f(v_k)$ be maximum and minimum values of f on $[\theta_{k-1}, \theta_k]$. From Figure 8.21, we have

$$\underbrace{\frac{1}{2}\left[f(u_k)\right]^2 \Delta\theta_k}_{\text{Area of the sector of radius} f(u_k)} \leq \Delta A_k \leq \underbrace{\frac{1}{2}\left[f(v_k)\right]^2 \Delta\theta_k}_{\text{Area of the sector of radius} f(v_k)}$$

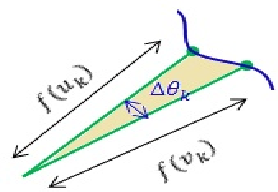


Figure 8.21

By summing from $k = 1$ to $k = n$, we obtain

$$\sum_{k=1}^{n} \frac{1}{2}\left[f(u_k)\right]^2 \Delta\theta_k f(u_k) \leq \underbrace{\sum_{k=1}^{n} \Delta A_k}_{=A} \leq \sum_{k=1}^{n} \frac{1}{2}\left[f(v_k)\right]^2 \Delta\theta_k f(v_k)$$

The limit of the sums as the norm $||P||$ approaches zero,

$$\lim_{||P||\to 0} \sum_{k=1}^{n} \frac{1}{2}\left[f(u_k)\right]^2 \Delta\theta_k f(u_k) = \lim_{||P||\to 0} \sum_{k=1}^{n} \frac{1}{2}\left[f(u_k)\right]^2 \Delta\theta_k f(v_k) = \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 d\theta \, .$$

Therefore,

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(f(\theta)\right)^2 \, d\theta$$

Similarly, assume f and g are continuous on the interval $[\alpha, \beta]$ such that $f(\theta) \geq g(\theta)$. The area of the polar region bounded by the graphs of f and g on the interval $[\alpha, \beta]$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right] d\theta$$

Example 8.23 Find the area of the region bounded by the graph of the polar equation.

(1) $r = 3$

(2) $r = 2\cos\,\theta$

(3) $r = 4\sin\,\theta$

(4) $r = 6 - 6\sin\,\theta$

Solution:

(1) The area is

$$A = \frac{1}{2}\int_0^{2\pi} 3^2 \, d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \frac{9}{2}\Big[\,\theta\,\Big]_0^{2\pi} = 9\pi.$$

Note that one can evaluate the area in the first quadrant and multiply the result by 4 to find the area of the whole region i.e.,

$$A = 4\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} 3^2 \, d\theta\Big) = 18\int_0^{\frac{\pi}{2}} d\theta = 18\Big[\,\theta\,\Big]_0^{\frac{\pi}{2}} = 9\pi.$$

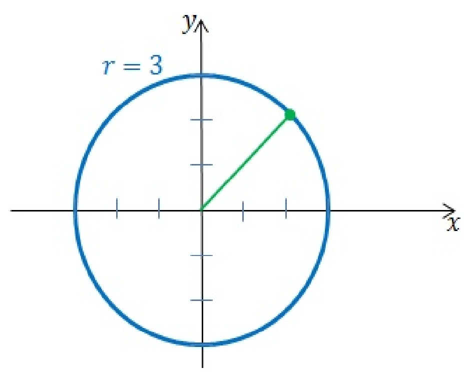


Figure 8.22

(2) We find the area of the upper half circle and multiply the result by 2 as follows:

$$\begin{aligned} A = 2\Big(\frac{1}{2}\int_0^{\frac{\pi}{2}} (2\cos\,\theta)^2 \, d\theta\Big) &= \int_0^{\frac{\pi}{2}} 4\cos^2\,\theta \, d\theta \\ &= 2\int_0^{\frac{\pi}{2}} (1 + \cos\,2\theta) \, d\theta \\ &= 2\Big[\theta + \frac{\sin\,2\theta}{2}\Big]_0^{\frac{\pi}{2}} \\ &= 2\Big[\frac{\pi}{2} - 0\Big] \\ &= \pi. \end{aligned}$$

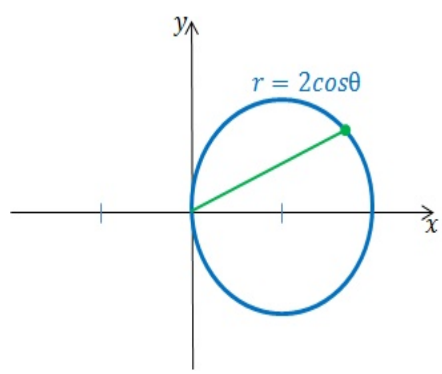


Figure 8.23

(3) The area of the region is

$$A = \frac{1}{2}\int_0^{\pi} (4\sin\theta)^2\, d\theta = \frac{16}{4}\int_0^{\pi} (1-\cos 2\theta)\, d\theta$$
$$= 4\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$
$$= 4\left[\pi - 0\right]$$
$$= 4\pi.$$

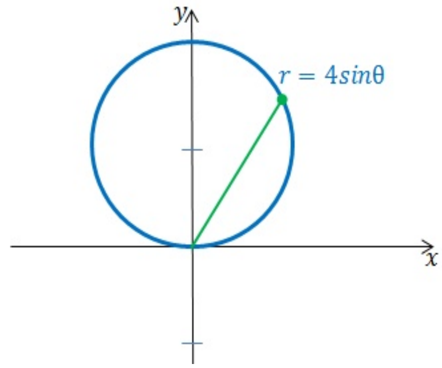


Figure 8.24

(4) The area of the region is

$$A = \frac{1}{2}\int_0^{2\pi} 36(1-\sin\theta)^2\, d\theta$$
$$= 18\int_0^{2\pi} (1-2\sin\theta+\sin^2\theta)\, d\theta$$
$$= 18\left[\theta + 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$
$$= 18\left[(2\pi+2+\pi)-2\right]$$
$$= 54\pi.$$

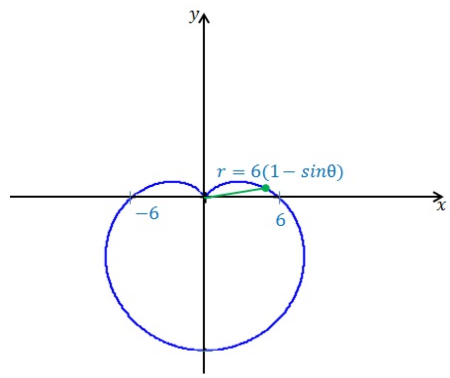


Figure 8.25

Example 8.24 Find the area of the region that is inside the graphs of the equations $r = \sin\theta$ and $r = \sqrt{3}\cos\theta$.

Solution:
First, we find the intersection points of the two curves

$$\sin\theta = \sqrt{3}\cos\theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}.$$

The origin O is in each circle, but it cannot be found by solving the equations. Therefore, when looking for the intersection points of the polar graphs, we sometimes take under consideration the graphs.

The region is divided into two small regions: below and above the line $\frac{\pi}{3}$.

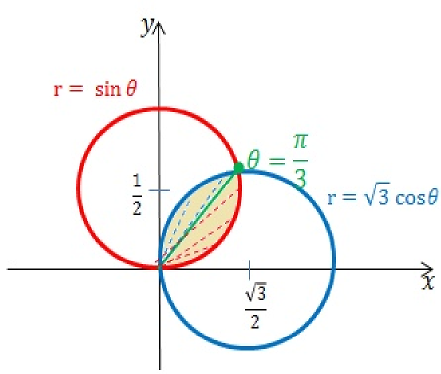


Figure 8.26

Region(1): below the line $\frac{\pi}{3}$.

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\,\theta\, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} (1-\cos\,2\theta)\, d\theta$$
$$= \frac{1}{4}\left[\theta - \frac{\sin\,2\theta}{2}\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sin\,\frac{2\pi}{3}}{2}\right]$$
$$= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right].$$

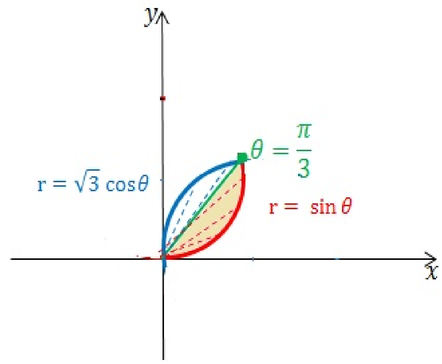


Figure 8.27

Region(2): above the line $\frac{\pi}{3}$.

$$A_2 = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sqrt{3}\cos\,\theta)^2\, d\theta = \frac{3}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos\,2\theta)\, d\theta$$
$$= \frac{3}{4}\left[\theta + \frac{\sin\,2\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$
$$= \frac{3}{4}\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)\right]$$
$$= \frac{3}{4}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right].$$

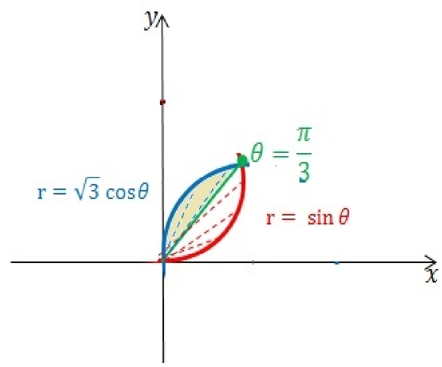


Figure 8.28

Total area $A = A_1 + A_2 = \frac{5\pi}{24} - \frac{\sqrt{3}}{4}$.

Example 8.25 Find the area of the region that is outside the graph of $r = 3$ and inside the graph of $r = 2+2\cos\theta$.

Solution: As shown in the figure, we find the area in the first quadrant and then we double the result to find the area of the whole region. The intersection point of the two curves in the first quadrant is

$$2+2\cos\,\theta = 3 \Rightarrow \cos\,\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

$$A = 2\left(\frac{1}{2}\int_0^{\frac{\pi}{3}} \left(4(1+\cos\,\theta)^2 - 9\right)\, d\theta\right)$$
$$= \int_0^{\frac{\pi}{3}} \left(4(1+2\cos\,\theta+\cos^2\,\theta) - 9\right)\, d\theta$$
$$= \int_0^{\frac{\pi}{3}} (8\cos\,\theta + 4\cos^2\,\theta - 5)\, d\theta$$
$$= \left[8\sin\,\theta + \sin\,2\theta - 3\theta\right]_0^{\frac{\pi}{3}}$$
$$= \frac{9}{2}\sqrt{3} - \pi.$$

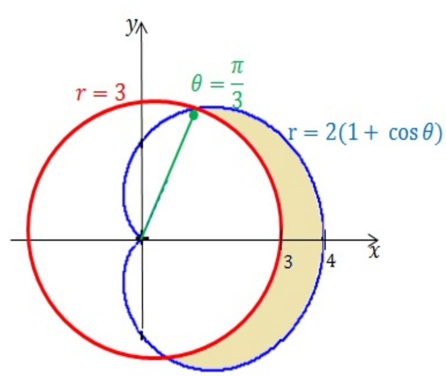


Figure 8.29

Exercise 8.3

1 - 8 Find the area of the region bounded by the graph of the polar equation.

1 $r = 4\sin\theta$

2 $r = 1 + \sin\theta$

3 $r = 5$

4 $r = 2\cos\theta$

5 $r = 6(1 + \sin\theta)$

6 $r = 2(1 - \cos\theta)$

7 $r = 3\cos 3\theta$

8 $r = 3 + 2\sin\theta$

9 - 18 Find the area of the region bounded by the graph of the polar equations.

9 inside $r = 1 + \cos\theta$ and outside $r = 3\cos\theta$

10 inside $r = 2 + 2\cos\theta$ and outside $r = 3$

11 outside $r = 2 - 2\cos\theta$ and inside $r = 4$

12 inside both graphs $r = 1 + \cos\theta$ and $r = 1$

13 inside $r = 1 + \sin\theta$ and outside $r = 1$

14 inside both graphs $r = 2\cos\theta$ and $r = 2\sin\theta$

15 outside $r = 3$ and inside $r = -6\cos\theta$

16 inside both graphs $r = \cos\theta$ and $r = -\sin\theta$

17 between the graphs $r = 1 + \sin\theta$ and $r = 3\sin\theta$

18 inside both graphs $r = 2$ and $r = 2 + 2\sin\theta$

19 inside the graph $r = 1 - \cos\theta$ in the first quadrant

20 between the graphs $r = 1 + \sin\theta$ and $r = 3\sin\theta$ in the second quadrant

8.3.1 Arc Length and Surface Area of Revolution in Polar Coordinates

Arc Length in Polar Coordinates

Let the polar function $r = f(\theta)$, $\alpha \le \theta \le \beta$ be smooth. We know that

$$x = f(\theta)\cos\theta \quad \text{and} \quad y = f(\theta)\sin\theta, \quad \alpha \le \theta \le \beta.$$

Thus,

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \big(f'(\theta)\cos\theta - f(\theta)\sin\theta\big)^2 + \big(f'(\theta)\sin\theta + f(\theta)\cos\theta\big)^2 \\
&= \big(f'(\theta)\big)^2\cos^2\theta - 2f(\theta)f'(\theta)\cos\theta\sin\theta + \big(f(\theta)\big)^2\sin^2\theta \\
&\quad + \big(f'(\theta)\big)^2\sin^2\theta + 2f(\theta)f'(\theta)\cos\theta\sin\theta + \big(f(\theta)\big)^2\cos^2\theta \\
&= \big(f'(\theta)\big)^2\left[\cos^2\theta + \sin^2\theta\right] + \big(f(\theta)\big)^2\left[\sin^2\theta + \cos^2\theta\right] \\
&= \big(f'(\theta)\big)^2 + \big(f(\theta)\big)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2.
\end{aligned}$$

Therefore, the arc length of the curve is

$$\boxed{L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

Example 8.26 Find the length of the curve.

(1) $r = 2$

(2) $r = 2\sin\theta$

(3) $r = e^{-\theta}$ where $0 \le \theta \le 2\pi$

(4) $r = 2 - 2\cos\theta$

Solution:

(1) $r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4$. Hence,

$$L = \int_0^{2\pi} \sqrt{4}\, d\theta = 2\Big[\,\theta\,\Big]_0^{2\pi} = 4\pi.$$

(2) $r^2+(\frac{dr}{d\theta})^2 = 4\sin^2\theta + 4\cos^2\theta = 4(\sin^2\theta+\cos^2\theta) = 4$. This implies

$$L = \int_0^{\pi} \sqrt{4}\, d\theta = 2\Big[\theta\Big]_0^{\pi} = 2\pi.$$

(3) $r^2+(\frac{dr}{d\theta})^2 = e^{-2\theta}+e^{-2\theta} = 2e^{-2\theta}$. Hence,

$$L = \int_0^{2\pi} \sqrt{2e^{-2\theta}}\, d\theta = \sqrt{2}\int_0^{2\pi} e^{-\theta}\, d\theta = \sqrt{2}\big[1-e^{-2\pi}\big].$$

(4) $r^2+(\frac{dr}{d\theta})^2 = 4-8\cos\theta+4\cos^2\theta+4\sin^2\theta = 8-8\cos\theta = 8(1-\cos\theta)$.

$$L = \int_0^{2\pi} \sqrt{8(1-\cos\theta)}\, d\theta = 2\sqrt{2}\int_0^{2\pi} \sqrt{1-\cos\theta}\, d\theta.$$

Since $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$, then

$$L = 4\int_0^{2\pi} \sqrt{\sin^2\frac{\theta}{2}}\, d\theta = 8\int_0^{2\pi} \frac{1}{2}\sin\frac{\theta}{2}\, d\theta = -8\Big[\cos\frac{\theta}{2}\Big]_0^{2\pi} = 16.$$

Surface Area of Revolution in Polar Coordinates

Let the polar function $r=f(\theta)$, $\alpha\le\theta\le\beta$ be smooth. We know that

$$x = f(\theta)\cos\theta \;\text{ and }\; y = f(\theta)\sin\theta,\;\; \alpha\le\theta\le\beta.$$

■ The surface area generated by revolving the curve about the polar axis (the x-axis) is

$$\boxed{S.A = 2\pi\int_{\alpha}^{\beta} |\, r\sin\theta\,| \sqrt{r^2+(\frac{dr}{d\theta})^2}\, d\theta}$$

■ The surface area generated by revolving the curve about the line $\theta=\frac{\pi}{2}$ (the y-axis) is

$$\boxed{S.A = 2\pi\int_{\alpha}^{\beta} |\, r\cos\theta\,| \sqrt{r^2+(\frac{dr}{d\theta})^2}\, d\theta}$$

Note that when choosing α and β, we must ensure that the surface does not retrace itself when the curve C is revolved.

Example 8.27 Find the area of the surface generated by revolving the curve $r=2\sin\theta$ about

(1) the polar axis.

(2) the line $\theta=\frac{\pi}{2}$.

Solution:

(1) We apply the formula $S.A = 2\pi\int_{\alpha}^{\beta} |\, r\sin\theta\,| \sqrt{r^2+(\frac{dr}{d\theta})^2}\, d\theta$.

$$r^2+(\frac{dr}{d\theta})^2 = 4\sin^2\theta+4\cos^2\theta = 4(\sin^2\theta+\cos^2\theta) = 4.$$

Thus,

$$S.A = 8\pi\int_0^{\pi} \sin^2\theta\, d\theta = 4\pi\int_0^{\pi}(1-\cos 2\theta)\, d\theta = 4\pi\Big[\theta-\frac{\sin 2\theta}{2}\Big]_0^{\pi} = 4\pi\Big[\pi-0\Big] = 4\pi^2.$$

(2) We apply the formula $S.A = 2\pi\int_{\alpha}^{\beta} |\, r\cos\theta\,| \sqrt{r^2+(\frac{dr}{d\theta})^2}\, d\theta$. Thus,

$$S.A = 8\pi\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta = -\frac{8\pi}{2}\Big[\cos^2\theta\Big]_0^{\frac{\pi}{2}} = -4\pi\Big[0-1\Big] = 4\pi.$$

Exercise 8.4

1 - 6 Find the length of the curve.

1 $r = 3\cos\ \theta$

2 $r = \sin\ \theta$

3 $r = 2(1 - \cos\ \theta)$

4 $r = 3$

5 $r = 3 + 3\cos\ \theta$

6 $r = \theta,\ 0 \leq \theta \leq 1$

7 - 12 Find the area of the surface generated by revolving the graph of the equation about the polar axis.

7 $r = 1 + \cos\ \theta$

8 $r = \cos\ \theta$

9 $r = 3 - 3\cos\ \theta$

10 $r = 4$

11 $r = 4\sin\ \theta$

12 $r = 6(1 + \cos\ \theta)$

13 - 18 Find the area of the surface generated by revolving the graph of the equation about the line $\theta = \frac{\pi}{2}$.

13 $r = 1 + \sin\ \theta$

14 $r = 2$

15 $r = 1 - \sin\ \theta$

16 $r = 2(1 + \sin\ \theta)$

17 $r = 4\cos\ \theta$

18 $r = \sin\ \theta$

Review Exercises

1 - 8 The curve C is given parametrically. Find an equation in x and y, then sketch the graph and indicate the orientation.

1 $x = 3t,\ y = 2t+1,\ 0 \le t \le 3$

2 $x = t^2,\ y = 2\ln t,\ t > 0$

3 $x = t^2 - 1,\ y = t^2 + 1,\ -2 \le t \le 2$

4 $x = e^t,\ y = e^{-t},\ t \in \mathbb{R}$

5 $x = 3\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$

6 $x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$

7 $x = \ln t,\ y = te^t,\ t > 0$

8 $x = \sqrt{t},\ y = 2t+4,\ 0 \le t \le 5$

9 - 16 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the indicated value.

9 $x = 5t,\ y = 4t+2,\ \text{at } t = 1$

10 $x = e^t,\ y = e^{2t},\ \text{at } t = \ln 3$

11 $x = \sqrt{t},\ y = \frac{4t+2}{3},\ \text{at } t = 2$

12 $x = t^3 + 1,\ y = t^2 - 2t,\ \text{at } t = 1$

13 $x = \sin t,\ y = \cos t,\ \text{at } t = \frac{\pi}{6}$

14 $x = \sin^2 t,\ y = \cos^2 t,\ \text{at } t = \frac{\pi}{4}$

15 $x = 1 - \sin t,\ y = 1 - \cos t,\ \text{at } t = \frac{\pi}{3}$

16 $x = 3t^2 - 6t,\ y = \ln t,\ \text{at } t = 2$

17 - 24 Find an equation of the tangent line at the indicated value.

17 $x = t^2,\ y = t+2,\ \text{at } t = 2$

18 $x = \cos t,\ y = \sin^2 t,\ \text{at } t = \frac{\pi}{4}$

19 $x = \sqrt{t},\ y = \frac{t+1}{3},\ \text{at } t = 2$

20 $x = 2 + \sec t,\ y = 1 + 2\tan t,\ \text{at } t = \frac{\pi}{6}$

21 $x = t^3 - 3t,\ y = t^2 - 5t - 1,\ \text{at } t = 2$

22 $x = e^t,\ y = e^{-t},\ \text{at } t = 0$

23 $x = 1 + \sin t,\ y = 1 - 2\cos t,\ \text{at } t = \frac{\pi}{3}$

24 $x = \ln(t+1),\ y = t^2,\ \text{at } t = 3$

25 - 32 Find the points on the curve C at which the tangent line is either horizontal or vertical.

25 $x = 3t^2 - 6t,\ y = \sqrt{t},\ t \ge 0$

26 $x = t^3 - 3t,\ y = t^2 - 5t,\ t \in \mathbb{R}$

27 $x = 1 - t,\ y = t^2,\ t \in \mathbb{R}$

28 $x = 1 - t,\ y = t^3 - 3t,\ t \in \mathbb{R}$

29 $x = \sin t,\ y = \cos t,\ t \in \mathbb{R}$

30 $x = 1 + \sin t,\ y = 2\cos t,\ t \in \mathbb{R}$

31 $x = 1 - t^2,\ y = t^2 - t,\ t \in \mathbb{R}$

32 $x = e^t,\ y = e^{-t},\ t \in \mathbb{R}$

33 - 40 Find the length of the curve.

33 $x = 5t^2,\ y = t,\ 0 \le t \le 1$

34 $x = t+1,\ y = 2t,\ 0 \le t \le 5$

35 $x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$

36 $x = 2\sin t,\ y = 2\cos t,\ 0 \le t \le 2\pi$

37 $x = t^2,\ y = 2t,\ 0 \le t \le 3$

38 $x = \sqrt{t},\ y = 2t,\ 1 \le t \le 3$

39 $x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \frac{\pi}{2}$

40 $x = 8t^{\frac{3}{2}},\ y = 3 + (8-t)^{\frac{3}{2}},\ 0 \le t \le 4$

41 - 48 Find the area of the surface generated by revolving the curve about the x-axis.

41 $x = t^2,\ y = t,\ 0 \le t \le 3$

42 $x = t^3,\ y = t^2,\ 0 \le t \le 1$

43 $x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$

44 $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le \frac{\pi}{2}$

45 $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le \pi$

46 $x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \frac{\pi}{2}$

47 $x = e^t,\ y = e^{\frac{t}{2}},\ 0 \le t \le 1$

48 $x = 9 + 2t^2,\ y = 4t,\ 0 \le t \le 2$

49 - 56 Find the area of the surface generated by revolving the curve about the y-axis.

49 $x = 4t^3,\ y = 2t,\ 0 \le t \le 5$

50 $x = 3t + 2,\ y = t,\ 0 \le t \le 3$

51 $x = \cos^2 t,\ y = \sin^2 t,\ 0 \le t \le \frac{\pi}{2}$

52 $x = 3 + \cos t,\ y = \sin t,\ 0 \le t \le \pi$

53 $x = e^t,\ y = t,\ 0 \le t \le e$

54 $x = \sqrt{9 - t^2},\ y = 3t,\ -2 \le t \le 2$

55 $x = 2t,\ y = 1 - t^2,\ 0 \le t \le 1$

56 $x = 2\sin t,\ y = 2\cos t,\ \frac{\pi}{4} \le t \le \frac{\pi}{2}$

57 - 64 Find the corresponding rectangular coordinates for the given polar coordinates.

57 $(2, \pi)$

58 $(4, -\pi)$

59 $(-2, \frac{\pi}{3})$

60 $(1, \frac{\pi}{6})$

61 $(8, \frac{\pi}{4})$

62 $(-2, \pi)$

63 $(5, \frac{3\pi}{2})$

64 $(2, \frac{3\pi}{4})$

65 - 72 Find the corresponding polar coordinates for the given rectangular coordinates for $r \ge 0$ and $0 \le \theta \le \pi$.

65 $(1, 1)$

66 $(-1, 0)$

67 $(3, 3\sqrt{3})$

68 $(-2, 2)$

69 $(1, 0)$

70 $(\sqrt{2}, \frac{1}{2})$

71 $(-3, 0)$

72 $(-3, 4)$

73 - 80 Find a polar equation that has the same graph as the equation in x and y.

73 $x = 3$

74 $y = -7$

75 $x^2 + y^2 = 1$

76 $x^2 + y^2 - 6x = 0$

77 $xy = 4$

78 $y^2 = 9x$

79 $x^2 + y^2 + 9y = 0$

80 $x^2 - y^2 = 25$

81 - 88 Find an equation in x and y that has the same graph as the polar equation.

81 $r = 3$

82 $r = \sin\theta$

83 $r = 2\cos\theta$

84 $r\sin\theta = 4$

85 $r = \sec\theta$

86 $r(\cos\theta - \sin\theta) = 4$

87 $r = \frac{2}{1 - \sin\theta}$

88 $r = \frac{3}{1 + 2\cos\theta}$

89 - 92 Sketch the graph of the polar equations.

89 $r = 2$

90 $r = 4\sin\theta$

91 $r = 2(1 - \cos\theta)$

92 $r = 3(1 + \sin\theta)$

93 - 109 Find the area of the region bounded by the graph of the polar equations.

93 $r = 4\cos\theta$

94 $r = 6\sin\theta$

95 $r = \sin\theta,\ 0 \le \theta \le \frac{\pi}{4}$

96 $r = 1 + \sin\theta,$

97 $r = 2(1 - \cos\theta),\ 0 \le \theta \le \frac{\pi}{2}$

98 $r = e^{-\theta},\ 0 \le \theta \le 2\pi$

99 outside $r = 2(1 + \cos\theta)$ and inside $r = 3$

100 inside $r = 2(1+\sin\,\theta)$ and outside $r = 2$

101 outside $r = \cos\,\theta$ and inside $r = \sin\,\theta$

102 inside both graphs $r = 1-\cos\,\theta$ and $r = 1+\cos\,\theta$

103 inside both graphs $r = 4\cos\,\theta$ and $r = 4\sin\,\theta$

104 inside both graphs $r = 1-\sin\,\theta$ and $r = 1+\sin\,\theta$

105 outside $r = 2-2\cos\,\theta$ and inside $r = 4$

106 outside $r = 2(1+\sin\,\theta)$ and inside $r = 2(1-\sin\,\theta)$

107 inside $r = \cos\,\theta$ and $r = \sqrt{3}\sin\,\theta$

108 inside $r = 3$ and outside $r = 2$

109 inside $r = 2\cos\,\theta$ and outside $r = 4\cos\,\theta$

110 - 115 Find the length of the curve.

110 $r = 2$

111 $r = 2\sin\,\theta$

112 $r = 1-\cos\,\theta$

113 $r = e^{\theta},\ 0 \le \theta \le 2\pi$

114 $r = 3(1+\sin\,\theta)$

115 $r = \sqrt{5}\cos\,\theta$

116 - 121 Find the area of the surface generated by revolving the graph of the equation about the polar axis.

116 $r = \cos\,\theta,\ 0 \le \theta \le \frac{\pi}{2}$

117 $r = \sin\,\theta,\ 0 \le \theta \le \frac{\pi}{2}$

118 $r = 2+2\cos\,\theta$

119 $r = 1-\cos\,\theta$

120 $r = e^{\theta},\ 0 \le \theta \le \pi$

121 $r = \sqrt{3}\cos\,\theta,\ 0 \le \theta \le \frac{\pi}{2}$

122 - 127 Find the area of the surface generated by revolving the graph of the equation about line $\theta = \frac{\pi}{2}$.

122 $r = \cos\,\theta$

123 $r = \sin\,\theta,\ 0 \le \theta \le \frac{\pi}{2}$

124 $r = 1+\sin\,\theta$

125 $r = 1-\sin\,\theta$

126 $r = 3,\ 0 \le \theta \le \frac{\pi}{2}$

127 $r = e^{\theta},\ 0 \le \theta \le \frac{\pi}{2}$

128 - 150 Choose the correct answer.

128 The slope of the tangent line at the point corresponding to $t = 1$ on the curve given parametrically equations $x = 2t^2+1$, $y = 5t^3-1,\ -2 \le t \le 2$ is
(a) $\frac{5}{2}$ (b) $-\frac{5}{2}$ (c) $\frac{2}{5}$ (d) $\frac{15}{4}$

129 If a graph has polar equation $r = 2\sec\,\theta$, then its equation in xy-system is
(a) $x = 2$ (b) $y = 2$ (c) $x+y+1 = 0$ (d) $y = \frac{1}{2}$

130 The length of the curve C: $x = \cos\,2t$, $y = \sin\,2t$, $0 \le t \le \pi$ is equal to
(a) 2 (b) 2π (c) π (d) 4π

131 The surface area resulting by revolving the graph of the parametric equation $x = 3t$, $y = 3t$, $0 \le t \le 1$ about the x-axis is equal to
(a) $9\sqrt{2}\pi$ (b) $18\sqrt{2}\pi$ (c) $24\sqrt{2}\pi$ (d) $\frac{9}{2}\sqrt{2}\pi$

132 If a point has xy-coordinates $(x,y) = (1,1)$, then one of its (r,θ)-coordinates is
(a) $(1,\frac{\pi}{2})$ (b) $(-1,\frac{5\pi}{4})$ (c) $(2,\frac{\pi}{4})$ (d) $(\sqrt{2},\frac{\pi}{4})$

133 The slope of the tangent line to the graph of the equation $r = 2$ at $\theta = -\frac{\pi}{4}$ is
(a) 1 (b) -1 (c) 0 (d) ∞

134 The graph of the curve C defined by the parametric equations $x = 2 + \cos\ 2t, y = -1 + \sin\ 2t, 0 \le t \le \pi$ is
(a) a line (b) parabola (c) cardioid (d) circle

135 The slope of the tangent line at the point corresponding to $t = \frac{\pi}{4}$ on the parametric curve given by the equations , $x = \sin\ t,\ y = \cos\ t,\ 0 \le t \le 2\pi$ is
(a) -1 (b) 1 (c) 0 (d) $\frac{1}{3}$

136 If a graph has polar equation $r = 2\csc\ \theta$, then its equation in xy-system is
(a) $x = 2$ (b) $x = \frac{1}{2}$ (c) $y = 2$ (d) $y = \frac{1}{2}$

137 The length of the curve $C : x = \cos\ 2t,\ y = \sin\ 2t, 0 \le t \le \frac{\pi}{2}$
(a) π (b) $\frac{\pi}{2}$ (c) 2π (d) $\frac{\pi}{4}$

138 If a point has $(r,\theta)-$ coordinates $(r,\theta) = (1, \frac{\pi}{6})$, then its $(x,y)-$ coordinates is
(a) $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ (b) $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ (c) $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ (d) $(1,0)$

139 The slope of the tangent line to the curve: $r = \cos\ \theta$ at $\theta = \frac{\pi}{4}$ is
(a) $\frac{\pi}{2}$ (b) 0 (c) $\frac{\pi}{4}$ (d) 1

140 Let C be the curve given parametrically by : $x = t^2 + t,\ y = t^2 + 3,\ t \in \mathbb{R}$. The point on C at which the slope of the tangent line equal to 2 is given by
(a) $(0,4)$ (b) $(2,4)$ (c) $(4,4)$ (d) $(\frac{3}{4}, \frac{13}{4})$

141 If a graph has polar equation $r = \csc\ \theta$, then its equation in xy-system is
(a) $x = 1$ (b) $x + 1 = 0$ (c) $y = 1$ (d) $y + 1 = 0$

142 The length of the curve $C : \cos\ 4t,\ y = \sin\ 4t,\ 0 \le t \le \frac{\pi}{4}$ is equal to
(a) $\frac{\pi}{2}$ (b) 2π (c) π (d) 4π

143 If a point has xy-coordinates $(x,y) = (1,1)$ then one of its $(r,\theta)-$ coordinates is
(a) $(1, \frac{\pi}{4})$ (b) $(2, \frac{\pi}{4})$ (c) $(\sqrt{2}, \frac{-\pi}{4})$ (d) $(-\sqrt{2}, \frac{5\pi}{4})$

144 The equation in polar coordinates for the line $y = x - 1$ is
(a) $r = \frac{1}{\cos\ \theta - \sin\ \theta}$ (b) $r = \frac{1}{\cos\ \theta + \sin\ \theta}$ (c) $r = \frac{1}{\cos\ \theta} + \frac{1}{\sin\ \theta}$ (d) $r = \cos\ \theta + \sin\ \theta$

145 The parametric equation of the circle centered at the origin with radius 5 is given by
(a) $x = \cos\ 5\theta, y = \sin\ 5\theta$
(b) $x = 5\cos\ \theta, y = 5\sin\ \theta$
(c) $5x = \cos\ \theta, 5y = \sin\ \theta$
(d) $x = \cos\ \theta, y = \sin\ \theta$

146 The slope of the tangent line at the point corresponding to $t = \frac{\pi}{2}$ on the parametric curve given by the equations , $x = \sin^2\ t,\ y = \cos\ t,\ \frac{\pi}{2} \le t \le 2\pi$ is
(a) $-\infty$ (b) -1 (c) 0 (d) 1

147 The length of the curve $C : x = 2\cos\ t,\ y = 2\sin\ t;\ 0 \le t \le 1$ is equal to
(a) 1 (b) $\sqrt{2}$ (c) 2 (d) 4

148 If a graph has a polar equation $r = \frac{1}{2\sin\ \theta + \cos\ \theta}$, then its equation in xy-system is
(a) $x + 2y + 1 = 0$ (b) $x + 2y - 1 = 0$ (c) $2x + y + 1 = 0$ (d) $2x + y - 1 = 0$

149 The slope tangent line to the graph of the equation $r = 2$ at $\theta = \frac{\pi}{4}$ is
(a) 1 (b) -1 (c) 0 (d) ∞

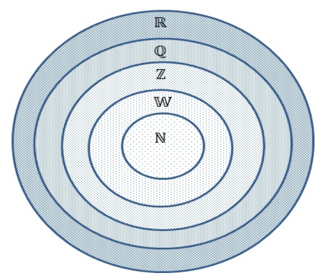
150 The polar equation that has the same graph as the equation $x^4 + 2x^2y^2 + y^4 = 2xy$ is
(a) $r^2 = \sin\ 2\theta$ (b) $r^2 = \cos\ 2\theta$ (c) $r^2 = \sin\ \theta\ \cos\ \theta$ (d) $r^2 = \frac{\sin\ 2\theta}{1 + \frac{1}{2}\sin\ 2\theta}$

Appendix

Appendix (1): Basic Mathematical Concepts

Mathematical Expressions$\Rightarrow$ is the symbol for implying. $\Leftrightarrow$ is the symbol for "$\Rightarrow$ and $\Leftarrow$. Also, the expression "iff" means if and only if . $b > a$ means b is greater than a and $a < b$ means a is less than b. $b \geq a$ to denote that b is greater than or equal to a.

Sets of Numbers & Notations



1. Natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$.
2. Whole numbers $\mathbb{W} = \{0, 1, 2, 3, ...\}$.
3. Integers $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
4. Rational numbers $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$.
5. Irrational numbers $\mathbb{I} = \{x \mid x \text{ is a real number that is not rational}\}$.
6. Real numbers $\mathbb{R}$ contains all the previous sets.

Fractions Operations

• Adding or subtracting two fractions

To add or subtract two fractions, we do the following steps:

1. Find the least common denominator.
2. Write both original fractions as equivalent fractions with the least common denominator.
3. Add (or subtract) the numerators.
4. Write the result with the denominator.

• Multiplying two fractions

To multiple two fractions, we do the following steps:

1. Multiply the numerator by the numerator.
2. Multiply the denominator by the denominator.

$$\frac{a}{b}.\frac{c}{d} = \frac{ac}{bd} \quad \text{where} \quad b \neq 0 \quad \text{and} \quad d \neq 0 \,.$$

• Dividing two fractions

To divide two fractions, we do the following steps:

1. Find the multiplicative inverse of the second fraction.
2. Multiply the two fractions.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b}.\frac{d}{c} = \frac{ad}{bc} \quad \text{where} \quad b \neq 0 \quad \text{and} \quad d \neq 0 \,.$$

Example 1

(1) $\frac{3}{7} + \frac{2}{5} = \frac{15}{35} + \frac{14}{35} = \frac{15+14}{35} = \frac{29}{35}$

(2) $\frac{4}{9} - \frac{3}{7} = \frac{28}{63} - \frac{27}{63} = \frac{28-27}{63} = \frac{1}{63}$

(3) $\frac{2}{5} \times \frac{4}{9} = \frac{2\times4}{5\times9} = \frac{8}{45}$

(4) $\frac{2}{5} \div \frac{4}{9} = \frac{2}{5} \times \frac{9}{4} = \frac{2\times9}{5\times4} = \frac{18}{20}$

Exponents

Assume n is a positive integer and a is a real number. The nth power of a is

$$a^n = a.a...a \,.$$

Basic Rules

For every $x, y > 0$ and $a, b \in \mathbb{R}$,

1. $x^0 = 1$
2. $x^a x^b = x^{a+b}$
3. $\frac{x^a}{x^b} = x^{a-b}$
4. $(x^a)^b = x^{ab}$
5. $(xy)^a = x^a y^a$
6. $x^{-a} = \frac{1}{x^a}$

Example 2

(1) $2^3 2^{-5} = 2^{3-5} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

(2) $\frac{3^2}{3^{-2}} = 3^{2-(-2)} = 3^4 = 81$

(3) $(5x)^2 = 25x^2$

(4) $\frac{x^2 y^3}{(yz)^5} = \frac{x^2 y^3}{y^5 z^5} = x^2 \frac{y^3}{y^5} \frac{1}{z^5} = x^2 y^{3-5} \frac{1}{z^5} = \frac{x^2}{y^2 z^5}$

Algebraic Expressions

Let a and b be real numbers. Then,

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)(a-b) = a^2 - b^2$
4. $(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$
5. $(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$
6. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
7. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
8. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2} b + a^{n-3} b^2 + \ldots + ab^{n-2} + b^{n-1})$

Example 3

(1) $(x \pm 2)^2 = x^2 \pm 4x + 4$

(2) $x^2 - 25 = (x-5)(x+5)$

(3) $(x \pm 2)^3 = x^3 \pm 6x^2 + 12x \pm 8$

(4) $x^3 \pm 27 = (x \pm 3)(x^2 \mp 3x + 9)$

Intervals

Let $a, b \in \mathbb{R}$ and $a < b$.

• Open interval (a, b).
It contains all real numbers between a and b, i.e.,

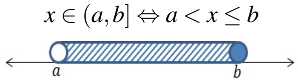
$$x \in (a, b) \Leftrightarrow a < x < b$$

• Closed interval $[a, b]$.
It contains all real numbers between a and b including a and b, i.e.,

$$x \in [a, b] \Leftrightarrow a \leq x \leq b$$

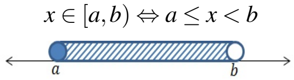
• Half-open interval $(a, b]$.
It contains all real numbers between a and b including b, i.e.,

$$x \in (a, b] \Leftrightarrow a < x \leq b$$



• Half-open interval $[a, b)$.
It contains all real numbers between a and b including a, i.e.,

$$x \in [a, b) \Leftrightarrow a \leq x < b$$



• Interval $[a, \infty)$
It contains all real numbers larger than or equal to a, i.e.,

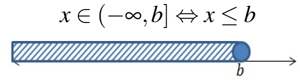
$$x \in [a, \infty) \Leftrightarrow a \leq x$$

• Interval (a, ∞)
It contains all real numbers larger than a, i.e.,

$$x \in (a, \infty) \Leftrightarrow a < x$$

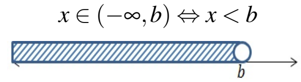
• Interval $(-\infty, b]$
It contains all real numbers less than or equal to b, i.e.,

$$x \in (-\infty, b] \Leftrightarrow x \leq b$$



• Interval $(-\infty, b)$
It contains all real numbers less than b, i.e.,

$$x \in (-\infty, b) \Leftrightarrow x < b$$



Example 4

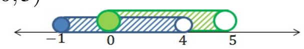
(1) $(2,5]$

(2) $[-2,4)\cap[1,6)$

$[1,4)$

(3) $[-1,\infty)$

(4) $[-1,4)\cup[0,5)$



$[-1,5)$

Absolute Value

The absolute value of x is defined as follows:

$$|\,x\,| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

Example 5 $|2|=2$, $|-2|=2$, $|0|=0$.

Equations and Inequalities

If $b>0$,

1. $|x-a|=b \Leftrightarrow x=a-b$ or $x=a+b$.
2. $|x-a|<b \Leftrightarrow a-b<x<a+b$.
3. $|x-a|>b \Leftrightarrow x<a-b$ or $x>a+b$.

Example 6 Solve for x.

(1) $|3x-4|=7$ **(2)** $|2x+1|<1$

Solution:

(1) $|3x-4|=7 \Leftrightarrow 3x-4=7$ or $3x-4=-7$. Thus, $x=\frac{11}{3}$ or $x=-1$.

(2) $|2x+1|<1 \Leftrightarrow -1<2x+1<1$. By subtracting 1 and then dividing by 2, we have $-1<x<0$.

Functions

A function $f: D \to S$ is a mapping that assigns each element in D to an element in S. The set D is called the domain of the function f. All values of $f(x)$ belong to a set $R \subseteq S$ called the range.

• Domains and Ranges

In the following, we show how to determine the domain and range of some functions.

1. Polynomials $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$.
   Domain: $\mathbb{R}$ Range: $\mathbb{R}$
2. Square Roots $f(x)=\sqrt{g(x)}$.
   Domain: $x\in\mathbb{R}$ such that $g(x)\geq 0$ Range: $\mathbb{R}^+$
3. Rational Functions $q(x)=\frac{f(x)}{g(x)}$.
   To determine the domain, we need to find the intersection of the domains of f and g. Then, we remove zeros of the function g.

Example 7 Find the domain of the function.

(1) $f(x)=\sqrt{x-1}$

(2) $q(x)=\frac{x+1}{2x-1}$

(3) $q(x)=\frac{3x^2+x+2}{\sqrt{x+2}}$

Solution:

(1) We need to find all $x\in\mathbb{R}$ such that $x-1\geq 0$. By solving the inequality, we have $x-1\geq 0 \Rightarrow x\geq 1$. Thus, the domain is $[1,\infty)$. Hence, $\forall x\in D(f)$, $f(x)=\sqrt{g(x)}\geq 0$ i.e., the range is $[0,\infty)$.

(2) The domain of the numerator and the denominator is $\mathbb{R}$. The denominator $g(x)=0$ if $x=\frac{1}{2}$. Thus, the domain of q is $\mathbb{R}\setminus\{\frac{1}{2}\}$.

(3) The domain of the numerator is $\mathbb{R}$ and the domain of the denominator is $[-2,\infty)$. The denominator $g(x)=0$ if $x=-2$. Thus, the domain of q is $(-2,\infty)$.

• Functions Operations

Let f and g be functions such that x belongs to their domains. Then

1. $(f\pm g)(x)=f(x)\pm g(x)$.
2. $(fg)(x)=f(x)g(x)$.
3. $(\frac{f}{g})(x)=\frac{f(x)}{g(x)}$ where $g(x)\neq 0$.

Example 8 If $f(x)=x^2-1$ and $g(x)=x-1$, find the following:

(1) $(f+g)(x)$ **(2)** $(fg)(x)$ **(3)** $(\frac{f}{g})(x)$

Solution:

(1) $(f+g)(x) = f(x)+g(x) = (x^2-1)+(x-1) = x^2+x-2$
(2) $(fg)(x) = f(x)g(x) = (x^2-1)(x-1) = x^3-x^2-x+1$
(3) $(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = x+1$

• Composite Functions

If f and g are two functions, the composite function $(f\circ g)(x) = f(g(x))$. The domain of $f\circ g$ is $\{\forall x \in D(g) : g(x) \in D(f)\}$.

Example 9 If $f(x) = x^2$ and $g(x) = x+2$, find $(f\circ g)(x)$.

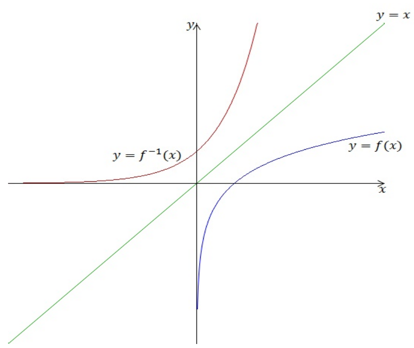
Solution:

$(f\circ g)(x) = f(g(x)) = \big(x+2\big)^2 = x^2+4x+4.$

• Inverse Functions

A function f has an inverse function f^{-1} if it is one to one: $y = f^{-1}(x) \Leftrightarrow x = f(y)$.[1]

Properties of inverse functions:
1. $D(f^{-1})$ is the range of f.
2. The range of f^{-1} is the domain of f.
3. $f^{-1}\big(f(x)\big) = x, \forall x \in D(f)$.
4. $f\big(f^{-1}(x)\big) = x, \forall x \in D(f^{-1})$.
5. $(f^{-1})^{-1}(x) = f(x) \forall x \in D(f)$.



• Even and Odd Functions

Let f be a function and $-x \in D(f)$.

1. If $f(-x) = -f(x)$ $\forall x \in D(f)$, the function f is odd.
2. If $f(-x) = f(x)$ $\forall x \in D(f)$, the function f is even.

Example 10

(1) The function $f(x) = 2x^3+x$ is odd because $f(-x) = 2(-x)^3+(-x) = -2x^3-x = -(2x^3+x) = -f(x)$.
(2) The function $f(x) = x^4+3x^2$ is even because $f(-x) = (-x)^4+3(-x)^2 = x^4+3x^2 = f(x)$.

Roots of Linear and Quadratic Equations

• Linear Equations

A linear equation is an equation that can be written in the form $ax+b=0$ where x is the unknown, and $a,b \in \mathbb{R}$ and $a \neq 0$. To solve the equation, we subtract b from both sides and then divide the result by a:

$$ax+b=0 \Rightarrow ax+b-b=0-b \Rightarrow ax=-b \Rightarrow x=\frac{-b}{a}\ .$$

Example 11 Solve for x the equation $x+2=5$.

Solution:

$$3x+2=5 \Rightarrow 3x=5-2 \Rightarrow 3x=3 \Rightarrow x=\frac{3}{3}=1\ .$$

[1]The -1 in f^{-1} is not exponent where $\frac{1}{f(x)}$ is written as $\big(f(x)\big)^{-1}$

• Quadratic Equations

A quadratic equation is an equation that can be written in the form $ax^2+bx+c=0$ where a, b, and c are constants and $a \neq 0$. The quadratic equations are solved by using the factorization method or the quadratic formula, or the completing the square.

Factorization Method

The factorization method depends on finding factors of c that add up to b. Then, we use the fact that if $x, y \in \mathbb{R}$, then

$$xy = 0 \Rightarrow x = 0 \text{ or } y = 0 \, .$$

Example 12 Solve for x the following quadratic equations:

(1) $x^2+2x-8=0$
(2) $x^2+5x+6=0$

Solution:

(1) Consider 2 and -4, we have $2\times(-4) = -8 = c$, but $2+(-4) = -2 \neq b$. Now, consider -2 and 4, then $-2\times 4 = -8 = c$ and $-2+4 = 2 = b$. Thus,

$$x^2+2x-8 = (x-2)(x+4) = 0 \Rightarrow x-2=0 \text{ or } x+4=0 \Rightarrow x=2 \text{ or } x=-4\, .$$

(2) By factoring the left side, we have

$$(x+2)(x+3) = 0 \Rightarrow x+2=0 \text{ or } x+3=0 \Rightarrow x=-2 \text{ or } x=-3\, .$$

Quadratic Formula Solutions

We can solve the quadratic equations by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\, .$$

Remark: The expression b^2-4ac is called the discriminant of the quadratic equation.

1. If $b^2-4ac>0$, then the equation has two distinct real solutions.

2. If $b^2-4ac=0$, then the equation has one distinct real solution.

3. If $b^2-4ac<0$, then the equation has no real solutions.

Example 13 Solve for x the following quadratic equations:

(1) $x^2+2x-8=0$
(2) $x^2+2x+1=0$
(3) $x^2+2x+8=0$

Solution:

(1) $a=1,\ b=2,\ c=-8$. Since $b^2-4ac = 2^2-4(1)(-8) = 36$, then there are two solutions $x=2$ and $x=-4$.
(2) $a=1,\ b=2,\ c=1$. Since $b^2-4ac = 2^2-4(1)(1) = 0$, then there is one solution $x=-1$.
(3) $a=1,\ b=2,\ c=8$. Since $b^2-4ac = 2^2-4(1)(8) < 0$, then there are no real solutions.

Completing the Square Method

To solve the quadratic equation by the completing the square method, we need to do the following steps:

Step 1: Divide all terms by a (the coefficient of x^2).

Step 2: Move the term $(\frac{c}{a})$ to the right side of the equation.

Step 3: Complete the square on the left side of the equation and balance this by adding the same value to the right side.

Step 4: Take the square root of both sides and subtract the number that remains on the left side.

Example 14 Solve for x the quadratic equation $x^2+2x-8=0$.

Solution: $a=1,\ b=2,\ c=-8$.

Step 1 can be skipped in this example since $a=1$.

Step 2: $x^2+2x=8$.

Step 3: To complete the square, we need to add $(\frac{b}{2})^2$ since $a=1$.

$$x^2+2x+1 = 8+1 \Rightarrow (x+1)^2 = 9\, .$$

Step 4: $x+1 = \pm 3 \Rightarrow x = \pm 3 - 1 \Rightarrow x=2$ or $x=-4$.

Systems of Equations

A system of equations consists of two or more equations with the same set of unknowns. The equations in the system can be linear or non-linear, but for the purpose of this book, we only consider the linear ones.

Consider a system of two equations in two unknowns x and y

$$ax + by = c$$

$$dx + ey = f\ .$$

To solve the system, we try to find values of the unknowns that will satisfy each equation in the system. To do this, we can use elimination or substitution.

Example 15 Solve the following system of equations:

$$x - 3y = 4 \rightarrow 1$$

$$2x + y = 6 \rightarrow 2$$

Solution:

• By using the elimination method.

Multiply equation 2 by 3, then add the result to equation 1. This implies $7x = 22 \Rightarrow x = \frac{22}{7}$. Substitute the value of x into the first or the second equation to obtain $y = -\frac{2}{7}$.

• By using the substitution method.

From the first equation, we have $x = 4 + 3y$. By substituting that into the second equation, we obtain

$$2(4 + 3y) + y = 6 \Rightarrow 7y + 8 = 6 \Rightarrow y = -\frac{2}{7}$$

Substitute value of y into $x = 4 + 3y$ to have $x = \frac{22}{7}$.

Pythagorean Theorem

If c denotes the length of the hypotenuse and a and b denote the lengths of the other two sides, the Pythagorean theorem can be expressed as follows:

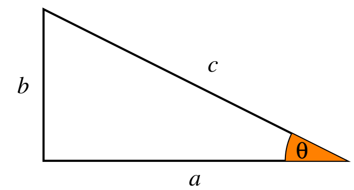
$$a^2+b^2=c^2 \text{ or } c=\sqrt{a^2+b^2}\,.$$

If a and c are known and b is unknown, then

$$b=\sqrt{c^2-a^2}\,.$$

Similarly, if b and c are known and a is unknown, then

$$a=\sqrt{c^2-b^2}$$

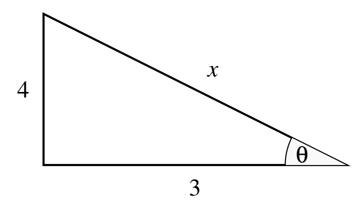


a is adjacent to the angle θ
b is opposite
c is hypotenuse

The trigonometric functions for a right triangle are

$$\cos\theta=\frac{a}{c} \qquad \sin\theta=\frac{b}{c} \qquad \tan\theta=\frac{b}{a}$$

Example 16 Find value of x. Then find $\cos\theta$, and $\sin\theta$.

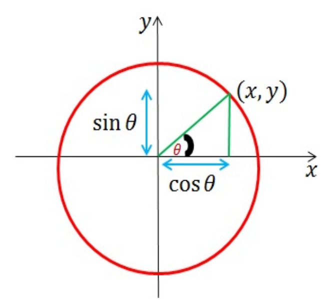


Solution:
$a=3,\ b=4 \Rightarrow c^2=4^2+3^2=25 \Rightarrow c=5$
$\cos\theta=\frac{3}{5}$
$\sin\theta=\frac{4}{5}$

Trigonometric Functions

- If (x,y) is a point on the unit circle, and if the ray from the origin $(0,0)$ to that point (x,y) makes an angle θ with the positive x-axis, then

$$\cos\theta=x\,,\quad \sin\theta=y\,,$$



- Each point (x,y) on the unit circle can be written as $(\cos\theta,\sin\theta)$.
- Since $x^2+y^2=1$, then $\cos^2\theta+\sin^2\theta=1$. Therefore,

$$1+\tan^2\theta=\sec^2\theta \text{ and } \cot^2\theta+1=\csc^2\theta.$$

Also,

$$\tan\theta=\frac{\sin\theta}{\cos\theta} \qquad \cot\theta=\frac{\cos\theta}{\sin\theta} \qquad \sec\theta=\frac{1}{\cos\theta} \qquad \csc\theta=\frac{1}{\sin\theta}$$

- Trigonometric functions of negative angles

$$\cos(-\theta)=\cos(\theta), \qquad \sin(-\theta)=-\sin(\theta), \qquad \tan(-\theta)=-\tan(\theta)$$

- Double and half angle formulas

$$\sin 2\theta=2\sin\theta\cos\theta, \qquad \cos 2\theta=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$$

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2},\ \cos^2\frac{\theta}{2}=\frac{1+\cos\theta}{2}$$

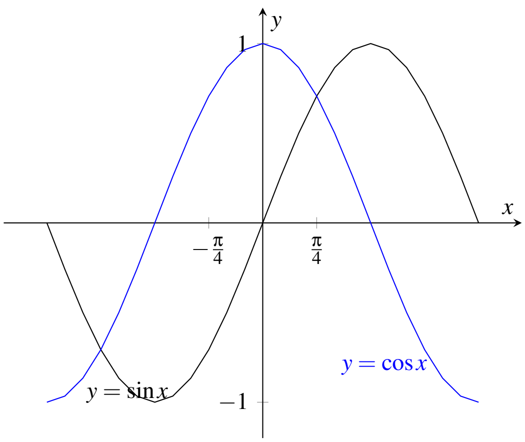
- Angle addition formulas

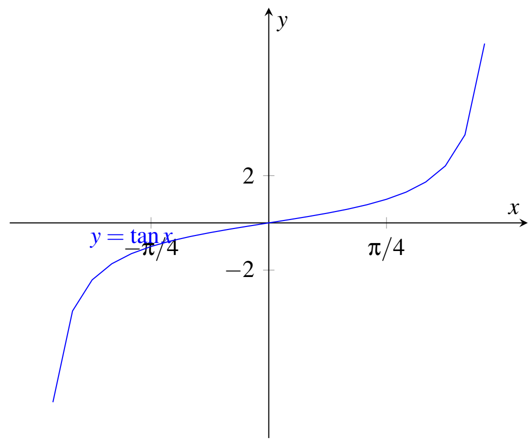
$$\sin(\theta_1 \pm \theta_2) = \sin\theta_1 \cos\theta_2 \pm \cos\theta_1 \sin\theta_2$$
$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2$$
$$\tan(\theta_1 \pm \theta_2) = \frac{\tan\theta_1 \pm \tan\theta_2}{1 \mp \tan\theta_1 \tan\theta_2}$$

• Values of trigonometric functions of most commonly used angles

Degrees	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

• Graphs of trigonometric functions



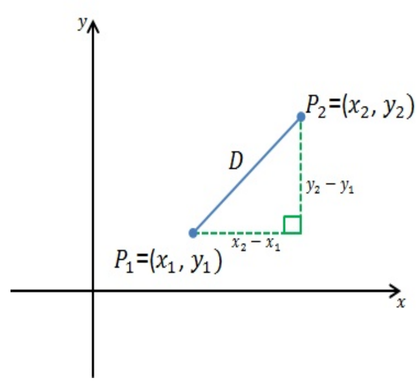


Distance Formula

Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in the Cartesian plane. The distance between P_1 and P_2 is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \,.$$

Example 17 Find the distance between the two points $P_1(1,1)$ and $P_2(-3,4)$.
Solution: $D = \sqrt{(-3-1)^2 + (4-1)^2} = \sqrt{16+9} = \sqrt{25} = 5$.



Differentiation of Functions

Differentiation Rules

$\frac{d}{dx}\big(f(x)+g(x)\big) = f'(x)+g'(x)$

$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x)+f(x)g'(x)$

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{\big(g(x)\big)^2}$

$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{-g'(x)}{\big(g(x)\big)^2}$

$\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

Elementary Derivatives

$\frac{d}{dx}x^r = rx^{r-1}$ $\qquad \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$ $\qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

Derivative of Composite Functions (Chain Rule)

If $y = f(u), u = g(x)$ such that dy/du and du/dx exist, then the derivative of the composite function $(f \circ g)(x)$ exists and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = f'(u)g'(x) = f'\big(g(x)\big)g'(x) \,.$$

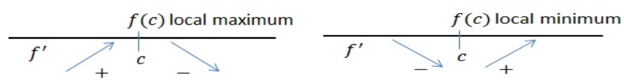
■ Derivative of Inverse Functions

If a function f has an inverse function f^{-1}, then $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$.

Graphs of Functions

• The First and Second Derivative Tests

1. Let f be continuous on $[a,b]$ and f' exists on (a,b).
   - If $f'(x) > 0, \forall x \in (a,b)$, then f is increasing on $[a,b]$.
   - If $f'(x) < 0, \forall x \in (a,b)$, then f is decreasing on $[a,b]$.
2. Let f be continuous at a critical number c and differentiable on an open interval (a,b), except possibly at c.
   - $f(c)$ is a local maximum of f if f' changes from positive to negative at c.
   - $f(c)$ is a local minimum of f if f' changes from negative to positive at c.



3. If f'' exists on an open interval I,
   - the graph of f is concave upward on I if $f''(x) > 0$ on I.
   - the graph of f is concave downward on I if $f''(x) < 0$ on I.

• Shifting Graphs

Let $y = f(x)$ is a function.

1. Replacing each x in the function with $x - c$ shifts the graph c units horizontally.
   - If $c > 0$, the shift will be to the right.
   - If $c < 0$, the shift will be to the left.
2. Replacing y in the function with $y - c$ shifts the graph c units vertically.
   - If $c > 0$, the shift will be upward.
   - If $c < 0$, the shift will be downward.

• Symmetry about the y-axis and the origin

1. If a function f is odd, the graph of f is symmetric about the origin.
2. If a function f is even, the graph of f is symmetric about the y-axis.

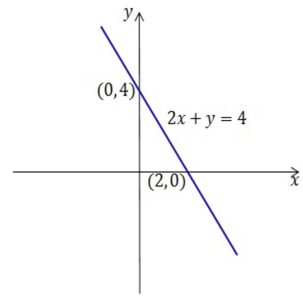
• Lines

The general linear equation in two variables x and y can be written in the form:

$$ax + by + c = 0 ,$$

where a, b and c are constants with a and b not both 0.

Example 18



$2x+y=4$
$a=2,\ b=-1,\ c=-4$
To plot the line, we rewrite the equation to become $y=-2x+4$. Then, we use the following table to make points on the plane:

x	0	2
y	4	0

The line $2x+y=4$ passes through the points $(0,4)$ and $(2,0)$.

Slope

1. The slope of a line passing through the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is $m=\frac{y_2-y_1}{x_2-x_1}$.
2. Point-Slope form: $y-y_1=m(x-x_1)$.
3. Slope-Intercept form:
   If $b\neq 0$, the general linear equation can be rewritten as

$$ax+by+c=0 \Rightarrow by=-ax-c \Rightarrow y=-\frac{a}{b}x-\frac{c}{b} \Rightarrow y=mx+d\ ,$$

   where m is the slope.

Example 19 Find the slope of the line $2x-5y+9=0$.

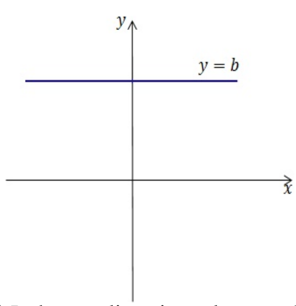
Solution: $2x-5y+9=0 \Rightarrow -5y=-2x-9 \Rightarrow y=\frac{2}{5}x+\frac{9}{5}$.

Thus, the slope is $\frac{2}{5}$. Alternatively, take any two points on that line say $(-2,1)$ and $(3,3)$. Then,

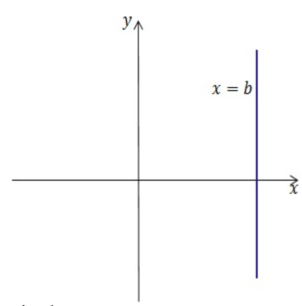
$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{3-(-2)}=\frac{2}{5}\ .$$

Special cases of lines in a plane

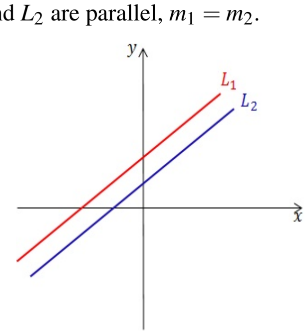
1. If m is undefined, the line is vertical.



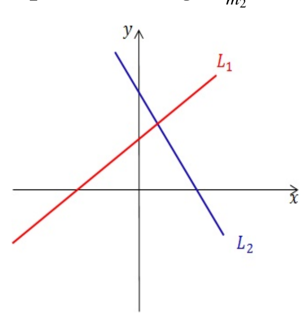
2. If $m=0$, the line is horizontal.



3. Let L_1 and L_2 be two lines in a plane, and let m_1 and m_2 be their slopes, respectively.
   - If L_1 and L_2 are parallel, $m_1=m_2$.



   - If L_1 and L_2 are vertical, $m_1=\frac{-1}{m_2}$.



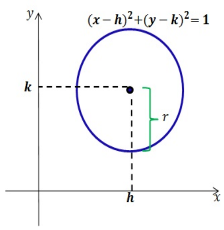
• Quadratic Functions

■ Circles

Let $C(h,k)$ be the center of a circle and r be the radius.
The equation of the circle is

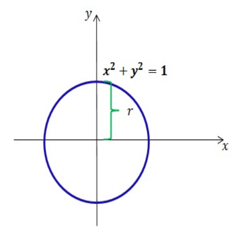
$$(x-h)^2+(y-k)^2=r^2$$

for $h,k>0$



If $h=k=0$, the center of the circle is the origin $(0,0)$ and the equation of the circle becomes

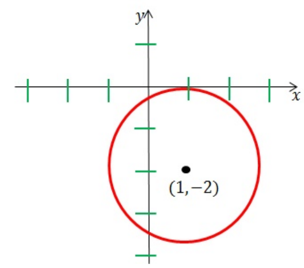
$$x^2+y^2=r^2 .$$



Example 20 Find the equation of the circle that has center at the point $(1,-2)$ and radius $r=2$.

Solution:

$$(x-1)^2+(y+2)^2=4$$
$$x^2+y^2-2x+4y=-1 .$$



■ Conic Sections

Parabola:

A parabola is the set of all points in the plane equidistant from a fixed point F (called the focus) and a fixed line D (called the directrix).

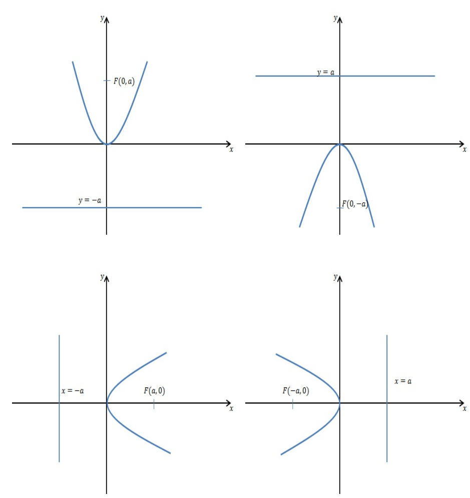
1. The vertex of the parabola is the origin $(0,0)$.

(A) $x^2=4ay$, $a>0$.
- The parabola opens upward.
- Focus: $F(0,a)$.
- Directrix equation: $y=-a$.
- Parabola axis: the y-axis.

(B) $x^2=-4ay$, $a>0$.
- The parabola opens downward.
- Focus: $F(0,-a)$.
- Directrix equation: $y=a$.
- Parabola axis: the y-axis.

(C) $y^2=4ax$, $a>0$.
- The parabola opens to the right.
- Focus: $F(a,0)$.
- Directrix equation: $x=-a$.
- Parabola axis: the x-axis.

(D) $y^2=-4ax$, $a>0$.
- The parabola opens to the left.
- Focus: $F(-a,0)$.
- Directrix equation: $x=a$.
- Parabola axis: the x-axis.



2. The general formula of a parabola $V(h,k)$:

(A) $(x-h)^2=4a(y-k)$, $a>0$.
- The parabola opens upwards.
- Focus: $F(h,k+a)$.
- Directrix equation: $y=k-a$.
- Parabola axis: parallel to the y-axis.

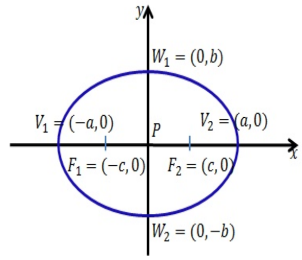
(B) $(x-h)^2=-4a(y-k)$, $a>0$.
- The parabola open downwards.
- Focus: $F(h,k-a)$.
- Directrix equation: $y=k+a$.
- Parabola axis: parallel to the y-axis.

(C) $(y-k)^2=4a(x-h)$, $a>0$
- The parabola opens to the right.
- Focus: $F(h+a,k)$.
- Directrix equation: $x=h-a$.
- Parabola axis: parallel to the x-axis.

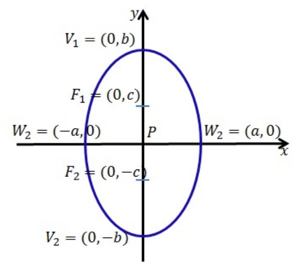
(D) $(y-k)^2=-4a(x-h)$, $a>0$
- The parabola opens to the left.
- Focus: $F(h-a,k)$.
- Directrix equation: $x=h+a$.
- Parabola axis: parallel to the x-axis.

Ellipse:
An ellipse is the set of all points in the plane for which the sum of the distances to two fixed points is constant.

1. The center of the ellipse is the origin $(0,0)$.
(A) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b$ and $c=\sqrt{a^2-b^2}$.
- Foci: $F_1(-c,0)$ and $F_2(c,0)$.
- Vertices: $V_1(-a,0)$ and $V_2(a,0)$.
- Major axis: the x-axis, its length is $2a$.
- Minor axis endpoints: $W_1(0,b)$ and $W_2(0,-b)$.



(B) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b>a$ and $c=\sqrt{b^2-a^2}$.
- Foci: $F_1(0,c)$ and $F_2(0,-c)$.
- Vertices: $V_1(0,b)$ and $V_2(0,-b)$.
- Major axis: the y-axis, its length is $2b$.
- Minor axis endpoints: $W_1(-a,0)$ and $W_2(a,0)$.

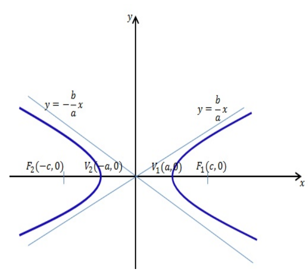


2. The general formula of the ellipse $P(h,k)$.
(A) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where $a>b$ and $c=\sqrt{a^2-b^2}$.
- Foci: $F_1(h-c,k)$ and $F_2(h+c,k)$.
- Vertices: $V_1(h-a,k)$ and $V_2(h+a,k)$.
- Major axis: parallel to the x-axis, its length is $2a$.
- Minor endpoints: $W_1(h,k+b)$ and $W_2(h,k-b)$.

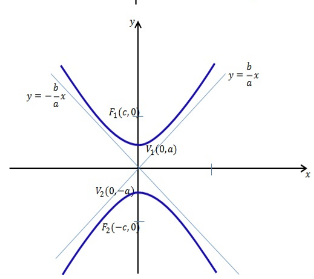
(B) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ where $b>a$ and $c=\sqrt{b^2-a^2}$.
- Foci: $F_1(h,k+c)$ and $F_2(h,k-c)$.
- Vertices: $V_1(h,k+b)$ and $V_2(h,k-b)$.
- Major axis: parallel to the y-axis, its length is $2b$.
- Minor endpoints: $W_1(h-a,k)$ and $W_2(h+a,k)$.

Hyperbola:
A hyperbola is the set of all points in the plane for which the absolute difference of the distances between two fixed points is constant.

1. The center of the hyperbola is the origin $(0,0)$.
(A) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $c=\sqrt{a^2+b^2}$.
- Foci: $F_1(-c,0)$ and $F_2(c,0)$.
- Vertices: $V_1(-a,0)$ and $V_2(a,0)$.
- Transverse axis: the x-axis, its length is $2a$.
- Asymptotes: $y=\pm\frac{b}{a}x$.



(B) $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ where $c=\sqrt{a^2+b^2}$.
- Foci: $F_1(0,c)$ and $F_2(0,-c)$.
- Vertices: $V_1(0,b)$ and $V_2(0,-b)$.
- Transverse axis: the y-axis, its length is $2b$.
- Asymptotes: $y=\pm\frac{b}{a}x$.

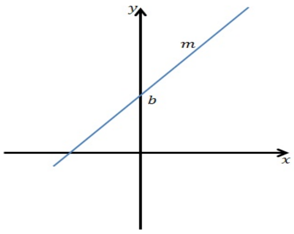


2. The general formula of the hyperbola $P(h,k)$.
(A) $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where $c=\sqrt{a^2+b^2}$.
- Foci: $F_1(h-c,k)$ and $F_2(h+c,k)$.
- Vertices: $V_1(h-a,k)$ and $V_2(h+a,k)$.
- Transverse axis: parallels to the x-axis, its length is $2a$.
- Asymptotes: $(y-k)=\pm\frac{b}{a}(x-h)$.

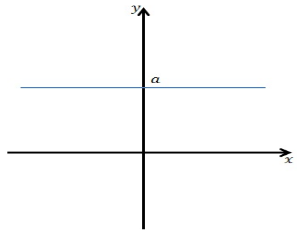
(B) $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ where $c=\sqrt{a^2+b^2}$.
- Foci: $F_1(h,k+c)$ and $F_2(h,k-c)$.
- Vertices: $V_1(h,k+b)$ and $V_2(h,k-b)$.
- Transverse axis: parallels to the y-axis, its length is $2b$.
- Asymptotes: $(y-k)=\pm\frac{b}{a}(x-h)$.

• **Graph of Some Functions**

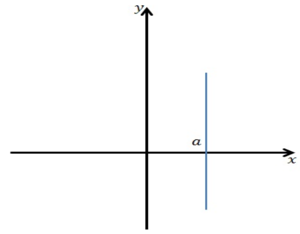
$y = mx + b$



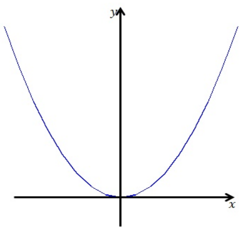
$y = a$



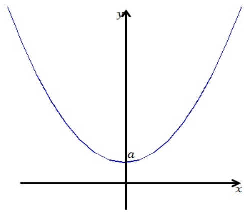
$x = a$



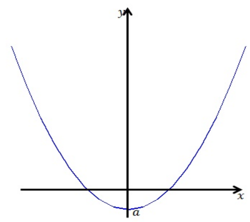
$y = x^2$



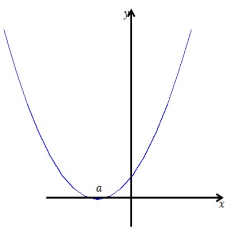
$y = x^2 + a$



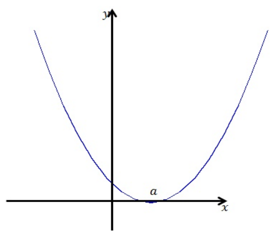
$y = x^2 - a$



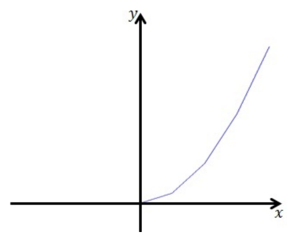
$y = (x+a)^2$



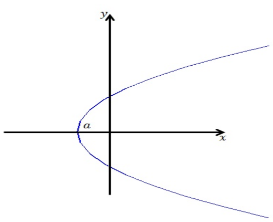
$y = (x-a)^2$



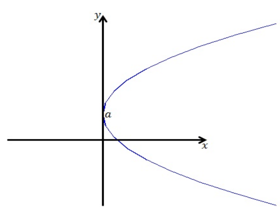
$x = \sqrt{y}$



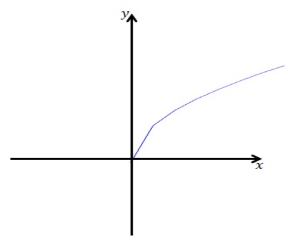
$x = y^2 - a$



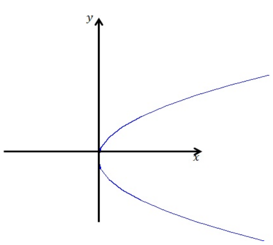
$x = (y-a)^2$



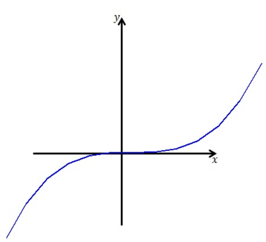
$y = \sqrt{x}$



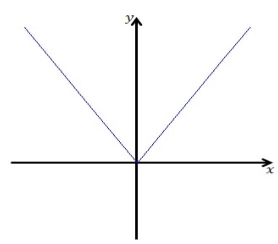
$x = y^2$



$y = x^3$

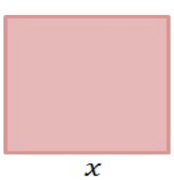


$y = |x|$

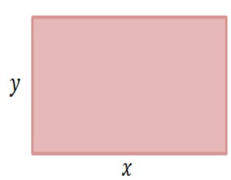


Areas and Volumes of Special Shapes

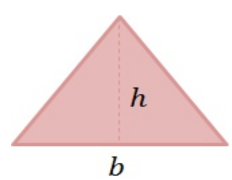
Area $= x^2$



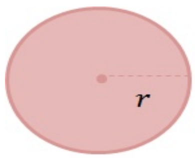
Area $= xy$



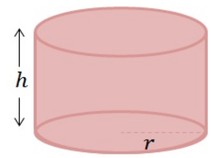
Area $= \frac{1}{2}bh$



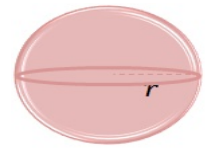
Area $= \pi r^2$



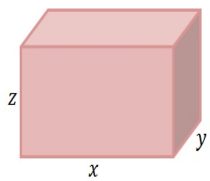
Volume $= \pi r^2 h$



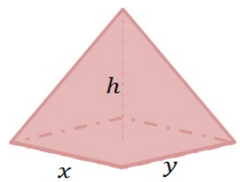
Volume $= \frac{4}{3}\pi r^3$



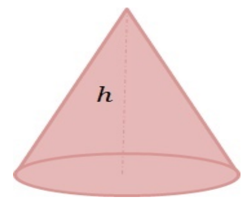
Volume $= xyz$



Volume $= \frac{1}{3}xyh$



Volume $= \frac{1}{3}\pi r^2 h$



Appendix

Appendix (1): Integration Rules and Integrals Table

Integration Rules:

$$\int \big(f(x) \pm g(x)\big)\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

$$\int k f(x)\ dx = k \int f(x)\ dx$$

$$\int f'\big(g(x)\big) g'(x)\ dx = f\big(g(x)\big) + c$$

$$\int_a^b f'(x)\ dx = f(b) - f(a)$$

Elementary Integrals:

$$\int x^r\ dx = \frac{x^{r+1}}{r+1} \text{ if } r \neq -1$$

$$\int \sin x\ dx = \cos x$$

$$\int \cos x\ dx = -\sin x$$

$$\int \sec^2 x\ dx = \tan x$$

$$\int \csc^2 x\ dx = -\cot x$$

$$\int \sec x\ \tan x\ dx = \sec x$$

$$\int \csc x\ \cot x\ dx = -\csc x$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\ dx = \sin^{-1} \frac{x}{a}$$

$$\int \frac{1}{a^2 + x^2}\ dx = \frac{1}{a}\ \tan^{-1} \frac{x}{a}$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\ dx = \frac{1}{a}\ \sec^{-1} \left| \frac{x}{a} \right|$$

Inverse Trigonometric Integrals:

$$\int \sin^{-1} x\ dx = x \sin^{-1} x + \sqrt{1 - x^2} + c$$

$$\int \tan^{-1} x\ dx = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + c$$

$$\int \sec^{-1} x\ dx = x \sec^{-1} x - \ln | x + \sqrt{x^2 - 1} | + c$$

$$\int x^n\ \sin^{-1} x\ dx = \frac{x^{n+1}}{n+1} \sin^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{\sqrt{1 - x^2}}\ dx + c \text{ if } n \neq -1$$

$$\int x^n\ \tan^{-1} x\ dx = \frac{x^{n+1}}{n+1} \tan^{-1} x - \frac{1}{n+1} \int \frac{x^{n+1}}{1 + x^2}\ dx + c \text{ if } n \neq -1$$

$$\int x^n\ \sec^{-1} x\ dx = \frac{x^{n+1}}{n+1} \sec^{-1} x - \frac{1}{n+1} \int \frac{x^n}{\sqrt{x^2 - 1}}\ dx + c \text{ if } n \neq -1$$

Trigonometric Integrals:

$$\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + c$$

$$\int \tan^2 x\, dx = \tan x - x + c$$

$$\int \cot^2 x\, dx = -\cot x - x + c$$

$$\int \sec^3 x\, dx = \frac{1}{2}\ \sec x\ \tan x + \frac{1}{2} \ln|\sec x\ +\ \tan x| + c$$

$$\int \sec^3 x\, dx = \frac{1}{2}\ \csc x\ \cot x + \frac{1}{2} \ln|\csc x\ -\ \cot x| + c$$

$$\int \sin^n x\, dx = -\frac{1}{n}\ \sin^{n-1} x\ \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx + c$$

$$\int \cos^n x\, dx = \frac{1}{n}\ \cos^{n-1} x\ \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx + c$$

$$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx + c \text{ if } n \neq 1$$

$$\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx + c \text{ if } n \neq 1$$

$$\int \sec^n x\, dx = \frac{1}{n-1}\ \sec^{n-2} x\ \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx + c \text{ if } n \neq 1$$

$$\int \csc^n x\, dx = -\frac{1}{n-1}\ \csc^{n-2} x\ \cot x + \frac{n-2}{n-1}\int \csc^{n-2} x\, dx + c \text{ if } n \neq 1$$

$$\int \sin^n x\ \cos^m x\, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2} x\ \cos^m x\, dx + c \text{ if } n \neq m$$

$$\int \sin^n x\ \cos^m x\, dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+m} + \frac{m-1}{n+m}\int \sin^n x\ \cos^{m-2} x\, dx + c \text{ if } m \neq n$$

$$\int x^n\ \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx + c$$

$$\int x^n\ \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx + c$$

Miscellaneous Integrals:

$$\int x(ax+b)^{-1}\, dx = \frac{x}{a} - \frac{b}{a^2} \ln|ax+b| + c$$

$$\int x(ax+b)^{-2}\, dx = \frac{1}{a^2}\left(\ln|ax+b| + \frac{b}{ax+b}\right) + c$$

$$\int x(ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a^2}\left(\frac{ax+b}{n+2} - \frac{b}{n-1}\right) + c$$

$$\int \frac{a}{(a^2 \pm x^2)^n}\, dx = \frac{1}{2a^2(n-1)}\left(\frac{x}{(a^2 \pm x^2)^{n-1}} + (2n-3)\int \frac{1}{(a^2 \pm x^2)^{n-1}}\, dx\right) \text{ if } n \neq -1$$

$$\int x\sqrt{ax+b}\, dx = \frac{2}{15a^2}(3ax-2b)(ax+b)^{3/2} + c$$

$$\int x^n\sqrt{ax+b}\, dx = \frac{2}{a(2n+3)}\left(x^n(ax+b)^{3/2} - nb\int x^{n-1}\sqrt{ax+b}\, dx\right)$$

$$\int \frac{x}{\sqrt{ax+b}}\, dx = \frac{2}{3a^2}(ax-2b)\sqrt{ax+b} + c$$

$$\int \frac{x^n}{\sqrt{ax+b}}\, dx = \frac{2}{a(2n+1)}\left(x^n\sqrt{ax+b} - nb\int \frac{x^{n-1}}{\sqrt{ax+b}}\, dx\right)$$

$$\int \frac{1}{x\sqrt{ax+b}}\, dx = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + c \text{ if } b > 0$$

$$\int \frac{1}{x\sqrt{ax+b}}\,dx = \frac{1}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}} + c \text{ if } b < 0$$

$$\int \frac{1}{x^n\sqrt{ax+b}}\,dx = -\frac{\sqrt{ax+b}}{b(n-1)x^{n-1}} - \frac{(2n-3)a}{2(n-1)b}\int \frac{1}{x^{n-1}\sqrt{ax+b}}\,dx \text{ if } n \neq 1$$

$$\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\cos^{-1}(\frac{a-x}{a}) + c$$

$$\int x\sqrt{2ax-x^2}\,dx = \frac{2x^2-ax-3a^3}{6}\sqrt{2ax-x^2} + \frac{a^3}{2}\cos^{-1}(\frac{a-x}{a}) + c$$

$$\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\cos^{-1}(\frac{a-x}{a}) + c$$

$$\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -\frac{2\sqrt{2ax-x^2}}{x} - \cos^{-1}(\frac{a-x}{a}) + c$$

$$\int \frac{dx}{\sqrt{2ax-x^2}} = \cos^{-1}(\frac{a-x}{a}) + c$$

$$\int \frac{x}{\sqrt{2ax-x^2}}\,dx = -\sqrt{2ax-x^2} + a\cos^{-1}(\frac{a-x}{a}) + c$$

$$\int \frac{x^2}{\sqrt{2ax-x^2}}\,dx = -\frac{(x+3a)}{2}\sqrt{2ax-x^2} + \frac{3a^2}{2}\cos^{-1}(\frac{a-x}{a}) + c$$

$$\int \frac{1}{x\sqrt{2ax-x^2}}\,dx = -\frac{\sqrt{2ax-x^2}}{ax} + c$$

Appendix (2): Answers to Exercises

Chapter 1:

Exercise 1.1

1 $2\sqrt{x}+c$

2 $-\frac{4}{\sqrt[4]{x}}+c$

3 $-\cot x+c$

4 $-\tan x+c$

5 $\frac{5}{4}x^{\frac{4}{5}}+c$

6 $\sec x+c$

7 $-\frac{2}{3\sqrt{x^3}}+c$

8 $-\cos x+c$

Exercise 1.2

1 $\frac{2}{7}x^{\frac{7}{2}}+c$

2 $\frac{4}{7}x^{\frac{7}{4}}+\frac{x^3}{3}+x+c$

3 $\frac{x^5}{5}+\frac{2}{3}x^3+\frac{x^2}{2}+c$

4 $\frac{x^3}{3}+\tan x+c$

5 $-\cot x-\frac{2}{3}x^{\frac{3}{2}}+c$

6 $3x-4\cot x+c$

7 $\frac{-1}{x}+\frac{1}{3x^3}+c$

8 $\frac{20}{7}x^{\frac{7}{5}}-\frac{6}{5}x^{\frac{5}{3}}+\frac{1}{2}x^2+c$

9 $-\frac{3}{2x^2}-\frac{2}{3x^3}+x+c$

10 $\frac{3}{8}x^{\frac{8}{3}}+\frac{3}{5}x^{\frac{5}{3}}+\frac{3}{2}x^{\frac{2}{3}}+c$

11 $\sqrt{\cos^3 x+1}$

12 $\sqrt{\cos^3 x+1}+c$

13 $f(x)=x^4+x^2+x+1$

14 $f(x)=-\sin x-2\cos x+4x+3$

15 $f(x)=\frac{2}{3}x^{\frac{3}{2}}$

16 $f(x)=\sin x+1$

17 $f(x)=\tan x-1$

Exercise 1.3

1 $\frac{(1+x^2)^{\frac{3}{2}}}{3}+c$

2 $\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$

3 $\frac{2}{7}(x-1)^{\frac{7}{2}}+\frac{4}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$

4 $\frac{\tan^2 x}{2}+c$

5 $\frac{\sin^6 x}{6}+c$

6 $\frac{1}{2}(2x^2+1)^{\frac{1}{2}}+c$

1 $-\frac{2}{3}(1-\sin t)^{\frac{3}{2}}+c$

2 $-\frac{\cos^4 x}{4}+c$

3 $\frac{1}{3}\sin(3x+4)+c$

4 $\frac{-2}{\sqrt{x+1}}+c$

5 $\frac{1}{4}\sec 4x+c$

6 $-\frac{2}{3}\cot^{\frac{3}{2}}x+c$

7 $-\frac{1}{2}(1+\frac{1}{t})^2+c$

8 $\frac{(2x-1)^{\frac{3}{2}}}{6}+\frac{(2x-1)^{\frac{1}{2}}}{2}+c$

9 $\frac{(4x^3-6)^8}{96}+c$

10 $\frac{1}{9}\sin^3(3x)+c$

Review Exercises

1 x^2+c

2 x^3+x+c

3 $\frac{x^4}{8}+\frac{x^2}{2}+c$

4 $\frac{x^5}{5}+\frac{x^4}{4}+c$

5 $\frac{x^3}{3}+\frac{3}{2}x^2-x+c$

6 $x-x^2-\frac{5}{4}x^4+c$

7 $\frac{-1}{x}+c$

8 $\frac{2}{7}x^{\frac{7}{2}}+c$

9 $\frac{-2}{\sqrt{x}}+c$

10 $\frac{x^3}{3}-x+c$

11 $\frac{x^4}{2}-2x^{\frac{3}{2}}-\frac{1}{x^4}+c$

12 $\frac{5(1+x)^{\frac{6}{5}}}{6}+c$

13 $\frac{x^5}{5}-\frac{x^4}{4}+\frac{x^2}{2}-x+c$

14 $\frac{x^2}{2}+x+c$

15 $\frac{2}{3}x^{\frac{3}{2}}-6x^{\frac{1}{2}}+c$

16 $\frac{3}{5}x^{\frac{5}{3}}+\frac{3}{2}x^{\frac{2}{3}}+c$

17 $\frac{x^7}{7}-\frac{x^4}{2}+x+c$

18 $-\cos x+x+c$

19 $\sin x-\frac{x^2}{2}+c$

20 $\tan x-4x+c$

21 $\sec x+\frac{x^2}{2}+c$

22 $-\cot x+\frac{x^3}{3}+x+c$

23 $\tan x+c$

24 $-\cot x+c$

25 $\sec x+c$

26 $\sec x-\tan x+c$

27 $\tan x+x+c$

28 $-\csc x+c$

29 $\frac{\tan^2 x}{2}+c$

30 $\sec x+c$

31 $-\csc x+c$

32 $\tan x+2\sec x+c$

33 $-\cot x-3\csc x+c$

34 $\frac{-2}{5}\cos^{\frac{5}{2}}x+c$

35 $\frac{(3x^5+1)^{11}}{165}+c$

36 $\frac{(x^2+1)^{\frac{3}{2}}}{3}+c$

37 $\frac{2}{3}(x^2+x+2)^{\frac{3}{2}}+c$

38 $\frac{(x^3-3x+2)^{\frac{4}{3}}}{4}+c$

39 $\frac{(5x^2+2x-5)^4}{8}+c$

40 $-2\cos\sqrt{x}+c$

41 $2\tan\sqrt{x}-2\sqrt{x}+c$

42 $-2\cot\sqrt{x}-2\sqrt{x}+c$

43 $\frac{1}{2}\sec 2x+c$

44 $-2\csc\sqrt{x}+c$

45 $-\frac{1}{2}\cos x^2+c$

46 $\frac{1}{2}\tan x^2+c$

47 $-\frac{1}{2}\cot(x^2+x-1)+c$

48 $-3\cot\sqrt[3]{x}$

49 $5\tan(\sqrt[5]{x}+1)+c$

50 $\sqrt{x^2+9}+c$

51 $\frac{3}{4}(x^2-1)^{\frac{2}{3}}+c$

52 $-\frac{\cos^3 x}{3}+c$

53 $\sin^2\sqrt{x}+c$

54 $-\frac{1}{2}\cos^4\sqrt{x}+c$

55 $-2\cot x-\csc x+c$

56 $\frac{5}{9}(x+1)^{\frac{9}{5}}-\frac{5}{4}(x+1)^{\frac{4}{5}}+c$

57 $\frac{2}{5}(x-3)^{\frac{5}{2}}+2(x-3)^{\frac{3}{2}}+c$

58 $-\frac{1}{(\sqrt{x}+1)^2}+c$

59 $\sqrt{4x-x^2}+c$

60 $-\frac{\cos^3 x}{3}-\cos x+c$

61 $\frac{1}{4(5+\cos 2x)^2}+c$

62 $\frac{1}{2}\sqrt{x^4-1}+c$

63 $\frac{2}{3}x^{\frac{3}{2}}+c$

64 $3\sec\sqrt[3]{x}+c$

65 (a)

66 (c)

67 (d)

68 (a)

69 (d)

70 (c)

Chapter 2:

Exercise 2.1

1 9

2 55

3 $\frac{163}{60}$

4 275

5 120

6 11

7 $\frac{n(n-1)}{2}$

8 $\frac{2n^3+3n^2+7n}{6}$

9 $\frac{n\left[(n+1)\left(3n^2+11n-2\right)+12\right]}{12}$

Exercise 2.2

1 2.7

2 1.5

3 3

4 2.3

5 2.1

6 2.1

7 0.5

8 $\frac{\pi}{4}$

9 $\{0,\frac{3}{5},\frac{6}{5},\frac{9}{5},\frac{12}{5},3\}$

10 $\{-1,\frac{-1}{6},\frac{2}{3},\frac{3}{2},\frac{7}{3},\frac{19}{6},4\}$

11 $\{-4,-3,-2,-1,0,1,2,3,4\}$

12 $\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\}$

13 15

14 39

15 24.5

16 $\frac{1}{2}$

17 $\frac{3}{2}$

18 $\frac{28}{3}$

19 60

Exercise 2.3

1 35

2 0

7 11

8 -5

9 0

10 -2

Exercise 2.4

1 $\frac{21}{2}$

2 $\frac{8}{3}$

3 $10(4\sqrt{10}+1)$

4 $4(\sqrt{2}-1)$

5 0

6 1

7 $1+\sqrt{3}-\sqrt{2}$

8 1

9 $\frac{10}{3}$

10 $\frac{1}{\sqrt{3}}$

11 $\sqrt[3]{2}-1$

12 $\sqrt[3]{-2}$

13 $\left(\frac{14}{9}\right)^2$

14 $\frac{9}{4}$

15 $\sin^{-1}(\frac{2}{\pi})$

16 $\cos^{-1}(\frac{2}{\pi})$

17 $\frac{7}{3}$

18 $\frac{3}{16}(3\sqrt[3]{3}-1)$

19 $\frac{3}{25}$

20 $\frac{3}{\pi}(2-\sqrt{3})$

21 $\sqrt{\sin x+1}\cos x+\sqrt{\cos x+1}\sin x$

22 $\frac{1}{2\sqrt{x}(x+1)}-\frac{1}{x^2+1}$

23 $x-1$

24 $\frac{3}{3x-4}-\frac{1}{x}$

25 $\cos x\int_1^x \sqrt{t}\,dt+\sqrt{x}\sin x$

26 $\sin(x+1)+2\sin(-2x+1)$

27 $-\frac{3x^2}{x^{12}+1}$

28 $\sqrt{1+\sec^4 x}\ \sec x\tan x-\sqrt{1+\tan^4 x}\ \sec^2 x$

29 $F(2)=0 \quad F'(2)=\sqrt{13} \quad F''(2)=\frac{6}{\sqrt{13}}$

30 $G(0)=0 \quad G'(0)=0 \quad G''(0)=-1$

31 $H'(2)=4\sqrt[5]{5}-\sqrt[5]{3}$

32 $F(0)=0 \quad F'(0)=0$

Exercise 2.5

1 2.3251, $|E_T|\le 0.0147$

2 3.046, $|E_T|\le 8\times 10^{-4}$

3 2.317, $|E_T|\le 0.0053$

4 1.8961, $|E_T|=0$

5 1.5, $|E_S|<5\times 10^{-4}$

6 0.5, $|E_S|<1\times 10^{-4}$

7 2, $|E_S|<9\times 10^{-7}$

8 4, $|E_S|<4\times 10^{-6}$

9 $n=99$

10 $n=4$

Review Exercises

1 $\frac{n(n-1)}{2}$

2 $n(n+2)$

3 $\frac{n\left(2(n^2-1)+6\right)}{6}$

4 $\frac{n\left((n+1)(n^2+n+4)+4\right)}{4}$

5 24

6 1.45

7 26

8 14

9 2

10 1.5

11 1.5

12 1.55

13 a. 20 b. 25 c. 22.5

14 a. 3 b. 10.5 c. 6.75

15 a. 20.375 b. 27.875 c. 23.9375

16 a. −164 b. −512 c. −299

17 10

18 5/2

19 2/3

20 28/3

21 3

22 8

23 2

24 1/3

25 0

26 9/2

27 20

28 14/3

29 12

30 $\frac{5}{2}$

31 $\frac{275}{6}$

32 0

33 $\frac{-11}{20}$

34 0

35 $\frac{17}{2}$

36 $\frac{9}{2}$

37 $\frac{2}{3}$

38 0

39 1

40 −2

41 0

42 2.0414

43 −2

44 4

45 7

46 −1

47 $\frac{65}{3}$

48 0

55 7

56 6

57 $\frac{-1}{3}$

58 $\frac{9}{4}$

59 16

60 $\frac{4}{\sqrt{3}}$

61 $\sqrt[3]{5}$

62 4

63 $\sqrt[3]{\frac{15}{4}}$

64 $\sin\sqrt{x}$

65 $\frac{1}{x}$

66 $3x^2\sin(x^9+1)^{10}-3\sin(27x^3+1)^{10}$

67 $\frac{1}{x^2+2x+2}$

68 $\cos x^2+\cos(\cos^2 x)\sin x$

69 $\sqrt{x^2+1}$

70 $-6\sqrt[3]{12x+2}$

71 $\frac{\tan x}{2\sqrt{x}}$

72 162

73 0.694

74 1.727

75 6.244

76 2.405

77 0.984

78 4.671

79 1.250

80 19

81 1891

82 800

83 157

84 d

85 d

86 c

87 a

88 a

89 b

90 b

91 c

92 d

93 d

94 c

95 b

96 d

97 c

98 b

99 d

100 a

101 b

102 b

103 d

104 b

105 d

106 c

Chapter 3:

Exercise 3.1

1 $\frac{1}{x+1}$

2 $\frac{3x^2+2}{x^3+2x-4}$

3 $\frac{1}{2x}$

4 $\frac{2}{3x}$

5 $\frac{-1}{x}$

6 $\frac{\cos x+1}{\sin x+x+1}$

7 $\frac{\sec x\tan x+2x}{\sec x+x^2}$

8 $-2\tan x$

9 $2\cot x$

10 $2\tan x+\cot x$

11 $-\csc x\ \cot x\ \ln x+\frac{\csc x}{x}$

12 $\frac{2\ln(x^3+1)}{3\sqrt[3]{x}}+\frac{3x^2\sqrt[3]{x^2}}{x^3+1}$

13 $\frac{x}{x^2-1}-\frac{1}{2(x+2)}$

14 $\frac{2x}{x^2+1}+\frac{1}{x-1}$

15 $-\frac{1}{2\sqrt{x(x+1)}}$

16 $\frac{\ln x^2-2}{(\ln x^2)^2}$

17 $\frac{3x^2}{x^3+1}$

18 $\frac{\cot x}{\ln\sin x}$

19 $\frac{1}{5}\left[\frac{2}{2x+1}-\frac{3}{3x-1}\right]\sqrt[5]{\frac{2x+1}{3x-1}}$

20 $\left[\frac{1}{x-1}+\frac{3x^2+2}{2(x^3+2x+1)}-\frac{3x^2+4x+1}{x^3+2x^2+x-1}\right]\frac{(x-1)\sqrt{x^3+2x+1}}{x^3+2x^2+x-1}$

21 $\left[\frac{2}{x}+\frac{7}{2(7x+3)}-\frac{6x}{(1+x^2)}\right]\frac{x^2\sqrt{7x+3}}{(1+x^2)^3}$

22 $\frac{1}{3}\left[\frac{2}{\cos x\ \sin x}+\cot x-\tan x-\frac{3}{2x}\right]\sqrt[3]{\frac{\tan^2 x\ \sin x\ \cos x}{\sqrt{x^3}}}$

23 $\frac{7}{2}\left[\frac{1-x}{2x(x+1)}+2x\tan x^2\right]\left(\frac{x\ \sec x^2}{\sqrt{x}(x+1)}\right)^{\frac{7}{2}}$

24 $\left[\frac{1}{3(x+1)}-2\tan x+3\tan 3x-\frac{2}{x+1}\right]\frac{\sqrt[3]{x+1}\ \cos^2 x}{(x+1)^2\cos(3x)}$

25 $\frac{3}{2}\ln(x^2+1)+c$

26 $\ln\sqrt{3}$

27 $\frac{1}{2}\ln\mid\ln(x^2)\mid+c$

28 $\ln(\sqrt{2}+1)$

29 $-\ln\mid 1+\cot x\mid+c$

30 $\frac{1}{2}\left[\ln 17-\ln 2\right]$

31 $-\ln\mid\csc x+\cot x\mid+c$

32 $2\sin\sqrt{x+1}+c$

33 $\frac{(\ln x^2)^{\frac{3}{2}}}{3}+c$

34 $\ln 2+\frac{3}{2}$

35 $\sin(\ln x)+c$

36 $-\frac{1}{4}\left[\frac{1}{(\ln 3)^4}-\frac{1}{(\ln 2)^4}\right]$

Exercise 3.2

1 1

2 $\sqrt[5]{x}$

3 x^2-4

4 $3+\ln x^2$

5 $x=\pm e^2$

6 $x=e^e$

7 $x=\sqrt{27}$

8 $x=1$ or $x=-3$

9 $e^{\sin x-3x^2}(\cos x-6x)$

10 $e^{x\sqrt{x}}\left[1+\frac{3x\sqrt{x}}{2}\right]$

11 $e^x\cos(\ln x)-\frac{e^x\sin(\ln x)}{x}$

12 $\frac{e^{\frac{1}{x}}(x-\ln x)}{x^2}$

13 $-1+\frac{1}{2\sqrt{x}(1+\sqrt{x})}$

14 $e^{\sqrt[3]{x}}\cos x+\frac{e^{\sqrt[3]{x}}\sin x}{3x^{2/3}}$

15 $\frac{e^x\sec^2(e^x)}{\tan e^x}$

16 $\frac{\sqrt{e^x}}{2}$

17 $\frac{1+2e^x-e^{-x}}{2\sqrt{e^{-x}+1}}$

18 $6e^{3x}\sec^2(e^{3x})\tan(e^{3x})$

19 $\frac{e}{2}(e^2-1)$

20 $2e^{\sqrt{x}}+c$

21 $e^{\sin x}+c$

22 $2e^{\sqrt{x}+\cos x}+c$

23 $-e^{\frac{1}{x}}+c$

24 $e^{\sqrt{2}}-e$

25 $-2e^{-\sqrt{x}}+c$

26 $-\frac{1}{4(1+e^x)^4}+c$

27 $\sin x+c$

28 $\ln(e^2+1)-\ln(e+1)$

Exercise 3.3

1 $3^x\ln 3$

2 $2^{\sin x\cos x}\ln 2\cos 2x$

3 $\ln 2$

4 $\frac{-\tan x}{\ln 2}$

5 $\frac{1}{3\ln 10\,(x+1)}$

6 $5^{\sqrt{x}\tan x}\left(\frac{\tan x}{2\sqrt{x}}+\sqrt{x}\sec^2 x\right)$

7 $4^{-2x}-2\ln(4)\,x\,4^{-2x}$

8 $\frac{1}{\ln 10\,(x+1)}$

9 $\tan 5^{x+1}\,(5^{x+1}\ln 5)$

10 $\frac{3}{2(\ln 5)\,x}$

11 $\left(\ln\sin x+x\cot x\right)(\sin x)^x$

12 $\left(\ln e^x+x\right)(e^x)^x$

13 $\left(e^x\ln x+\frac{e^x}{x}\right)x^{e^x}$

14 $\left(\frac{\ln(x^2-x)}{x}+\frac{(2x-1)\ln x}{x^2-x}\right)(x^2-x)^{\ln x}$

15 $\frac{1}{3\ln 5}5^{x^3}+c$

16 $\frac{1}{\ln 2}\sin(2^x+1)+c$

17 $\frac{\ln(10)}{2}\ln|\log x^2|+c$

18 $\frac{2\sqrt{3^x+1}}{\ln 3}+c$

19 $\frac{2}{9\ln 7}(7^{3x}+1)^{\frac{3}{2}}+c$

20 $\frac{\ln 2}{2}(\log_2\sin x)^2+c$

Review Exercises

1 2

2 e

3 $3/64$

4 $\pm 2\sqrt{2}$

5 $(1+\sqrt{5})/2$

6 $\ln 2$

7 0

8 0

9 1

10 ∞

11 ∞

12 $-\infty$

13 $\frac{2}{x}$

14 $\frac{2x+3}{x^2+3x+1}$

15 $-3\tan x$

16 $2x\cot x^2$

17 $\frac{3x^2+1}{2(x^3+x-1)}$

18 $-\frac{1}{2\sqrt{x(x-1)}}$

19 $\cos x\ln(\cos x)-\frac{\sin^2 x}{\cos x}$

20 $\frac{2}{x}+\cot x-\frac{1}{2(x+1)}$

21 $-\frac{1}{x}\left[\frac{1}{(\ln x)^2}+1\right]$

22 $\frac{6\ln x^3}{x}$

23 $\frac{\ln(x^2+x-2)}{2\sqrt{x}}+\frac{(2x+1)\sqrt{x}}{x^2+x-2}$

24 $e^x\sec x\,(1+\tan x)$

25 $2x+1$

26 $e^{x+1}\sin^2 x\left(3\cos x+\sin x\right)$

27 $\frac{1}{(x+1)^2}e^{\frac{x}{x+1}}$

28 $e^x \cot e^x$

29 $2e^{2x+1}$

30 $e^{\sin x} \cos x$

31 $2\sec^2 x \, \tan x \, e^{\sec^2 x}$

32 $(6x^2+1)\, e^{2x^3+x-1} \, \cos(e^{2x^3+x-1})$

33 $2e^{2x+1}$

34 $\frac{xe^x}{(x+1)^2}$

35 $\frac{e^x(x \, \ln x - 1)}{x \, (\ln x)^2}$

36 $e^{x\tan x}\left(\tan x + x\sec^2 x\right)$

37 $e^x \ln x + \frac{e^x}{x}$

38 $\frac{xe^{\sqrt{x}}}{2}(4+\sqrt{x})$

39 $-\pi^{\cos x} \sin x \, \ln \pi$

40 $2^{\sin^2 x} \, \ln 2 \, \left(2\sin x \cos x\right)$

41 $3 \, \ln(10) \, 10^{3x}$

42 $\sec^2(2^{\sin x})(2^{\sin x} \, \cos x \, \ln 2)$

43 $\frac{1}{\ln 3}\left(\frac{6}{6x+1} - \frac{2}{2x-1}\right)$

44 $\frac{1}{10x \ln x}$

45 $\sec^2 x \left[\ln(\tan x)+1\right](\tan x)^{\tan x}$

46 $\left[\ln x + 1\right]x^x$

47 $\left[\frac{\ln x + 2}{2\sqrt{x}}\right]x^{\sqrt{x}}$

48 $4(\ln x+1)x^{4x}$

49 $\left[\cos x \, \ln x + \frac{\sin x}{x}\right]x^{\sin x}$

50 $\left[\sec^2 x \, \ln(\ln x) + \frac{\tan x}{x \, \ln x}\right](\ln x)^{\tan x}$

51 $\frac{1}{3}\ln|\,x^3+2\,| + c$

52 $-\ln|\,\cos x\,| + c$

53 $\frac{1}{2}\ln|\,x^2+2x\,| + c$

54 $\frac{2(\ln x)^{3/2}}{3} + c$

55 $\ln\sqrt{2}$

56 $\frac{1}{2}(\ln 3 - \ln 7)$

57 $\sin(\ln x)$

58 $\frac{x^2}{2} + 2x + \ln|x| + c$

59 $\frac{-1}{\ln x} + c$

60 $-\ln|\,\sin x + \cos x\,| + c$

61 $-\frac{1}{2\ln 3}3^{-x^2} + c$

62 $e^{x^2} + c$

63 $\ln(e^x + e^{-x}) + c$

64 $\sin x + c$

65 $e^{\tan x} + c$

66 $\frac{2}{\ln 5}\, 5^{\sqrt{x}} + c$

67 $-\frac{5}{2}$

68 $\frac{1}{4}\ln|\,x^4+1\,| + c$

69 $\frac{4^{3x}}{6\ln 2} + c$

70 $\frac{1}{-2\ln 3}\left(\frac{1}{3^9} - 1\right)$

71 $\frac{1}{2\ln 10}10^{x^2+1} + c$

72 $\frac{2a^{\sqrt{x+1}}}{\ln a}$

73 b

74 c

75 a

76 b

77 c

78 d

79 a

80 a

81 b

82 b

83 c

84 c

85 a

86 c

87 a

88 a

89 c

Chapter 4:

Exercise 4.1

1 $\frac{1}{x\sqrt{1-(\ln x)^2}}$

2 $\frac{-8x}{\sqrt{1-16x^4}}$

3 $\frac{1}{2\sqrt{x}(x+1)}$

4 $\frac{1}{|x|\sqrt{\frac{25}{9}x^2-1}}$

5 $\frac{2x+1}{\sqrt{1-(x^2+x-1)^2}}$

6 $\frac{-1}{1+x^2}$

7 $\frac{e^{\frac{1}{x}}}{x^2\left(e^{\frac{2}{x}}+1\right)}$

8 $\frac{1}{3x\ln(\sqrt[3]{x})\sqrt{(\ln\sqrt[3]{x})^2-1}}$

9 $\sin^{-1}(\frac{x}{3}) + c$

10 $\frac{1}{9}\tan^{-1}(\frac{x}{9})+c$

11 $\frac{1}{2}\sec^{-1}(\frac{e^x}{2})+c$

12 $\tan^{-1}(\sin x)+c$

13 $\frac{1}{12}\sec^{-1}(\frac{x^4}{3})+c$

14 $\tan^{-1}(e^x)+c$

15 $\sin^{-1}(\ln x)+c$

16 $\frac{1}{\sqrt{3}}\sec^{-1}(\frac{\tan x}{\sqrt{3}})+c$

Exercise 4.2

1 $\frac{3}{2}\sqrt{x}\cosh(\sqrt{x^3})$

2 $5\text{sech}^2(5x)$

3 $-e^{-x}\cosh x+e^{-x}\sinh x$

4 $2e^{\sinh(2x)}\cosh(2x)$

5 $\frac{-\text{csch}^2 x}{\coth x}$

6 $-\frac{1}{2}\sqrt{\text{csch}x}\ \coth x$

7 $\cosh(\tan x)\ \sec^2 x$

8 $\frac{e^{\sqrt{x}}\sinh(e^{\sqrt{x}})}{2\ \sqrt{x}}$

9 $\frac{\text{sech}^2(\ln x)}{x}$

10 $\text{csch}x\ \left[\frac{1-2(x+1)\coth x}{2\ \sqrt{x+1}}\right]$

11 $2\cosh(\sqrt{x})+c$

12 $\sinh(\ln x)+c$

13 $\ln(\cosh e^x)+c$

14 $\frac{(1+\tanh x)^4}{4}+c$

15 $e^{\sinh x}+c$

16 $-\ln(1+\text{sech}x)+c$

17 $\frac{2(3+\cosh x)^{3/2}}{3}+c$

18 $2\big(-\text{sech}\sqrt{x}+\ln(\cosh\sqrt{x})\big)+c$

19 $\ln|\tanh x|+c$

20 $\frac{-\left(\ln(\coth x)\right)^2}{2}+c$

Exercise 4.3

1 $\sec x$

2 $\frac{e^{\sqrt{x}}}{2\sqrt{x(e^{2\sqrt{x}}-1)}}$

3 $\frac{1}{x(1-(\ln x)^2)}$

4 $\frac{\text{csch}^{-1}x}{2\sqrt{x+1}}+\frac{-\sqrt{x+1}}{|x|\sqrt{x^2+1}}$

5 $\sec^2 x\ \tanh^{-1}x+\frac{\tan x}{1-x^2}$

6 $6(2x-1)^2\sinh^{-1}(\sqrt{x})+\frac{(2x-1)^3}{2\sqrt{x(x+1)}}$

7 $\frac{1}{\sqrt{2}}\cosh^{-1}x+c$

8 $\tanh^{-1}(e^x)+c$

9 $\frac{-1}{2}\text{sech}^{-1}x^2+c$

10 $\sinh^{-1}(\frac{x}{5})+c$

11 $\cosh^{-1}(\frac{x}{5})+c$

12 $\tanh^{-1}(\sin x)+c$

13 $-\frac{1}{3\sqrt{2}}\text{csch}^{-1}(\frac{|x^3|}{\sqrt{2}})+c$

14 $-\frac{1}{2}\text{sech}^{-1}(\frac{e^x}{2})+c$

Review Exercises

1 $\frac{3}{\sqrt{1-(3x+1)^2}}$

2 $\frac{-1}{2\sqrt{x(1-x)}}$

3 $\frac{2}{3+4x^2/3}$

4 $\frac{1}{x\sqrt{9x^2-1}}$

5 $4\cosh(4x+1)$

6 $e^x\sinh(e^x)$

7 $\frac{1}{2\sqrt{x}}\tanh(\sqrt{x})+\frac{\text{sech}^2(\sqrt{x})}{2}$

8 $e^{3x}\big(3\cosh(2x)+2\sinh(2x)\big)$

9 $\frac{3\cosh(3x)+5\sinh(5x)}{2\ \sqrt{\sinh(3x)+\cosh(5x)}}$

10 $\text{sech}x$

11 $e^x\cosh(\cosh x)+e^x\sinh(\cosh x)\ \sinh x$

12 $\text{sech}^2 x$

13 $\frac{-1}{x\sqrt{1-9x^2}}$

14 $\frac{1}{2\ \sqrt{x}(1-x)}$

15 $4x^3\cosh^{-1}x+\frac{x^4}{\sqrt{x^2-1}}$

16 $e^x \tanh^{-1}(\sqrt[3]{x}) + \frac{e^x}{3(x^{\frac{2}{3}} - x^{\frac{4}{3}})}$

17 $\frac{\operatorname{sech}^2 x}{\sqrt{\tanh^2 x+1}}$

18 $-\frac{1}{2x}$

19 0

20 ∞

21 ∞

22 1

23 $\frac{\sinh^4 x}{4} + c$

24 $\frac{\tanh^5 x}{5} + c$

25 $e^{\sinh x} + c$

26 $\ln|e^{2x} - 1| + c$

27 $2\sinh(\sqrt{x}) + c$

28 $-\frac{1}{2}\operatorname{sech} x^2 + c$

29 $\frac{1}{3}\sinh 3x + c$

30 $\ln|\cosh x| + c$

31 $\frac{1}{\sqrt{3}}\tan^{-1}(\frac{x}{\sqrt{3}}) + c$

32 $\frac{1}{4}\sec^{-1}(\frac{x^2}{2}) + c$

33 $\sec^{-1}(e^x) + c$

34 $-\sqrt{4-x^2} - \sin^{-1}(\frac{x}{2}) + c$

35 $\sin^{-1}(\frac{x}{3}) - \frac{1}{5}sech^{-1}(\frac{|x|}{5}) + c$

36 $\frac{1}{16}\sec^{-1}(\frac{x^4}{4}) + c$

37 $\frac{1}{2}\sinh^{-1}(2x) + c$

38 $\frac{1}{6}\tanh^{-1}(\frac{3x}{2}) + c$

39 $-\frac{1}{8}\big(\coth^{-1}(16) - \coth^{-1}(4)\big)$

40 $\cosh^{-1}(3) - \cosh^{-1}(2)$

41 $\frac{1}{3}\sinh^{-1}(\frac{3x}{5}) + c$

42 $\frac{1}{4}\sec^{-1}(\frac{e^x}{4}) + c$

43 a

44 a

45 a

46 c

47 b

48 b

49 c

50 c

51 d

52 c

53 c

54 a

55 d

Chapter 5:

Exercise 5.1

1 $\frac{x^4}{4}(\ln x - \frac{1}{4}) + c$

2 $\frac{1}{2}(1 - \ln\frac{1}{2})$

3 $\sqrt{1-x^2} + x\sin^{-1}x + c$

4 $\frac{1}{5}(4-x^2)^{5/2} - \frac{4}{3}(4-x^2)^{3/2} + c$

5 $\sin x - x\cos x + c$

6 $(x^2-2)\sin x + 2x\cos x + c$

7 $\frac{e^x}{5}\big(\sin(2x) - 2\cos(2x)\big) + c$

8 $\frac{\pi}{6\sqrt{3}} - \ln\frac{2}{\sqrt{3}}$

9 $\frac{e^{2x}}{5}(\sin x + 2\cos x) + c$

10 $x\big((\ln^2 x - 2)\ln x + 2\big) + c$

11 $-\frac{\ln x}{x} - \frac{1}{x} + c$

12 $\frac{\sin 2x - 2x\cos 2x}{8} + c$

13 $-\frac{1}{2\ln^2 x} + c$

14 $e - 2$

15 $\frac{(x^2+1)\tan^{-1}x - x}{2} + c$

16 $-(x+1)e^{-x} + c$

Exercise 5.2

1 $\frac{1}{8}\sin^3 x\,\cos^5 x + \frac{5}{8}\big(\frac{x}{16} - \frac{1}{64}\sin 4x\big) + \frac{1}{6}\sin^3 x\,\cos^3 x + c$

2 $-\frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$

3 $\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x + c$

4 $\frac{1}{24}\sin 4x\,\cos^5 4x + \frac{5}{24}\big(\frac{3}{2}x - \frac{3}{32}\sin 16x\big) + \frac{1}{4}\sin 4x\,\cos^3 4x] + c$

5 $x + \frac{1}{3}\tan^3 x - \tan x + c$

6 $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \ln|\sin x| + c$

7 $\sqrt{x} - \frac{1}{2}\sin(2\sqrt{x}) + c$

8 $-\frac{1}{4}\cot x\,\csc^3 x + \frac{1}{4}\big(-\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2}\cot^2\frac{x}{2} - 2\ln|\tan x|\big) + \frac{3}{16} + c$

9 $-\frac{1}{5}\cot^5 x + c$

10 $\frac{1}{15}\sec^3 x(3\sec^2 x - 5) + c$

11 $\frac{\tan^3 x}{3} + c$

12 $-\frac{1}{2}\sec x \tan x + \frac{1}{4}\sec^3 x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + \frac{3}{8}\big(\sec x \tan x + \ln|\sec x + \tan x|\big) + c$

13 $\frac{1}{4}\tan x \sec^3 x + \frac{3}{4}\big(\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|\big) + c$

14 $-x + \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x + c$

15 $\frac{1}{40}\big(-5\cos(4x) - 2\cos(10x)\big) + c$

16 $\frac{1}{2}\big(\sin x + \frac{\sin 7x}{7}\big) + c$

17 $\frac{1}{16}\big(4\sin 2x - \sin 8x\big) + c$

18 $\frac{1}{16}\big(4\cos 2x - \cos 8x\big) + c$

Exercise 5.3

1 $\frac{\sqrt{x^2-16}}{16x} + c$

2 $\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}(\frac{x}{3}) + c$

3 $-\frac{x}{\sqrt{9x^2-1}} + c$

4 $\sinh^{-1}(\frac{x}{3}) + c$

5 $-\frac{\sqrt{x^2+4}}{4x} + c$

6 $\frac{1}{16}\ln|4-x| - \frac{1}{16}\ln|x+4| - \frac{1}{4(x-4)} + c$

7 $\frac{1}{4}\sin^{-1}(x^4) + c$

8 $\operatorname{csch}^{-1}(3\cot x) + c$

9 $\frac{1}{2}\left[\frac{x}{x^2+1} + \tan^{-1} x\right] + c$

10 $\frac{x}{2}\sqrt{x^2-16} - 8\cosh^{-1}(\frac{x}{4}) + c$

11 $\sqrt{e^{2x}-25} - 5\tan^{-1}(\frac{1}{5}\sqrt{e^{2x}-25}) + c$

12 $\sin^{-1}\left(\frac{\sin x}{\sqrt{2}}\right) + c$

13 $\ln\left|\frac{\sqrt{x^2+2}}{\sqrt{2}} + \frac{x}{\sqrt{x^2+2}}\right| + c$

14 $\frac{2x}{3\sqrt{1-x^2}} + \frac{x}{3(1-x^2)^{3/2}} + c$

15 $\frac{1}{2}\left[e^x\sqrt{1-e^{2x}} + \sin^{-1}(e^x)\right] + c$

16 $-\frac{\sqrt{9-x^2}}{x} - \sin^{-1}(\frac{x}{3}) + c$

Exercise 5.4

1 $\ln|x-1| - \ln|x| + c$

2 $\ln 3 - \frac{1}{2}\ln 5$

3 $-\frac{1}{2}\tanh^{-1}(\frac{x}{2}) + c$

4 $\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + c$

5 $\frac{5}{4}\ln|x+6| - \frac{1}{4}\ln|x+2| + c$

6 $4\ln|x+4| - 3\ln|x+3| + c$

7 $4 - 6\ln 3 + 3\ln 5$

8 $\frac{1}{2}\big(25\ln|x^2-25| + x^2 - 25\big) + c$

9 $\frac{5}{6}\ln|x+6| - \frac{1}{5}\ln|x+1| + c$

10 $\frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{2x+3}{3\sqrt{3}}\right)$

11 $-\frac{1}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1} x + \frac{1}{2}\ln|x-1| + c$

12 $-\frac{2}{3}\tanh^{-1}(\frac{2x-1}{3}) + c$

13 $\frac{x^2}{2} + 3x + \frac{11}{7}\ln|x+2| + \frac{136}{7}\ln|x-5| + c$

14 $\tan^{-1} x$

15 $\ln|x| + \frac{2}{\sqrt{5}}\tanh^{-1}(\frac{2x+1}{\sqrt{5}}) + \ln|5-(2x+1)^2| - \frac{4}{\sqrt{5}}\tanh^{-1}(\frac{2x-1}{\sqrt{5}}) + c$

16 $\frac{2}{3}\ln 2$

17 $-3\ln|x| - \frac{2}{x} + 3\ln|x+1| + c$

18 $1 - \ln(e+1) + \ln(2)$

19 $-\frac{1}{4}\tanh^{-1}(\frac{e^x-1}{4}) + c$

20 $\frac{1}{x} - \tanh^{-1} x + c$

Exercise 5.5

1 $\tan^{-1}(3) - \tan^{-1}(2)$

2 $-\frac{1}{2\sqrt{2}}\tanh^{-1}\left(\frac{x-3}{2\sqrt{2}}\right) + c$

3 $\tanh^{-1}(\frac{x+1}{2}) + \ln|(x+1)^2 - 4| - \frac{3}{2}\tanh^{-1}(\frac{x+1}{2}) + c$

4 $1 - x - 5\tanh(1-x) + c$

5 $-\sin^{-1}(1/3) + \sin^{-1}(2/3)$

6 $-\frac{1}{5}\tanh^{-1}(\frac{x+4}{5}) + c$

7 $5\sin^{-1}(\frac{x+2}{\sqrt{5}}) + c$

8 $-\frac{1}{\sqrt{2}}\tanh^{-1}(\frac{e^x+1}{\sqrt{2}}) + c$

9 $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{2x+3}{\sqrt{21}}\right) + c$

10 $\frac{1}{2}\left(\sin^{-1}(x-1) + (x-1)\sqrt{1-(x-1)^2}\right) + c$

11 $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\sqrt{x}-3}{\sqrt{3}}\right) + c$

12 $\frac{9}{2}\left(\sin^{-1}(\frac{x+1}{3}) + \frac{(x+1)\sqrt{9-(x+1)^2}}{9}\right) + c$

Exercise 5.6

1 $4(\frac{\sqrt{x}}{2}-\sqrt[4]{x})-4\ln(\sqrt[4]{x}+1)+c$

2 $10(\frac{x^{9/10}}{9}-\frac{x^{3/10}}{3})+\frac{10}{3}\tan^{-1}(x^{3/10})+c$

3 $\sqrt{2}\sinh\left(\tan(\frac{x}{2})\right)+c$

4 $(\sqrt{x}+4)^2-16(\sqrt{x}+4)+32\ln|\sqrt{x}+4|+c$

5 $-\frac{1}{\sqrt{2}}\tanh^{-1}\left(\frac{\tan(x/2)+3}{2\sqrt{2}}\right)+c$

6 $-x-\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{1-3\tan(x/2)}{2\sqrt{2}}\right)+c$

7 $\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan(\frac{x}{2})\right)+c$

8 $2(x^{\frac{1}{6}}+1)^3-9(x^{\frac{1}{6}}+1)^2+18(x^{\frac{1}{6}}+1)-6\ln|x^{\frac{1}{6}}+1|+c$

9 $4\left(\frac{x^{\frac{5}{4}}}{5}+\frac{x^{\frac{3}{4}}}{3}-\frac{x}{4}-\frac{\sqrt{x}}{2}+x^{\frac{1}{4}}-\ln|x^{\frac{1}{4}}+1|\right)+c$

10 $-10\left(\frac{x^{2/5}}{4}+\frac{x^{3/10}}{3}+\frac{x^{1/5}}{2}+x^{1/10}+\ln|x^{1/10}-1|\right)+c$

11 $-\frac{2}{\sqrt{3}}\tanh^{-1}\left(\sqrt{3}\tan(x/2)\right)+c$

12 $\sqrt{2}\tanh^{-1}\left(\frac{\tan(x/2)-1}{\sqrt{2}}\right)+c$

Review Exercises

1 $\frac{(2x-1)}{4}e^{2x}+c$

2 $\frac{e^{x^2}}{2}+c$

3 $\sin x-x\cos x+c$

4 $\frac{1}{16}\left(4x\sin(4x)+\cos(4x)\right)+c$

5 $\frac{2}{9}x^{3/2}(3\ln|x|-2)+c$

6 $x\cos^{-1}x-\sqrt{1-x^2}+c$

7 $x\tan x+\ln|\cos x|+c$

8 $-\frac{xe^{-4x}}{4}-\frac{e^{-4x}}{16}+c$

9 $\sqrt{5}-\sqrt{2}$

10 $x\ln^3 x-3x\ln^2 x+6(x\ln x-x)+c$

11 $\frac{\sin^7 x}{7}-\frac{2\sin^7 x}{7}+\frac{\sin^3 x}{3}+c$

12 $\frac{1}{32}\left(\frac{3x}{4}-\frac{3}{16}\sin(4x)-\frac{1}{4}\sin^3 2x\,\cos 2x\right)+c$

13 $\frac{\sec^3 x}{3}+c$

14 $\frac{\sec^5 x}{5}-\frac{\sec^3 x}{3}+c$

15 $-\frac{1}{4}\cot x\csc^3 x+\frac{1}{4}\left(-\frac{1}{2}\tan^2(\frac{x}{2})+\frac{1}{2}\cot^2(\frac{x}{2})-2\ln|\tan(\frac{x}{2})|\right)+c$

16 $-\frac{1}{7}\cot^7 x-\frac{1}{5}\cot^5 x+c$

17 $\frac{1}{2}\left(-\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right)+c$

18 $\frac{1}{2}\left(\frac{1}{4}\cos(4x)-\frac{1}{10}\cos(10x)\right)+c$

19 $\frac{1}{12}\left(3\sin(2x)+\sin(6x)\right)+c$

20 $\frac{x\sqrt{25-x^2}}{2}+\frac{25}{2}\sin^{-1}(\frac{x}{5})+c$

21 $\sin^{-1}(\frac{x}{5})+c$

22 $\sqrt{x^2-16}-4\tan^{-1}\left(\frac{\sqrt{x^2-16}}{4}\right)+c$

23 $\frac{1}{2(16-x^2)}+c$

24 $-\frac{x^2+2}{(x^2+3)^{\frac{3}{2}}}+c$

25 $\frac{81}{16}\pi$

26 $\tanh^{-1}(1-x)+c$

27 $\frac{1}{2}\ln\left(\frac{1}{4}(x-2)^2+1\right)+\tan^{-1}(\frac{x-2}{2})+c$

28 $\frac{3}{2}\ln\left(\frac{1}{4}(x-3)^2+1\right)+5\tan^{-1}(\frac{x-3}{2})+c$

29 $-\frac{2}{5}\tanh^{-1}(\frac{2x+3}{5})+c$

30 $\frac{1}{2}\ln|5-(2x+1)^2|-\frac{1}{\sqrt{5}}\tanh^{-1}(\frac{2x+1}{\sqrt{5}})+c$

31 $\frac{1}{3}\Big[\ln|x-1|+5\ln|x+2|\Big]+c$

32 $4-\frac{\ln 2}{3}+\frac{4\ln 5}{3}$

33 $6\big(x+2\ln|x-2|\big)-\frac{3x^2-10}{x-2}+c$

34 $\frac{1}{2}(x-1)^2+2(x-1)-8\ln|x-1|+c$

35 $x^2-x+\frac{1}{x+1}+\frac{1}{2}\ln(x-3)-\frac{3}{2}\ln(x+1)-6+c$

36 $x-\ln(e^x+1)+c$

37 $\frac{4}{5(x+2)}+\frac{9}{25}\ln|x-3|+\frac{16}{25}\ln|x+2|+c$

38 $\frac{x-3}{7(x^2+x+2)}+\frac{1}{7\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+c$

39 $\frac{1}{27}\left[\frac{3(55x-107)}{(x-2)^2}+\ln|x-2|+53\ln|x+1|\right]+c$

40 $\frac{2}{9}(x^3-2)\sqrt{x^3+1}+c$

41 $\frac{2}{3}\tan^{-1}(\sqrt{x^3-1})+c$

42 $(\sqrt{x}+1)\big((\sqrt{x}+1)-4\big)+2\ln(\sqrt{x}+1)+c$

43 $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{2}}\right)+c$

44 $-\frac{2}{\tan(x/2)-1}+c$

45 $\frac{1}{5}\Big[\ln|\tan(\frac{x}{2})+2|-\ln|2\tan(\frac{x}{2})-1|\Big]+c$

46 $\frac{1}{2}\ln\left(\frac{2+\sqrt{3}}{2}\right)$

47 c
48 b
49 c
50 a
51 d
52 c
53 b
54 b
55 a
56 c
57 c
58 d
59 a
60 b
61 d
62 b
63 a
64 c
65 a
66 b
67 a
68 c
69 a
70 d
71 d
72 c

Chapter 6:

Exercise 6.1

1 0
2 6
3 $-\infty$
4 -1
5 $-\infty$
6 $-\infty$
7 e^2
8 1
9 0
10 0
11 1
12 1
13 0
14 1

Exercise 6.2

1 Divergent
2 Convergent
3 Divergent
4 Convergent
5 Divergent
6 Divergent
7 Divergent
8 Divergent
9 Convergent
10 Convergent
11 Convergent
12 Convergent
13 Divergent
14 Divergent
15 Convergent
16 Divergent

Review Exercises

1 ∞
2 $-\infty$
3 0
4 0
5 1
6 $\ln(3)$
7 1
8 2
9 $1/e$
10 e^2
11 Convergent
12 Convergent
13 Divergent
14 Convergent
15 Divergent
16 Divergent

17 Divergent
18 Convergent
19 b
20 b
21 a
22 c
23 b
24 b
25 b
26 d
27 c
28 c
29 d
30 c
31 b
32 c
33 d
34 c
35 b

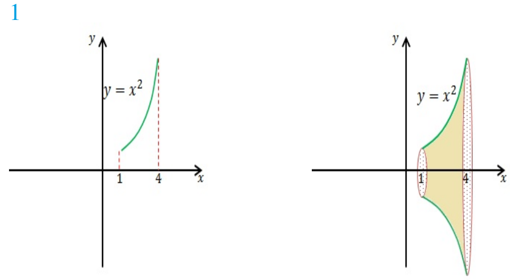
Chapter 7:

Exercise 7.1

1 $13/3$
2 4
3 $27/2$
4 $14/3$
5 $5/4$
6 2
7 $\ln(2)/2$
8 $1/4$
9 $7/6$
10 $5/6$
11 4
12 $4/3$
13 63
14 4
15 $3/\sqrt[3]{4}$
16 $5/9$
17 $4\sqrt{2}/3$
18 10
19 $3/2$
20 $e^3 - e^{-2}$
21 $e(e-1)$
22 $5\ln 5 - 4$
23 $(\sqrt{2}-1)/\sqrt{2}$
24 $\sqrt{2}-1$
25 $\sqrt{2}$
26 $14/3$
27 1

Exercise 7.2

1



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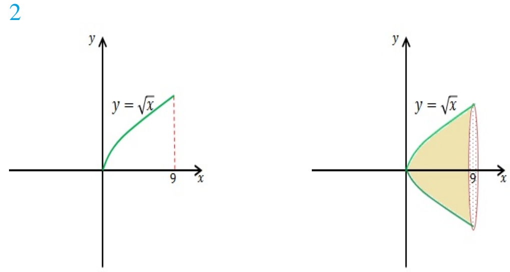

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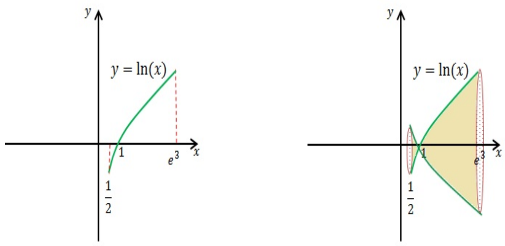

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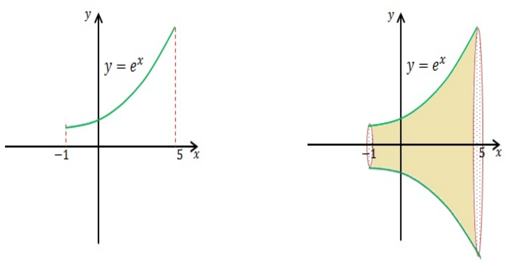

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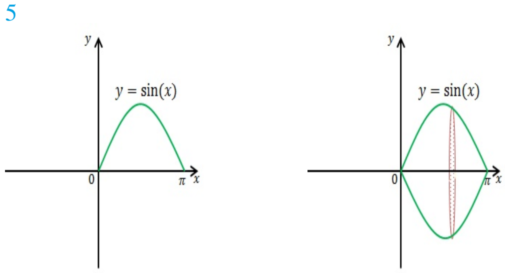

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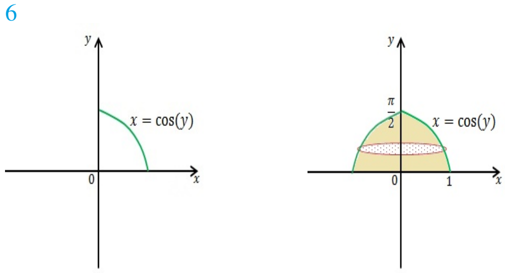

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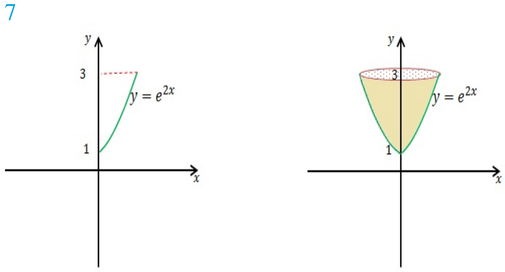

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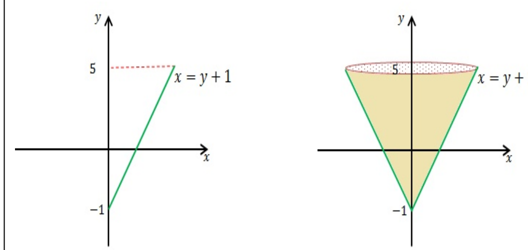

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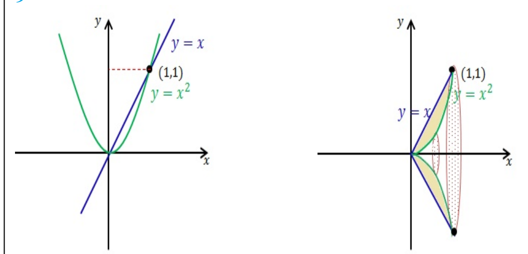

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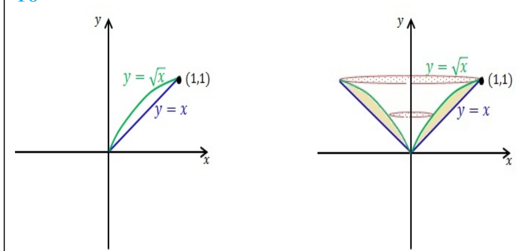


Exercise 7.3

1 $\frac{7}{3}\pi$

2 $\frac{206}{15}\pi$

3 $\frac{128}{7}\pi$

4 8π

5 $\frac{\pi}{6}$

6 $\frac{\pi^2}{4}$

7 $\frac{2}{3}\pi$

8 π

9 $\frac{15}{2}\pi$

10 $\frac{243}{5}\pi$

11 $\frac{\pi^2}{4}$

12 $(e-2)\pi$

13 $\frac{\pi}{2}$

14 $2e^2\pi$

15 9π

16 $\frac{24}{5}\pi$

17 $\frac{8}{3}\pi$

18 $\frac{8}{5}\pi$

19 $\frac{29}{30}\pi$

20 $\frac{256}{15}\pi$

21 $\frac{38}{15}\pi$

22 $\frac{\pi}{2}$

23 $\frac{17}{6}\pi$

24 $\frac{67}{6}\pi$

25 24π

26 $\frac{120+60\pi-11\pi}{15\pi}$

27 $\frac{21\pi}{2}$

28 $\frac{4\pi}{5}$

29 $\frac{768\pi}{7}$

30 2π

31 $\frac{\pi}{2}$

32 $\frac{2\pi}{15}$

33 $\frac{\pi(\pi\sqrt{2}-4)}{2}$

34 $\frac{\pi}{6}$

35 $8\pi(3-\ln 4)$

Exercise 7.4

1 $2(\sqrt{5}-1)+\tanh^{-1}(\sqrt{2})-\tanh^{-1}(\sqrt{10})$

2 $\sqrt{1+e^2}-\tanh^{-1}(\sqrt{1+e^2})-\sqrt{2}+\tanh^{-1}(\sqrt{2})$

3 $\frac{1}{4}\left(-2\sqrt{5}+6\sqrt{37}-\sinh^{-1}(2)+\sinh^{-1}(6)\right)$

4 $\frac{1}{4}\left(-2\sqrt{5}+4\sqrt{17}-\coth^{-1}(\frac{2}{\sqrt{5}})+\coth^{-1}(\frac{4}{\sqrt{17}})\right)$

5 $\frac{1}{2}\left(\sqrt{2}+\sinh^{-1}(1)\right)$

6 $\ln(2+\sqrt{3})-\sinh^{-1}(1)$

7 $\frac{14}{3}$

8 $\frac{\pi}{3}$

9 $2\sqrt{5}$

10 $\sinh(3)-\sinh(1)$

11 $\frac{1}{12}\left(-2\sqrt{13}+8\sqrt{73}-9\ln(2+\sqrt{13})+9\ln(8+\sqrt{73})\right)$

12 $\frac{1}{4}\left(2\sqrt{5}+\sinh^{-1}(2)\right)$

13 $\ln(\sqrt{2}+1)$

14 8π

15 $\frac{\pi}{6}(17\sqrt{17}-5\sqrt{5})$

16 $\pi\left(-\sqrt{2}+e\sqrt{1+e^2}-\sinh^{-1}(1)+\sinh^{-1}(e)\right)$

17 $\pi\left(\sqrt{2}(3\sqrt{5}-1)-\sinh^{-1}(1)+\sinh^{-1}(3)\right)$

18 $\pi(\sqrt{2}+\sinh^{-1}(1))$

19 $\pi\left(-e\sqrt{1+e^2}+e^2\sqrt{1+e^4}-\sinh^{-1}(e)+\sinh^{-1}(e^2)\right)$

20 $36\sqrt{82}\pi$

21 $\frac{\pi}{27}(145\sqrt{145}-1)$

22 $\frac{\pi}{4}\left(2\sqrt{3}+\ln(2+\sqrt{3})\right)$

23 $\frac{5\pi}{27}(29\sqrt{145}-2\sqrt{10})$

24 $\frac{\pi}{6}(5\sqrt{5}-1)$

Review Exercises

1 $\frac{20\sqrt{5}}{3}$

2 4

3 $\frac{1}{4}$

4 3

5 4π

6 12

7 $\frac{1}{3}$

8 44

9 0

10 1

11 10

12 $\frac{11}{6}$

13 $\frac{2-\sqrt{2}}{\sqrt{2}}$

14 3

15 $\frac{79}{225}$

16 $\frac{e^3-1}{e^2}$

17 $2\sqrt{2}-1$

18 $\frac{1}{2}$

19 $\frac{1}{\sqrt{2}}$

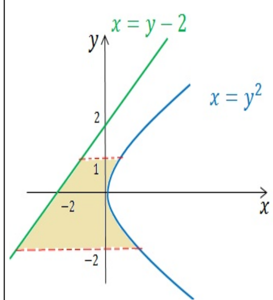
20 $\ln(3+2\sqrt{2})$

21 $1+\ln(3)\left(\ln(\ln 3)-1\right)$

22 $2\left(\sqrt{2}+\sinh^{-1}(1)\right)$

23 $\frac{\ln(2)}{2}$

24



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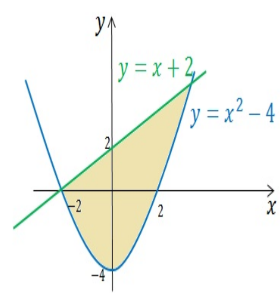

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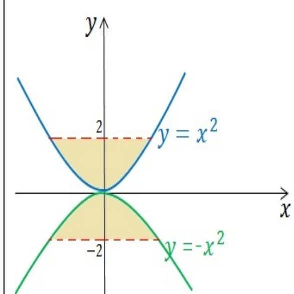


27 $\frac{2}{3}\pi$

28 $\frac{64\sqrt{2}}{3}\pi$

29 $\frac{16}{3}\pi$

30 $\frac{2\sqrt{2}}{3}\pi$

31 $\frac{1944}{5}\pi$

32 $\frac{2}{15}\pi$

33 $\frac{2}{35}\pi$

34 $\frac{373}{14}\pi$

35 $\frac{512}{15}\pi$

36 $\frac{(e^4-1)\pi}{2}$

37 $\left(6+4\ln^2(4)-16\ln(2)\right)\pi$

38 $\frac{\pi^2}{4}$

39 $\frac{\pi}{2\sqrt{2}}$

40 8π

41 $\frac{2}{5}\pi$

42 $\frac{64}{15}\pi$

43 $\frac{72}{5}\pi$

44 $\frac{16}{15}\pi$

45 8π

46 $\pi(\pi-2)$

47 $\frac{\pi}{2}(\sqrt{2}\pi-4)$

48 $\frac{16}{3}\pi$

49 $\frac{\pi}{6}$

50 $\frac{3}{2}\pi$

51 8π

52 $\frac{25}{2}\pi$

53 $\sqrt{17}+\frac{\sinh^{-1}(4)}{4}$

54 $4\sqrt{2}$

55 $\frac{1}{27}(22\sqrt{22}-13\sqrt{13})$

56 $\frac{2}{27}(37\sqrt{37}-1)$

57 $\frac{1}{27}\left((4+18\sqrt[3]{2})^{\frac{3}{2}}-(4+9(2)^{\frac{3}{2}})^{\frac{3}{2}}\right)$

58 45

59 $\frac{2}{3}(2\sqrt{2}-1)$

60 $\frac{14}{3}$

61 $-\frac{\sqrt{17}}{4}+\frac{\sqrt{1+e^2}}{e}+\sinh^{-1}(4)-\sinh^{-1}(e)$

62 74

63 $-\sqrt{5}+\tanh^{-1}(\sqrt{5})+\sqrt{17}-\tanh^{-1}(\sqrt{17})$

64 $\ln(\sqrt{2}+1)$

65 $\frac{3\pi}{2}-3\cos^{-1}(\frac{4}{3})$

66 $6\sqrt{5}\pi$

67 16π

68 $\frac{\pi}{9}(82\sqrt{82}-1)$

69 $\frac{\pi}{6}(5\sqrt{5}-1)$

70 $\pi\left(e\sqrt{1+e^2}+\ln(e+\sqrt{1+e^2})-\sqrt{2}-\ln(\sqrt{2}+1)\right)$

71 $\left(\sqrt{2}+sinh^{-1}(1)\right)\pi$

72 $\frac{\pi}{32}\left(438\sqrt{37}-\sinh^{-1}(6)\right)$

73 8π

74 $\frac{2\pi}{3}(10\sqrt{10}-1)$

75 $a^2\pi$

76 $\frac{\pi}{6}(5\sqrt{5}-1)$

77 $2\pi(\sqrt{2}+\sinh^{-1}(1))$

78 d

79 a

80 a

81 a

82 a

83 a

84 d

Chapter 8:

Exercise 8.1

1 $y=2x+1$

2 $x+2y^2=1$

3 $y=x^2$

4 $(x-1)^2+(y-1)^2=1$

5 $x=\ln(\ln y)$

6 $x^2+y^2=9$

7 $x=3y+5$

8 $y=x^3$

9 $\frac{3}{2},\ \frac{3}{4}$

10 $18,\ 54$

11 $\frac{4}{3},\ -\frac{4}{9}$

12 $-\frac{9}{2},\ -\frac{1}{4}$

13 $-1,\ 2$

14 $\frac{1}{\sqrt{2}-1},\ \frac{\sqrt{2}}{\sqrt{2}-1}$

15 $0,\ 2$

16 $6,\ 30$

17 4

18 $\frac{2\sqrt{2}}{3}$

19 $-\frac{9}{2}$

20 $-\frac{1}{2\sqrt{3}}$

21 $\frac{1}{3}$

22 $\sqrt{3}$

23 3

24 $30\sqrt[3]{25}$

25 Horizontal line at $(0,0)$ and no vertical line.

26 Horizontal line at $(0,0)$ and no vertical line.

27 There are no horizontal or vertical lines.

28 Horizontal lines at $(1,2)$ and $(1,-2)$ and vertical line at $(0,0)$.

29 Vertical line at $(-3,1)$ and no horizontal lines.

30 Horizontal lines at $(1,2)$ and $(1,-2)$, and vertical lines at $(0,0)$ and $(2,0)$

31 $4\sqrt{10}$

32 $2(5\sqrt{5}-1)$

33 $\frac{1}{4}\left(-2\sqrt{5}+8\sqrt{65}-\sinh^{-1}(2)+\sinh^{-1}(8)\right)$

34 $\frac{\pi}{12}$

35 $-\sqrt{2}+\sqrt{17}+\tanh^{-1}(\sqrt{2})-\tanh^{-1}(\sqrt{17})$

36 π

37 $\frac{3\pi}{4}$

38 $\frac{61}{216}$

39 $\frac{\pi}{6}(5\sqrt{5}-1)$

40 $\frac{2\sqrt{2}\pi(1+2e^{\pi})}{5}$

41 $\frac{5\sqrt{5}\pi}{6}(13\sqrt{13}-1)$

42 $\frac{13\pi}{6}$

43 $\frac{\pi}{6}(17\sqrt{17}-1)$

44 $2\pi^2$

45 $\sqrt{2}\pi$

46 $\frac{\pi}{54}(145\sqrt{145}-1)$

Exercise 8.2

1 $(0,1)$

2 $(0,-1)$

3 $(\sqrt{2},\sqrt{2})$

4 $(-3,0)$

5 $(0,-\frac{1}{2})$

6 $(-3,0)$

7 $(-\frac{7}{\sqrt{2}},\frac{7}{\sqrt{2}})$

8 $(\frac{3\sqrt{3}}{2},\frac{3}{2})$

9 $(\sqrt{2},\frac{\pi}{4})$

10 $(2,\frac{\pi}{6})$

11 $(\sqrt{2},\frac{3\pi}{4})$

12 $(2\sqrt{3},\frac{\pi}{3})$

13 $(\sqrt{6},35.26)$

14 $(3,0)$

15 $(2\sqrt{5},26.57)$

16 $(3\sqrt{2},\frac{5\pi}{4})$

17 $r=9\sec\theta$

18 $r=1$

19 $y=1$

20 $x^2+y^2-2x=0$

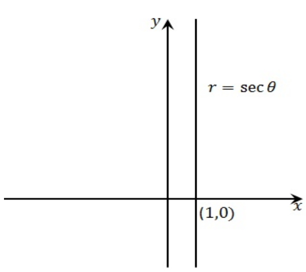
21 $r=3\tan\theta\sec\theta$

22 $r=4\sqrt{\sec 2\theta}$

23 $\sqrt{x^2+y^2}-y=3$

24 $x^2+y^2+2y-3\sqrt{x^2+y^2}=0$

25



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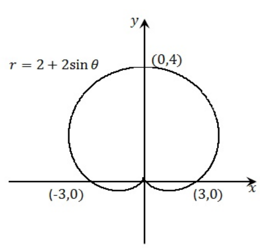

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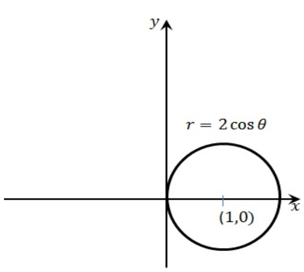

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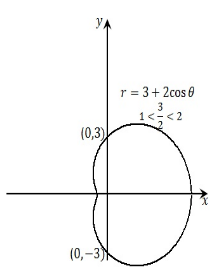


29 $-\sqrt{3}$

30 $-\frac{3}{3+2\sqrt{2}}$

31 The curve has a vertical tangent line.

32 $-(1+\sqrt{2})$

33 1

Exercise 8.3

1 4π

2 $\frac{3}{2}\pi$

3 25π

4 2π

5 54π

6 6π

7 $\frac{27}{4}\pi$

8 11π

9 $\frac{4-\pi}{2}$

10 $\frac{9\sqrt{3}}{2}-\pi$

11 10π

12 $\frac{5}{4}\pi-2$

13 $\frac{8+\pi}{4}$

14 $\frac{(\pi+2)}{2}$

15 $\frac{9\sqrt{3}}{2}+3\pi$

16 $\frac{\pi-2}{8}$

17 π

18 $5\pi-8$

19 $\frac{3\pi}{8}-1$

20 $\frac{\pi}{2}$

Exercise 8.4

1 3π

2 π

3 16

4 6π

5 24

6 $\frac{1}{2}\left(\sqrt{2}+\sinh^{-1}(1)\right)$

7 $\frac{64\pi}{\sqrt{5}}$

8 2π

9 $\frac{288\pi}{5}$

10 128π

11 $16\pi^2$

12 $\frac{2304\pi}{5}$

13 $\frac{64}{5}\pi$

14 32π

15 $\frac{64}{5}\pi$

16 $\frac{256}{5}\pi$

17 $32\pi^2$

18 2π

Review Exercises

1 $y=\frac{2x}{3}+1$

2 $y=\ln(x)$

3 $y=x+2$

4 $y=\frac{1}{x}$

5 $\frac{x^2}{9}+\frac{y^2}{4}=1$

6 $x^2+y^2=1$

7 $y=e^x e^{e^x}$

8 $y=2x^2+4$

9 $\frac{4}{5},\ 0$

10 $6,\ 2$

11 $\frac{8\sqrt{2}}{3},\ \frac{8}{3}$

12 $0,\ \frac{2}{3}$

13 $-\frac{1}{\sqrt{3}},\ -\frac{8}{3\sqrt{3}}$

14 $-1,\ 0$

15 $-\sqrt{3},\ 8$

16 $\frac{1}{12},\ -\frac{1}{48}$

17 $y = \frac{1}{4}x + 3$

18 $y = -\sqrt{2}x + \frac{3}{2}$

19 $y = \frac{2\sqrt{2}}{3}x - \frac{1}{3}$

20 $y = 4x - \frac{2}{\sqrt{3}}$

21 $y = -\frac{1}{9}x - \frac{61}{9}$

22 $y = -x + 2$

23 $y = 2\sqrt{3}x - (2\sqrt{3} + 3)$

24 $y = 24x - (24\ln(4) - 9)$

25 Vertical line at $(-3, 1)$ and no horizontal lines.

26 Horizontal line at $(\frac{65}{8}, -\frac{25}{4})$ and vertical line at $(-2, -4)$ and $(2, 6)$.

27 Horizontal line at $(1, 0)$ and no vertical lines.

28 Horizontal line at $(0, -2)$ and $(2, 2)$ and no vertical lines.

29 Horizontal line at $(0, 1)$ and $(0, -1)$ and vertical line at $(1, 0)$ and $(-1, 0)$.

30 Horizontal line at $(1, 2)$ and $(1, -2)$ and vertical line at $(2, 0)$ and $(0, 0)$.

31 Horizontal line at $(\frac{3}{4}, -\frac{1}{4})$ and vertical line at $(1, 0)$.

32 There are no horizontal or vertical lines.

33 $\frac{1}{20}\left(10\sqrt{101} + \sinh^{-1}(10)\right)$

34 $5\sqrt{5}$

35 π

36 4π

37 $3\sqrt{10} + \sinh^{-1}(3)$

38 $\frac{1}{16}\left(28\sqrt{3} - 4\sqrt{17} + \tanh^{-1}(\frac{7}{4\sqrt{3}}) - \coth^{-1}(\frac{4}{\sqrt{17}})\right)$

39 $\sqrt{2}(e^{\frac{\pi}{2}} - 1)$

40 $\frac{8}{63}(65\sqrt{65} - 2\sqrt{2})$

41 $\frac{\pi}{6}(37\sqrt{37} - 1)$

42 $\frac{2(64+247\sqrt{13})\pi}{1215}$

43 4π

44 $\frac{\pi}{3}(32 - 20\sqrt{2})$

45 16π

46 $\frac{2\sqrt{2}\pi}{5}(1 + 2e^{\pi})$

47 $\frac{4\pi}{3\sqrt{2}}\left((1+e)^{\frac{3}{2}} - 2\sqrt{2}\right)$

48 $\frac{32\pi}{3}(5\sqrt{5} - 1)$

49 $\frac{1250\pi}{3}$

50 $39\sqrt{10}\,\pi$

51 $\sqrt{2}\,\pi$

52 $6\,\pi^2$

53 $\pi\left(-\sqrt{2} + e^e\sqrt{e^{2e}+1} - \sinh^{-1}(1) + \sinh^{-1}(e^e)\right)$

54 $\left(\frac{28\sqrt{2}+81\sin^{-1}(\frac{4\sqrt{2}}{9})}{\sqrt{2}}\right)\pi$

55 $\frac{8\pi}{3}(2\sqrt{2} - 1)$

56 $\frac{8\pi}{\sqrt{2}}$

57 $(-2, 0)$

58 $(-4, 0)$

59 $(-1, -\sqrt{3})$

60 $(\frac{\sqrt{3}}{2}, \frac{1}{2})$

61 $(4\sqrt{2}, 4\sqrt{2})$

62 $(2, 0)$

63 $(0, -5)$

64 $(-\sqrt{2}, \sqrt{2})$

65 $(\sqrt{2}, \frac{\pi}{4})$

66 $(1, \pi)$

67 $(6, \frac{\pi}{3})$

68 $(2\sqrt{2}, \frac{3\pi}{4})$

69 $(1, 0)$

70 $(\frac{3}{2}, 19.47)$

71 $(3, \pi)$

72 $(5, 126.87)$

73 $r = 3\sec\theta$

74 $r = -7\csc\theta$

75 $r = 1$

76 $r = 6\cos\theta$

77 $r^2 = 8\csc 2\theta$

78 $r = 9\cot\theta\csc\theta$

79 $r = -9\sin\theta$

80 $r^2 = 25\sec 2\theta$

81 $x^2 + y^2 = 9$

82 $x^2 + y^2 - y = 0$

83 $x^2 + y^2 - 2x = 0$

84 $y = 4$

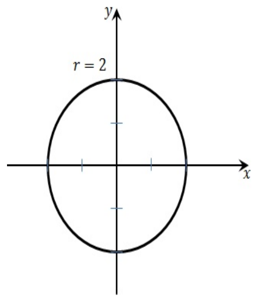
85 $x = 1$

86 $x-y=4$

87 $\sqrt{x^2+y^2}-y=2$

88 $\sqrt{x^2+y^2}+2x=3$

89



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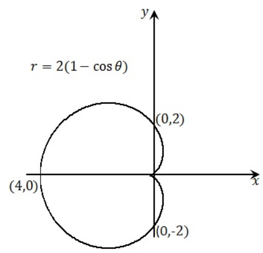

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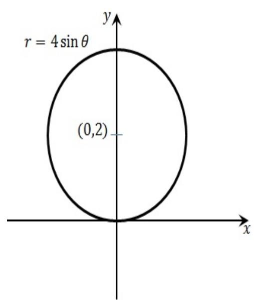

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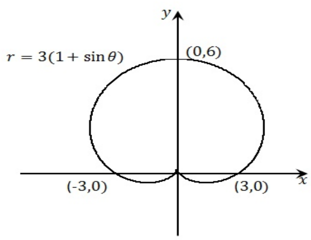

93 4π

94 9π

95 $\frac{\pi-2}{16}$

96 $\frac{3\pi}{2}$

97 $\frac{3\pi}{2}-4$

98 $\frac{1}{4}(1-e^{-4\pi})$

99 $\frac{9\sqrt{3}}{2}+2\pi$

100 $\pi+8$

101 $\frac{2+\pi}{8}$

102 $\frac{3\pi-8}{2}$

103 $2(\pi-2)$

104 $4+\frac{3\pi}{2}$

105 10π

106 16

107 $\frac{5\pi-6\sqrt{3}}{24}$

108 5π

109 3π

110 4π

111 2π

112 8

113 $\sqrt{2}(e^{2\pi}-1)$

114 $12\sqrt{2}$

115 $2\sqrt{5}\pi$

116 π

117 $\frac{\pi^2}{2}$

118 $\frac{128}{5}\pi$

119 $\frac{32}{5}\pi$

120 $\frac{2\sqrt{2}\pi}{5}(1+e^{2\pi})$

121 3π

122 π^2

123 π

124 $\frac{32}{5}\pi$

125 $\frac{32}{5}\pi$

126 18π

127 $\frac{2\sqrt{2}\pi}{5}(e^{\pi}-2)$

128 d

129 a

130 b

131 a

132 d

133 a

134 d

135 a

136 c

137 a

138 a

139 b

140 a

141 c

142 c

143 d

144 a

145 b

146 a

147 c

148 b

149 b

150 a

Homework

Chapter 1:
Exercise 1.1 : 1, 6
Exercise 1.2 : 2, 7, 13
Exercise 1.3 : 1, 2, 3, 16
Review Exercises : 41, 62, 65, 66

Chapter 2:
Exercise 2.1 : 1, 7
Exercise 2.2 : 1, 9, 13, 14, 15
Exercise 2.3 : 7, 10
Exercise 2.4 : 1, 7, 9, 12, 17, 24, 29
Exercise 2.5 : 1, 5
Review Exercises : 17, 19, 84, 87, 92

Chapter 3:
Exercise 3.1 : 2, 15, 21, 27, 35
Exercise 3.2 : 9, 18, 19, 25
Exercise 3.3 : 2, 5, 15, 18
Review Exercises : 73, 75

Chapter 4:
Exercise 4.1 : 1, 3, 9
Exercise 4.2 : 1, 10, 11, 16
Exercise 4.3 : 1, 4, 7, 8
Review Exercises : 43, 55

Chapter 5:
Exercise 5.1 : 1, 5
Exercise 5.2 : 1, 13
Exercise 5.3 : 1, 6
Exercise 5.4 : 1, 3, 8, 16
Exercise 5.5 : 2, 10
Exercise 5.6 : 6
Review Exercises : 51, 70, 72

Chapter 6:
Exercise 6.1 : 3, 7

Exercise 6.2 : 1, 9
Review Exercises : 20, 26, 35

Chapter 7:

Exercise 7.1 : 2, 9, 17
Exercise 7.2 : 1
Exercise 7.3 : 1, 2, 9, 17
Exercise 7.4 : 1, 2, 14
Review Exercises : 78, 79

Chapter 8:

Exercise 8.1 : 1, 9, 17, 25, 31, 39
Exercise 8.2 : 1, 9, 18, 27, 29
Exercise 8.3 : 1, 9, 11
Exercise 8.4 : 1, 7, 13
Review Exercises : 128, 130, 143, 148, 150