Introduction to Probability Models
10th Edition
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Introduction to Probability Theory

Exercises

1. A box contains three marbles: one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box then replacing it in the box and drawing a second marble from the box. What is the sample space? If, at all times, each marble in the box is equally likely to be selected, what is the probability of each point in the sample space?

*2. Repeat Exercise 1 when the second marble is drawn without replacing the first marble.

3. A coin is to be tossed until a head appears twice in a row. What is the sample space for this experiment? If the coin is fair, what is the probability that it will be tossed exactly four times?

4. Let E, F, G be three events. Find expressions for the events that of E, F, G
   (a) only F occurs,
   (b) both E and F but not G occur,
   (c) at least one event occurs,
   (d) at least two events occur,
   (e) all three events occur,
   (f) none occurs,
   (g) at most one occurs,
   (h) at most two occur.

*5. An individual uses the following gambling system at Las Vegas. He bets \$1 that the roulette wheel will come up red. If he wins, he quits. If he loses then he makes the same bet a second time only this time he bets \$2; and then regardless of the outcome, quits. Assuming that he has a probability of $\frac{1}{2}$ of winning each bet, what is the probability that he goes home a winner? Why is this system not used by everyone?

6. Show that $E(F \cup G) = EF \cup EG$.

7. Show that $(E \cup F)^c = E^c F^c$.

8. If $P(E) = 0.9$ and $P(F) = 0.8$, show that $P(EF) \geqslant 0.7$. In general, show that

$$P(EF) \geqslant P(E) + P(F) - 1$$

This is known as Bonferroni's inequality.

*9. We say that $E \subset F$ if every point in E is also in F. Show that if $E \subset F$, then

$$P(F) = P(E) + P(FE^c) \geqslant P(E)$$

10. Show that

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leqslant \sum_{i=1}^{n} P(E_i)$$

This is known as Boole's inequality.

Hint: Either use Equation (1.2) and mathematical induction, or else show that $\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$, where $F_1 = E_1$, $F_i = E_i \bigcap_{j=1}^{i-1} E_j^c$, and use property (iii) of a probability.

11. If two fair dice are tossed, what is the probability that the sum is i, $i = 2, 3, \ldots, 12$?

12. Let E and F be mutually exclusive events in the sample space of an experiment. Suppose that the experiment is repeated until either event E or event F occurs. What does the sample space of this new super experiment look like? Show that the probability that event E occurs before event F is $P(E)/\,[P(E) + P(F)]$.

Hint: Argue that the probability that the original experiment is performed n times and E appears on the nth time is $P(E) \times (1-p)^{n-1}$, $n = 1, 2, \ldots$, where $p = P(E) + P(F)$. Add these probabilities to get the desired answer.

13. The dice game craps is played as follows. The player throws two dice, and if the sum is seven or eleven, then she wins. If the sum is two, three, or twelve, then she loses. If the sum is anything else, then she continues throwing until she either throws that number again (in which case she wins) or she throws a seven (in which case she loses). Calculate the probability that the player wins.

14. The probability of winning on a single toss of the dice is p. A starts, and if he fails, he passes the dice to B, who then attempts to win on her toss. They continue tossing the dice back and forth until one of them wins. What are their respective probabilities of winning?

15. Argue that $E = EF \cup EF^c$, $E \cup F = E \cup FE^c$.

16. Use Exercise 15 to show that $P(E \cup F) = P(E) + P(F) - P(EF)$.

*17. Suppose each of three persons tosses a coin. If the outcome of one of the tosses differs from the other outcomes, then the game ends. If not, then the persons start over and retoss their coins. Assuming fair coins, what is the probability that the game will end with the first round of tosses? If all three coins are biased and have probability $\frac{1}{4}$ of landing heads, what is the probability that the game will end at the first round?

18. Assume that each child who is born is equally likely to be a boy or a girl. If a family has two children, what is the probability that both are girls given that (a) the eldest is a girl, (b) at least one is a girl?

*19. Two dice are rolled. What is the probability that at least one is a six? If the two faces are different, what is the probability that at least one is a six?

20. Three dice are thrown. What is the probability the same number appears on exactly two of the three dice?

21. Suppose that 5 percent of men and 0.25 percent of women are color-blind. A color-blind person is chosen at random. What is the probability of this person being male? Assume that there are an equal number of males and females.

22. A and B play until one has 2 more points than the other. Assuming that each point is independently won by A with probability p, what is the probability they will play a total of $2n$ points? What is the probability that A will win?

23. For events $E_1, E_2, \ldots, E_n$ show that

$$P(E_1E_2\cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$$

24. In an election, candidate A receives n votes and candidate B receives m votes, where $n > m$. Assume that in the count of the votes all possible orderings of the $n + m$ votes are equally likely. Let $P_{n,m}$ denote the probability that from the first vote on A is always in the lead. Find

(a) $P_{2,1}$ (b) $P_{3,1}$ (c) $P_{n,1}$ (d) $P_{3,2}$ (e) $P_{4,2}$
(f) $P_{n,2}$ (g) $P_{4,3}$ (h) $P_{5,3}$ (i) $P_{5,4}$
(j) Make a conjecture as to the value of $P_{n,m}$.

*25. Two cards are randomly selected from a deck of 52 playing cards.
(a) What is the probability they constitute a pair (that is, that they are of the same denomination)?
(b) What is the conditional probability they constitute a pair given that they are of different suits?

26. A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each. Define events E_1, E_2, E_3, and E_4 as follows:

$E_1 = \{$the first pile has exactly 1 ace$\}$,
$E_2 = \{$the second pile has exactly 1 ace$\}$,
$E_3 = \{$the third pile has exactly 1 ace$\}$,
$E_4 = \{$the fourth pile has exactly 1 ace$\}$

Use Exercise 23 to find $P(E_1E_2E_3E_4)$, the probability that each pile has an ace.

*27. Suppose in Exercise 26 we had defined the events E_i, $i = 1, 2, 3, 4$, by

$E_1 = \{$one of the piles contains the ace of spades$\}$,
$E_2 = \{$the ace of spades and the ace of hearts are in different piles$\}$,
$E_3 = \{$the ace of spades, the ace of hearts, and the ace of diamonds are in different piles$\}$,
$E_4 = \{$all 4 aces are in different piles$\}$

Now use Exercise 23 to find $P(E_1E_2E_3E_4)$, the probability that each pile has an ace. Compare your answer with the one you obtained in Exercise 26.

28. If the occurrence of B makes A more likely, does the occurrence of A make B more likely?

29. Suppose that $P(E) = 0.6$. What can you say about $P(E|F)$ when
    (a) E and F are mutually exclusive?
    (b) $E \subset F$?
    (c) $F \subset E$?

*30. Bill and George go target shooting together. Both shoot at a target at the same time. Suppose Bill hits the target with probability 0.7, whereas George, independently, hits the target with probability 0.4.
    (a) Given that exactly one shot hit the target, what is the probability that it was George's shot?
    (b) Given that the target is hit, what is the probability that George hit it?

31. What is the conditional probability that the first die is six given that the sum of the dice is seven?

*32. Suppose all n men at a party throw their hats in the center of the room. Each man then randomly selects a hat. Show that the probability that none of the n men selects his own hat is

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - + \cdots \frac{(-1)^n}{n!}$$

Note that as $n \to \infty$ this converges to e^{-1}. Is this surprising?

33. In a class there are four freshman boys, six freshman girls, and six sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

34. Mr. Jones has devised a gambling system for winning at roulette. When he bets, he bets on red, and places a bet only when the ten previous spins of the roulette have landed on a black number. He reasons that his chance of winning is quite large since the probability of eleven consecutive spins resulting in black is quite small. What do you think of this system?

35. A fair coin is continually flipped. What is the probability that the first four flips are
    (a) H, H, H, H?
    (b) T, H, H, H?
    (c) What is the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H?

36. Consider two boxes, one containing one black and one white marble, the other, two black and one white marble. A box is selected at random and a marble is drawn at random from the selected box. What is the probability that the marble is black?

37. In Exercise 36, what is the probability that the first box was the one selected given that the marble is white?

38. Urn 1 contains two white balls and one black ball, while urn 2 contains one white ball and five black balls. One ball is drawn at random from urn 1 and placed in urn 2. A ball is then drawn from urn 2. It happens to be white. What is the probability that the transferred ball was white?

39. Stores A, B, and C have 50, 75, and 100 employees, and, respectively, 50, 60, and 70 percent of these are women. Resignations are equally likely among all employees,

regardless of sex. One employee resigns and this is a woman. What is the probability that she works in store C?

*40. (a) A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random, and when he flips it, it shows heads. What is the probability that it is the fair coin?
(b) Suppose that he flips the same coin a second time and again it shows heads. Now what is the probability that it is the fair coin?
(c) Suppose that he flips the same coin a third time and it shows tails. Now what is the probability that it is the fair coin?

41. In a certain species of rats, black dominates over brown. Suppose that a black rat with two black parents has a brown sibling.
(a) What is the probability that this rat is a pure black rat (as opposed to being a hybrid with one black and one brown gene)?
(b) Suppose that when the black rat is mated with a brown rat, all five of their offspring are black. Now, what is the probability that the rat is a pure black rat?

42. There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

43. Suppose we have ten coins which are such that if the ith one is flipped then heads will appear with probability $i/10$, $i = 1, 2, \ldots, 10$. When one of the coins is randomly selected and flipped, it shows heads. What is the conditional probability that it was the fifth coin?

44. Urn 1 has five white and seven black balls. Urn 2 has three white and twelve black balls. We flip a fair coin. If the outcome is heads, then a ball from urn 1 is selected, while if the outcome is tails, then a ball from urn 2 is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails?

*45. An urn contains b black balls and r red balls. One of the balls is drawn at random, but when it is put back in the urn c additional balls of the same color are put in with it. Now suppose that we draw another ball. Show that the probability that the first ball drawn was black given that the second ball drawn was red is $b/(b + r + c)$.

46. Three prisoners are informed by their jailer that one of them has been chosen at random to be executed, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from $\frac{1}{3}$ to $\frac{1}{2}$, since he would then be one of two prisoners. What do you think of the jailer's reasoning?

47. For a fixed event B, show that the collection $P(A|B)$, defined for all events A, satisfies the three conditions for a probability. Conclude from this that

$$P(A|B) = P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$$

Then directly verify the preceding equation.

*48. Sixty percent of the families in a certain community own their own car, thirty percent own their own home, and twenty percent own both their own car and their own home. If a family is randomly chosen, what is the probability that this family owns a car or a house but not both?

Chapter 1

1. $S = \{(R,R),(R,G),(R,B),(G,R),(G,G),(G,B),$
$(B,R),(B,G),(B,B)\}$
The probability of each point in S is 1/9.

2. $S = \{(R,G),(R,B),(G,R),(G,B),(B,R),(B,G)\}$

3. $S = \{(e_1, e_2, \ldots, e_n),\ n \geq 2\}$ where $e_i \in$ (heads, tails}. In addition, $e_n = e_{n-1}$ = heads and for $i = 1, \ldots, n-2$ if e_i = heads, then e_{i+1} = tails.

$$P\{4 \text{ tosses}\} = P\{(t,t,h,h)\} + P\{(h,t,h,h)\}$$
$$= 2\left[\frac{1}{2}\right]^4 = \frac{1}{8}$$

4. (a) $F(E \cup G)^c = FE^cG^c$
(b) EFG^c
(c) $E \cup F \cup G$
(d) $EF \cup EG \cup FG$
(e) EFG
(f) $(E \cup F \cup G)^c = E^cF^cG^c$
(g) $(EF)^c(EG)^c(FG)^c$
(h) $(EFG)^c$

5. $\frac{3}{4}$. If he wins, he only wins \$1, while if he loses, he loses \$3.

6. If $E(F \cup G)$ occurs, then E occurs and either F or G occur; therefore, either EF or EG occurs and so

$$E(F \cup G) \subset EF \cup EG$$

Similarly, if $EF \cup EG$ occurs, then either EF or EG occurs. Thus, E occurs and either F or G occurs; and so $E(F \cup G)$ occurs. Hence,

$$EF \cup EG \subset E(F \cup G)$$

which together with the reverse inequality proves the result.

7. If $(E \cup F)^c$ occurs, then $E \cup F$ does not occur, and so E does not occur (and so E^c does); F does not occur (and so F^c does) and thus E^c and F^c both occur. Hence,

$$(E \cup F)^c \subset E^cF^c$$

If E^cF^c occurs, then E^c occurs (and so E does not), and F^c occurs (and so F does not). Hence, neither E or F occurs and thus $(E \cup F)^c$ does. Thus,

$$E^cF^c \subset (E \cup F)^c$$

and the result follows.

8. $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$

9. $F = E \cup FE^c$, implying since E and FE^c are disjoint that $P(F) = P(E) + P(FE)^c$.

10. Either by induction or use

$$\bigcup_1^n E_i = E_1 \cup E_1^cE_2 \cup E_1^cE_2^cE_3 \cup \cdots \cup E_1^c \cdots E_{n-1}^cE_n$$

and as each of the terms on the right side are mutually exclusive:

$$P(\bigcup_i E_i) = P(E_1) + P(E_1^cE_2) + P(E_1^cE_2^cE_3) + \cdots$$
$$+ P(E_1^c \cdots E_{n-1}^cE_n)$$
$$\leq P(E_1) + P(E_2) + \cdots + P(E_n) \quad \text{(why?)}$$

11. $$P\{\text{sum is } i\} = \begin{cases} \dfrac{i-1}{36}, & i = 2, \ldots, 7 \\ \dfrac{13-i}{36}, & i = 8, \ldots, 12 \end{cases}$$

12. Either use hint or condition on initial outcome as:

$P\{E \text{ before } F\}$

$$= P\{E \text{ before } F \mid \text{initial outcome is } E\}P(E)$$
$$+ P\{E \text{ before } F \mid \text{initial outcome is } F\}P(F)$$
$$+ P\{E \text{ before } F \mid \text{initial outcome neither E}$$
$$\text{or } F\}[1 - P(E) - P(F)]$$

$$= 1 \cdot P(E) + 0 \cdot P(F) + P\{E \text{ before } F\}$$
$$= [1 - P(E) - P(F)]$$

Therefore, $P\{E \text{ before } F\} = \dfrac{P(E)}{P(E) + P(F)}$

13. Condition an initial toss

$$P\{\text{win}\} = \sum_{i=2}^{12} P\{\text{win} \mid \text{throw } i\} P\{\text{throw } i\}$$

Now,

$$P\{\text{win} \mid \text{throw } i\} = P\{i \text{ before } 7\}$$

$$= \begin{cases} 0 & i = 2, 12 \\ \dfrac{i-1}{5+1} & i = 3, \ldots, 6 \\ 1 & i = 7, 11 \\ \dfrac{13-i}{19-1} & i = 8, \ldots, 10 \end{cases}$$

where above is obtained by using Problems 11 and 12.

$P\{\text{win}\} \approx .49.$

14. $$P\{A \text{ wins}\} = \sum_{n=0}^{\infty} P\{A \text{ wins on } (2n+1)\text{st toss}\}$$
$$= \sum_{n=0}^{\infty} (1-P)^{2n} P$$
$$= P \sum_{n=0}^{\infty} [(1-P)^2]^n$$
$$= P \frac{1}{1-(1-P)^2}$$
$$= \frac{P}{2P - P^2}$$
$$= \frac{1}{2-P}$$
$$P\{B \text{ wins}\} = 1 - P\{A \text{ wins}\}$$
$$= \frac{1-P}{2-P}$$

16. $$P(E \cup F) = P(E \cup FE^c)$$
$$= P(E) + P(FE^c)$$

since E and FE^c are disjoint. Also,

$$P(F) = P(FE \cup FE^c)$$
$$= P(FE) + P(FE^c) \text{ by disjointness}$$

Hence,

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

17. $$\text{Prob}\{\text{end}\} = 1 - \text{Prob}\{\text{continue}\}$$
$$= 1 - P(\{H, H, H\} \cup \{T, T, T\})$$
$$= 1 - [\text{Prob}(H, H, H) + \text{Prob}(T, T, T)].$$

Fair coin: $$\text{Prob}\{\text{end}\} = 1 - \left[\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right] = \frac{3}{4}$$

Biased coin: $$P\{\text{end}\} = 1 - \left[\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}\right] = \frac{9}{16}$$

18. Let B = event both are girls; E = event oldest is girl; L = event at least one is a girl.

(a) $$P(B|E) = \frac{P(BE)}{P(E)} = \frac{P(B)}{P(E)} = \frac{1/4}{1/2} = \frac{1}{2}$$

(b) $$P(L) = 1 - P(\text{no girls}) = 1 - \frac{1}{4} = \frac{3}{4},$$

$$P(B|L) = \frac{P(BL)}{P(L)} = \frac{P(B)}{P(L)} = \frac{1/4}{3/4} = \frac{1}{3}$$

19. E = event at least 1 six $P(E)$

$$= \frac{\text{number of ways to get } E}{\text{number of sample pts}} = \frac{11}{36}$$

D = event two faces are different $P(D)$

$$= 1 - \text{Prob(two faces the same)}$$

$$= 1 - \frac{6}{36} = \frac{5}{6} P(E|D) = \frac{P(ED)}{P(D)} = \frac{10/36}{5/6} = \frac{1}{3}$$

20. Let E = event same number on exactly two of the dice; S = event all three numbers are the same; D = event all three numbers are different. These three events are mutually exclusive and define the whole sample space. Thus, $1 = P(D) + P(S) + P(E)$, $P(S) = 6/216 = 1/36$; for D have six possible values for first die, five for second, and four for third.

$\therefore$ Number of ways to get $D = 6 \cdot 5 \cdot 4 = 120$.

$$P(D) = 120/216 = 20/36$$

$$\therefore P(E) = 1 - P(D) - P(S)$$
$$= 1 - \frac{20}{36} - \frac{1}{36} = \frac{5}{12}$$

21. Let C = event person is color blind.

$P(\text{Male}|C)$

$$= \frac{P(C|\text{Male})\,P(\text{Male})}{P(C|\text{Male}\,P(\text{Male}) + P(C|\text{Female})\,P(\text{Female})}$$

$$= \frac{.05 \times .5}{.05 \times .5 + .0025 \times .5}$$

$$= \frac{2500}{2625} = \frac{20}{21}$$

22. Let trial 1 consist of the first two points; trial 2 the next two points, and so on. The probability that each player wins one point in a trial is $2p(1-p)$. Now a total of $2n$ points are played if the first $(a-1)$ trials all result in each player winning one of the points in that trial and the n^{th} trial results in one of the players winning both points. By independence, we obtain

$P\{2n \text{ points are needed}\}$

$= (2p(1-p))^{n-1}(p^2 + (1-p)^2), \quad n \geq 1$

The probability that A wins on trial n is $(2p(1-p))^{n-1}p^2$ and so

$$P\{A \text{ wins}\} = p^2 \sum_{n=1}^{\infty} (2p(1-p))^{n-1}$$

$$= \frac{p^2}{1 - 2p(1-p)}$$

23. $P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$

$$= P(E_1)\frac{P(E_1E_2)}{P(E_1)}\frac{P(E_1E_2E_3)}{P(E_1E_2)}\cdots\frac{P(E_1\cdots E_n)}{P(E_1\cdots E_{n-1})}$$

$$= P(E_1\cdots E_n)$$

24. Let a signify a vote for A and b one for B.

(a) $P_{2,1} = P\{a,a,b\} = 1/3$

(b) $P_{3,1} = P\{a,a\} = (3/4)(2/3) = 1/2$

(c) $P_{3,2} = P\{a,a,a\} + P\{a,a,b,a\}$

$= (3/5)(2/4)[1/3 + (2/3)(1/2)] = 1/5$

(d) $P_{4,1} = P\{a,a\} = (4/5)(3/4) = 3/5$

(e) $P_{4,2} = P\{a,a,a\} + P\{a,a,b,a\}$

$= (4/6)(3/5)[2/4 + (2/4)(2/3)] = 1/3$

(f) $P_{4,3} = P\{\text{always ahead}|a,a\}(4/7)(3/6)$

$= (2/7)[1 - P\{a,a,a,b,b,b|a,a\}$

$- P\{a,a,b,b|a,a\} - P\{a,a,b,a,b,b|a,a\}]$

$= (2/7)[1 - (2/5)(3/4)(2/3)(1/2)$

$- (3/5)(2/4) - (3/5)(2/4)(2/3)(1/2)]$

$= 1/7$

(g) $P_{5,1} = P\{a,a\} = (5/6)(4/5) = 2/3$

(h) $P_{5,2} = P\{a,a,a\} + P\{a,a,b,a\}$

$= (5/7)(4/6)[(3/5) + (2/5)(3/4)] = 3/7$

By the same reasoning we have

(i) $P_{5,3} = 1/4$

(j) $P_{5,4} = 1/9$

(k) In all the cases above, $P_{n,m} = \dfrac{n-n}{n+n}$

25. (a) $P\{pair\} = P\{\text{second card is same denomination as first}\}$

$= 3/51$

(b) $P\{\text{pair}|\text{different suits}\}$

$$= \frac{P\{\text{pair, different suits}\}}{P\{\text{different suits}\}}$$

$= P\{\text{pair}\}/P\{\text{different suits}\}$

$$= \frac{3/51}{39/51} = 1/13$$

26. $P(E_1) = \binom{4}{1}\binom{48}{12}\Big/\binom{52}{13} = \dfrac{39 \cdot 38 \cdot 37}{51 \cdot 50 \cdot 49}$

$P(E_2|E_1) = \binom{3}{1}\binom{36}{12}\Big/\binom{39}{13} = \dfrac{26 \cdot 25}{38 \cdot 37}$

$P(E_3|E_1E_2) = \binom{2}{1}\binom{24}{12}\Big/\binom{26}{13} = 13/25$

$P(E_4|E_1E_2E_3) = 1$

$$P(E_1E_2E_3E_4) = \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49}$$

27. $P(E_1) = 1$

$P(E_2|E_1) = 39/51$, since 12 cards are in the ace of spades pile and 39 are not.

$P(E_3|E_1E_2) = 26/50$, since 24 cards are in the piles of the two aces and 26 are in the other two piles.

$P(E_4|E_1E_2E_3) = 13/49$

So

$P\{\text{each pile has an ace}\} = (39/51)(26/50)(13/49)$

28. Yes. $P(A|B) > P(A)$ is equivalent to $P(AB) > P(A)P(B)$, which is equivalent to $P(B|A) > P(B)$.

29. (a) $P(E|F) = 0$

(b) $P(E|F) = P(EF)/P(F) = P(E)/P(F) \geq P(E) = .6$

(c) $P(E|F) = P(EF)/P(F) = P(F)/P(F) = 1$

30. (a) $P\{\text{George}|\text{exactly 1 hit}\}$

$$= \frac{P\{\text{George, not Bill}\}}{P\{\text{exactly } 1\}}$$
$$= \frac{P\{G, \text{ not } B\}}{P\{G, \text{ not } B\} + P\{B, \text{ not } G)\}}$$
$$= \frac{(.4)(.3)}{(.4)(.3) + (.7)(.6)}$$
$$= 2/9$$

(b) $P\{G|\text{hit}\}$

$$= P\{G, \text{hit}\}/P\{\text{hit}\}$$
$$= P\{G\}/P\{\text{hit}\} = .4/[1 - (.3)(.6)]$$
$$= 20/41$$

31. Let S = event sum of dice is 7; F = event first die is 6.

$$P(S) = \frac{1}{6}P(FS) = \frac{1}{36}P(F|S) = \frac{P(F|S)}{P(S)}$$
$$= \frac{1/36}{1/6} = \frac{1}{6}$$

32. Let E_i = event person i selects own hat.
P (no one selects own hat)

$$= 1 - P(E_1 \cup E_2 \cup \cdots \cup E_n)$$
$$= 1 - \left[\sum_{i_1} P(Ei_1) - \sum_{i_1<i_2} P(Ei_1 Ei_2) + \cdots + (-1)^{n+1} P(E_1 E_2 E_n)\right]$$
$$= 1 - \sum_{i_1} P(Ei_1) - \sum_{i_1<i_2} P(Ei_1 Ei_2) - \sum_{i_1<i_2<i_3} P(Ei_1 Ei_2 Ei_3) + \cdots + (-1)^n P(E_1 E_2 E_n)$$

Let $k \in \{1, 2, \ldots, n\}$. $P(Ei_1\, EI_2\, Ei_k)$ = number of ways k specific men can select own hats $\div$ total number of ways hats can be arranged $= (n-k)!/n!$. Number of terms in summation $\sum_{i_1<i_2<\cdots<i_k}$ = number of ways to choose k variables out of n variables $= \begin{bmatrix} n \\ k \end{bmatrix} = n!/k!(n-k)!$.

Thus,

$$\sum_{i_1<\cdots<i_k} P(Ei_1 Ei_2 \cdots Ei_k)$$
$$= \sum_{i_1<\cdots<i_k} \frac{(n-k)!}{n!}$$
$$= \begin{bmatrix} n \\ k \end{bmatrix} \frac{(n-k)!}{n!} = \frac{1}{k!}$$

$\therefore P$(no one selects own hat)

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$
$$= \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}$$

33. Let S = event student is sophomore; F = event student is freshman; B = event student is boy; G = event student is girl. Let x = number of sophomore girls; total number of students = $16 + x$.

$$P(F) = \frac{10}{16+x} P(B) = \frac{10}{16+x} P(FB) = \frac{4}{16+x}$$
$$\frac{4}{16+x} = P(FB) = P(F)P(B) = \frac{10}{16+x}$$
$$\frac{10}{16+x} \Rightarrow x = 9$$

34. Not a good system. The successive spins are independent and so

$$P\{11^{\text{th}} \text{ is red}|\text{1st 10 black}\} = P\{11^{\text{th}} \text{ is red}\}$$
$$= P\left[= \frac{18}{38}\right]$$

35. (a) 1/16

(b) 1/16

(c) 15/16, since the only way in which the pattern H, H, H, H can appear before the pattern T, H, H, H is if the first four flips all land heads.

36. Let B = event marble is black; B_i = event that box i is chosen. Now

$$B = BB_1 \cup BB_2 P(B) = P(BB_1) + P(BB_2)$$
$$= P(B|B_1)P(B_1) + P(B|B_2)P(B_2)$$
$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{7}{12}$$

37. Let W = event marble is white.

$$P(B_1|W) = \frac{P(W|B_1)P(B_1)}{P(W|B_1)P(B_1) + P(W|B_2)P(B_2)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5}$$

38. Let T_W = event transfer is white; T_B = event transfer is black; W = event white ball is drawn from urn 2.

$$P(T_W|W) = \frac{P(W|T_W)P(T_W)}{P(W|T_W)P(T_W) + P(W|T_B)P(T_B)} = \frac{\frac{2}{7}\cdot\frac{2}{3}}{\frac{2}{7}\cdot\frac{2}{3} + \frac{1}{7}\cdot\frac{1}{3}} = \frac{\frac{4}{21}}{\frac{5}{21}} = \frac{4}{5}$$

39. Let W = event woman resigns; A, B, C are events the person resigning works in store A, B, C, respectively.

$P(C|W)$

$$= \frac{P(W|C)P(C)}{P(W|C)P(C) + P(W|B)P(B) + P(W|A)P(A)}$$

$$= \frac{.70 \times \frac{100}{225}}{.70 \times \frac{100}{225} + .60 \times \frac{75}{225} + .50\ \frac{50}{225}}$$

$$= \frac{70}{225}\Big/\frac{140}{225} = \frac{1}{2}$$

40. (a) F = event fair coin flipped; U = event two-headed coin flipped.

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

(b) $$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH|F)P(F) + P(HH|U)P(U)} = \frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{\frac{1}{8}}{\frac{5}{8}} = \frac{1}{5}$$

(c) $P(F|HHT)$

$$= \frac{P(HHT|F)P(F)}{P(HHT|F)P(F) + P(HHT|U)P(U)}$$

$$= \frac{P(HHT|F)P(F)}{P(HHT|F)P(F) + 0} = 1$$

since the fair coin is the only one that can show tails.

41. Note first that since the rat has black parents and a brown sibling, we know that both its parents are hybrids with one black and one brown gene (for if either were a pure black then all their offspring would be black). Hence, both of their offspring's genes are equally likely to be either black or brown.

(a) P(2 black genes | at least one black gene)

$$= \frac{P(\text{2 black genes})}{P(\text{at least one black gene})}$$

$$= \frac{1/4}{3/4} = 1/3$$

(b) Using the result from part (a) yields the following:

P(2 black genes | 5 black offspring)

$$= \frac{P(\text{2 black genes})}{P(\text{5 black offspring})}$$

$$= \frac{1/3}{1(1/3) + (1/2)^5(2/3)}$$

$$= 16/17$$

where P(5 black offspring) was computed by conditioning on whether the rat had 2 black genes.

42. Let B = event biased coin was flipped; F and U (same as above).

$P(U|H)$

$$= \frac{P(H|U)P(U)}{P(H|U)P(U) + P(H|B)P(B) + P(H|F)P(F)}$$

$$= \frac{1\cdot\frac{1}{3}}{1\cdot\frac{1}{3} + \frac{3}{4}\cdot\frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{9}{12}} = \frac{4}{9}$$

43. Let i = event coin was selected; $P(H|i) = \frac{i}{10}$.

$$P(5|H) = \frac{P(H|5)P(5)}{\sum_{i=1}^{10} P(H|i)P(i)} = \frac{\frac{5}{10}\cdot\frac{1}{10}}{\sum_{i=1}^{10}\frac{1}{10}\cdot\frac{1}{10}}$$

$$= \frac{5}{\sum_{i=1}^{10} i} = \frac{1}{11}$$

44. Let W = event white ball selected.

$$P(T|W) = \frac{P(W|T)P(T)}{P(W|T)P(T) + P(W|H)P(H)}$$
$$= \frac{\frac{1}{5}\cdot\frac{1}{2}}{\frac{1}{5}\cdot\frac{1}{2} + \frac{5}{12}\cdot\frac{1}{2}} = \frac{12}{37}$$

45. Let B_i = event i^{th} ball is black; R_i = event i^{th} ball is red.

$$P(B_1|R_2) = \frac{P(R_2|B_1)P(B_1)}{P(R_2|B_1)P(B_1) + P(R_2|R_1)P(R_1)}$$
$$= \frac{\frac{r}{b+r+c}\cdot\frac{b}{b+r}}{\frac{r}{b+r+c}\cdot\frac{b}{b+r} + \frac{r+c}{b+r+c}\cdot\frac{r}{b+r}}$$
$$= \frac{rb}{rb + (r+c)r}$$
$$= \frac{b}{b+r+c}$$

46. Let $X(= B \text{ or } = C)$ denote the jailer's answer to prisoner A. Now for instance,

$$P\{A \text{ to be executed}|X = B\}$$
$$= \frac{P\{A \text{ to be executed}, X = B\}}{P\{X = B\}}$$
$$= \frac{P\{A \text{ to be executed}\}P\{X = B|A \text{ to be executed}\}}{P\{X = B\}}$$
$$= \frac{(1/3)P\{X = B|A \text{ to be executed}\}}{1/2}.$$

Now it is reasonable to suppose that if A is to be executed, then the jailer is equally likely to answer either B or C. That is,

$$P\{X = B|\,A \text{ to be executed}\} = \frac{1}{2}$$

and so,

$$P\{A \text{ to be executed}|X = B\} = \frac{1}{3}$$

Similarly,

$$P\{A \text{ to be executed}|X = C\} = \frac{1}{3}$$

and thus the jailer's reasoning is invalid. (It is true that if the jailer were to answer B, then A knows that the condemned is either himself or C, but it is twice as likely to be C.)

47. 1. $0 \le P(A|B) \le 1$

2. $P(S|B) = \dfrac{P(SB)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

3. For disjoint events A and D

$$P(A \cup D|B) = \frac{P((A \cup D)B)}{P(B)}$$
$$= \frac{P(AB \cup DB)}{P(B)}$$
$$= \frac{P(AB) + P(DB)}{P(B)}$$
$$= P(A|B) + P(D|B)$$

Direct verification is as follows:

$$P(A|BC)P(C|B) + P(A|BC^c)P(C^c|B)$$
$$= \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(B)} + \frac{P(ABC^c)}{P(BC^c)}\frac{P(BC^c)}{P(B)}$$
$$= \frac{P(ABC)}{P(B)} + \frac{P(ABC^c)}{P(B)}$$
$$= \frac{P(AB)}{P(B)}$$
$$= P(A|B)$$

Random Variables

Exercises

1. An urn contains five red, three orange, and two blue balls. Two balls are randomly selected. What is the sample space of this experiment? Let X represent the number of orange balls selected. What are the possible values of X? Calculate $P\{X = 0\}$.
2. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?
3. In Exercise 2, if the coin is assumed fair, then, for $n = 2$, what are the probabilities associated with the values that X can take on?

*4. Suppose a die is rolled twice. What are the possible values that the following random variables can take on?
   (a) The maximum value to appear in the two rolls.
   (b) The minimum value to appear in the two rolls.
   (c) The sum of the two rolls.
   (d) The value of the first roll minus the value of the second roll.

5. If the die in Exercise 4 is assumed fair, calculate the probabilities associated with the random variables in (i)–(iv).
6. Suppose five fair coins are tossed. Let E be the event that all coins land heads. Define the random variable I_E

$$I_E = \begin{cases} 1, & \text{if } E \text{ occurs} \\ 0, & \text{if } E^c \text{ occurs} \end{cases}$$

   For what outcomes in the original sample space does I_E equal 1? What is $P\{I_E = 1\}$?
7. Suppose a coin having probability 0.7 of coming up heads is tossed three times. Let X denote the number of heads that appear in the three tosses. Determine the probability mass function of X.

8. Suppose the distribution function of X is given by

$$F(b) = \begin{cases} 0, & b < 0 \\ \frac{1}{2}, & 0 \leq b < 1 \\ 1, & 1 \leq b < \infty \end{cases}$$

What is the probability mass function of X?

9. If the distribution function of F is given by

$$F(b) = \begin{cases} 0, & b < 0 \\ \frac{1}{2}, & 0 \leq b < 1 \\ \frac{3}{5}, & 1 \leq b < 2 \\ \frac{4}{5}, & 2 \leq b < 3 \\ \frac{9}{10}, & 3 \leq b < 3.5 \\ 1, & b \geq 3.5 \end{cases}$$

calculate the probability mass function of X.

10. Suppose three fair dice are rolled. What is the probability at most one six appears?

*11. A ball is drawn from an urn containing three white and three black balls. After the ball is drawn, it is then replaced and another ball is drawn. This goes on indefinitely. What is the probability that of the first four balls drawn, exactly two are white?

12. On a multiple-choice exam with three possible answers for each of the five questions, what is the probability that a student would get four or more correct answers just by guessing?

13. An individual claims to have extrasensory perception (ESP). As a test, a fair coin is flipped ten times, and he is asked to predict in advance the outcome. Our individual gets seven out of ten correct. What is the probability he would have done at least this well if he had no ESP? (Explain why the relevant probability is $P\{X \geq 7\}$ and not $P\{X = 7\}$.)

14. Suppose X has a binomial distribution with parameters 6 and $\frac{1}{2}$. Show that $X = 3$ is the most likely outcome.

15. Let X be binomially distributed with parameters n and p. Show that as k goes from 0 to n, $P(X = k)$ increases monotonically, then decreases monotonically reaching its largest value
   (a) in the case that $(n + 1)p$ is an integer, when k equals either $(n + 1)p - 1$ or $(n + 1)p$,
   (b) in the case that $(n + 1)p$ is not an integer, when k satisfies $(n + 1)p - 1 < k < (n + 1)p$.

   Hint: Consider $P\{X = k\}/P\{X = k - 1\}$ and see for what values of k it is greater or less than 1.

*16. An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

17. Suppose that an experiment can result in one of r possible outcomes, the ith outcome having probability p_i, $i = 1, \ldots, r$, $\sum_{i=1}^{r} p_i = 1$. If n of these experiments are performed, and if the outcome of any one of the n does not affect the outcome of the other $n-1$ experiments, then show that the probability that the first outcome appears x_1 times, the second x_2 times, and the rth x_r times is

$$\frac{n!}{x_1!x_2!\ldots x_r!}p_1^{x_1}p_2^{x_2}\cdots p_r^{x_r} \qquad \text{when } x_1 + x_2 + \cdots + x_r = n$$

This is known as the *multinomial* distribution.

18. Show that when $r = 2$ the multinomial reduces to the binomial.

19. In Exercise 17, let X_i denote the number of times the ith outcome appears, $i = 1, \ldots, r$. What is the probability mass function of $X_1 + X_2 + \ldots + X_k$?

20. A television store owner figures that 50 percent of the customers entering his store will purchase an ordinary television set, 20 percent will purchase a color television set, and 30 percent will just be browsing. If five customers enter his store on a certain day, what is the probability that two customers purchase color sets, one customer purchases an ordinary set, and two customers purchase nothing?

21. In Exercise 20, what is the probability that our store owner sells three or more televisions on that day?

22. If a fair coin is successively flipped, find the probability that a head first appears on the fifth trial.

*23. A coin having probability p of coming up heads is successively flipped until the rth head appears. Argue that X, the number of flips required, will be n, $n \geq r$, with probability

$$P\{X = n\} = \binom{n-1}{r-1}p^r(1-p)^{n-r}, \qquad n \geq r$$

This is known as the negative binomial distribution.

Hint: How many successes must there be in the first $n-1$ trials?

24. The probability mass function of X is given by

$$p(k) = \binom{r+k-1}{r-1}p^r(1-p)^k, \qquad k = 0, 1, \ldots$$

Give a possible interpretation of the random variable X.

Hint: See Exercise 23.

In Exercises 25 and 26, suppose that two teams are playing a series of games, each of which is independently won by team A with probability p and by team B with probability $1-p$. The winner of the series is the first team to win i games.

25. If $i = 4$, find the probability that a total of 7 games are played. Also show that this probability is maximized when $p = 1/2$.

26. Find the expected number of games that are played when
    (a) $i = 2$;
    (b) $i = 3$.
    In both cases, show that this number is maximized when $p = 1/2$.

*27. A fair coin is independently flipped n times, k times by A and $n-k$ times by B. Show that the probability that A and B flip the same number of heads is equal to the probability that there are a total of k heads.

28. Suppose that we want to generate a random variable X that is equally likely to be either 0 or 1, and that all we have at our disposal is a biased coin that, when flipped, lands on heads with some (unknown) probability p. Consider the following procedure:

    1. Flip the coin, and let 0_1, either heads or tails, be the result.
    2. Flip the coin again, and let 0_2 be the result.
    3. If 0_1 and 0_2 are the same, return to step 1.
    4. If 0_2 is heads, set $X = 0$, otherwise set $X = 1$.

    (a) Show that the random variable X generated by this procedure is equally likely to be either 0 or 1.
    (b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different, and then sets $X = 0$ if the final flip is a head, and sets $X = 1$ if it is a tail?

29. Consider n independent flips of a coin having probability p of landing heads. Say a changeover occurs whenever an outcome differs from the one preceding it. For instance, if the results of the flips are $H\ H\ T\ H\ T\ H\ H\ T$, then there are a total of five changeovers. If $p = 1/2$, what is the probability there are k changeovers?

30. Let X be a Poisson random variable with parameter λ. Show that $P\{X = i\}$ increases monotonically and then decreases monotonically as i increases, reaching its maximum when i is the largest integer not exceeding λ.

    Hint: Consider $P\{X = i\}/P\{X = i-1\}$.

31. Compare the Poisson approximation with the correct binomial probability for the following cases:
    (a) $P\{X = 2\}$ when $n = 8$, $p = 0.1$.
    (b) $P\{X = 9\}$ when $n = 10$, $p = 0.95$.
    (c) $P\{X = 0\}$ when $n = 10$, $p = 0.1$.
    (d) $P\{X = 4\}$ when $n = 9$, $p = 0.2$.

32. If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is $\frac{1}{100}$, what is the (approximate) probability that you will win a prize (a) at least once, (b) exactly once, (c) at least twice?

33. Let X be a random variable with probability density

$$f(x) = \begin{cases} c(1-x^2), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

    (a) What is the value of c?
    (b) What is the cumulative distribution function of X?

34. Let the probability density of X be given by

$$f(x) = \begin{cases} c(4x-2x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) $P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = ?$

35. The density of X is given by

$$f(x) = \begin{cases} 10/x^2, & \text{for } x > 10 \\ 0, & \text{for } x \leq 10 \end{cases}$$

What is the distribution of X? Find $P\{X > 20\}$.

36. A point is uniformly distributed within the disk of radius 1. That is, its density is

$$f(x, y) = C, \qquad 0 \leq x^2 + y^2 \leq 1$$

Find the probability that its distance from the origin is less than x, $0 \leq x \leq 1$.

37. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each having a uniform distribution over (0,1). Let $M = \text{maximum } (X_1, X_2, \ldots, X_n)$. Show that the distribution function of M, $F_M(\cdot)$, is given by

$$F_M(x) = x^n, \qquad 0 \leq x \leq 1$$

What is the probability density function of M?

*38. If the density function of X equals

$$f(x) = \begin{cases} ce^{-2x}, & 0 < x < \infty \\ 0, & x < 0 \end{cases}$$

find c. What is $P\{X > 2\}$?

39. The random variable X has the following probability mass function:

$$p(1) = \tfrac{1}{2}, \qquad p(2) = \tfrac{1}{3}, \qquad p(24) = \tfrac{1}{6}$$

Calculate $E[X]$.

40. Suppose that two teams are playing a series of games, each of which is independently won by team A with probability p and by team B with probability $1-p$. The winner of the series is the first team to win four games. Find the expected number of games that are played, and evaluate this quantity when $p = 1/2$.

41. Consider the case of arbitrary p in Exercise 29. Compute the expected number of changeovers.

42. Suppose that each coupon obtained is, independent of what has been previously obtained, equally likely to be any of m different types. Find the expected number of coupons one needs to obtain in order to have at least one of each type.

Hint: Let X be the number needed. It is useful to represent X by

$$X = \sum_{i=1}^{m} X_i$$

where each X_i is a geometric random variable.

43. An urn contains $n + m$ balls, of which n are red and m are black. They are withdrawn from the urn, one at a time and without replacement. Let X be the number of red balls removed before the first black ball is chosen. We are interested in determining $E[X]$. To obtain this quantity, number the red balls from 1 to n. Now define the random variables X_i, $i = 1, \ldots, n$, by

$$X_i = \begin{cases} 1, & \text{if red ball } i \text{ is taken before any black ball is chosen} \\ 0, & \text{otherwise} \end{cases}$$

(a) Express X in terms of the X_i.
(b) Find $E[X]$.

44. In Exercise 43, let Y denote the number of red balls chosen after the first but before the second black ball has been chosen.
(a) Express Y as the sum of n random variables, each of which is equal to either 0 or 1.
(b) Find $E[Y]$.
(c) Compare $E[Y]$ to $E[X]$ obtained in Exercise 43.
(d) Can you explain the result obtained in part (c)?

45. A total of r keys are to be put, one at a time, in k boxes, with each key independently being put in box i with probability p_i, $\sum_{i=1}^{k} p_i = 1$. Each time a key is put in a nonempty box, we say that a collision occurs. Find the expected number of collisions.

46. If X is a nonnegative integer valued random variable, show that

(a) $$E[X] = \sum_{n=1}^{\infty} P\{X \geq n\} = \sum_{n=0}^{\infty} P\{X > n\}$$

Hint: Define the sequence of random variables I_n, $n \geq 1$, by

$$I_n = \begin{cases} 1, & \text{if } n \leq X \\ 0, & \text{if } n > X \end{cases}$$

Now express X in terms of the I_n.

(b) If X and Y are both nonnegative integer valued random variables, show that

$$E[XY] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(X \geq n, Y \geq m)$$

*47. Consider three trials, each of which is either a success or not. Let X denote the number of successes. Suppose that $E[X] = 1.8$.
(a) What is the largest possible value of $P\{X = 3\}$?
(b) What is the smallest possible value of $P\{X = 3\}$?
In both cases, construct a probability scenario that results in $P\{X = 3\}$ having the desired value.

*48. If X is a nonnegative random variable, and g is a differential function with $g(0) = 0$, then

$$E[g(X)] = \int_0^\infty P(X > t)g'(t)dt$$

Prove the preceding when X is a continuous random variable.

*49. Prove that $E[X^2] \geq (E[X])^2$. When do we have equality?

50. Let c be a constant. Show that
(a) $\text{Var}(cX) = c^2\text{Var}(X)$;
(b) $\text{Var}(c + X) = \text{Var}(X)$.

51. A coin, having probability p of landing heads, is flipped until a head appears for the rth time. Let N denote the number of flips required. Calculate $E[N]$.

Hint: There is an easy way of doing this. It involves writing N as the sum of r geometric random variables.

52. (a) Calculate $E[X]$ for the maximum random variable of Exercise 37.
(b) Calculate $E[X]$ for X as in Exercise 33.
(c) Calculate $E[X]$ for X as in Exercise 34.

53. If X is uniform over $(0, 1)$, calculate $E[X^n]$ and $\text{Var}(X^n)$.

54. Let X and Y each take on either the value 1 or -1. Let

$$\begin{aligned} p(1, 1) &= P\{X = 1, Y = 1\}, \\ p(1, -1) &= P\{X = 1, Y = -1\}, \\ p(-1, 1) &= P\{X = -1, Y = 1\}, \\ p(-1, -1) &= P\{X = -1, Y = -1\} \end{aligned}$$

Suppose that $E[X] = E[Y] = 0$. Show that
(a) $p(1, 1) = p(-1, -1)$;
(b) $p(1, -1) = p(-1, 1)$.
Let $p = 2p(1, 1)$. Find
(c) $\text{Var}(X)$;
(d) $\text{Var}(Y)$;
(e) $\text{Cov}(X, Y)$.

55. Suppose that the joint probability mass function of X and Y is

$$P(X = i, Y = j) = \binom{j}{i} e^{-2\lambda}\lambda^j/j!, \quad 0 \leq i \leq j$$

(a) Find the probability mass function of Y.
(b) Find the probability mass function of X.
(c) Find the probability mass function of $Y - X$.

56. There are n types of coupons. Each newly obtained coupon is, independently, type i with probability p_i, $i = 1, \ldots, n$. Find the expected number and the variance of the number of distinct types obtained in a collection of k coupons.

57. Suppose that X and Y are independent binomial random variables with parameters (n, p) and (m, p). Argue probabilistically (no computations necessary) that $X + Y$ is binomial with parameters $(n + m, p)$.

58. An urn contains $2n$ balls, of which r are red. The balls are randomly removed in n successive pairs. Let X denote the number of pairs in which both balls are red.
    (a) Find $E[X]$.
    (b) Find $\text{Var}(X)$.

59. Let X_1, X_2, X_3, and X_4 be independent continuous random variables with a common distribution function F and let
    $p = P\{X_1 < X_2 > X_3 < X_4\}$
    (a) Argue that the value of p is the same for all continuous distribution functions F.
    (b) Find p by integrating the joint density function over the appropriate region.
    (c) Find p by using the fact that all 4! possible orderings of $X_1, \ldots, X_4$ are equally likely.

60. Calculate the moment generating function of the uniform distribution on $(0, 1)$. Obtain $E[X]$ and $\text{Var}[X]$ by differentiating.

61. Let X and W be the working and subsequent repair times of a certain machine. Let $Y = X + W$ and suppose that the joint probability density of X and Y is

$$f_{X,Y}(x, y) = \lambda^2 e^{-\lambda y}, \quad 0 < x < y < \infty$$

    (a) Find the density of X.
    (b) Find the density of Y.
    (c) Find the joint density of X and W.
    (d) Find the density of W.

62. In deciding upon the appropriate premium to charge, insurance companies sometimes use the exponential principle, defined as follows. With X as the random amount that it will have to pay in claims, the premium charged by the insurance company is

$$P = \frac{1}{a} \ln\left(E[e^{aX}]\right)$$

where a is some specified positive constant. Find P when X is an exponential random variable with parameter λ, and $a = \alpha\lambda$, where $0 < \alpha < 1$.

63. Calculate the moment generating function of a geometric random variable.

*64. Show that the sum of independent identically distributed exponential random variables has a gamma distribution.

65. Consider Example 2.48. Find $\text{Cov}(X_i, X_j)$ in terms of the a_{rs}.

66. Use Chebyshev's inequality to prove the *weak law of large numbers*. Namely, if $X_1, X_2, \ldots$ are independent and identically distributed with mean μ and variance σ^2 then, for any $\varepsilon > 0$,

$$P\left\{\left|\frac{X_1 + X_2 + \cdots + X_n}{n} - \mu\right| > \varepsilon\right\} \to 0 \qquad \text{as } n \to \infty$$

67. Suppose that X is a random variable with mean 10 and variance 15. What can we say about $P\{5 < X < 15\}$?

68. Let $X_1, X_2, \ldots, X_{10}$ be independent Poisson random variables with mean 1.

(a) Use the Markov inequality to get a bound on $P\{X_1 + \cdots + X_{10} \geq 15\}$.

(b) Use the central limit theorem to approximate $P\{X_1 + \cdots + X_{10} \geq 15\}$.

69. If X is normally distributed with mean 1 and variance 4, use the tables to find $P\{2 < X < 3\}$.

*70. Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$$

Hint: Let X_n be Poisson with mean n. Use the central limit theorem to show that $P\{X_n \leq n\} \to \frac{1}{2}$.

71. Let X denote the number of white balls selected when k balls are chosen at random from an urn containing n white and m black balls.

(a) Compute $P\{X = i\}$.

(b) Let, for $i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n,$

$$X_i = \begin{cases} 1, & \text{if the } i\text{th ball selected is white} \\ 0, & \text{otherwise} \end{cases}$$

$$Y_j = \begin{cases} 1, & \text{if white ball } j \text{ is selected} \\ 0, & \text{otherwise} \end{cases}$$

Compute $E[X]$ in two ways by expressing X first as a function of the X_is and then of the Y_js.

*72. Show that $\mathrm{Var}(X) = 1$ when X is the number of men who select their own hats in Example 2.31.

73. For the multinomial distribution (Exercise 17), let N_i denote the number of times outcome i occurs. Find

(a) $E[N_i]$;

(b) $\mathrm{Var}(N_i)$;

(c) $\mathrm{Cov}(N_i, N_j)$;

(d) Compute the expected number of outcomes that do not occur.

74. Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed continuous random variables. We say that a record occurs at time n if $X_n > \max(X_1, \ldots, X_{n-1})$. That is, X_n is a record if it is larger than each of $X_1, \ldots, X_{n-1}$. Show

(a) $P\{\text{a record occurs at time } n\} = 1/n$;

(b) $E[\text{number of records by time } n] = \sum_{i=1}^{n} 1/i$;

(c) $\mathrm{Var}(\text{number of records by time } n) = \sum_{i=1}^{n} (i-1)/i^2$;

(d) Let $N = \min\{n\colon n > 1 \text{ and a record occurs at time } n\}$. Show $E[N] = \infty$.

Hint: For (ii) and (iii) represent the number of records as the sum of indicator (that is, Bernoulli) random variables.

75. Let $a_1 < a_2 < \cdots < a_n$ denote a set of n numbers, and consider any permutation of these numbers. We say that there is an inversion of a_i and a_j in the permutation if $i < j$ and a_j precedes a_i. For instance the permutation 4, 2, 1, 5, 3 has 5 inversions—(4, 2), (4, 1), (4, 3), (2, 1), (5, 3). Consider now a random permutation of $a_1, a_2, \ldots, a_n$—in the sense that each of the $n!$ permutations is equally likely to be chosen—and let N denote the number of inversions in this permutation. Also, let

$$N_i = \text{number of } k\colon k < i,\ a_i \text{ precedes } a_k \text{ in the permutation}$$

and note that $N = \sum_{i=1}^{n} N_i$.

(a) Show that $N_1, \ldots, N_n$ are independent random variables.
(b) What is the distribution of N_i?
(c) Compute $E[N]$ and $\text{Var}(N)$.

76. Let X and Y be independent random variables with means μ_x and μ_y and variances σ_x^2 and σ_y^2. Show that

$$\text{Var}(XY) = \sigma_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2$$

77. Let X and Y be independent normal random variables, each having parameters μ and σ^2. Show that $X + Y$ is independent of $X - Y$.

Hint: Find their joint moment generating function.

78. Let $\phi(t_1, \ldots, t_n)$ denote the joint moment generating function of $X_1, \ldots, X_n$.
(a) Explain how the moment generating function of X_i, $\phi_{X_i}(t_i)$, can be obtained from $\phi(t_1, \ldots, t_n)$.
(b) Show that $X_1, \ldots, X_n$ are independent if and only if

$$\phi(t_1, \ldots, t_n) = \phi_{x_1}(t_1) \cdots \phi_{X_n}(t_n)$$

79. With $K(t) = \log(E\left[e^{tX}\right])$, show that

$$K'(0) = E[X], \quad K''(0) = \text{Var}(X)$$

80. Let X denote the number of the events $A_1, \ldots, A_n$, that occur. Express $E[X]$, $\text{Var}(X)$, and $E\left[\binom{X}{k}\right]$ in terms of the quantities $S_k = \sum_{i_1 < \ldots < i_k} P(A_{i_1} \ldots A_{i_k})$, $k = 1, \ldots, n$.

Chapter 2

1. $P\{X=0\} = \begin{bmatrix}7\\2\end{bmatrix} \Big/ \begin{bmatrix}10\\2\end{bmatrix} = \frac{14}{30}$

2. $-n, -n+2, -n+4, \ldots, n-2, n$

3. $P\{X=-2\} = \frac{1}{4} = P\{X=2\}$

   $P\{X=0\} = \frac{1}{2}$

4. (a) $1, 2, 3, 4, 5, 6$
   (b) $1, 2, 3, 4, 5, 6$
   (c) $2, 3, \ldots, 11, 12$
   (d) $-5, -4, \ldots, 4, 5$

5. $P\{\max = 6\} = \frac{11}{36} = P\{\min = 1\}$

   $P\{\max = 5\} = \frac{1}{4} = P\{\min = 2\}$

   $P\{\max = 4\} = \frac{7}{36} = P\{\min = 3\}$

   $P\{\max = 3\} = \frac{5}{36} = P\{\min = 4\}$

   $P\{\max = 2\} = \frac{1}{12} = P\{\min = 5\}$

   $P\{\max = 1\} = \frac{1}{36} = P\{\min = 6\}$

6. $(H, H, H, H, H), p^5$ if $p = P\{\text{heads}\}$

7. $p(0) = (.3)^3 = .027$

   $p(1) = 3(.3)^2(.7) = .189$

   $p(2) = 3(.3)(.7)^2 = .441$

   $p(3) = (.7)^3 = .343$

8. $p(0) = \frac{1}{2}, \quad p(1) = \frac{1}{2}$

9. $p(0) = \frac{1}{2}, \quad p(1) = \frac{1}{10}, \quad p(2) = \frac{1}{5},$

   $p(3) = \frac{1}{10}, \quad p(3.5) = \frac{1}{10}$

10. $1 - \begin{bmatrix}3\\2\end{bmatrix} \begin{bmatrix}\frac{1}{6}\end{bmatrix}^2 \begin{bmatrix}\frac{5}{6}\end{bmatrix} - \begin{bmatrix}3\\3\end{bmatrix} \begin{bmatrix}\frac{1}{6}\end{bmatrix}^3 = \frac{200}{216}$

11. $\frac{3}{8}$

12. $\begin{bmatrix}5\\4\end{bmatrix} \begin{bmatrix}\frac{1}{3}\end{bmatrix}^4 \begin{bmatrix}\frac{2}{3}\end{bmatrix} + \begin{bmatrix}5\\5\end{bmatrix} \begin{bmatrix}\frac{1}{3}\end{bmatrix}^5 = \frac{10+1}{243} = \frac{11}{243}$

13. $\sum_{i=7}^{10} \binom{10}{i} \begin{bmatrix}\frac{1}{2}\end{bmatrix}^{10}$

14. $P\{X=0\} = P\{X=6\} = \begin{bmatrix}\frac{1}{2}\end{bmatrix}^6 = \frac{1}{64}$

    $P\{X=1\} = P\{X=5\} = 6\begin{bmatrix}\frac{1}{2}\end{bmatrix}^6 = \frac{6}{64}$

    $P\{X=2\} = P\{X=4\} = \begin{bmatrix}6\\2\end{bmatrix} \begin{bmatrix}\frac{1}{2}\end{bmatrix}^6 = \frac{15}{64}$

    $P\{X=3\} = \begin{bmatrix}6\\3\end{bmatrix} \begin{bmatrix}\frac{1}{2}\end{bmatrix}^6 = \frac{20}{64}$

15. $$\frac{P\{X=k\}}{P\{X=k-1\}} = \frac{\frac{n!}{(n-k)!\,k!}p^k(1-p)^{n-k}}{\frac{n!}{(n-k+1)!(k-1)!}p^{k-1}(1-p)^{n-k+1}} = \frac{n-k+1}{k}\frac{p}{1-p}$$

    Hence,

    $$\frac{P\{X=k\}}{P\{X=k-1\}} \geq 1 \leftrightarrow (n-k+1)p > k(1-p) \leftrightarrow (n+1)p \geq k$$

    The result follows.

16. $1 - (.95)^{52} - 52(.95)^{51}(.05)$

17. Follows since there are $\frac{n!}{x_1! \cdots x_r!}$ permutations of n objects of which x_1 are alike, x_2 are alike, $\ldots, x_r$ are alike.

18. Follows immediately.

19. $P\{X_1 + \cdots + X_k = m\}$

$$= \begin{bmatrix} n \\ m \end{bmatrix} (p_1 + \cdots + p_k)^m (p_{k+1} + \cdots + p_r)^{n-m}$$

20. $\dfrac{5!}{2!1!2!} \left[\dfrac{1}{5}\right]^2 \left[\dfrac{3}{10}\right]^2 \left[\dfrac{1}{2}\right]^1 = .054$

21. $1 - \left[\dfrac{3}{10}\right]^5 - 5\left[\dfrac{3}{10}\right]^4 \left[\dfrac{7}{10}\right] - \left[\dfrac{5}{2}\right] \left[\dfrac{3}{10}\right]^3 \left[\dfrac{7}{10}\right]^2$

22. $\dfrac{1}{32}$

23. In order for X to equal n, the first $n-1$ flips must have $r-1$ heads, and then the n^{th} flip must land heads. By independence the desired probability is thus

$$\begin{bmatrix} n-1 \\ r-1 \end{bmatrix} p^{r-1}(1-p)^{n-r} xp$$

24. It is the number of tails before heads appears for the r^{th} time.

25. A total of 7 games will be played if the first 6 result in 3 wins and 3 losses. Thus,

$$P\{7 \text{ games}\} = \binom{6}{3} p^3 (1-p)^3$$

Differentiation yields

$$\begin{aligned} \frac{d}{dp} P\{7\} &= 20\left[3p^2(1-p)^3 - p^3 3(1-p)^2\right] \\ &= 60p^2(1-p)^2\,[1 - 2p] \end{aligned}$$

Thus, the derivative is zero when $p = 1/2$. Taking the second derivative shows that the maximum is attained at this value.

26. Let X denote the number of games played.

(a) $P\{X = 2\} = p^2 + (1-p)^2$

$P\{X = 3\} = 2p(1-p)$

$$\begin{aligned} E[X] &= 2\left\{p^2 + (1-p)^2\right\} + 6p(1-p) \\ &= 2 + 2p(1-p) \end{aligned}$$

Since $p(1-p)$ is maximized when $p = 1/2$, we see that $E[X]$ is maximized at that value of p.

(b) $P\{X = 3\} = p^3 + (1-p)^3$

$P\{X = 4\}$

$= P\{X = 4, \text{ I has 2 wins in first 3 games}\}$

$\quad + P\{X = 4, \text{ II has 2 wins in first 3 games}\}$

$= 3p^2(1-p)p + 3p(1-p)^2(1-p)$

$P\{X = 5\}$

$= P\{\text{each player has 2 wins in the first 4 games}\}$

$= 6p^2(1-p)^2$

$$E[X] = 3\left[p^3 + (1-p)^3\right] + 12p(1-p)\left[p^2 + (1-p)^2\right] + 30p^2(1-p)^2$$

Differentiating and setting equal to 0 shows that the maximum is attained when $p = 1/2$.

27. $P\{\text{same number of heads}\} = \sum_i P\{A = i, B = i\}$

$$\begin{aligned} &= \sum_i \binom{k}{i} (1/2)^k \binom{n-k}{i} (1/2)^{n-k} \\ &= \sum_i \binom{k}{i} \binom{n-k}{i} (1/2)^n \\ &= \sum_i \binom{k}{k-i} \binom{n-k}{i} (1/2)^n \\ &= \binom{n}{k} (1/2)^n \end{aligned}$$

Another argument is as follows:

$P\{\# \text{ heads of } A = \# \text{ heads of } B\}$

$= P\{\# \text{ tails of } A = \# \text{ heads of } B\}$

since coin is fair

$= P\{k - \# \text{ heads of } A = \# \text{ heads of } B\}$

$= P\{k = \text{total } \# \text{ heads}\}$

28. (a) Consider the first time that the two coins give different results. Then

$$\begin{aligned} P\{X = 0\} &= P\{(t,h) | (t,h) \text{ or } (h,t)\} \\ &= \frac{p(1-p)}{2p(1-p)} = \frac{1}{2} \end{aligned}$$

(b) No, with this procedure

$P\{X = 0\} = P\{\text{first flip is a tail}\} = 1 - p$

29. Each flip after the first will, independently, result in a changeover with probability 1/2. Therefore,

$$P\{k \text{ changeovers}\} = \binom{n-1}{k}(1/2)^{n-1}$$

30. $$\frac{P\{X=i\}}{P\{X=i-1\}} = \frac{e^{-\lambda}\lambda^i/i!}{e^{-\lambda}\lambda^{i-1}/(i-1)!} = \lambda/i$$

Hence, $P\{X = i\}$ is increasing for $\lambda \geq i$ and decreasing for $\lambda < i$.

32. (a) .394 (b) .303 (c) .091

33. $$c\int_{-1}^{1}\left(1-x^2\right)dx = 1$$
$$c\left[x - \frac{x^3}{3}\right]\Big|_{-1}^{1} = 1$$
$$c = \frac{3}{4}$$
$$F(y) = \frac{3}{4}\int_{-1}^{1}(1-x^2)dx$$
$$= \frac{3}{4}\left[y - \frac{y^3}{3} + \frac{2}{3}\right], \qquad -1 < y < 1$$

34. $$c\int_0^2\left(4x-2x^2\right)dx = 1$$
$$c(2x^2 - 2x^3/3) = 1$$
$$8c/3 = 1$$
$$c = \frac{3}{8}$$
$$P\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = \frac{3}{8}\int_{1/2}^{3/2}\left(4x-2x^2\right)dx$$
$$= \frac{11}{16}$$

35. $$P\{X > 20\} = \int_{20}^{\infty}\frac{10}{x^2}dx = \frac{1}{2}$$

36. $$P\{D \leq x\} = \frac{\text{area of disk of radius } x}{\text{area of disk of radius } 1}$$
$$= \frac{\pi x^2}{\pi} = x^2$$

37. $$P\{M \leq x\} = P\{\max(X_1, \ldots, X_n) \leq x\}$$
$$= P\{X_1 \leq x, \ldots, X_n \leq x\}$$
$$= \prod_{i=1}^{n} P\{X_i \leq x\}$$
$$= x^n$$
$$f_M(x) = \frac{d}{dx}P\{M \leq x\} = nx^{n-1}$$

38. $c = 2$

39. $$E[X] = \frac{31}{6}$$

40. Let X denote the number of games played.

$$P\{X=4\} = p^4 + (1-p)^4$$
$$P\{X=5\} = P\{X=5, \text{ I wins 3 of first 4}\}$$
$$+ P\{X=5, \text{ II wins 3 of first 4}\}$$
$$= 4p^3(1-p)p + 4(1-p)^3p(1-p)$$
$$P\{X=6\} = P\{X=6, \text{ I wins 3 of first 5}\}$$
$$+ P\{X=6, \text{ II wins 3 of first 5}\}$$
$$= 10p^3(1-p)^2p + 10p^2(1-p)^3(1-p)$$
$$P\{X=7\} = P\{\text{first 6 games are split}\}$$
$$= 20p^3(1-p)^3$$
$$E[X] = \sum_{i=4}^{7} iP\{X=i\}$$

When $p = 1/2$, $E[X] = 93/16 = 5.8125$

41. Let X_i equal 1 if a changeover results from the i^{th} flip and let it be 0 otherwise. Then

$$\text{number of changeovers} = \sum_{i=2}^{n} X_i$$

As,

$$E[X_i] = P\{X_i = 1\} = P\{\text{flip } i-1 \neq \text{flip } i\}$$
$$= 2p(1-p)$$

we see that

$$E[\text{number of changeovers}] = \sum_{i=2}^{n} E[X_i]$$
$$= 2(n-1)p(1-p)$$

42. Suppose the coupon collector has i different types. Let X_i denote the number of additional coupons collected until the collector has $i + 1$ types. It is easy to see that the X_i are independent geometric random variables with respective parameters $(n-i)/n$, $i = 0, 1, \ldots, n-1$. Therefore,

$$\sum\left[\sum_{i=0}^{n-1} X_i\right] = \sum_{i=0}^{n-1}\sum[X_i] = \sum_{i=0}^{n-1} n/(n-i)$$
$$= n\sum_{j=1}^{n} 1/j$$

43. (a) $X = \sum_{i=1}^{n} X_i$

(b) $E[X_i] = P\{X_i = 1\}$

$= P\{$red ball i is chosen before all n black balls$\}$

$= 1/(n+1)$ since each of these $n+1$ balls is equally likely to be the one chosen earliest

Therefore,

$$E[X] = \sum_{i=1}^{n} E[X_i] = n/(n+1)$$

44. (a) Let Y_i equal 1 if red ball i is chosen after the first but before the second black ball, $i = 1, \ldots, n$. Then

$Y = \sum_{i=1}^{n} Y_i$

(b) $E[Y_i] = P\{Y_i = 1\}$

$= P\{$red ball i is the second chosen from a set of $n+1$ balls$\}$

$= 1/(n+1)$ since each of the $n+1$ is equally likely to be the second one chosen.

Therefore,

$$E[Y] = n/(n+1)$$

(c) Answer is the same as in Problem 41.

(d) We can let the outcome of this experiment be the vector $(R_1, R_2, \ldots, R_n)$ where R_i is the number of red balls chosen after the $(i-1)^{st}$ but before the i^{th} black ball. Since all orderings of the $n+m$ balls are equally likely it follows that all different orderings of $R_1, \ldots, R_n$ will have the same probability distribution.

For instance,

$$P\{R_1 = a, R_2 = b\} = P\{R_2 = a, R_1 = b\}$$

From this it follows that all the R_i have the same distribution and thus the same mean.

45. Let N_i denote the number of keys in box i, $i = 1, \ldots, k$. Then, with X equal to the number of collisions we have that $X = \sum_{i=1}^{k} (N_i - 1)^+ = \sum_{i=1}^{k} (N_i - 1 + I\{N_i = 0\})$ where $I\{N_i = 0\}$ is equal to 1 if $N_i = 0$ and is equal to 0 otherwise. Hence,

$$E[X] = \sum_{i=1}^{k} (rp_i - 1 + (1-p_i)^r) = r - k + \sum_{i=1}^{k} (1-p_i)^r$$

Another way to solve this problem is to let Y denote the number of boxes having at least one key, and then use the identity $X = r - Y$, which is true since only the first key put in each box does not result in a collision. Writing $Y = \sum_{i=1}^{k} I\{N_i > 0\}$ and taking expectations yields

$$E[X] = r - E[Y] = r - \sum_{i=1}^{k} [1 - (1-p_i)^r]$$
$$= r - k + \sum_{i=1}^{k} (1-p_i)^r$$

46. Using that $X = \sum_{n=1}^{\infty} I_n$, we obtain

$$E[X] = \sum_{n=1}^{\infty} E[I_n] = \sum_{n=1}^{\infty} P\{X \geq n\}$$

Making the change of variables $m = n - 1$ gives

$$E[X] = \sum_{m=0}^{\infty} P\{X \geq m+1\} = \sum_{m=0}^{\infty} P\{X > m\}$$

(b) Let

$$I_n = \begin{cases} 1, & \text{if } n \leq X \\ 0, & \text{if } n > X \end{cases}$$

$$J_m = \begin{cases} 1, & \text{if } m \leq Y \\ 0, & \text{if } m > Y \end{cases}$$

Then

$$XY = \sum_{n=1}^{\infty} I_n \sum_{m=1}^{\infty} J_m = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} I_n J_m$$

Taking expectations now yields the result

$$E[XY] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E[I_n J_m]$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P(X \geq n, Y \geq m)$$

47. Let X_i be 1 if trial i is a success and 0 otherwise.

(a) The largest value is .6. If $X_1 = X_2 = X_3$, then

$$1.8 = E[X] = 3E[X_1] = 3P\{X_1 = 1\}$$

and so

$$P\{X = 3\} = P\{X_1 = 1\} = .6$$

That this is the largest value is seen by Markov's inequality, which yields

$$P\{X \geq 3\} \leq E[X]/3 = .6$$

(b) The smallest value is 0. To construct a probability scenario for which $P\{X = 3\} = 0$ let U be a uniform random variable on (0, 1), and define

$$X_1 = \begin{matrix} 1 & \text{if } U \leq .6 \\ 0 & \text{otherwise} \end{matrix}$$

$$X_2 = \begin{matrix} 1 & \text{if } U \geq .4 \\ 0 & \text{otherwise} \end{matrix}$$

$$X_3 = \begin{matrix} 1 & \text{if either } U \leq .3 \quad \text{or} \quad U \geq .7 \\ 0 & \text{otherwise} \end{matrix}$$

It is easy to see that

$$P\{X_1 = X_2 = X_3 = 1\} = 0$$

49. $E[X^2] - (E[X])^2 = Var(X) = E(X - E[X])^2 \geq 0$. Equality when $Var(X) = 0$, that is, when X is constant.

50. $Var(cX) = E[(cX - E[cX])^2]$

$$= E[c^2(X - E(X))^2]$$

$$= c^2 Var(X)$$

$$Var(c + X) = E[(c + X - E[c + X])^2]$$

$$= E[(X - E[X])^2]$$

$$= Var(X)$$

51. $N = \sum_{i=1}^{r} X_i$ where X_i is the number of flips between the $(i-1)^{st}$ and i^{th} head. Hence, X_i is geometric with mean $1/p$. Thus,

$$E[N] = \sum_{i=1}^{r} E[X_i] = \frac{r}{p}$$

52. (a) $\dfrac{n}{n+1}$

(b) 0

(c) 1

53. $\dfrac{1}{n+1}, \quad \dfrac{1}{2n+1} - \left[\dfrac{1}{n+1}\right]^2$.

54. (a) Using the fact that $E[X + Y] = 0$ we see that $0 = 2p(1, 1) - 2p(-1, -1)$, which gives the result.

(b) This follows since

$$0 = E[X - Y] = 2p(1, -1) - 2p(-1, 1)$$

(c) $Var(X) = E[X^2] = 1$

(d) $Var(Y) = E[Y^2] = 1$

(e) Since

$$1 = p(1,1) + p(-1,1) + p(1,-1) + p(-1,1)$$

$$= 2p(1,1) + 2p(1,-1)$$

we see that if $p = 2p(1,1)$ then $1 - p = 2p(1,-1)$

Now,

$$Cov(X, Y) = E[XY]$$

$$= p(1,1) + p(-1,-1)$$

$$-p(1,-1) - p(-1,1)$$

$$= p - (1 - p) = 2p - 1$$

55. (a) $P(Y = j) = \sum_{i=0}^{j} \binom{j}{i} e^{-2\lambda} \lambda^j / j!$

$$= e^{-2\lambda} \frac{\lambda^j}{j!} \sum_{i=0}^{j} \binom{j}{i} 1^i 1^{j-i}$$

$$= e^{-2\lambda} \frac{(2\lambda)^j}{j!}$$

(b) $P(X = i) = \sum_{j=i}^{\infty} \binom{j}{i} e^{-2\lambda} \lambda^j / j!$

$$= \frac{1}{i!} e^{-2\lambda} \sum_{j=i}^{\infty} \frac{1}{(j-i)!} \lambda^j$$

$$= \frac{\lambda^i}{i!} e^{-2\lambda} \sum_{k=0}^{\infty} \lambda^k / k!$$

$$= e^{-\lambda} \frac{\lambda^i}{i!}$$

(c) $P(X = i,\ Y - X = k) = P(X = i, Y = k + i)$

$$= \binom{k+i}{i} e^{-2\lambda} \frac{\lambda^{k+i}}{(k+i)!}$$

$$= e^{-\lambda} \frac{\lambda^i}{i!} e^{-\lambda} \frac{\lambda^k}{k!}$$

showing that X and $Y - X$ are independent Poisson random variables with mean λ. Hence,

$$P(Y - X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

56. Let X_j equal 1 if there is a type i coupon in the collection, and let it be 0 otherwise. The number of distinct types is $X = \sum_{i=1}^{n} X_i$.

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{X_i = 1\} = \sum_{i=1}^{n} (1 - p_i)^k$$

To compute $Cov(X_i, X_j)$ when $i \neq j$, note that $X_i X_j$ is either equal to 1 or 0 if either X_i or X_j is equal to 0, and that it will equal 0 if there is either no type i or type j coupon in the collection. Therefore,

$$\begin{aligned} P\{X_i X_j = 0\} &= P\{X_i = 0\} + P\{X_j = 0\} \\ &\quad - P\{X_i = X_j = 0\} \\ &= (1 - p_i)^k + (1 - p_j)^k \\ &\quad -(1 - p_i - p_j)^k \end{aligned}$$

Consequently, for $i \neq j$

$$\begin{aligned} Cov(X_i, X_j) &= P\{X_i X_j = 1\} - E[X_i]E[X_j] \\ &= 1 - [(1 - p_i)^k + (1 - p_j)^k \\ &\quad -(1 - p_i - p_j)^k] - (1 - p_i)^k(1 - p_j)^k \end{aligned}$$

Because $Var(X_i) = (1 - p_i)^k[1 - (1 - p_i)^k]$
we obtain

$$\begin{aligned} Var(X) &= \sum_{i=1}^{n} Var(X_i) + 2\sum\sum_{i<j} Cov(X_i, X_j) \\ &= \sum_{i=1}^{n} (1 - p_i)^k[1 - (1 - p_i)^k] \\ &\quad + 2\sum_{j}\sum_{i<j} [1 - [(1 - p_i)^k \\ &\quad + (1 - p_j)^k - (1 - p_i - p_j)^k] \\ &\quad - (1 - p_i)^k(1 - p_j)^k \end{aligned}$$

57. It is the number of successes in $n + m$ independent p-trials.

58. Let X_i equal 1 if both balls of the i^{th} withdrawn pair are red, and let it equal 0 otherwise. Because

$$E[X_i] = P\{X_i = 1\} = \frac{r(r-1)}{2n(2n-1)}$$

we have

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} E[X_i] \\ &= \frac{r(r-1)}{(4n-2)} \end{aligned}$$

because

$$E[X_i X_j] = \frac{r(r-1)(r-2)(r-3)}{2n(2n-1)(2n-2)2n-3)}$$

For $Var(X)$ use

$$\begin{aligned} Var(X) &= \sum_{i} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j) \\ &= n\, Var(X_1) + n(n-1)\, Cov(X_1, X_2) \end{aligned}$$

where

$$Var(X_1) = E[X_1](1 - E[X_1])$$

$$\begin{aligned} Cov(X_1, X_2) &= \frac{r(r-1)(r-2)(r-3)}{2n(2n-1)(2n-2)(2n-3)} \\ &\quad - (E[X_1])^2 \end{aligned}$$

59. (a) Use the fact that $F(X_i)$ is a uniform $(0, 1)$ random variable to obtain

$$\begin{aligned} p &= P\{F(X_1) < F(X_2) > F(X_3) < F(X_4)\} \\ &= P\{U_1 < U_2 > U_3 < U_4\} \end{aligned}$$

where the $U_i, i = 1, 2, 3, 4,$ are independent uniform $(0, 1)$ random variables.

(b)
$$\begin{aligned} p &= \int_0^1 \int_{x_1}^1 \int_0^{x_2} \int_{x_3}^1 dx_4 dx_3 dx_2 dx_1 \\ &= \int_0^1 \int_{x_1}^1 \int_0^{x_2} (1 - x_3) dx_3 dx_2 dx_1 \\ &= \int_0^1 \int_{x_1}^1 (x_2 - x_2^2/2) dx_2 dx_1 \\ &= \int_0^1 (1/3 - x_1^2/2 + x_1^3/6) dx_1 \\ &= 1/3 - 1/6 + 1/24 = 5/24 \end{aligned}$$

(c) There are 5 (of the 24 possible) orderings such that $X_1 < X_2 > X_3 < X_4$. They are as follows:

$X_2 > X_4 > X_3 > X_1$
$X_2 > X_4 > X_1 > X_3$
$X_2 > X_1 > X_4 > X_3$
$X_4 > X_2 > X_3 > X_1$
$X_4 > X_2 > X_1 > X_3$

60. $E[e^{tX}] = \int_0^1 e^{tx}dx = \frac{e^t - 1}{t}$

$$\frac{d}{dt}E[e^{tX}] = \frac{te^t - e^t + 1}{t^2}$$

$$\frac{d^2}{dt^2}E[e^{tX}] = \frac{[t^2(te^2 + e^t - e^t) - 2t(te^t - e^t + 1)]}{t^4}$$

$$= \frac{t^2e^t - 2(te^t - e^t + 1)}{t^3}$$

To evaluate at $t = 0$, we must apply l'Hospital's rule.

This yields

$$E[X] = \lim_{t=0} \frac{te^t + e^t - e^t}{2t} = \lim_{t=0} \frac{e^t}{2} = \frac{1}{2}$$

$$E[X^2] = \lim_{t=0} \frac{2te^t + t^2e^t - 2te^t - 2e^t + 2e^t}{3t^2}$$

$$= \lim_{t=0} \frac{e^t}{3} = \frac{1}{3}$$

Hence, $Var(X) = \frac{1}{3} - \left[\frac{1}{2}\right]^2 = \frac{1}{12}$

61. (a) $f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y}dy$

$= \lambda e^{-\lambda x}$

(b) $f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y}dx$

$= \lambda^2 y e^{-\lambda y}$

(c) Because the Jacobian of the transformation $x = x, w = y - x$ is 1, we have

$$f_{X,W}(x,w) = f_{X,Y}(x, x + w) = \lambda^2 e^{-\lambda(x+w)}$$
$$= \lambda e^{-\lambda x} \lambda e^{-\lambda w}$$

(d) It follows from the preceding that X and W are independent exponential random variables with rate λ.

62. $E[e^{\alpha\lambda X}] = \int e^{\alpha\lambda x}\lambda e^{-\lambda x}dx = \frac{1}{1-\alpha}$

Therefore,

$$P = -\frac{1}{\alpha\lambda}\ln(1-\alpha)$$

The inequality $\ln(1 - x) \leq -x$ shows that $P \geq 1/\lambda$.

63. $\phi(t) = \sum_{n=1}^{\infty} e^{tn}(1-p)^{n-1}p$

$$= pe^t \sum_{n=1}^{\infty} ((1-p)e^t)^{n-1}$$

$$= \frac{pe^t}{1-(1-p)e^t}$$

64. (See Section 2.3 of Chapter 5.)

65. $Cov(X_i, X_j) = Cov(\mu_i + \sum_{k=1}^{n} a_{ik}Z_k, \mu_j + \sum_{t=1}^{n} a_{jt}Z_t)$

$$= \sum_{t=1}^{n}\sum_{k=1}^{n} Cov(a_{jk}Z_k, a_{jt}Z_t)$$

$$= \sum_{t=1}^{n}\sum_{k=1}^{n} a_{ik}a_{jt}Cov(Z_k, Z_t)$$

$$= \sum_{k=1}^{n} a_{ik}a_{jk}$$

where the last equality follows since

$$Cov(Z_k, Z_t) = \begin{matrix} 1 & \text{if } k = t \\ 0 & \text{if } k \neq t \end{matrix}$$

66. $P\left\{\left|\frac{X_1 + \cdots + X_n - n\mu}{n}\right| > \in\right\}$

$= P\{|X_1 + \cdots + X_n - n\mu| > n \in\}$

$\leq Var\{X_1 + \cdots + X_n\}/n^2 \in^2$

$= n\sigma^2/n^2 \in^2$

$\to 0$ as $n \to \infty$

67. $P\{5 < X < 15\} \geq \frac{2}{5}$

68. (a) $P\{X_1 + \cdots + X_{10} > 15\} \leq \frac{2}{3}$

(b) $P\{X_1 + \cdots + X_{10} > 15\} \approx 1 - \Phi\left[\frac{5}{\sqrt{10}}\right]$

69. $\Phi(1) - \Phi\left[\frac{1}{2}\right] = .1498$

70. Let X_i be Poisson with mean 1. Then

$$P\left\{\sum_1^n X_i \leq n\right\} = e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}$$

But for n large $\sum_1^n X_i - n$ has approximately a normal distribution with mean 0, and so the result follows.

71. (a) $P\{X=i\} = \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ k-i \end{bmatrix} \Big/ \begin{bmatrix} n+m \\ k \end{bmatrix}$

$$i = 0, 1, \ldots, min(k, n)$$

(b) $X = \sum_{i=1}^{k} X_i$

$$E[X] = \sum_{i=1}^{K} E[X_i] = \frac{kn}{n+m}$$

since the i^{th} ball is equally likely to be either of the $n + m$ balls, and so $E[X_i] = P\{X_i = 1\} = \dfrac{n}{n+m}$

$$X = \sum_{i=1}^{n} Y_i$$

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} E[Y_i] \\ &= \sum_{i=1}^{n} P\{i^{th} \text{ white ball is selected}\} \\ &= \sum_{i=1}^{n} \frac{k}{n+m} = \frac{nk}{n+m} \end{aligned}$$

72. For the matching problem, letting

$$X = X_1 + \cdots + X_N$$

where

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ man selects his own hat} \\ 0 & \text{otherwise} \end{cases}$$

we obtain

$$Var(X) = \sum_{i=1}^{N} Var(X_i) + 2 \sum_{i<j} \sum Cov(X_i, X_j)$$

Since $P\{X_i = 1\} = 1/N$, we see

$$Var(X_i) = \frac{1}{N}\left[1 - \frac{1}{N}\right] = \frac{N-1}{N^2}$$

Also

$$Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j]$$

Now,

$$X_i X_j = \begin{cases} 1 & \text{if the } i^{th} \text{ and } j^{th} \text{ men both select their own hats} \\ 0 & \text{otherwise} \end{cases}$$

and thus

$$\begin{aligned} E[X_i X_j] &= P\{X_i = 1, X_j = 1\} \\ &= P\{X_i = 1\}P\{X_j = 1 | X_i = 1\} \\ &= \frac{1}{N}\frac{1}{N-1} \end{aligned}$$

Hence,

$$Cov(X_i, X_j) = \frac{1}{N(N-1)} - \left[\frac{1}{N}\right]^2 = \frac{1}{N^2(N-1)}$$

and

$$\begin{aligned} Var(X) &= \frac{N-1}{N} + 2\begin{bmatrix} N \\ 2 \end{bmatrix} \frac{1}{N^2(N-1)} \\ &= \frac{N-1}{N} + \frac{1}{N} \\ &= 1 \end{aligned}$$

73. As N_i is a binomial random variable with parameters (n, P_i), we have (a) $E[N_i] = nP_{ji}$ (b) $Var(X_i) = nP_i = (1 - P_i)$; (c) for $i \neq j$, the covariance of N_i and N_j can be computed as

$$Cov(N_i, N_j) = Cov\left[\sum_k X_k, \sum_k Y_k\right]$$

where $X_k(Y_k)$ is 1 or 0, depending upon whether or not outcome k is type $i(j)$. Hence,

$$Cov(N_i, N_j) = \sum_k \sum_\ell Cov(X_k, Y_\ell)$$

Now for $k \neq \ell$, $Cov(X_k, Y_\ell) = 0$ by independence of trials and so

$$\begin{aligned} Cov(N_i, N_j) &= \sum_k Cov(X_k, Y_k) \\ &= \sum_k (E[X_k Y_k] - E[X_k]E[Y_k]) \\ &= -\sum_k E[X_k]E[Y_k] \text{ (since } X_k Y_k = 0) \\ &= -\sum_k P_i P_j \\ &= -nP_i P_j \end{aligned}$$

(d) Letting

$$Y_i = \begin{cases} 1, & \text{if no type } i\text{'s occur} \\ 0, & \text{otherwise} \end{cases}$$

we have that the number of outcomes that never occur is equal to $\sum_1^r Y_i$ and thus,

$$\begin{aligned} E\left[\sum_1^r Y_i\right] &= \sum_1^r E[Y_i] \\ &= \sum_1^r P\{\text{outcomes } i \text{ does not occur}\} \\ &= \sum_1^r (1 - P_i)^n \end{aligned}$$

74. (a) As the random variables are independent, identically distributed, and continuous, it follows that, with probability 1, they will all have

different values. Hence the largest of $X_1, \ldots, X_n$ is equally likely to be either X_1 or $X_2 \ldots$ or X_n. Hence, as there is a record at time n when X_n is the largest value, it follows that

$$P\{\text{a record occurs at } n\} = \frac{1}{n}$$

(b) Let $I_j = \begin{cases} 1, & \text{if a record occurs at } j \\ 0, & \text{otherwise} \end{cases}$

Then

$$E\left[\sum_1^n I_j\right] = \sum_1^n E[I_j] = \sum_1^n \frac{1}{j}$$

(c) It is easy to see that the random variables $I_1, I_2, \ldots, I_n$ are independent. For instance, for $j < k$

$$P\{I_j = 1/I_k = 1\} = P\{I_j = 1\}$$

since knowing that X_k is the largest of $X_1, \ldots, X_j, \ldots, X_k$ clearly tells us nothing about whether or not X_j is the largest of $X_1, \ldots, X_j$. Hence,

$$Var \sum_1^n I_j = \sum_1^n Var(I_j) = \sum_{j=1}^n \left[\frac{1}{j}\right]\left[\frac{j-1}{j}\right]$$

(d) $P\{N > n\}$

$$= P\{X_1 \text{ is the largest of } X_1, \ldots, X_n\} = \frac{1}{n}$$

Hence,

$$E[N] = \sum_{n=1}^\infty P\{N > n\} = \sum_{n=1}^\infty \frac{1}{n} = \infty$$

75. (a) Knowing the values of $N_1, \ldots, N_j$ is equivalent to knowing the relative ordering of the elements $a_1, \ldots, a_j$. For instance, if $N_1 = 0, N_2 = 1, N_3 = 1$ then in the random permutation a_2 is before a_3, which is before a_1. The independence result follows for clearly the number of $a_1, \ldots, a_i$ that follow a_{i+1} does not probabilistically depend on the relative ordering of $a_1, \ldots, a_i$.

(b) $P\{N_i = k\} = \dfrac{1}{i}, \quad k = 0, 1, \ldots, i-1$

which follows since of the elements $a_1, \ldots, a_{i+1}$ the element a_{i+1} is equally likely to be first or second or … or $(i+1)^{st}$.

(c) $E[N_i] = \dfrac{1}{i}\displaystyle\sum_{k=0}^{i-1} k = \dfrac{i-1}{2}$

$$E[N_i^2] = \frac{1}{i}\sum_{k=0}^{i-1} k^2 = \frac{(i-1)(2i-1)}{6}$$

and so

$$Var(N_i) = \frac{(i-1)(2i-1)}{6} - \frac{(i-1)^2}{4} = \frac{i^2-1}{12}$$

76. $E[XY] = \mu_x \mu_y$

$$E[(XY)^2] = (\mu_x^2 + \sigma_x^2)(\mu_y^2 + \sigma_y^2)$$

$$Var(XY) = E[(XY)^2] - (E[XY])^2$$

77. If $g_1(x, y) = x + y$, $g_2(x, y) = x - y$, then

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = 2$$

Hence, if $U = X + Y$, $V = X - Y$, then

$$f_{U,V}(u, v) = \frac{1}{2} f_{X,Y}\left[\frac{u+v}{2}, \frac{u-v}{2}\right]$$

$$= \frac{2}{4\tau\sigma^2} exp\left[-\frac{1}{2\sigma^2}\left[\left[\frac{u+v}{2} - \mu\right]^2 + \left[\frac{u-v}{2} - \mu\right]^2\right]\right]$$

$$= \frac{e-\mu^2/\sigma^2}{4\tau\sigma^2} exp\left[\frac{u\mu}{\sigma^2} - \frac{u^2}{4\sigma^2}\right] exp\left\{-\frac{v^2}{4\sigma^2}\right\}$$

78. (a) $\phi_{x_i}(t_i) = \phi(0, 0 \ldots 0, 1, 0 \ldots 0)$ with the 1 in the i^{th} place.

(b) If independent, then $E\left[e^{\sum t_i x_i}\right] = \pi_i \left[e^{t_i x_i}\right]$

On the other hand, if the above is satisfied, then the joint moment generating function is that of the sum of n independent random variables the i^{th} of which has the same distribution as x_i. As the joint moment generating function uniquely determines the joint distribution, the result follows.

79. $K'(t) = \dfrac{E\left[Xe^{tX}\right]}{E\left[e^{tX}\right]}$

$$K''(t) = \frac{E\left[e^{tX}\right]E\left[X^2e^{tX}\right] - E^2\left[Xe^{tX}\right]}{E^2\left[e^{tX}\right]}$$

Hence,

$$K'(0) = E[X]$$

$$K''(0) = E[X^2] - E^2[X] = \text{Var}(X)$$

80. Let I_i be the indicator variable for the event that A_i occurs. Then

$$\binom{X}{k} = \sum_{i_1 < \ldots < i_k} I_{i_1} \cdots I_{i_k}$$

Taking expectations yields

$$E\left[\binom{X}{k}\right] = S_k$$

Hence,

$$E[X] = S_1, \quad E\left[\frac{X(X-1)}{2}\right] = S_2$$

giving that

$$Var(X) = E[X^2] - S_1^2 = 2S_2 + S_1 - S_1^2$$

Conditional Probability and Conditional Expectation

Exercises

1. If X and Y are both discrete, show that $\sum_x p_{X|Y}(x|y) = 1$ for all y such that $p_Y(y) > 0$.

*2. Let X_1 and X_2 be independent geometric random variables having the same parameter p. Guess the value of

$$P\{X_1 = i|X_1 + X_2 = n\}$$

Hint: Suppose a coin having probability p of coming up heads is continually flipped. If the second head occurs on flip number n, what is the conditional probability that the first head was on flip number i, $i = 1, \ldots, n-1$?

Verify your guess analytically.

3. The joint probability mass function of X and Y, $p(x, y)$, is given by

$$\begin{aligned} &p(1,1) = \tfrac{1}{9}, \quad p(2,1) = \tfrac{1}{3}, \quad p(3,1) = \tfrac{1}{9}, \\ &p(1,2) = \tfrac{1}{9}, \quad p(2,2) = 0, \quad p(3,2) = \tfrac{1}{18}, \\ &p(1,3) = 0, \quad p(2,3) = \tfrac{1}{6}, \quad p(3,3) = \tfrac{1}{9} \end{aligned}$$

Compute $E[X|Y = i]$ for $i = 1, 2, 3$.

4. In Exercise 3, are the random variables X and Y independent?

5. An urn contains three white, six red, and five black balls. Six of these balls are randomly selected from the urn. Let X and Y denote respectively the number of white and black balls selected. Compute the conditional probability mass function of X given that $Y = 3$. Also compute $E[X|Y = 1]$.

*6. Repeat Exercise 5 but under the assumption that when a ball is selected its color is noted, and it is then replaced in the urn before the next selection is made.

7. Suppose $p(x, y, z)$, the joint probability mass function of the random variables X, Y, and Z, is given by

$$\begin{aligned} &p(1,1,1) = \tfrac{1}{8}, \quad p(2,1,1) = \tfrac{1}{4}, \\ &p(1,1,2) = \tfrac{1}{8}, \quad p(2,1,2) = \tfrac{3}{16}, \\ &p(1,2,1) = \tfrac{1}{16}, \quad p(2,2,1) = 0, \\ &p(1,2,2) = 0, \quad p(2,2,2) = \tfrac{1}{4} \end{aligned}$$

What is $E[X|Y = 2]$? What is $E[X|Y = 2, Z = 1]$?

8. An unbiased die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a six and a five. Find (a) $E[X]$, (b) $E[X|Y = 1]$, (c) $E[X|Y = 5]$.
9. Show in the discrete case that if X and Y are independent, then

$$E[X|Y = y] = E[X] \quad \text{for all } y$$

10. Suppose X and Y are independent continuous random variables. Show that

$$E[X|Y = y] = E[X] \quad \text{for all } y$$

11. The joint density of X and Y is

$$f(x, y) = \frac{(y^2 - x^2)}{8} e^{-y}, \quad 0 < y < \infty, \quad -y \leqslant x \leqslant y$$

Show that $E[X|Y = y] = 0$.

12. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-x/y} e^{-y}}{y}, \quad 0 < x < \infty, \quad 0 < y < \infty$$

Show $E[X|Y = y] = y$.

*13. Let X be exponential with mean $1/\lambda$; that is,

$$f_X(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty$$

Find $E[X|X > 1]$.

14. Let X be uniform over (0, 1). Find $E[X|X < \frac{1}{2}]$.
15. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-y}}{y}, \quad 0 < x < y, \quad 0 < y < \infty$$

Compute $E[X^2|Y = y]$.

16. The random variables X and Y are said to have a bivariate normal distribution if their joint density function is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}} \exp\left\{ -\frac{1}{2(1 - \rho^2)} \right. \\ \left. \times \left[\left(\frac{x - \mu_x}{\sigma_x} \right)^2 - \frac{2\rho(x - \mu_x)(y - \mu_y)}{\sigma_x\sigma_y} + \left(\frac{y - \mu_y}{\sigma_y} \right)^2 \right] \right\}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, where σ_x, σ_y, μ_x, μ_y, and ρ are constants such that $-1 < \rho < 1$, $\sigma_x > 0$, $\sigma_y > 0$, $-\infty < \mu_x < \infty$, $-\infty < \mu_y < \infty$.

(a) Show that X is normally distributed with mean μ_x and variance σ_x^2, and Y is normally distributed with mean μ_y and variance σ_y^2.

(b) Show that the conditional density of X given that $Y = y$ is normal with mean $\mu_x + (\rho\sigma_x/\sigma_y)(y - \mu_y)$ and variance $\sigma_x^2(1 - \rho^2)$.

The quantity ρ is called the correlation between X and Y. It can be shown that

$$\begin{aligned}\rho &= \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x\sigma_y} \\ &= \frac{\text{Cov}(X, Y)}{\sigma_x\sigma_y}\end{aligned}$$

17. Let Y be a gamma random variable with parameters (s, α). That is, its density is

$$f_Y(y) = Ce^{-\alpha y}y^{s-1}, \quad y > 0$$

where C is a constant that does not depend on y. Suppose also that the conditional distribution of X given that $Y = y$ is Poisson with mean y. That is,

$$P\{X = i|Y = y\} = e^{-y}y^i/i!, \quad i \geqslant 0$$

Show that the conditional distribution of Y given that $X = i$ is the gamma distribution with parameters $(s + i, \alpha + 1)$.

18. Let $X_1, \ldots, X_n$ be independent random variables having a common distribution function that is specified up to an unknown parameter θ. Let $T = T(\mathbf{X})$ be a function of the data $\mathbf{X} = (X_1, \ldots, X_n)$. If the conditional distribution of $X_1, \ldots, X_n$ given $T(\mathbf{X})$ does not depend on θ then $T(\mathbf{X})$ is said to be a *sufficient statistic* for θ. In the following cases, show that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.
(a) The X_i are normal with mean θ and variance 1.
(b) The density of X_i is $f(x) = \theta e^{-\theta x}$, $x > 0$.
(c) The mass function of X_i is $p(x) = \theta^x(1 - \theta)^{1-x}$, $x = 0, 1$, $0 < \theta < 1$.
(d) The X_i are Poisson random variables with mean θ.

*19. Prove that if X and Y are jointly continuous, then

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)\,dy$$

20. An individual whose level of exposure to a certain pathogen is x will contract the disease caused by this pathogen with probability $P(x)$. If the exposure level of a randomly chosen member of the population has probability density function f, determine the conditional probability density of the exposure level of that member given that he or she
(a) has the disease.
(b) does not have the disease.
(c) Show that when $P(x)$ increases in x, then the ratio of the density of part (a) to that of part (b) also increases in x.

21. Consider Example 3.13, which refers to a miner trapped in a mine. Let N denote the total number of doors selected before the miner reaches safety. Also, let T_i denote the travel time corresponding to the ith choice, $i \geqslant 1$. Again let X denote the time when the miner reaches safety.

(a) Give an identity that relates X to N and the T_i.
(b) What is $E[N]$?
(c) What is $E[T_N]$?
(d) What is $E[\sum_{i=1}^{N} T_i | N = n]$?
(e) Using the preceding, what is $E[X]$?

22. Suppose that independent trials, each of which is equally likely to have any of m possible outcomes, are performed until the same outcome occurs k consecutive times. If N denotes the number of trials, show that

$$E[N] = \frac{m^k - 1}{m - 1}$$

Some people believe that the successive digits in the expansion of $\pi = 3.14159\ldots$ are "uniformly" distributed. That is, they believe that these digits have all the appearance of being independent choices from a distribution that is equally likely to be any of the digits from 0 through 9. Possible evidence against this hypothesis is the fact that starting with the 24,658,601st digit there is a run of nine successive 7s. Is this information consistent with the hypothesis of a uniform distribution?

To answer this, we note from the preceding that if the uniform hypothesis were correct, then the expected number of digits until a run of nine of the same value occurs is

$$(10^9 - 1)/9 = 111{,}111{,}111$$

Thus, the actual value of approximately 25 million is roughly 22 percent of the theoretical mean. However, it can be shown that under the uniformity assumption the standard deviation of N will be approximately equal to the mean. As a result, the observed value is approximately 0.78 standard deviations less than its theoretical mean and is thus quite consistent with the uniformity assumption.

*23. A coin having probability p of coming up heads is successively flipped until two of the most recent three flips are heads. Let N denote the number of flips. (Note that if the first two flips are heads, then $N = 2$.) Find $E[N]$.

24. A coin, having probability p of landing heads, is continually flipped until at least one head and one tail have been flipped.
(a) Find the expected number of flips needed.
(b) Find the expected number of flips that land on heads.
(c) Find the expected number of flips that land on tails.
(d) Repeat part (a) in the case where flipping is continued until a total of at least two heads and one tail have been flipped.

25. Independent trials, resulting in one of the outcomes 1, 2, 3 with respective probabilities p_1, p_2, p_3, $\sum_{i=1}^{3} p_i = 1$, are performed.
(a) Let N denote the number of trials needed until the initial outcome has occurred exactly 3 times. For instance, if the trial results are 3, 2, 1, 2, 3 ,2, 3 then $N = 7$. Find $E[N]$.
(b) Find the expected number of trials needed until both outcome 1 and outcome 2 have occurred.

26. You have two opponents with whom you alternate play. Whenever you play A, you win with probability p_A; whenever you play B, you win with probability p_B,

where $p_B > p_A$. If your objective is to minimize the expected number of games you need to play to win two in a row, should you start with A or with B?

Hint: Let $E[N_i]$ denote the mean number of games needed if you initially play i. Derive an expression for $E[N_A]$ that involves $E[N_B]$; write down the equivalent expression for $E[N_B]$ and then subtract.

27. A coin that comes up heads with probability p is continually flipped until the pattern T, T, H appears. (That is, you stop flipping when the most recent flip lands heads, and the two immediately preceding it lands tails.) Let X denote the number of flips made, and find $E[X]$.

28. Polya's urn model supposes that an urn initially contains r red and b blue balls. At each stage a ball is randomly selected from the urn and is then returned along with m other balls of the same color. Let X_k be the number of red balls drawn in the first k selections.
    (a) Find $E[X_1]$.
    (b) Find $E[X_2]$.
    (c) Find $E[X_3]$.
    (d) Conjecture the value of $E[X_k]$, and then verify your conjecture by a conditioning argument.
    (e) Give an intuitive proof for your conjecture.

    Hint: Number the initial r red and b blue balls, so the urn contains one type i red ball, for each $i = 1, \ldots, r$; as well as one type j blue ball, for each $j = 1, \ldots, b$. Now suppose that whenever a red ball is chosen it is returned along with m others of the same type, and similarly whenever a blue ball is chosen it is returned along with m others of the same type. Now, use a symmetry argument to determine the probability that any given selection is red.

29. Two players take turns shooting at a target, with each shot by player i hitting the target with probability p_i, $i = 1, 2$. Shooting ends when two consecutive shots hit the target. Let μ_i denote the mean number of shots taken when player i shoots first, $i = 1, 2$.
    (a) Find μ_1 and μ_2.
    (b) Let h_i denote the mean number of times that the target is hit when player i shoots first, $i = 1, 2$. Find h_1 and h_2.

30. Let X_i, $i \geqslant 0$ be independent and identically distributed random variables with probability mass function

$$p(j) = P\{X_i = j\}, \quad j = 1, \ldots, m, \quad \sum_{j=1}^{m} P(j) = 1$$

    Find $E[N]$, where $N = \min\{n > 0 : X_n = X_0\}$.

31. Each element in a sequence of binary data is either 1 with probability p or 0 with probability $1 - p$. A maximal subsequence of consecutive values having identical outcomes is called a run. For instance, if the outcome sequence is $1, 1, 0, 1, 1, 1, 0$, the first run is of length 2, the second is of length 1, and the third is of length 3.
    (a) Find the expected length of the first run.
    (b) Find the expected length of the second run.

32. Independent trials, each resulting in success with probability p, are performed.

(a) Find the expected number of trials needed for there to have been both at least n successes or at least m failures.

Hint: Is it useful to know the result of the first $n + m$ trials?

(b) Find the expected number of trials needed for there to have been either at least n successes or at least m failures.

Hint: Make use of the result from part (a).

33. If R_i denotes the random amount that is earned in period i, then $\sum_{i=1}^{\infty} \beta^{i-1} R_i$, where $0 < \beta < 1$ is a specified constant, is called the total discounted reward with discount factor β. Let T be a geometric random variable with parameter $1 - \beta$ that is independent of the R_i. Show that the expected total discounted reward is equal to the expected total (undiscounted) reward earned by time T. That is, show that

$$E\left[\sum_{i=1}^{\infty} \beta^{i-1} R_i\right] = E\left[\sum_{i=1}^{T} R_i\right]$$

34. A set of n dice is thrown. All those that land on six are put aside, and the others are again thrown. This is repeated until all the dice have landed on six. Let N denote the number of throws needed. (For instance, suppose that $n = 3$ and that on the initial throw exactly two of the dice land on six. Then the other die will be thrown, and if it lands on six, then $N = 2$.) Let $m_n = E[N]$.
    (a) Derive a recursive formula for m_n and use it to calculate m_i, $i = 2, 3, 4$ and to show that $m_5 \approx 13.024$.
    (b) Let X_i denote the number of dice rolled on the ith throw. Find $E[\sum_{i=1}^{N} X_i]$.

35. Consider n multinomial trials, where each trial independently results in outcome i with probability p_i, $\sum_{i=1}^{k} p_i = 1$. With X_i equal to the number of trials that result in outcome i, find $E[X_1 | X_2 > 0]$.

36. Let $p_0 = P\{X = 0\}$ and suppose that $0 < p_0 < 1$. Let $\mu = E[X]$ and $\sigma^2 = \text{Var}(X)$.
    (a) Find $E[X | X \neq 0]$.
    (b) Find $\text{Var}(X | X \neq 0)$.

37. A manuscript is sent to a typing firm consisting of typists A, B, and C. If it is typed by A, then the number of errors made is a Poisson random variable with mean 2.6; if typed by B, then the number of errors is a Poisson random variable with mean 3; and if typed by C, then it is a Poisson random variable with mean 3.4. Let X denote the number of errors in the typed manuscript. Assume that each typist is equally likely to do the work.
    (a) Find $E[X]$.
    (b) Find $\text{Var}(X)$.

38. Let U be a uniform $(0, 1)$ random variable. Suppose that n trials are to be performed and that conditional on $U = u$ these trials will be independent with a common success probability u. Compute the mean and variance of the number of successes that occur in these trials.

39. A deck of n cards, numbered 1 through n, is randomly shuffled so that all $n!$ possible permutations are equally likely. The cards are then turned over one at a time until card number 1 appears. These upturned cards constitute the first cycle. We now

determine (by looking at the upturned cards) the lowest numbered card that has not yet appeared, and we continue to turn the cards face up until that card appears. This new set of cards represents the second cycle. We again determine the lowest numbered of the remaining cards and turn the cards until it appears, and so on until all cards have been turned over. Let m_n denote the mean number of cycles.

(a) Derive a recursive formula for m_n in terms of m_k, $k = 1, \ldots, n-1$.
(b) Starting with $m_0 = 0$, use the recursion to find m_1, m_2, m_3, and m_4.
(c) Conjecture a general formula for m_n.
(d) Prove your formula by induction on n. That is, show it is valid for $n = 1$, then assume it is true for any of the values $1, \ldots, n-1$ and show that this implies it is true for n.
(e) Let X_i equal 1 if one of the cycles ends with card i, and let it equal 0 otherwise, $i = 1, \ldots, n$. Express the number of cycles in terms of these X_i.
(f) Use the representation in part (e) to determine m_n.
(g) Are the random variables $X_1, \ldots, X_n$ independent? Explain.
(h) Find the variance of the number of cycles.

40. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that returns him to his cell after two days of travel. The second leads to a tunnel that returns him to his cell after three days of travel. The third door leads immediately to freedom.
(a) Assuming that the prisoner will always select doors 1, 2, and 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?
(b) Assuming that the prisoner is always equally likely to choose among those doors that he has not used, what is the expected number of days until he reaches freedom? (In this version, for instance, if the prisoner initially tries door 1, then when he returns to the cell, he will now select only from doors 2 and 3.)
(c) For parts (a) and (b) find the variance of the number of days until the prisoner reaches freedom.

41. A rat is trapped in a maze. Initially it has to choose one of two directions. If it goes to the right, then it will wander around in the maze for three minutes and will then return to its initial position. If it goes to the left, then with probability $\frac{1}{3}$ it will depart the maze after two minutes of traveling, and with probability $\frac{2}{3}$ it will return to its initial position after five minutes of traveling. Assuming that the rat is at all times equally likely to go to the left or the right, what is the expected number of minutes that it will be trapped in the maze?

*42. If $X_i, i = 1, \ldots, n$ are independent normal random variables, with X_i having mean μ_i and variance 1, then the random variable $\sum_{i=1}^{n} X_i^2$ is said to be a *noncentral chi-squared* random variable.
(a) if X is a normal random variable having mean μ and variance 1 show, for $|t| < 1/2$, that the moment generating function of X^2 is

$$(1-2t)^{-1/2} e^{\frac{t\mu^2}{1-2t}}$$

(b) Derive the moment generating function of the noncentral chi-squared random variable $\sum_{i=1}^{n} X_i^2$, and show that its distribution depends on the sequence of

means $\mu_1, \ldots, \mu_n$ only through the sum of their squares. As a result, we say that $\sum_{i=1}^n X_i^2$ is a noncentral chi-squared random variable with parameters n and $\theta = \sum_{i=1}^n \mu_i^2$.

(c) If all $\mu_i = 0$, then $\sum_{i=1}^n X_i^2$ is called a chi-squared random variable with n degrees of freedom. Determine, by differentiating its moment generating function, its expected value and variance.

(d) Let K be a Poisson random variable with mean $\theta/2$, and suppose that conditional on $K = k$, the random variable W has a chi-squared distribution with $n + 2k$ degrees of freedom. Show, by computing its moment generating function, that W is a noncentral chi-squared random variable with parameters n and θ.

(e) Find the expected value and variance of a noncentral chi-squared random variable with parameters n and θ.

43. The density function of a chi-squared random variable having n degrees of freedom can be shown to be

$$f(x) = \frac{\frac{1}{2}e^{-x/2}(x/2)^{\frac{n}{2}-1}}{\Gamma(n/2)}, \quad x > 0$$

where $\Gamma(t)$ is the gamma function defined by

$$\Gamma(t) = \int_0^\infty e^{-x}x^{t-1}dx, \quad t > 0$$

Integration by parts can be employed to show that $\Gamma(t) = (t-1)\Gamma(t-1)$, when $t > 1$. If Z and χ_n^2 are independent random variables with Z having a standard normal distribution and χ_n^2 having a chi-square distribution with n degrees of freedom, then the random variable T defined by

$$T = \frac{Z}{\sqrt{\chi_n^2/n}}$$

is said to have a *t-distribution* with n degrees of freedom. Compute its mean and variance when $n > 2$.

44. The number of customers entering a store on a given day is Poisson distributed with mean $\lambda = 10$. The amount of money spent by a customer is uniformly distributed over $(0, 100)$. Find the mean and variance of the amount of money that the store takes in on a given day.

45. An individual traveling on the real line is trying to reach the origin. However, the larger the desired step, the greater is the variance in the result of that step. Specifically, whenever the person is at location x, he next moves to a location having mean 0 and variance βx^2. Let X_n denote the position of the individual after having taken n steps. Supposing that $X_0 = x_0$, find

(a) $E[X_n]$;

(b) $\mathrm{Var}(X_n)$.

46. (a) Show that

$$\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, E[Y \mid X])$$

(b) Suppose, that, for constants a and b,

$$E[Y \mid X] = a + bX$$

Show that

$$b = \mathrm{Cov}(X, Y)/\mathrm{Var}(X)$$

*47. If $E[Y \mid X] = 1$, show that

$$\mathrm{Var}(X\,Y) \geqslant \mathrm{Var}(X)$$

48. Suppose that we want to predict the value of a random variable X by using one of the predictors $Y_1, \ldots, Y_n$, each of which satisfies $E[Y_i|X] = X$. Show that the predictor Y_i that minimizes $E[(Y_i - X)^2]$ is the one whose variance is smallest.

Hint: Compute $\mathrm{Var}(Y_i)$ by using the conditional variance formula.

49. A and B play a series of games with A winning each game with probability p. The overall winner is the first player to have won two more games than the other.
(a) Find the probability that A is the overall winner.
(b) Find the expected number of games played.

50. There are three coins in a barrel. These coins, when flipped, will come up heads with respective probabilities 0.3, 0.5, 0.7. A coin is randomly selected from among these three and is then flipped ten times. Let N be the number of heads obtained on the ten flips.
(a) Find $P\{N = 0\}$.
(b) Find $P\{N = n\}, n = 0, 1, \ldots, 10$.
(c) Does N have a binomial distribution?
(d) If you win \$1 each time a head appears and you lose \$1 each time a tail appears, is this a fair game? Explain.

51. If X is geometric with parameter p, find the probability that X is even.

52. Suppose that X and Y are independent random variables with probability density functions f_X and f_Y. Determine a one-dimensional integral expression for $P\{X + Y < x\}$.

*53. Suppose X is a Poisson random variable with mean λ. The parameter λ is itself a random variable whose distribution is exponential with mean 1. Show that $P\{X = n\} = (\frac{1}{2})^{n+1}$.

54. A coin is randomly selected from a group of ten coins, the nth coin having a probability $n/10$ of coming up heads. The coin is then repeatedly flipped until a head appears. Let N denote the number of flips necessary. What is the probability distribution of N? Is N a geometric random variable? When would N be a geometric random variable; that is, what would have to be done differently?

55. You are invited to a party. Suppose the times at which invitees are independent uniform (0,1) random variables. Suppose that, aside from yourself, the number of other people who are invited is a Poisson random variable with mean 10.
(a) Find the expected number of people who arrive before you.
(b) Find the probability that you are the nth person to arrive.

56. Data indicate that the number of traffic accidents in Berkeley on a rainy day is a Poisson random variable with mean 9, whereas on a dry day it is a Poisson random variable with mean 3. Let X denote the number of traffic accidents tomorrow. If it will rain tomorrow with probability 0.6, find
(a) $E[X]$;
(b) $P\{X = 0\}$;
(c) $\mathrm{Var}(X)$.

57. The number of storms in the upcoming rainy season is Poisson distributed but with a parameter value that is uniformly distributed over $(0, 5)$. That is, Λ is uniformly distributed over $(0, 5)$, and given that $\Lambda = \lambda$, the number of storms is Poisson with mean λ. Find the probability there are at least three storms this season.

58. A collection of n coins is flipped. The outcomes are independent, and the ith coin comes up heads with probability $\alpha_i, i = 1, \ldots, n$. Suppose that for some value of j, $1 \leqslant j \leqslant n, \alpha_j = \frac{1}{2}$. Find the probability that the total number of heads to appear on the n coins is an even number.

59. Suppose each new coupon collected is, independent of the past, a type i coupon with probability p_i. A total of n coupons is to be collected. Let A_i be the event that there is at least one type i in this set. For $i \neq j$, compute $P(A_iA_j)$ by
(a) conditioning on N_i, the number of type i coupons in the set of n coupons;
(b) conditioning on F_i, the first time a type i coupon is collected;
(c) using the identity $P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_iA_j)$.

*60. Two players alternate flipping a coin that comes up heads with probability p. The first one to obtain a head is declared the winner. We are interested in the probability that the first player to flip is the winner. Before determining this probability, which we will call $f(p)$, answer the following questions.
(a) Do you think that $f(p)$ is a monotone function of p? If so, is it increasing or decreasing?
(b) What do you think is the value of $\lim_{p\to 1} f(p)$?
(c) What do you think is the value of $\lim_{p\to 0} f(p)$?
(d) Find $f(p)$.

61. Suppose in Exercise 29 that the shooting ends when the target has been hit twice. Let m_i denote the mean number of shots needed for the first hit when player i shoots first, $i = 1, 2$. Also, let P_i, $i = 1, 2$, denote the probability that the first hit is by player 1, when player i shoots first.
(a) Find m_1 and m_2.
(b) Find P_1 and P_2.
For the remainder of the problem, assume that player 1 shoots first.
(c) Find the probability that the final hit was by 1.
(d) Find the probability that both hits were by 1.
(e) Find the probability that both hits were by 2.
(f) Find the mean number of shots taken.

62. A, B, and C are evenly matched tennis players. Initially A and B play a set, and the winner then plays C. This continues, with the winner always playing the waiting player, until one of the players has won two sets in a row. That player is then declared the overall winner. Find the probability that A is the overall winner.

63. Suppose there are n types of coupons, and that the type of each new coupon obtained is independent of past selections and is equally likely to be any of the n types. Suppose one continues collecting until a complete set of at least one of each type is obtained.
(a) Find the probability that there is exactly one type i coupon in the final collection.

Hint: Condition on T, the number of types that are collected before the first type i appears.

(b) Find the expected number of types that appear exactly once in the final collection.

64. A and B roll a pair of dice in turn, with A rolling first. A's objective is to obtain a sum of 6, and B's is to obtain a sum of 7. The game ends when either player reaches his or her objective, and that player is declared the winner.
(a) Find the probability that A is the winner.
(b) Find the expected number of rolls of the dice.
(c) Find the variance of the number of rolls of the dice.

65. The number of red balls in an urn that contains n balls is a random variable that is equally likely to be any of the values $0, 1, \ldots, n$. That is,

$$P\{i \text{ red}, n - i \text{ non-red}\} = \frac{1}{n+1}, \quad i = 0, \ldots, n$$

The n balls are then randomly removed one at a time. Let Y_k denote the number of red balls in the first k selections, $k = 1, \ldots, n$.
(a) Find $P\{Y_n = j\}, j = 0, \ldots, n$.
(b) Find $P\{Y_{n-1} = j\}$, $j = 0, \ldots, n$.
(c) What do you think is the value of $P\{Y_k = j\}, j = 0, \ldots, n$?
(d) Verify your answer to part (c) by a backwards induction argument. That is, check that your answer is correct when $k = n$, and then show that whenever it is true for k it is also true for $k - 1, k = 1, \ldots, n$.

66. The opponents of soccer team A are of two types: either they are a class 1 or a class 2 team. The number of goals team A scores against a class i opponent is a Poisson random variable with mean λ_i, where $\lambda_1 = 2$, $\lambda_2 = 3$. This weekend the team has two games against teams they are not very familiar with. Assuming that the first team they play is a class 1 team with probability 0.6 and the second is, independently of the class of the first team, a class 1 team with probability 0.3, determine
(a) the expected number of goals team A will score this weekend.
(b) the probability that team A will score a total of five goals.

*67. A coin having probability p of coming up heads is continually flipped. Let $P_j(n)$ denote the probability that a run of j successive heads occurs within the first n flips.
(a) Argue that

$$P_j(n) = P_j(n-1) + p^j(1-p)[1 - P_j(n-j-1)]$$

(b) By conditioning on the first non-head to appear, derive another equation relating $P_j(n)$ to the quantities $P_j(n-k), k = 1, \ldots, j$.

68. In a knockout tennis tournament of 2^n contestants, the players are paired and play a match. The losers depart, the remaining 2^{n-1} players are paired, and they play a match. This continues for n rounds, after which a single player remains unbeaten and is declared the winner. Suppose that the contestants are numbered 1 through 2^n, and that whenever two players contest a match, the lower numbered one wins with probability p. Also suppose that the pairings of the remaining players are always done at random so that all possible pairings for that round are equally likely.
    (a) What is the probability that player 1 wins the tournament?
    (b) What is the probability that player 2 wins the tournament?

    Hint: Imagine that the random pairings are done in advance of the tournament. That is, the first-round pairings are randomly determined; the 2^{n-1} first-round pairs are then themselves randomly paired, with the winners of each pair to play in round 2; these 2^{n-2} groupings (of four players each) are then randomly paired, with the winners of each grouping to play in round 3, and so on. Say that players i and j are scheduled to meet in round k if, provided they both win their first $k-1$ matches, they will meet in round k. Now condition on the round in which players 1 and 2 are scheduled to meet.

69. In the match problem, say that $(i, j), i < j$, is a pair if i chooses j's hat and j chooses i's hat.
    (a) Find the expected number of pairs.
    (b) Let Q_n denote the probability that there are no pairs, and derive a recursive formula for Q_n in terms of $Q_j, j < n$.

    Hint: Use the cycle concept.

    (c) Use the recursion of part (b) to find Q_8.

70. Let N denote the number of cycles that result in the match problem.
    (a) Let $M_n = E[N]$, and derive an equation for M_n in terms of $M_1, \ldots, M_{n-1}$.
    (b) Let C_j denote the size of the cycle that contains person j. Argue that

$$N = \sum_{j=1}^{n} 1/C_j$$

    and use the preceding to determine $E[N]$.
    (c) Find the probability that persons $1, 2, \ldots, k$ are all in the same cycle.
    (d) Find the probability that $1, 2, \ldots, k$ is a cycle.

71. Use Equation (3.14) to obtain Equation (3.10).

    Hint: First multiply both sides of Equation (3.14) by n, then write a new equation by replacing n with $n-1$, and then subtract the former from the latter.

72. In Example 3.28 show that the conditional distribution of N given that $U_1 = y$ is the same as the conditional distribution of M given that $U_1 = 1 - y$. Also, show that

$$E[N|U_1 = y] = E[M|U_1 = 1 - y] = 1 + e^y$$

*73. Suppose that we continually roll a die until the sum of all throws exceeds 100. What is the most likely value of this total when you stop?

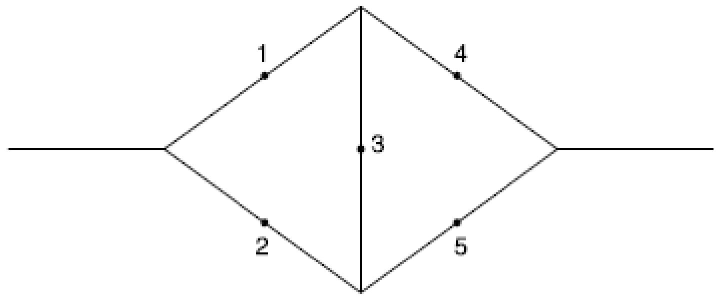


Figure 3.7

74. There are five components. The components act independently, with component i working with probability p_i, $i = 1, 2, 3, 4, 5$. These components form a system as shown in Figure 3.7.

    The system is said to work if a signal originating at the left end of the diagram can reach the right end, where it can pass through a component only if that component is working. (For instance, if components 1 and 4 both work, then the system also works.) What is the probability that the system works?

75. This problem will present another proof of the ballot problem of Example 3.27.

    (a) Argue that

    $$P_{n,m} = 1 - P\{A \text{ and } B \text{ are tied at some point}\}$$

    (b) Explain why

    $$P\{A \text{ receives first vote and they are eventually tied}\}$$
    $$= P\{B \text{ receives first vote and they are eventually tied}\}$$

    Hint: Any outcome in which they are eventually tied with A receiving the first vote corresponds to an outcome in which they are eventually tied with B receiving the first vote. Explain this correspondence.

    (c) Argue that $P\{\text{eventually tied}\} = 2m/(n+m)$, and conclude that $P_{n,m} = (n-m)/(n+m)$.

76. Consider a gambler who on each bet either wins 1 with probability 18/38 or loses 1 with probability 20/38. (These are the probabilities if the bet is that a roulette wheel will land on a specified color.) The gambler will quit either when he or she is winning a total of 5 or after 100 plays. What is the probability he or she plays exactly 15 times?

77. Show that

    (a) $E[XY|Y = y] = yE[X|Y = y]$

    (b) $E[g(X, Y)|Y = y] = E[g(X, y)|Y = y]$

    (c) $E[XY] = E[YE[X|Y]]$

78. In the ballot problem (Example 3.27), compute $P\{A \text{ is never behind}\}$.

79. An urn contains n white and m black balls that are removed one at a time. If $n > m$, show that the probability that there are always more white than black balls

in the urn (until, of course, the urn is empty) equals $(n-m)/(n+m)$. Explain why this probability is equal to the probability that the set of withdrawn balls always contains more white than black balls. (This latter probability is $(n-m)/(n+m)$ by the ballot problem.)

80. A coin that comes up heads with probability p is flipped n consecutive times. What is the probability that starting with the first flip there are always more heads than tails that have appeared?

81. Let $X_i, i \geqslant 1$, be independent uniform $(0, 1)$ random variables, and define N by

$$N = \min\{n: X_n < X_{n-1}\}$$

where $X_0 = x$. Let $f(x) = E[N]$.

(a) Derive an integral equation for $f(x)$ by conditioning on X_1.
(b) Differentiate both sides of the equation derived in part (a).
(c) Solve the resulting equation obtained in part (b).
(d) For a second approach to determining $f(x)$ argue that

$$P\{N \geqslant k\} = \frac{(1-x)^{k-1}}{(k-1)!}$$

(e) Use part (d) to obtain $f(x)$.

82. Let $X_1, X_2, \ldots$ be independent continuous random variables with a common distribution function F and density $f = F'$, and for $k \geqslant 1$ let

$$N_k = \min\{n \geqslant k: X_n = k\text{th largest of } X_1, \ldots, X_n\}$$

(a) Show that $P\{N_k = n\} = \frac{k-1}{n(n-1)}, n \geqslant k$.
(b) Argue that

$$f_{X_{N_k}}(x) = f(x)(\bar{F}(x))^{k-1} \sum_{i=0}^{\infty} \binom{i+k-2}{i} (F(x))^i$$

(c) Prove the following identity:

$$a^{1-k} = \sum_{i=0}^{\infty} \binom{i+k-2}{i} (1-a)^i, \quad 0 < a < 1, k \geqslant 2$$

Hint: Use induction. First prove it when $k = 2$, and then assume it for k. To prove it for $k + 1$, use the fact that

$$\sum_{i=1}^{\infty} \binom{i+k-1}{i} (1-a)^i = \sum_{i=1}^{\infty} \binom{i+k-2}{i} (1-a)^i + \sum_{i=1}^{\infty} \binom{i+k-2}{i-1} (1-a)^i$$

where the preceding used the combinatorial identity

$$\binom{m}{i} = \binom{m-1}{i} + \binom{m-1}{i-1}$$

Now, use the induction hypothesis to evaluate the first term on the right side of the preceding equation.

(d) Conclude that X_{N_k} has distribution F.

83. An urn contains n balls, with ball i having weight $w_i, i = 1, \ldots, n$. The balls are withdrawn from the urn one at a time according to the following scheme: When S is the set of balls that remains, ball $i, i \in S$, is the next ball withdrawn with probability $w_i / \sum_{j \in S} w_j$. Find the expected number of balls that are withdrawn before ball $i, i = 1, \ldots, n$.

84. In the list example of Section 3.6.1 suppose that the initial ordering at time $t = 0$ is determined completely at random; that is, initially all $n!$ permutations are equally likely. Following the front-of-the-line rule, compute the expected position of the element requested at time t.

Hint: To compute $P\{e_j \text{ precedes } e_i \text{ at time } t\}$ condition on whether or not either e_i or e_j has ever been requested prior to t.

85. In the list problem, when the P_i are known, show that the best ordering (best in the sense of minimizing the expected position of the element requested) is to place the elements in decreasing order of their probabilities. That is, if $P_1 > P_2 > \cdots > P_n$, show that $1, 2, \ldots, n$ is the best ordering.

86. Consider the random graph of Section 3.6.2 when $n = 5$. Compute the probability distribution of the number of components and verify your solution by using it to compute $E[C]$ and then comparing your solution with

$$E[C] = \sum_{k=1}^{5} \binom{5}{k} \frac{(k-1)!}{5^k}$$

87. (a) From the results of Section 3.6.3 we can conclude that there are $\binom{n+m-1}{m-1}$ nonnegative integer valued solutions of the equation $x_1 + \cdots + x_m = n$. Prove this directly.

(b) How many positive integer valued solutions of $x_1 + \cdots + x_m = n$ are there?

Hint: Let $y_i = x_i - 1$.

(c) For the Bose–Einstein distribution, compute the probability that exactly k of the X_i are equal to 0.

88. In Section 3.6.3, we saw that if U is a random variable that is uniform on $(0, 1)$ and if, conditional on $U = p$, X is binomial with parameters n and p, then

$$P\{X = i\} = \frac{1}{n+1}, \quad i = 0, 1, \ldots, n$$

For another way of showing this result, let $U, X_1, X_2, \ldots, X_n$ be independent uniform (0, 1) random variables. Define X by

$$X = \#i\colon X_i < U$$

That is, if the $n+1$ variables are ordered from smallest to largest, then U would be in position $X+1$.

(a) What is $P\{X=i\}$?

(b) Explain how this proves the result of Section 3.6.3.

89. Let $I_1, \ldots, I_n$ be independent random variables, each of which is equally likely to be either 0 or 1. A well-known nonparametric statistical test (called the signed rank test) is concerned with determining $P_n(k)$ defined by

$$P_n(k) = P\left\{\sum_{j=1}^{n} jI_j \leqslant k\right\}$$

Justify the following formula:

$$P_n(k) = \tfrac{1}{2}P_{n-1}(k) + \tfrac{1}{2}P_{n-1}(k-n)$$

90. The number of accidents in each period is a Poisson random variable with mean 5. With $X_n, n \geqslant 1$, equal to the number of accidents in period n, find $E[N]$ when

(a) $N = \min(n\colon X_{n-2}=2, X_{n-1}=1, X_n=0)$;

(b) $N = \min(n\colon X_{n-3}=2, X_{n-2}=1, X_{n-1}=0, X_n=2)$.

91. Find the expected number of flips of a coin, which comes up heads with probability p, that are necessary to obtain the pattern h, t, h, h, t, h, t, h.

92. The number of coins that Josh spots when walking to work is a Poisson random variable with mean 6. Each coin is equally likely to be a penny, a nickel, a dime, or a quarter. Josh ignores the pennies but picks up the other coins.

(a) Find the expected amount of money that Josh picks up on his way to work.

(b) Find the variance of the amount of money that Josh picks up on his way to work.

(c) Find the probability that Josh picks up exactly 25 cents on his way to work.

*93. Consider a sequence of independent trials, each of which is equally likely to result in any of the outcomes $0, 1, \ldots, m$. Say that a round begins with the first trial, and that a new round begins each time outcome 0 occurs. Let N denote the number of trials that it takes until all of the outcomes $1, \ldots, m-1$ have occurred in the same round. Also, let T_j denote the number of trials that it takes until j distinct outcomes have occurred, and let I_j denote the jth distinct outcome to occur. (Therefore, outcome I_j first occurs at trial T_j.)

(a) Argue that the random vectors $(I_1, \ldots, I_m)$ and $(T_1, \ldots, T_m)$ are independent.

(b) Define X by letting $X = j$ if outcome 0 is the jth distinct outcome to occur. (Thus, $I_X = 0$.) Derive an equation for $E[N]$ in terms of $E[T_j], j = 1, \ldots, m-1$ by conditioning on X.

(c) Determine $E[T_j], j = 1, \ldots, m-1$.

Hint: See Exercise 42 of Chapter 2.

(d) Find $E[N]$.

94. Let N be a hypergeometric random variable having the distribution of the number of white balls in a random sample of size r from a set of w white and b blue balls.

That is,

$$P\{N = n\} = \frac{\binom{w}{n}\binom{b}{r-n}}{\binom{w+b}{r}}$$

where we use the convention that $\binom{m}{j} = 0$ if either $j < 0$ or $j > m$. Now, consider a compound random variable $S_N = \sum_{i=1}^{N} X_i$, where the X_i are positive integer valued random variables with $\alpha_j = P\{X_i = j\}$.

(a) With M as defined as in Section 3.7, find the distribution of $M - 1$.

(b) Suppressing its dependence on b, let $P_{w,r}(k) = P\{S_N = k\}$, and derive a recursion equation for $P_{w,r}(k)$.

(c) Use the recursion of (b) to find $P_{w,r}(2)$.

95. For the left skip free random walk of Section 3.6.6 let $\beta = P(S_n \leqslant 0$ for all $n)$ be the probability that the walk is never positive. Find β when $E[X_i] < 0$.

96. Consider a large population of families, and suppose that the number of children in the different families are independent Poisson random variables with mean λ. Show that the number of siblings of a randomly chosen child is also Poisson distributed with mean λ.

*97. Use the conditional variance formula to find the variance of a geometric random variable.

Chapter 3

1. $\sum_x p_{X|Y}(x|y) = \dfrac{\sum_x p(x,y)}{p_Y(y)} = \dfrac{p_Y(y)}{p_Y(y)} = 1$

2. Intuitively it would seem that the first head would be equally likely to occur on either of trials $1, \ldots, n-1$. That is, it is intuitive that

$$P\{X_1 = i | X_1 + X_2 = n\} = 1/(n-1),$$
$$i = 1, \ldots, n-1$$

Formally,

$$\begin{aligned} &P\{X_1 = i | X_1 + X_2 = n\} \\ &= \frac{P\{X_1 = i, X_1 + X_2 = n\}}{P\{X_1 + X_2 = n\}} \\ &= \frac{P\{X_1 = i, X_2 = n - i\}}{P\{X_1 + X_2 = n\}} \\ &= \frac{p(1-p)^{i-1} p(1-p)^{n-i-1}}{\binom{n-1}{1} p(1-p)^{n-2} p} \\ &= 1/(n-1) \end{aligned}$$

In the above, the next to last equality uses the independence of X_1 and X_2 to evaluate the numerator and the fact that $X_1 + X_2$ has a negative binomial distribution to evaluate the denominator.

3. $E[X|Y = 1] = 2$

$$E[X|Y = 2] = \frac{5}{3}$$

$$E[X|Y = 3] = \frac{12}{5}$$

4. No.

5. (a) $P\{X = i | Y = 3\} = P\{i$ white balls selected when choosing 3 balls from 3 white and 6 red$\}$

$$= \frac{\begin{bmatrix} 3 \\ i \end{bmatrix} \begin{bmatrix} 6 \\ 3-i \end{bmatrix}}{\begin{bmatrix} 9 \\ 3 \end{bmatrix}}, \quad i = 0, 1, 2, 3$$

(b) By same reasoning as in (a), if $Y = 1$, then X has the same distribution as the number of white balls chosen when 5 balls are chosen from 3 white and 6 red. Hence,

$$E[X|Y = 1] = 5\frac{3}{9} = \frac{5}{3}$$

6.
$$\begin{aligned} p_{X|Y}(1|3) &= P\{X = 1, Y = 3\}/P\{Y = 3\} \\ &= P\{1 \text{ white}, 3 \text{ black}, 2 \text{ red}\} \\ &\quad /P\{3 \text{ black}\} \\ &= \frac{6!}{1!3!2!} \left[\frac{3}{14}\right]^1 \left[\frac{5}{14}\right]^3 \left[\frac{6}{14}\right]^2 \\ &\quad \Big/ \frac{6!}{3!3!} \left[\frac{5}{14}\right]^3 \left[\frac{9}{14}\right]^3 \\ &= \frac{4}{9} \end{aligned}$$

$$p_{X|Y}(0|3) = \frac{8}{27}$$

$$p_{X|Y}(2|3) = \frac{2}{9}$$

$$p_{X|Y}(3|3) = \frac{1}{27}$$

$$E[X|Y = 1] = \frac{5}{3}$$

7. Given $Y = 2$, the conditional distribution of X and Z is

$$P\{(X, Z) = (1, 1)|Y = 2\} = \frac{1}{5}$$

$$P\{(1, 2)|Y = 2\} = 0$$

$$P\{(2, 1)|Y = 2\} = 0$$

$$P\{(2, 2)|Y = 2\} = \frac{4}{5}$$

So,

$$E[X|Y = 2] = \frac{1}{5} + \frac{8}{5} = \frac{9}{5}$$

$$E[X|Y = 2, Z = 1] = 1$$

8. (a) $E[X] = E[X|\text{first roll is } 6]\frac{1}{6}$

$$+ E[X|\text{first roll is not } 6]\frac{5}{6}$$

$$= \frac{1}{6} + (1 + E[X])\frac{5}{6}$$

implying that $E[X] = 6$.

(b) $E[X|Y = 1] = 1 + E[X] = 7$

(c) $E[X|Y = 5]$

$$= 1\left[\frac{1}{5}\right] + 2\left[\frac{4}{5}\right]\left[\frac{1}{5}\right] + 3\left[\frac{4}{5}\right]^2\left[\frac{1}{5}\right]$$

$$+ 4\left[\frac{4}{5}\right]^3\left[\frac{1}{5}\right] + 6\left[\frac{4}{5}\right]^4\left[\frac{1}{6}\right]$$

$$+ 7\left[\frac{4}{5}\right]^4\left[\frac{5}{6}\right]\left[\frac{1}{6}\right] + \cdots$$

9. $E[X|Y = y] = \sum_x xP\{X = x|Y = y\}$

$$= \sum_x xP\{X = x\} \quad \text{by independence}$$

$$= E[X]$$

10. (Same as in Problem 8.)

11. $E[X|Y = y] = C\int_{-y}^{y} x(y^2 - x^2)dx = 0$

12. $f_{X|Y}(x|y) = \dfrac{\frac{1}{y}\exp^{-x/y}\exp^{-y}}{\exp^{-y}\int \frac{1}{y}\exp^{-x/y}dx} = \frac{1}{y}\exp^{-x/y}$

Hence, given $Y = y$, X is exponential with mean y.

13. The conditional density of X given that $X > 1$ is

$$f_{X|X>1}(x) = \frac{f(x)}{P\{X > 1\}} = \frac{\lambda\exp^{-\lambda x}}{\exp^{-\lambda}} \text{ when } x > 1$$

$$E[X|X > 1] = \exp^{\lambda}\int_1^{\infty} x\lambda\exp^{-\lambda x}\,dx = 1 + 1/\lambda$$

by integration by parts.

14. $f_{X|X<\frac{1}{2}}(x) = \dfrac{f(x)}{P\{X < 1\}}, \quad x < \dfrac{1}{2}$

$$= \frac{1}{1/2} = 2$$

Hence, $E\left[X|X < \frac{1}{2}\right] = \int_0^{1/2} 2x\,dx = \frac{1}{4}$

15. $f_{X|Y=y}(x|y) = \dfrac{\frac{1}{y}\exp^{-y}}{f_y(y)} = \dfrac{\frac{1}{y}\exp^{-y}}{\int_0^y \frac{1}{y}\exp^{-y}\,dx}$

$$= \frac{1}{y}, \quad 0 < x < y$$

$$E[X^2|Y = y] = \frac{1}{y}\int_0^y x^2dx = \frac{y^2}{3}$$

17. With $K = 1/P\{X = i\}$, we have that

$$f_{Y|X}(y|i) = KP\{X = i|Y = y\}f_Y(y)$$

$$= K_1 e^{-y}y^i e^{-\alpha y}y^{a-1}$$

$$= K_1 e^{-(1+\alpha)y}y^{a+i-1}$$

where K_1 does not depend on y. But as the preceding is the density function of a gamma random variable with parameters $(s + i, 1 + \alpha)$ the result follows.

18. In the following $t = \sum_{i=1}^n x_i$, and C does not depend on θ. For (a) use that T is normal with mean $n\theta$ and variance n; in (b) use that T is gamma with parameters (n, θ); in (c) use that T is binomial with parameters (n, θ); in (d) use that T is Poisson with mean $n\theta$.

(a) $f(x_1, \ldots, x_n|T = t)$

$$= \frac{f(x_1, \ldots, x_n,\ T = t)}{f_T(t)}$$

$$= \frac{f(x_1, \ldots, x_n)}{f_T(t)}$$

$$= C\frac{\exp\{-\sum(x_i - \theta)^2/2\}}{\exp\{-(t - n\theta)^2/2n\}}$$

$$= C\exp\{(t - n\theta)^2/2n - \sum(x_i - \theta)^2/2\}$$

$$= C\exp\{t^2/2n - \theta t\ +\ n\theta^2/2 - \sum x_i^2/2$$

$$+\ \theta t - n\theta^2/2\}$$

$$= C\exp\{(\sum x_i)^2/2n - \sum x_i^2/2\}$$

(b) $f(x_1, \ldots, x_n|T = t) = \dfrac{f(x_1, \ldots, x_n)}{f_T(t)}$

$$= \frac{\theta^n e^{-\theta\sum x_i}}{\theta e^{-\theta t}(\theta t)^{n-1}/(n-1)!}$$

$$= (n-1)!t^{1-n}$$

Parts (c) and (d) are similar.

19. $\int E[X|Y=y]f_Y(y)dy$

$$= \int\int x f_{X|Y}(x|y)dx f_Y(Y)dy$$

$$= \int\int x \frac{f(x,y)}{f_Y(y)} dx f_Y(y)dy$$

$$= \int x \int f(x \cdot y)dydx$$

$$= \int x f_X(x)dx$$

$$= E[X]$$

20. (a) $f(x|\text{disease}) = \dfrac{P\{\text{disease}|x\}f(x)}{\int P\{\text{disease}|x\}f(x)dx}$

$$= \frac{P(x)f(x)}{\int P(x)f(x)dx}$$

(b) $f(x|\text{no disease}) = \dfrac{[1-P(x)]f(x)}{\int[1-P(x)]f(x)dx}$

(c) $\dfrac{f(x|\text{disease})}{f(x|\text{no disease})} = C\dfrac{P(x)}{1-P(x)}$

where C does not depend on x.

21. (a) $X = \sum_{i=1}^{N} T_i$

(b) Clearly N is geometric with parameter 1/3; thus, $E[N] = 3$.

(c) Since T_N is the travel time corresponding to the choice leading to freedom it follows that $T_N = 2$, and so $E[T_N] = 2$.

(d) Given that $N = n$, the travel times $T_i i = 1, \ldots, n-1$ are each equally likely to be either 3 or 5 (since we know that a door leading back to the nine is selected), whereas T_n is equal to 2 (since that choice led to safety). Hence,

$$E\left[\sum_{i=1}^{N} T_i|N=n\right] = E\left[\sum_{i=1}^{n-1} T_i|N=n\right] + E[T_n|N=n]$$
$$= 4(n-1)+2$$

(e) Since part (d) is equivalent to the equation

$$E\left[\sum_{i=1}^{N} T_i|N\right] = 4N-2$$

we see from parts (a) and (b) that

$$E[X] = 4E[N]-2$$
$$= 10$$

22. Letting N_i denote the time until the same outcome occurs i consecutive times we obtain, upon conditioning N_{i-1}, that

$$E[N_i] = E[E[N_i|N_{i-1}]]$$

Now,

$$E[N_i|N_{i-1}]$$
$$= N_{i-1} + \begin{array}{l} 1 \text{ with probability } 1/n \\ E[N_i] \text{ with probability} (n-1)/n \end{array}$$

The above follows because after a run of $i-1$ either a run of i is attained if the next trial is the same type as those in the run or else if the next trial is different then it is exactly as if we were starting all over at that point.

From the above equation we obtain

$$E[N_i] = E[N_{i-1}] + 1/n + E[N_i](n-1)/n$$

Solving for $E[N_i]$ gives

$$E[N_i] = 1 + nE[N_{i-1}]$$

Solving recursively now yields

$$E[N_i] = 1 + n\{1 + nE[N_{i-2}]\}$$
$$= 1 + n + n^2E[N_{i-2}]$$
$$\cdot$$
$$\cdot$$
$$= 1 + n + \cdots + n^{k-1}E[N_1]$$
$$= 1 + n + \cdots + n^{k-1}$$

23. Let X denote the first time a head appears. Let us obtain an equation for $E[N|X]$ by conditioning on the next two flips after X. This gives

$$E[N|X] = E[N|X,h,h]p^2 + E[N|X,h,t]pq + E[N|X,t,h]pq + E[N|X,t,t]q^2$$

where $q = 1-p$. Now

$$E[N|X,h,h] = X+1,\ E[N|X,h,t] = X+1$$
$$E[N|X,t,h] = X+2,\ E[N|X,t,t] = X+2+E[N]$$

Substituting back gives

$$E[N|X] = (X+1)(p^2+pq) + (X+2)pq + (X+2+E[N])q^2$$

Taking expectations, and using the fact that X is geometric with mean $1/p$, we obtain

$$E[N] = 1 + p + q + 2pq + q^2/p + 2q^2 + q^2E[N]$$

Solving for $E[N]$ yields

$$E[N] = \frac{2 + 2q + q^2/p}{1 - q^2}$$

24. In all parts, let X denote the random variable whose expectation is desired, and start by conditioning on the result of the first flip. Also, h stands for heads and t for tails.

(a) $$\begin{aligned} E[X] &= E[X|h]p + E[X|t](1-p) \\ &= \left(1 + \frac{1}{1-p}\right)p + \left(1 + \frac{1}{p}\right)(1-p) \\ &= 1 + p/(1-p) + (1-p)/p \end{aligned}$$

(b) $$\begin{aligned} E[X] &= (1 + E[\text{number of heads before first tail}])p + 1(1-p) \\ &= 1 + p(1/(1-p) - 1) \\ &= 1 + p/(1-p) - p \end{aligned}$$

(c) Interchanging p and $1-p$ in (b) gives result: $1 + (1-p)/p - (1-p)$

(d) $$\begin{aligned} E[X] &= (1 + \text{answer from } (a))p + (1 + 2/p)(1-p) \\ &= (2 + p/(1-p) + (1-p)/p)p + (1 + 2/p)(1-p) \end{aligned}$$

25. (a) Let F be the initial outcome.

$$E[N] = \sum_{i=1}^{3} E[N|F=i]p_i = \sum_{i=1}^{3}\left(1 + \frac{2}{p_i}\right)p_i = 1 + 6 = 7$$

(b) Let $N_{1,2}$ be the number of trials until both outcome 1 and outcome 2 have occurred. Then

$$\begin{aligned} E[N_{1,2}] &= E[N_{1,2}|F=1]p_1 + E[N_{1,2}|F=2]p_2 + E[N_{1,2}|F=3]p_3 \\ &= \left(1 + \frac{1}{p_2}\right)p_1 + \left(1 + \frac{1}{p_1}\right)p_2 + (1 + E[N_{1,2}])p_3 \\ &= 1 + \frac{p_1}{p_2} + \frac{p_2}{p_1} + p_3 E[N_{1,2}] \end{aligned}$$

Hence,

$$E[N_{1,2}] = \frac{1 + \frac{p_1}{p_2} + \frac{p_2}{p_1}}{p_1 + p_2}$$

26. Let N_A and N_B denote the number of games needed given that you start with A and given that you start with B. Conditioning on the outcome of the first game gives

$$E[N_A] = E[N_A|w]p_A + E[N_A|l](1 - p_A)$$

Conditioning on the outcome of the next game gives

$$\begin{aligned} E[N_A|w] &= E[N_A|ww]p_B + E[N_A|wl](1 - p_B) \\ &= 2p_B + (2 + E[N_A])(1 - p_B) \\ &= 2 + (1 - p_B)E[N_A] \end{aligned}$$

As $E[N_A|l] = 1 + E[N_B]$ we obtain

$$\begin{aligned} E[N_A] &= (2 + (1-p_B)E[N_A])p_A + (1 + E[N_B])(1 - p_A) \\ &= 1 + p_A + p_A(1 - p_B)E[N_A] + (1 - p_A)E[N_B] \end{aligned}$$

Similarly,

$$E[N_B] = 1 + p_B + p_B(1 - p_A)E[N_B] + (1 - p_B)E[N_A]$$

Subtracting gives

$$\begin{aligned} &E[N_A] - E[N_B] \\ &\quad = p_A - p_B + (p_A - 1)(1 - p_B)E[N_A] + (1 - p_B)(1 - p_A)E[N_B] \end{aligned}$$

or

$$[1 + (1 - p_A)(1 - p_B)](E[N_A] - E[N_B]) = p_A - p_B$$

Hence, if $p_B > p_A$ then $E[N_A] - E[N_B] < 0$, showing that playing A first is better.

27. Condition on the outcome of the first flip to obtain

$$\begin{aligned} E[X] &= E[X|H]p + E[X|T](1-p) \\ &= (1 + E[X])p + E[X|T](1-p) \end{aligned}$$

Conditioning on the next flip gives

$$\begin{aligned} E[X|T] &= E[X|TH]p + E[X|TT](1-p) \\ &= (2 + E[X])p + (2 + 1/p)(1-p) \end{aligned}$$

where the final equality follows since given that the first two flips are tails the number of additional flips is just the number of flips needed to obtain a head. Putting the preceding together yields

$$E[X] = (1 + E[X])p + (2 + E[X])p(1-p) + (2 + 1/p)(1-p)^2$$

or

$$E[X] = \frac{1}{p(1-p)^2}$$

28. Let Y_i equal 1 if selection i is red, and let it equal 0 otherwise. Then

$$E[X_k] = \sum_{i=1}^{k} E[Y_i]$$

$$E[Y_1] = \frac{r}{r+b}$$

$$E[X_1] = \frac{r}{r+b}$$

$$\begin{aligned}
E[Y_2] &= E[E[Y_2|X_1]] \\
&= E\left[\frac{r+mX_1}{r+b+m}\right] \\
&= \frac{r+m\frac{r}{r+b}}{r+b+m} \\
&= \frac{r}{r+b+m} + \frac{m}{r+b+m}\frac{r}{r+b} \\
&= \frac{r}{r+b+m}\left(1+\frac{m}{r+b}\right) \\
&= \frac{r}{r+b}
\end{aligned}$$

$$E[X_2] = 2\frac{r}{r+b}$$

To prove by induction that $E[Y_k] = \frac{r}{r+b}$, assume that for all $i < k, E[Y_i] = \frac{r}{r+b}$.

Then

$$\begin{aligned}
E[Y_k] &= E[E[Y_k|X_{k-1}]] \\
&= E\left[\frac{r+mX_{k-1}}{r+b+(k-1)m}\right] \\
&= \frac{r+mE\left[\sum_{i<k} Y_i\right]}{r+b+(k-1)m} \\
&= \frac{r+m(k-1)\frac{r}{r+b}}{r+b+(k-1)m} \\
&= \frac{r}{r+b}
\end{aligned}$$

The intuitive argument follows because each selection is equally likely to be any of the $r+b$ types.

29. Let $q_i = 1 - p_i$, $i = 1.2$. Also, let h stand for hit and m for miss.

(a)
$$\begin{aligned}
\mu_1 &= E[N|h]p_1 + E[N|m]q_1 \\
&= p_1(E[N|h,h]p_2 + E[N|h,m]q_2) \\
&\quad + (1+\mu_2)q_1 \\
&= 2p_1p_2 + (2+\mu_1)p_1q_2 + (1+\mu_2)q_1
\end{aligned}$$

The preceding equation simplifies to

$$\mu_1(1 - p_1q_2) = 1 + p_1 + \mu_2 q_1$$

Similarly, we have that

$$\mu_2(1 - p_2q_1) = 1 + p_2 + \mu_1 q_2$$

Solving these equations gives the solution.

$$\begin{aligned}
h_1 &= E[H|h]p_1 + E[H|m]q_1 \\
&= p_1(E[H|h,h]p_2 + E[H|h,m]q_2) + h_2q_1 \\
&= 2p_1p_2 + (1+h_1)p_1q_2 + h_2q_1
\end{aligned}$$

Similarly, we have that

$$h_2 = 2p_1p_2 + (1+h_2)p_2q_1 + h_1q_2$$

and we solve these equations to find h_1 and h_2.

30. $E[N] = \sum_{j=1}^{m} E[N|X_o = j]p(j) = \sum_{j=1}^{m} \frac{1}{p(j)}p(j) = m$

31. Let L_i denote the length of run i. Conditioning on X, the initial value gives

$$\begin{aligned}
E[L_1] &= E[L_1|X=1]p + E[L_1|X=0](1-p) \\
&= \frac{1}{1-p}p + \frac{1}{p}(1-p) \\
&= \frac{p}{1-p} + \frac{1-p}{p}
\end{aligned}$$

and

$$\begin{aligned}
E[L_2] &= E[L_2|X=1]p + E[L_2|X=0](1-p) \\
&= \frac{1}{p}p + \frac{1}{1-p}(1-p) \\
&= 2
\end{aligned}$$

32. Let T be the number of trials needed for both at least n successes and m failures. Condition on N, the number of successes in the first $n+m$ trials, to obtain

$$E[T] = \sum_{i=0}^{n+m} E[T|N=i]\binom{n+m}{i}p^i(1-p)^{n+m-i}$$

Now use

$$E[T|N=i] = n + m + \frac{n-i}{p}, \quad i \le n$$

$$E[T|N = i] = n + m + \frac{i-n}{1-p}, \quad i > n$$

Let S be the number of trials needed for n successes, and let F be the number needed for m failures. Then $T = \max(S, F)$. Taking expectations of the identity

$$\min(S, F) + \max(S, F) = S + F$$

yields the result

$$E[\min(S, F)] = \frac{n}{p} + \frac{m}{1-p} - E[T]$$

33. Let $I(A)$ equal 1 if the event A occurs and let it equal 0 otherwise.

$$\begin{aligned} E\left[\sum_{i=1}^{T} R_i\right] &= E\left[\sum_{i=1}^{\infty} I(T \geq i) R_i\right] \\ &= \sum_{i=1}^{\infty} E[I(T \geq i) R_i] \\ &= \sum_{i=1}^{\infty} E[I(T \geq i)] E[R_i] \\ &= \sum_{i=1}^{\infty} P\{T \geq i\} E[R_i] \\ &= \sum_{i=1}^{\infty} \beta^{i-1} E[R_i] \\ &= E\left[\sum_{i=1}^{\infty} \beta^{i-1} R_i\right] \end{aligned}$$

34. Let X denote the number of dice that land on six on the first roll.

(a) $$\begin{aligned} m_n &= \sum_{i=0}^{n} E[N|X = i] \binom{n}{i} (1/6)^i (5/6)^{n-i} \\ &= \sum_{i=0}^{n} (1 + m_{n-i}) \binom{n}{i} (1/6)^i (5/6)^{n-i} \\ &= 1 + m_n (5/6)^n + \sum_{i=1}^{n-1} m_{n-i} \binom{n}{i} (1/6)^i \\ &\quad (5/6)^{n-i} \end{aligned}$$

implying that

$$m_n = \frac{1 + \sum_{i=1}^{n-1} m_{n-i} \binom{n}{i} (1/6)^i (5/6)^{n-i}}{1 - (5/6)^n}$$

Starting with $m_0 = 0$ we see that

$$m_1 = \frac{1}{1 - 5/6} = 6$$

$$m_2 = \frac{1 + m_1 (2)(1/6)(5/6)}{1 - (5/6)^2} = 96/11$$

and so on.

(b) Since each die rolled will land on six with probability $1/6$, the total number of dice rolled will equal the number of times one must roll a die until six appears n times. Therefore, $E\left[\sum_{i=1}^{N} X_i\right] = 6n$

35. $$\begin{aligned} np_1 &= E[X_1] \\ &= E[X_1|X_2 = 0](1 - p_2)^n \\ &\quad + E[X_1|X_2 > 0][1 - (1 - p_2)^n] \\ &= n \frac{p_1}{1 - p_2} (1 - p_2)^n \\ &\quad + E[X_1|X_2 > 0][1 - (1 - p_2)^n] \end{aligned}$$

yielding the result

$$E[X_1|X_2 > 0] = \frac{np_1(1 - (1 - p_2)^{n-1})}{1 - (1 - p_2)^n}$$

36. $E[X] = E[X|X \neq 0](1 - p_0) + E[X|X = 0]p_0$

yielding

$$E[X|X \neq 0] = \frac{E[X]}{1 - p_0}$$

Similarly,

$$E[X^2] = E[X^2|X \neq 0](1 - p_0) + E[X^2|X = 0]p_0$$

yielding

$$E[X^2|X \neq 0] = \frac{E[X^2]}{1 - p_0}$$

Hence,

$$\begin{aligned} Var(X|X \neq 0) &= \frac{E[X^2]}{1 - p_0} - \frac{E^2[X]}{(1 - p_0)^2} \\ &= \frac{\mu^2 + \sigma^2}{1 - p_0} - \frac{\mu^2}{(1 - p_0)^2} \end{aligned}$$

37. (a) $E[X] = (2.6 + 3 + 3.4)/3 = 3$

(b) $E[X^2] = [2.6 + 2.6^2 + 3 + 9 + 3.4 + 3.4^2]/3$
$= 12.1067$, and $Var(X) = 3.1067$

38. Let X be the number of successes in the n trials. Now, given that $U = u$, X is binomial with parameters (n, u). As a result,

$$E[X|U] = nU$$

$$E[X^2|U] = n^2U^2 + nU(1 - U) = nU + (n^2 - n)U^2$$

Hence,

$$E[X] = nE[U]$$

$$= E[X^2] = E[nU + (n^2 - n)U^2]$$

$$= n/2 + (n^2 - n)[(1/2)^2 + 1/12]$$

$$= n/6 + n^2/3$$

Hence,

$$Var(X) = n/6 + n^2/12$$

39. Let N denote the number of cycles, and let X be the position of card 1.

(a) $m_n = \frac{1}{n}\sum_{i=1}^{n} E[N|X=i] = \frac{1}{n}\sum_{i=1}^{n}(1 + m_{n-1})$

$$= 1 + \frac{1}{n}\sum_{j=1}^{n-1} m_j$$

(b) $m_1 = 1$

$$m_2 = 1 + \frac{1}{2} = 3/2$$

$$m_3 = 1 + \frac{1}{3}(1 + 3/2) = 1 + 1/2 + 1/3$$
$$= 11/6$$

$$m_4 = 1 + \frac{1}{4}(1 + 3/2 + 11/6) = 25/12$$

(c) $m_n = 1 + 1/2 + 1/3 + \cdots + 1/n$

(d) Using recursion and the induction hypothesis gives

$$m_n = 1 + \frac{1}{n}\sum_{j=1}^{n-1}(1 + \cdots + 1/j)$$

$$= 1 + \frac{1}{n}(n - 1 + (n-2)/2 + (n-3)/3$$
$$+ \cdots + 1/(n-1))$$

$$= 1 + \frac{1}{n}[n + n/2 + \cdots + n/(n-1)$$
$$- (n-1)]$$

$$= 1 + 1/2 + \cdots + 1/n$$

(e) $N = \sum_{i=1}^{n} X_i$

(f) $m_n = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{i \text{ is last of } 1,\ldots,i\}$

$$= \sum_{i=1}^{n} 1/i$$

(g) Yes, knowing for instance that $i + 1$ is the last of all the cards $1, \ldots, i + 1$ to be seen tells us nothing about whether i is the last of $1, \ldots, i$.

(h) $Var(N) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} (1/i)(1 - 1/i)$

40. Let X denote the number of the door chosen, and let N be the total number of days spent in jail.

(a) Conditioning on X, we get

$$E[N] = \sum_{i=1}^{3} E\{N|X = i\}P\{X = 1\}$$

The process restarts each time the prisoner returns to his cell. Therefore,

$$E(N|X = 1) = 2 + E(N)$$

$$E(N|X = 2) = 3 + E(N)$$

$$E(N|X = 3) = 0$$

and

$$E(N) = (.5)(2 + E(N)) + (.3)(3 + E(N))$$
$$+ (.2)(0)$$

or

$$E(N) = 9.5 \text{ days}$$

(b) Let N_i denote the number of additional days the prisoner spends after having initially chosen cell i.

$$E[N] = \frac{1}{3}(2 + E[N_1]) + \frac{1}{3}(3 + E[N_2]) + \frac{1}{3}(0)$$
$$= \frac{5}{3} + \frac{1}{3}(E[N_1] + E[N_2])$$

Now,

$$E[N_1] = \frac{1}{2}(3) + \frac{1}{2}(0) = \frac{3}{2}$$

$$E[N_2] = \frac{1}{2}(2) + \frac{1}{2}(0) = 1$$

and so,

$$E[N] = \frac{5}{3} + \frac{1}{3}\frac{5}{2} = \frac{5}{2}$$

41. Let N denote the number of minutes in the maze. If L is the event the rat chooses its left, and R the event it chooses its right, we have by conditioning on the first direction chosen:

$$E(N) = \frac{1}{2}E(N|L) + \frac{1}{2}E(N|R)$$
$$= \frac{1}{2}\left[\frac{1}{3}(2) + \frac{2}{3}(5 + E(N))\right] + \frac{1}{2}[3 + E(N)]$$
$$= \frac{5}{6}E(N) + \frac{21}{6}$$
$$= 21$$

43. $E[T|\chi_n^2] = \frac{1}{\sqrt{\chi_n^2/n}}E[Z|\chi_n^2] = \frac{1}{\sqrt{\chi_n^2/n}}E[Z] = 0$

$$E[T^2|\chi_n^2] = \frac{n}{\chi_n^2}E[Z^2|\chi_n^2] = \frac{n}{\chi_n^2}E[Z^2] = \frac{n}{\chi_n^2}$$

Hence, $E[T] = 0$, and

$$\begin{aligned}\mathrm{Var}(T) = E[T^2] &= E\left[\frac{n}{\chi_n^2}\right]\\ &= n\int_0^\infty \frac{1}{x}\,\frac{\frac{1}{2}e^{-x/2}(x/2)^{\frac{n}{2}-1}}{\Gamma(n/2)}\,dx\\ &= \frac{n}{2\Gamma(n/2)}\int_0^\infty \frac{1}{2}e^{-x/2}(x/2)^{\frac{n-2}{2}-1}\,dx\\ &= \frac{n\Gamma(n/2-1)}{2\Gamma(n/2)}\\ &= \frac{n}{2(n/2-1)}\\ &= \frac{n}{n-2}\end{aligned}$$

44. From Examples 4d and 4e, mean $= 500$, variance $= E[N]Var(X) + E^2(X)Var(N)$

$$\begin{aligned}&= \frac{10(100)^2}{12} + (50)^2(10)\\ &= 33,333\end{aligned}$$

45. Now

$$E[X_n|X_{n-1}] = 0, \quad Var(X_n|X_{n-1}) = \beta X_{n-1}^2$$

(a) From the above we see that

$$E[X_n] = 0$$

(b) From (a) we have that $Var(x_n) = E[X_n^2]$. Now

$$\begin{aligned}E[X_n^2] &= E\{E[X_n^2|X_{n-1}]\}\\ &= E[\beta X_{n-1}^2]\\ &= \beta E[X_{n-1}^2]\\ &= \beta^2 E[X_{n-2}^2]\\ &\cdot\\ &= \beta^n X_0^2\end{aligned}$$

46. (a) This follows from the identity $Cov(U,V) = E[UV] - E[U]E[V]$ upon noting that

$$E[XY] = E[E[XY|X]] = E[XE[Y|X]],$$
$$E[Y] = E[E[Y|X]]$$

(b) From part (a) we obtain

$$\begin{aligned}Cov(X,Y) &= Cov(a + bX, X)\\ &= b\,Var(X)\end{aligned}$$

47. $$\begin{aligned}E[X^2Y^2|X] &= X^2E[Y^2|X]\\ &\geq X^2(E[Y|X])^2 = X^2\end{aligned}$$

The inequality following since for any random variable U, $E[U^2] \geq (E[U])^2$ and this remains true when conditioning on some other random variable X. Taking expectations of the above shows that

$$E[(XY)^2] \geq E[X^2]$$

As

$$E[XY] = E[E[XY|X]] = E[XE[Y|X]] = E[X]$$

the result follows.

48. $$\begin{aligned}Var(Y_i) &= E[Var(Y_i|X)] + Var(E[Y_i|X])\\ &= E[Var(Y_i|X)] + Var(X)\\ &= E[E[(Y_i - E[Y_i|X])^2|X]] + Var(X)\\ &= E[E[(Y_i - X)^2|X]] + Var(X)\\ &= E[(Y_i - X)^2] + Var(X)\end{aligned}$$

49. Let A be the event that A is the overall winner, and let X be the number of games played. Let Y equal the number of wins for A in the first two games.

$$\begin{aligned}P(A) &= P(A|Y=0)P(Y=0)\\ &\quad + P(A|Y=1)P(Y=1)\\ &\quad + P(A|Y=2)P(Y=2)\\ &= 0 + P(A)2p(1-p) + p^2\end{aligned}$$

Thus,

$$P(A) = \frac{p^2}{1-2p(1-p)}$$

$$\begin{aligned}E[X] &= E[X|Y=0]P(Y=0)\\ &\quad + E[X|Y=1]P(Y=1)\\ &\quad + E[X|Y=2]P(Y=2)\\ &= 2(1-p)^2 + (2 + E[X])2p(1-p) + 2p^2\\ &= 2 + E[X]2p(1-p)\end{aligned}$$

Thus,

$$E[X] = \frac{2}{1-2p(1-p)}$$

50. $$\begin{aligned}P\{N=n\} = \frac{1}{3}\Bigg[&\begin{bmatrix}10\\ n\end{bmatrix}(.3)^n(.7)^{10-n}\\ &+ \begin{bmatrix}10\\ n\end{bmatrix}(.5)^n(.5)^{10-n}\\ &+ \begin{bmatrix}10\\ n\end{bmatrix}(.7)^n(.3)^{10-n}\Bigg]\end{aligned}$$

N is not binomial.

$$E[N] = 3\left[\frac{1}{3}\right] + 5\left[\frac{1}{3}\right] + 7\left[\frac{1}{3}\right] = 5$$

51. Let α be the probability that X is even. Conditioning on the first trial gives

$$\alpha = P(\text{even}|X = 1)p + P(\text{even}|X > 1)(1-p)$$
$$= (1-\alpha)(1-p)$$

Thus,

$$\alpha = \frac{1-p}{2-p}$$

More computationally

$$\alpha = \sum_{n=1}^{\infty} P(X = 2n) = \frac{p}{1-p}\sum_{n=1}^{\infty}(1-p)^{2n}$$
$$= \frac{p}{1-p}\frac{(1-p)^2}{1-(1-p)^2} = \frac{1-p}{2-p}$$

52. $$P\{X + Y < x\} = \int P\{X + Y < x|X = s\} f_X(s)ds$$
$$= \int P\{X + Y < x|X = s\} f_X(s)ds$$
$$= \int P\{Y < x - s|X = s\} f_X(s)ds$$
$$= \int P\{Y < x - s\} f_X(s)ds$$
$$= \int F_Y\{x - s\} f_X(s)ds$$

53. $$P\{X = n\} = \int_0^\infty P\{X = n|\lambda\} e^{-\lambda} d\lambda$$
$$= \int_0^\infty \frac{e^{-\lambda}\lambda^n}{n!} e^{-\lambda} d\lambda$$
$$= \int_0^\infty e^{-2\lambda}\lambda^n \frac{d\lambda}{n!}$$
$$= \int_0^\infty e^{-t} t^n \frac{dt}{n!}\left[\frac{1}{2}\right]^{n+1}$$

The result follows since

$$\int_0^\infty e^{-t} t^n dt = \Gamma(n+1) = n!$$

54. $$P\{N = k\} = \sum_{n=1}^{10}\left[\frac{10-n}{10}\right]^{k-1}\frac{n}{10}\frac{1}{10}$$

N is not geometric. It would be if the coin was reselected after each flip.

56. Let $Y = 1$ if it rains tomorrow, and let $Y = 0$ otherwise.

$$E[X] = E[X|Y = 1]P\{Y = 1\}$$
$$+ E[X|Y = 0]P\{Y = 0\}$$
$$= 9(.6) + 3(.4) = 6.6$$

$$P\{X = 0\} = P\{X = 0|Y = 1\}P\{Y = 1\}$$
$$+ P\{X = 0|Y = 0\}P\{Y = 0\}$$
$$= .6e^{-9} + .4e^{-3}$$

$$E[X^2] = E[X^2|Y = 1]P\{Y = 1\}$$
$$+ E[X^2|Y = 0]P\{Y = 0\}$$
$$= (81 + 9)(.6) + (9 + 3)(.4) = 58.8$$

Therefore,

$$Var(X) = 58.8 - (6.6)^2 = 15.24$$

57. Let X be the number of storms.

$$P\{X \geq 3\} = 1 - P\{X \leq 2\}$$
$$= 1 - \int_0^5 P\{X \leq 2|\Lambda = x\}\frac{1}{5}dx$$
$$= 1 - \int_0^5 [e^{-x} + xe^{-x} + e^{-x}x^2/2]\frac{1}{5}dx$$

58. Conditioning on whether the total number of flips, excluding the j^{th} one, is odd or even shows that the desired probability is 1/2.

59. (a) $$P(A_iA_j) = \sum_{k=0}^{n} P(A_iA_j|N_i = k)\binom{n}{k}p_i^k(1-p_i)^{n-k}$$
$$= \sum_{k=1}^{n} P(A_j|N_i = k)\binom{n}{k}p_i^k(1-p_i)^{n-k}$$
$$= \sum_{k=1}^{n-1}\left[1 - \left(1 - \frac{p_j}{1-p_i}\right)^{n-k}\right]\binom{n}{k}$$
$$\times p_i^k(1-p_i)^{n-k}$$
$$= \sum_{k=1}^{n-1}\binom{n}{k}p_i^k(1-p_i)^{n-k} - \sum_{k=1}^{n-1}$$
$$\times\left(1 - \frac{p_j}{1-p_i}\right)^{n-k}\binom{n}{k}$$
$$\times p_i^k(1-p_i)^{n-k}$$

$$= 1-(1-p_i)^n - p_i^n - \sum_{k=1}^{n-1}\binom{n}{k} \times p_i^k(1-p_i-p_j)^{n-k}$$

$$= 1-(1-p_i)^n - p_i^n - [(1-p_j)^n - (1-p_i-p_j)^n - p_i^n]$$

$$= 1+(1-p_i-p_j)^n - (1-p_i)^n - (1-p_j)^n$$

where the preceding used that conditional on $N_i = k$, each of the other $n-k$ trials independently results in outcome j with probability $\frac{p_j}{1-p_i}$.

(b) $$P(A_iA_j) = \sum_{k=1}^{n} P(A_iA_j|F_i = k)\, p_i(1-p_i)^{k-1} + P(A_iA_j|F_i > n)\,(1-p_i)^n$$

$$= \sum_{k=1}^{n} P(A_j|F_i = k)\, p_i(1-p_i)^{k-1}$$

$$= \sum_{k=1}^{n}\left[1-\left(1-\frac{p_j}{1-p_i}\right)^{k-1}(1-p_j)^{n-k}\right] \times p_i(1-p_i)^{k-1}$$

(c) $$P(A_iA_j) = P(A_i) + P(A_j) - P(A_i \cup A_j)$$

$$= 1-(1-p_i)^n + 1-(1-p_j)^n - [1-(1-p_i-p_j)^n]$$

$$= 1+(1-p_i-p_j)^n - (1-p_i)^n - (1-p_j)^n$$

60. (a) Intuitive that $f(p)$ is increasing in p, since the larger p is the greater is the advantage of going first.

(b) 1

(c) 1/2 since the advantage of going first becomes nil.

(d) Condition on the outcome of the first flip:

$$f(p) = P\{\text{I wins}|h\}p + P\{\text{I wins}|t\}(1-p)$$

$$= p + [1-f(p)](1-p)$$

Therefore,

$$f(p) = \frac{1}{2-p}$$

61. (a) $m_1 = E[X|h]p_1 + E[H|m]q_1 = p_1 + (1+m_2)$ $q_1 = 1 + m_2q_1$.

Similarly, $m_2 = 1 + m_1q_2$. Solving these equations gives

$$m_1 = \frac{1+q_1}{1-q_1q_2}, \quad m_2 = \frac{1+q_2}{1-q_1q_2}$$

(b) $P_1 = p_1 + q_1P_2$

$P_2 = q_2P_1$

implying that

$$P_1 = \frac{p_1}{1-q_1q_2}, \quad P_2 = \frac{p_1q_2}{1-q_1q_2}$$

(c) Let f_i denote the probability that the final hit was by 1 when i shoots first. Conditioning on the outcome of the first shot gives

$$f_1 = p_1P_2 + q_1f_2 \quad \text{and} \quad f_2 = p_2P_1 + q_2f_1$$

Solving these equations gives

$$f_1 = \frac{p_1P_2 + q_1p_2P_1}{1-q_1q_2}$$

(d) and (e) Let B_i denote the event that both hits were by i. Condition on the outcome of the first two shots to obtain

$$P(B_1) = p_1q_2P_1 + q_1q_2P(B_1) \rightarrow P(B_1) = \frac{p_1q_2P_1}{1-q_1q_2}$$

Also,

$$P(B_2) = q_1p_2(1-P_1) + q_1q_2P(B_2) \rightarrow P(B_2) = \frac{q_1p_2(1-P_1)}{1-q_1q_2}$$

(f) $$E[N] = 2p_1p_2 + p_1q_2(2+m_1) + q_1p_2(2+m_1) + q_1q_2(2+E[N])$$

implying that

$$E[N] = \frac{2 + m_1p_1q_2 + m_1q_1p_2}{1-q_1q_2}$$

62. Let W and L stand for the events that player A wins a game and loses a game, respectively. Let $P(A)$ be the probability that A wins, and let $P(C)$ be the probability that C wins, and note that this is equal

to the conditional probability that a player about to compete against the person who won the last round is the overall winner.

$$\begin{aligned} P(A) &= (1/2)P(A|W) + (1/2)P(A|L) \\ &= (1/2)[1/2 + (1/2)P(A|WL)] \\ &\quad + (1/2)(1/2)P(C) \\ &= 1/4 + (1/4)(1/2)P(C) \\ &\quad + (1/4)P(C) = 1/4 + (3/8)P(C) \end{aligned}$$

Also,

$$P(C) = (1/2)P(A|W) = 1/4 + (1/8)P(C)$$

and so

$$P(C) = 2/7, \quad P(A) = 5/14,$$
$$P(B) = P(A) = 5/14$$

63. Let S_i be the event there is only one type i in the final set.

$$\begin{aligned} P\{S_i = 1\} &= \sum_{j=0}^{n-1} P\{S_i = 1|T = j\}P\{T = j\} \\ &= \frac{1}{n}\sum_{j=0}^{n-1} P\{S_i = 1|T = j\} \\ &= \frac{1}{n}\sum_{j=0}^{n-1} \frac{1}{n-j} \end{aligned}$$

The final equality follows because given that there are still $n-j-1$ uncollected types when the first type i is obtained, the probability starting at that point that it will be the last of the set of $n-j$ types consisting of type i along with the $n-j-1$ yet uncollected types to be obtained is, by symmetry, $1/(n-j)$. Hence,

$$E\left[\sum_{i=1}^{n} S_i\right] = nE[S_i] = \sum_{k=1}^{n} \frac{1}{k}$$

64. (a) $P(A) = 5/36 + (31/36)(5/6)P(A)$
$$\to P(A) = 30/61$$

(b) $E[X] = 5/36 + (31/36)[1 + 1/6 + (5/6)$
$$(1 + E[X])] \to E[X] = 402/61$$

(c) Let Y equal 1 if A wins on her first attempt, let it equal 2 if B wins on his first attempt, and let it equal 3 otherwise. Then

$$Var(X|Y = 1) = 0, \quad Var(X|Y = 2) = 0,$$
$$Var(X|Y = 3) = Var(X)$$

Hence,

$$E[Var(X|Y)] = (155/216)Var(X)$$

Also,

$$E[X|Y = 1] = 1, \quad E[X|Y = 2] = 2,$$
$$E[X|Y = 3] = 2 + E[X] = 524/61$$

and so

$$\begin{aligned} Var(E[X|Y]) &= 1^2(5/36) + 2^2(31/216) \\ &\quad + (524/61)^2(155/216) \\ &\quad - (402/61)^2 \approx 10.2345 \end{aligned}$$

Hence, from the conditional variance formula we see that

$$\begin{aligned} Var(X) &\approx z(155/216)Var(X) + 10.2345 \\ &\to Var(X) \approx 36.24 \end{aligned}$$

65. (a) $P\{Y_n = j\} = 1/(n+1), \quad j = 0, \ldots, n$

(b) For $j = 0, \ldots, n-1$

$$\begin{aligned} P\{Y_{n-1} = j\} &= \sum_{i=0}^{n} \frac{1}{n+1} P\{Y_{n-1} = j|Y_n = i\} \\ &= \frac{1}{n+1}(P\{Y_{n-1} = j|Y_n = j\} \\ &\quad + P\{Y_{n-1} = j|Y_n = j+1\}) \\ &= \frac{1}{n+1}(P(\text{last is nonred}|\,j\text{ red}) \\ &\quad + P(\text{last is red}|\,j+1\text{ red}) \\ &= \frac{1}{n+1}\left(\frac{n-j}{n} + \frac{j+1}{n}\right) = 1/n \end{aligned}$$

(c) $P\{Y_k = j\} = 1/(k+1), \quad j = 0, \ldots, k$

(d) For $j = 0, \ldots, k-1$

$$\begin{aligned} P\{Y_{k-1} = j\} &= \sum_{i=0}^{k} P\{Y_{k-1} = j|Y_k = i\} \\ &\quad P\{Y_k = i\} \\ &= \frac{1}{k+1}(P\{Y_{k-1} = j|Y_k = j\} \\ &\quad + P\{Y_{k-1} = j|Y_k = j+1\}) \\ &= \frac{1}{k+1}\left(\frac{k-j}{k} + \frac{j+1}{k}\right) = 1/k \end{aligned}$$

where the second equality follows from the induction hypothesis.

66. (a) $E[G_1 + G_2] = E[G_1] + E[G_2]$
$$= (.6)2 + (.4)3 + (.3)2 + (.7)3 = 5.1$$

(b) Conditioning on the types and using that the sum of independent Poissons is Poisson gives the solution

$$P\{5\} = (.18)e^{-4}4^5/5! + (.54)e^{-5}5^5/5! + (.28)e^{-6}6^5/5!$$

67. A run of j successive heads can occur in the following mutually exclusive ways: (i) either there is a run of j in the first $n-1$ flips, or (ii) there is no j-run in the first $n-j-1$ flips, flip $n-j$ is a tail, and the next j flips are all heads. Consequently, (a) follows. Condition on the time of the first tail:

$$P_j(n) = \sum_{k=1}^{j} P_j(n-k)p^{k-1}(.1-p) + p^j, \quad j \le n$$

68. (a) p^n

(b) After the pairings have been made there are 2^{k-1} players that I could meet in round k. Hence, the probability that players 1 and 2 are scheduled to meet in round k is $2^{k-1}/(2^n - 1)$. Therefore, conditioning on the event R that player I reaches round k gives

$$\begin{aligned} P\{W_2\} &= P\{W_2|R\}p^{k-1} \\ &\quad + P\{W_2|R^c\}(1-p^{k-1}) \\ &= p^{n-1}(1-p)p^{k-1} + p^n(1-p^{k-1}) \end{aligned}$$

69. (a) Let $I(i,j)$ equal 1 if i and j are a pair and 0 otherwise. Then

$$E\left[\sum_{i<j} I(i,j)\right] = \binom{n}{2}\frac{1}{n}\frac{1}{n-1} = 1/2$$

Let X be the size of the cycle containing person 1. Then

$$Q_n = \sum_{i=1}^{n} P\{\text{no pairs}|X=i\}1/n = \frac{1}{n}\sum_{i \ne 2} Q_{n-i}$$

70. (a) Condition on X, the size of the cycle containing person 1, to obtain

$$M_n = \sum_{i=1}^{n} \frac{1}{n}(1 + M_{n-i}) = 1 + \frac{1}{n}\sum_{j=1}^{n-1} M_j$$

(b) Any cycle containing, say, r people is counted only once in the sum since each of the r people contributes $1/r$ to the sum. The identity gives

$$E[C] = nE[1/C_1] = n\sum_{i=1}^{n}(1/i)(1/n) = \sum_{i=1}^{n} 1/i$$

(c) Let p be the desired probability. Condition on X

$$p = \frac{1}{n}\sum_{i=k}^{n}\frac{\binom{n-k}{i-k}}{\binom{n-1}{i-1}}$$

(d) $\dfrac{(n-k)!}{n!}$

72. For $n \ge 2$

$$\begin{aligned} &P\{N > n|U_1 = y\} \\ &\quad = P\{y \ge U_2 \ge U_3 \ge \cdots \ge U_n\} \\ &\quad = P\{U_i \le y, i = 2, \ldots, n\} \\ &\qquad P\{U_2 \ge U_3 \ge \cdots \mathbf{geq} U_n| \\ &\qquad U_i \le y, i = 2, \ldots, n\} \\ &\quad = y^{n-1}/(n-1)! \end{aligned}$$

$$\begin{aligned} E[N|U_1 = y] &= \sum_{n=0}^{\infty} P\{N > n|U_1 = y\} \\ &= 2 + \sum_{n=2}^{\infty} y^{n-1}/(n-1)! = 1 + e^y \end{aligned}$$

Also,

$$\begin{aligned} P\{M > n|U_1 = 1-y\} &= P\{M(y) > n-1\} \\ &= y^{n-1}/(n-1)! \end{aligned}$$

73. Condition on the value of the sum prior to going over 100. In all cases the most likely value is 101. (For instance, if this sum is 98 then the final sum is equally likely to be either 101, 102, 103, or 104. If the sum prior to going over is 95 then the final sum is 101 with certainty.)

74. Condition on whether or not component 3 works. Now

$$\begin{aligned} &P\{\text{system works}|3 \text{ works}\} \\ &\quad = P\{\text{either 1 or 2 works}\}P\{\text{either 4 or 5 works}\} \\ &\quad = (p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5) \end{aligned}$$

Also,

$$\begin{aligned} &P\{\text{system works}|3 \text{ is failed}\} \\ &\quad = P\{\text{1 and 4 both work, or 2 and 5 both work}\} \\ &\quad = p_1p_4 - p_2p_5 - p_1p_4p_2p_5 \end{aligned}$$

Therefore, we see that

$P\{\text{system works}\}$
$$= p_3(p_1 + p_2 - p_1p_2)(p_4 + p_5 - p_4p_5) + (1 - p_3)(p_1p_4 + p_2p_5 - p_1p_4p_2p_5)$$

75. (a) Since A receives more votes than B (since $a > a$) it follows that if A is not always leading then they will be tied at some point.

(b) Consider any outcome in which A receives the first vote and they are eventually tied, say $a, a, b, a, b, a, b, b \ldots$. We can correspond this sequence to one that takes the part of the sequence until they are tied in the reverse order. That is, we correspond the above to the sequence $b, b, a, b, a, b, a, a \ldots$ where the remainder of the sequence is exactly as in the original. Note that this latter sequence is one in which B is initially ahead and then they are tied. As it is easy to see that this correspondence is one to one, part (b) follows.

(c) Now,

$P\{B$ receives first vote and they are eventually tied$\}$
$= P\{B \text{ receives first vote}\} = n/(n + m)$

Therefore, by part (b) we see that

$P\{\text{eventually tied}\} = 2n/(n + m)$

and the result follows from part (a).

76. By the formula given in the text after the ballot problem we have that the desired probability is

$$\frac{1}{3}\binom{15}{5}(18/38)^{10}(20/38)^5$$

77. We will prove it when X and Y are discrete.

(a) This part follows from (b) by taking $g(x, y) = xy$.

(b) $$E[g(X, Y)|Y = \overline{y}] = \sum_y \sum_x g(x, y) P\{X = x, Y = y|Y = \overline{y}\}$$

Now,

$$P\{X = x, Y = y|Y = \overline{y}\} = \begin{cases} 0, & \text{if } y \neq \overline{y} \\ P\{X = x,\ Y = \overline{y}\}, & \text{if } y = \overline{y} \end{cases}$$

So,

$$E[g(X, Y)|Y = \overline{y}] = \sum_k g(x, \overline{y}) P\{X = x|Y = \overline{y}\} = E[g(x, \overline{y})|Y = \overline{y}$$

(c) $$E[XY] = E[E[XY|Y]] = E[YE[X|Y]] \qquad \text{by (a)}$$

78. Let $Q_{n,m}$ denote the probability that A is never behind, and $P_{n,m}$ the probability that A is always ahead. Computing $P_{n,m}$ by conditioning on the first vote received yields

$$P_{n,m} = \frac{n}{n + m} Q_{n-1,m}$$

But as $P_{n,m} = \dfrac{n - m}{n + m}$, we have

$$Q_{n-1,m} = \frac{n + m}{n}\frac{n - m}{n + m} = \frac{n - m}{n}$$

and so the desired probability is

$$Q_{n,m} = \frac{n + 1 - m}{n + 1}$$

This also can be solved by conditioning on who obtains the last vote. This results in the recursion

$$Q_{n,m} = \frac{n}{n + m} Q_{n-1,m} + \frac{m}{n + m} Q_{n,m} - 1$$

which can be solved to yield

$$Q_{n,m} = \frac{n + 1 - m}{n + 1}$$

79. Let us suppose we take a picture of the urn before each removal of a ball. If at the end of the experiment we look at these pictures in reverse order (i.e., look at the last taken picture first), we will see a set of balls increasing at each picture. The set of balls seen in this fashion always will have more white balls than black balls if and only if in the original experiment there were always more white than black balls left in the urn. Therefore, these two events must have same probability, i.e., $n - m/n + m$ by the ballot problem.

80. Condition on the total number of heads and then use the result of the ballot problem. Let p denote the desired probability, and let j be the smallest integer that is at least $n/2$.

$$p = \sum_{i=j}^{n} \binom{n}{i} p^i (1 - p)^{n-i} \frac{2i - n}{n}$$

81. (a) $f(x) = E[N] = \int_0^1 E[N|X_1 = y]dy$

$$E[N|X_1 = y] = \begin{cases} 1 & \text{if } y < x \\ 1 + f(y) & \text{if } y > x \end{cases}$$

Hence,

$$f(x) = 1 + \int_x^1 f(y)dy$$

(b) $f'(x) = -f(x)$

(c) $f(x) = ce^{-x}$. Since $f(1) = 1$, we obtain that $c = e$, and so $f(x) = e^{1-x}$.

(d) $P\{N > n\} = P\{x < X_1 < X_2 < \cdots < X_n\} = (1-x)^n/n!$ since in order for the above event to occur all of the n random variables must exceed x (and the probability of this is $(1-x)^n$), and then among all of the $n!$ equally likely orderings of this variables the one in which they are increasing must occur.

(e) $$E[N] = \sum_{n=0}^{\infty} P\{N > n\}$$
$$= \sum_n (1-x)^n/n! = e^{1-x}$$

82. (a) Let A_i denote the event that X_i is the k^{th} largest of $X_1, \ldots, X_i$. It is easy to see that these are independent events and $P(A_i) = 1/i$.

$$P\{N_k = n\} = P(A_k^c A_{k+1}^c \cdots A_{n-1}^c A_n)$$
$$= \frac{k-1}{k}\frac{k}{k+1}\cdots\frac{n-2}{n-1}\frac{1}{n}$$
$$= \frac{k-1}{n(n-1)}$$

(b) Since knowledge of the set of values $\{X_1, \ldots, X_n\}$ gives us no information about the order of these random variables it follows that given $N_k = n$, the conditional distribution of X_{N_k} is the same as the distribution of the k^{th} largest of n random variables having distribution F. Hence,

$$f_{X_{N_k}}(x) = \sum_{n=k}^{\infty} \frac{k-1}{n(n-1)} \frac{n!}{(n-k)!(k-1)!} \times (F(x))^{n-k}(\overline{F}(x))^{k-1} f(x)$$

Now make the change of variable $i = n - k$. (c) Follow the hint. (d) It follows from (b) and (c) that $f_{X_{N_k}}(x) = f(x)$.

83. Let I_j equal 1 if ball j is drawn before ball i and let it equal 0 otherwise. Then the random variable of interest is $\sum_{j \neq i} I_j$. Now, by considering the first time that either i or j is withdrawn we see that $P\{j \text{ before } i\} = w_j/(w_i + w_j)$. Hence,

$$E\left[\sum_{j \neq i} I_j\right] = \sum_{j \neq i} \frac{w_j}{w_i + w_j}$$

84. We have

E[Position of element requested at time t]

$$= \sum_{i=i}^{n} E[\text{Position at time } t \mid e_i \text{ selected}] P_i$$
$$= \sum_{i=1}^{n} E[\text{Position of } e_i \text{ at time } t] P_i$$

$$\text{with } I_j = \begin{cases} 1, & \text{if } e_j \text{ precedes } e_i \text{ at time } t \\ 0, & \text{otherwise} \end{cases}$$

We have

$$\text{Position of } e_i \text{ at time } t = 1 + \sum_{j \neq i} I_j$$

and so,

E[Position of e_i at time t]

$$= 1 + \sum_{j \neq i} E(I_j)$$
$$= 1 + \sum_{j \neq i} P\{e_j \text{ precedes } e_i \text{ at time } t\}$$

Given that a request has been made for either e_i or e_j, the probability that the most recent one was for e_j is $P_j/(P_i + P_j)$. Therefore,

$P\{e_j$ precedes e_i at time $t | e_i$ or e_j was requested$\}$

$$= \frac{P_j}{P_i + P_j}$$

On the other hand,

$P\{e_j$ precedes e_i at time t | neither was ever requested$\}$

$$= \frac{1}{2}$$

As

$P\{$Neither e_i or e_j was ever requested by time $t\}$

$$= (1 - P_i - P_j)^{t-1}$$

we have

E[Position of e_i at time t]

$$= 1 + \sum_{j \neq i} \left[\frac{1}{2}(1 - P_i - P_j)^{t-1} + \frac{P_j}{P_j + P_i}(1 - (1 - P_i - P_j)^{t-1})\right]$$

and

E[Position of element requested at t]

$$= \sum P_j E[\text{Position of } e_i \text{ at time } t]$$

85. Consider the following ordering:

$e_1, e_2, \ldots, e_{l-1}, i, j, e_{l+1}, \ldots, e_n$ where $P_i < P_j$

We will show that we can do better by interchanging the order of i and j, i.e., by taking $e_1, e_2, \ldots, e_{l-1}, j, i, e_{l+2}, \ldots, e_n$. For the first ordering, the expected position of the element requested is

$$E_{i,j} = P_{e_1} + 2P_{e_2} + \cdots + (l-1)P_{e_{l-1}} + lp_i + (l+1)P_j + (l+2)P_{e_{l+2}} + \cdots$$

Therefore,

$$E_{i,j} - E_{j,i} = l(P_i - P_j) + (l+1)(P_j - P_i) = P_j - P_i > 0$$

and so the second ordering is better. This shows that every ordering for which the probabilities are not in decreasing order is not optimal in the sense that we can do better. Since there are only a finite number of possible orderings, the ordering for which $p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n$ is optimum.

87. (a) This can be proved by induction on m. It is obvious when $m = 1$ and then by fixing the value of x_1 and using the induction hypothesis, we see that there are $\sum_{i=0}^{n} \begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix}$ such solutions. As $\begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix}$ equals the number of ways of choosing $m-1$ items from a set of size $n + m - 1$ under the constraint that the lowest numbered item selected is number $i + 1$ (that is, none of $1, \ldots, i$ are selected where $i + 1$ is), we see that

$$\sum_{i=0}^{n} \begin{bmatrix} n-i+m-2 \\ m-2 \end{bmatrix} = \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}$$

It also can be proven by noting that each solution corresponds in a one-to-one fashion with a permutation of n ones and $(m-1)$ zeros. The correspondence being that x_1 equals the number of ones to the left of the first zero, x_2 the number of ones between the first and second zeros, and so on. As there are $(n+m-1)!/n!(m-1)!$ such permutations, the result follows.

(b) The number of positive solutions of $x_1 + \cdots + x_m = n$ is equal to the number of nonnegative solutions of $y_1 + \cdots + y_m = n - m$, and thus there are $\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$ such solutions.

(c) If we fix a set of k of the x_i and require them to be the only zeros, then there are by (b) (with m replaced by $m-k$) $\begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix}$ such solutions. Hence, there are $\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix}$ outcomes such that exactly k of the X_i are equal to zero, and so the desired probability is $\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n-1 \\ m-k-1 \end{bmatrix} \Big/ \begin{bmatrix} n+m-1 \\ m-1 \end{bmatrix}$.

88. (a) Since the random variables $U, X_1, \ldots, X_n$ are all independent and identically distributed it follows that U is equally likely to be the i^{th} smallest for each $i + 1, \ldots, n + 1$. Therefore,

$$P\{X = i\} = P\{U \text{ is the } (i+1)^{st} \text{ smallest}\} = 1/(n+1)$$

(b) Given U, each X_i is less than U with probability U, and so X is binomial with parameters n, U. That is, given that $U < p$, X is binomial with parameters n, p. Since U is uniform on $(0, 1)$ this is exactly the scenario in Section 6.3.

89. Condition on the value of I_n. This gives

$$\begin{aligned} P_n(K) &= P\left\{\sum_{j=1}^{n} jI_j \leq K | I_n = 1\right\} 1/2 \\ &\quad + P\left\{\sum_{j=1}^{n} jI_j \leq K | I_n = 0\right\} 1/2 \\ &= P\left\{\sum_{j=1}^{n-1} jI_j + n \leq K\right\} 1/2 \\ &\quad + P\left\{\sum_{j=1}^{n-1} jI_j \leq K\right\} 1/2 \\ &= [P_{n-1}(k-n) + P_{n-1}(K)]/2 \end{aligned}$$

90. (a) $\dfrac{1}{e^{-5}5^2/2! \cdot 5e^{-5} \cdot e^{-5}}$

(b) $\dfrac{1}{e^{-5}5^2/2! \cdot 5e^{-5} \cdot e^{-5} \cdot e^{-5}5^2/2!} + \dfrac{1}{e^{-5}5^2/2!}$

91. $\dfrac{1}{p^5(1-p)^3} + \dfrac{1}{p^2(1-p)} + \dfrac{1}{p}$

92. Let X denote the amount of money Josh picks up when he spots a coin. Then

$$E[X] = (5 + 10 + 25)/4 = 10,$$

$$E[X^2] = (25 + 100 + 625)/4 = 750/4$$

Therefore, the amount he picks up on his way to work is a compound Poisson random variable with mean $10 \cdot 6 = 60$ and variance $6 \cdot 750/4 = 1125$. Because the number of pickup coins that Josh spots is Poisson with mean $6(3/4) = 4.5$, we can also view the amount picked up as a compound Poisson random variable $S = \sum_{i=1}^{N} X_i$ where N is Poisson with mean 4.5, and (with 5 cents as the unit of measurement) the X_i are equally likely to be 1, 2, 3. Either use the recursion developed in the text or condition on the number of pickups to determine $P(S = 5)$. Using the latter approach, with $P(N = i) = e^{-4.5}(4.5)^i/i!$, gives

$$P(S = 5) = (1/3)P(N = 1) + 3(1/3)^3P(N = 3) + 4(1/3)^4P(N = 4) + 5(1/3)^5P(N = 5)$$

94. Using that $E[N] = rw/(w + b)$ yields

$$P\{M - 1 = n\}$$

$$= \frac{(n + 1)P\{N = n + 1\}}{E[N]}$$

$$= \frac{(n + 1)\binom{w}{n+1}\binom{b}{r-n-1}(w + b)}{rw\binom{w+b}{r}}$$

Using that

$$\frac{(n + 1)\binom{w}{n+1}}{w} = \binom{w-1}{n}\frac{w + b}{r\binom{w+b}{r}} = \frac{1}{\binom{w+b-1}{r-1}}$$

shows that

$$P\{M - 1 = n\} = \frac{\binom{w-1}{n}\binom{b}{r-n-1}}{\binom{w+b-1}{r-1}}$$

$$P_{w,r}(k) = \frac{rw}{k(w + b)}\sum_{i=1}^{k} i\alpha_i P_{w-1,r-1}(k - i)$$

When $k = 1$

$$P_{w,r}(1) = \frac{rw}{w + b}\alpha_1\frac{\binom{b}{r-1}}{\binom{w+b-1}{r-1}}$$

95. With $\alpha = P(S_n < 0 \text{ for all } n > 0)$, we have

$$-E[X] = \alpha = p_{-1}\beta$$

96. With $P_j = e^{-\lambda}\lambda^j/j!$, we have that N, the number of children in the family of a randomly chosen family is

$$P(N = j) = \frac{jP_j}{\lambda} = e^{-\lambda}\lambda^{j-1}/(j - 1)!, \quad j > 0$$

Hence,

$$P(N - 1 = k) = e^{-\lambda}\lambda^k/k!, \quad k \geq 0$$

Markov Chains

Exercises

*1. Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state $i, i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the nth step. Explain why $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain and calculate its transition probability matrix.

2. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov chain. How many states are needed?

3. In Exercise 2, suppose that if it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine **P** for this Markov chain.

*4. Consider a process $\{X_n, n = 0, 1, \ldots\}$, which takes on the values 0, 1, or 2. Suppose

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = \begin{cases} P_{ij}^{\mathrm{I}}, & \text{when } n \text{ is even} \\ P_{ij}^{\mathrm{II}}, & \text{when } n \text{ is odd} \end{cases}$$

where $\sum_{j=0}^{2} P_{ij}^{\mathrm{I}} = \sum_{j=0}^{2} P_{ij}^{\mathrm{II}} = 1$, $i = 0, 1, 2$. Is $\{X_n, n \geqslant 0\}$ a Markov chain? If not, then show how, by enlarging the state space, we may transform it into a Markov chain.

5. A Markov chain $\{X_n, n \geqslant 0\}$ with states 0, 1, 2, has the transition probability matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

If $P\{X_0 = 0\} = P\{X_0 = 1\} = \frac{1}{4}$, find $E[X_3]$.

6. Let the transition probability matrix of a two-state Markov chain be given, as in Example 4.2, by

$$\mathbf{P} = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}$$

Show by mathematical induction that

$$\mathbf{P}^{(n)} = \begin{Vmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{Vmatrix}$$

7. In Example 4.4 suppose that it has rained neither yesterday nor the day before yesterday. What is the probability that it will rain tomorrow?

8. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tails, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

9. If in Example 4.10 we had defined X_n to equal 1 if the nth selection were red and to equal 0 if it were blue, would $X_n, n \geqslant 1$ be a Markov chain?

10. In Example 4.3, Gary is currently in a cheerful mood. What is the probability that he is not in a glum mood on any of the following three days?

11. In Example 4.3, Gary was in a glum mood four days ago. Given that he hasn't felt cheerful in a week, what is the probability he is feeling glum today?

12. For a Markov chain $\{X_n, n \geqslant 0\}$ with transition probabilities $P_{i,j}$, consider the conditional probability that $X_n = m$ given that the chain started at time 0 in state i and has not yet entered state r by time n, where r is a specified state not equal to either i or m. We are interested in whether this conditional probability is equal to the n stage transition probability of a Markov chain whose state space does not include state r and whose transition probabilities are

$$Q_{i,j} = \frac{P_{i,j}}{1 - P_{i,r}}, \quad i, j \neq r$$

Either prove the equality

$$P\{X_n = m | X_0 = i, X_k \neq r, k = 1, \ldots, n\} = Q_{i,m}^n$$

or construct a counterexample.

13. Let $\mathbf{P}$ be the transition probability matrix of a Markov chain. Argue that if for some positive integer r, $\mathbf{P}^r$ has all positive entries, then so does $\mathbf{P}^n$, for all integers $n \geqslant r$.

14. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

$$\mathbf{P}_1 = \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix}, \qquad \mathbf{P}_2 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix},$$

$$\mathbf{P}_3 = \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix}, \quad \mathbf{P}_4 = \begin{Vmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

15. Prove that if the number of states in a Markov chain is M, and if state j can be reached from state i, then it can be reached in M steps or less.

*16. Show that if state i is recurrent and state i does not communicate with state j, then $P_{ij} = 0$. This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a *closed* class.

17. For the random walk of Example 4.18 use the strong law of large numbers to give another proof that the Markov chain is transient when $p \neq \frac{1}{2}$.

Hint: Note that the state at time n can be written as $\sum_{i=1}^{n} Y_i$ where the Y_is are independent and $P\{Y_i = 1\} = p = 1 - P\{Y_i = -1\}$. Argue that if $p > \frac{1}{2}$, then, by the strong law of large numbers, $\sum_1^n Y_i \to \infty$ as $n \to \infty$ and hence the initial state 0 can be visited only finitely often, and hence must be transient. A similar argument holds when $p < \frac{1}{2}$.

18. Coin 1 comes up heads with probability 0.6 and coin 2 with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.
    (a) What proportion of flips use coin 1?
    (b) If we start the process with coin 1 what is the probability that coin 2 is used on the fifth flip?
19. For Example 4.4, calculate the proportion of days that it rains.
20. A transition probability matrix $\mathbf{P}$ is said to be doubly stochastic if the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \quad \text{for all } j$$

If such a chain is irreducible and aperiodic and consists of $M+1$ states $0, 1, \ldots, M$, show that the limiting probabilities are given by

$$\pi_j = \frac{1}{M+1}, \quad j = 0, 1, \ldots, M$$

*21. A DNA nucleotide has any of four values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability $1 - 3\alpha$, and if it does change then it is equally likely to change to any of the other three values, for some $0 < \alpha < \frac{1}{3}$.
    (a) Show that $P^n_{1,1} = \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n$.
    (b) What is the long-run proportion of time the chain is in each state?
22. Let Y_n be the sum of n independent rolls of a fair die. Find

$$\lim_{n\to\infty} P\{Y_n \text{ is a multiple of } 13\}$$

Hint: Define an appropriate Markov chain and apply the results of Exercise 20.
23. In a good weather year the number of storms is Poisson distributed with mean 1; in a bad year it is Poisson distributed with mean 3. Suppose that any year's weather conditions depends on past years only through the previous year's condition. Suppose that a good year is equally likely to be followed by either a good or a bad year, and that a bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year—call it year 0—was a good year.
    (a) Find the expected total number of storms in the next two years (that is, in years 1 and 2).
    (b) Find the probability there are no storms in year 3.
    (c) Find the long-run average number of storms per year.
24. Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is

the same as that of the ball previously selected and is then returned to that urn. In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?

25. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of k pairs of running shoes, what proportion of the time does he run barefooted?

26. Consider the following approach to shuffling a deck of n cards. Starting with any initial ordering of the cards, one of the numbers $1, 2, \ldots, n$ is randomly chosen in such a manner that each one is equally likely to be selected. If number i is chosen, then we take the card that is in position i and put it on top of the deck—that is, we put that card in position 1. We then repeatedly perform the same operation. Show that, in the limit, the deck is perfectly shuffled in the sense that the resultant ordering is equally likely to be any of the $n!$ possible orderings.

*27. Each individual in a population of size N is, in each period, either active or inactive. If an individual is active in a period then, independent of all else, that individual will be active in the next period with probability α. Similarly, if an individual is inactive in a period then, independent of all else, that individual will be inactive in the next period with probability β. Let X_n denote the number of individuals that are active in period n.
(a) Argue that $X_n, n \geqslant 0$ is a Markov chain.
(b) Find $E[X_n | X_0 = i]$.
(c) Derive an expression for its transition probabilities.
(d) Find the long-run proportion of time that exactly j people are active.

Hint for (d): Consider first the case where $N = 1$.

28. Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game, it wins its next game with probability 0.3. If the team wins a game, then it has dinner together with probability 0.7, whereas if the team loses then it has dinner together with probability 0.2. What proportion of games result in a team dinner?

29. An organization has N employees where N is a large number. Each employee has one of three possible job classifications and changes classifications (independently) according to a Markov chain with transition probabilities

$$\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

What percentage of employees are in each classification?

30. Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

31. A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally

likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. In the long run, what proportion of days are sunny? What proportion are cloudy?

*32. Each of two switches is either on or off during a day. On day n, each switch will independently be on with probability

$$[1 + \text{ number of on switches during day } n-1]/4$$

For instance, if both switches are on during day $n-1$, then each will independently be on during day n with probability 3/4. What fraction of days are both switches on? What fraction are both off?

33. A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done well or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type $i, i = 1, 2, 3$?

34. A flea moves around the vertices of a triangle in the following manner: Whenever it is at vertex i it moves to its clockwise neighbor vertex with probability p_i and to the counterclockwise neighbor with probability $q_i = 1 - p_i$, $i = 1, 2, 3$.
    (a) Find the proportion of time that the flea is at each of the vertices.
    (b) How often does the flea make a counterclockwise move that is then followed by five consecutive clockwise moves?

35. Consider a Markov chain with states 0, 1, 2, 3, 4. Suppose $P_{0,4} = 1$; and suppose that when the chain is in state $i, i > 0$, the next state is equally likely to be any of the states $0, 1, \ldots, i-1$. Find the limiting probabilities of this Markov chain.

36. The state of a process changes daily according to a two-state Markov chain. If the process is in state i during one day, then it is in state j the following day with probability $P_{i,j}$, where

$$P_{0,0} = 0.4, \quad P_{0,1} = 0.6, \quad P_{1,0} = 0.2, \quad P_{1,1} = 0.8$$

Every day a message is sent. If the state of the Markov chain that day is i then the message sent is "good" with probability p_i and is "bad" with probability $q_i = 1 - p_i$, $i = 0, 1$
    (a) If the process is in state 0 on Monday, what is the probability that a good message is sent on Tuesday?
    (b) If the process is in state 0 on Monday, what is the probability that a good message is sent on Friday?
    (c) In the long run, what proportion of messages are good?
    (d) Let Y_n equal 1 if a good message is sent on day n and let it equal 2 otherwise. Is $\{Y_n, n \geqslant 1\}$ a Markov chain? If so, give its transition probability matrix. If not, briefly explain why not.

37. Show that the stationary probabilities for the Markov chain having transition probabilities $P_{i,j}$ are also the stationary probabilities for the Markov chain whose transition probabilities $Q_{i,j}$ are given by

$$Q_{i,j} = P^k_{i,j}$$

for any specified positive integer k.

38. Recall that state i is said to be positive recurrent if $m_{i,i} < \infty$, where $m_{i,i}$ is the expected number of transitions until the Markov chain, starting in state i, makes a transition back into that state. Because π_i, the long-run proportion of time the Markov chain, starting in state i, spends in state i, satisfies

$$\pi_i = \frac{1}{m_{i,i}}$$

it follows that state i is positive recurrent if and only if $\pi_i > 0$. Suppose that state i is positive recurrent and that state i communicates with state j. Show that state j is also positive recurrent by arguing that there is an integer n such that

$$\pi_j \geqslant \pi_i P^n_{i,j} > 0$$

39. Recall that a recurrent state that is not positive recurrent is called null recurrent. Use the result of Exercise 38 to prove that null recurrence is a class property. That is, if state i is null recurrent and state i communicates with state j, show that state j is also null recurrent.

40. It follows from the argument made in Exercise 38 that state i is null recurrent if it is recurrent and $\pi_i = 0$. Consider the one-dimensional symmetric random walk of Example 4.18.
    (a) Argue that $\pi_i = \pi_0$ for all i.
    (b) Argue that all states are null recurrent.

*41. Let π_i denote the long-run proportion of time a given irreducible Markov chain is in state i.
    (a) Explain why π_i is also the proportion of transitions that are into state i as well as being the proportion of transitions that are from state i.
    (b) $\pi_i P_{ij}$ represents the proportion of transitions that satisfy what property?
    (c) $\sum_i \pi_i P_{ij}$ represent the proportion of transitions that satisfy what property?
    (d) Using the preceding explain why

$$\pi_j = \sum_i \pi_i P_{ij}$$

42. Let A be a set of states, and let A^c be the remaining states.
    (a) What is the interpretation of

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij}?$$

(b) What is the interpretation of

$$\sum_{i \in A^c} \sum_{j \in A} \pi_i P_{ij}?$$

(c) Explain the identity

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij} = \sum_{i \in A^c} \sum_{j \in A} \pi_i P_{ij}$$

43. Each day, one of n possible elements is requested, the ith one with probability $P_i, i \geqslant 1, \sum_1^n P_i = 1$. These elements are at all times arranged in an ordered list that is revised as follows: The element selected is moved to the front of the list with the relative positions of all the other elements remaining unchanged. Define the state at any time to be the list ordering at that time and note that there are $n!$ possible states.
(a) Argue that the preceding is a Markov chain.
(b) For any state $i_1, \ldots, i_n$ (which is a permutation of $1, 2, \ldots, n$), let $\pi(i_1, \ldots, i_n)$ denote the limiting probability. In order for the state to be $i_1, \ldots, i_n$, it is necessary for the last request to be for i_1, the last non-i_1 request for i_2, the last non-i_1 or i_2 request for i_3, and so on. Hence, it appears intuitive that

$$\pi(i_1, \ldots, i_n) = P_{i_1} \frac{P_{i_2}}{1 - P_{i_1}} \frac{P_{i_3}}{1 - P_{i_1} - P_{i_2}} \cdots \frac{P_{i_{n-1}}}{1 - P_{i_1} - \cdots - P_{i_{n-2}}}$$

Verify when $n = 3$ that the preceding are indeed the limiting probabilities.

44. Suppose that a population consists of a fixed number, say, m, of genes in any generation. Each gene is one of two possible genetic types. If exactly i (of the m) genes of any generation are of type 1, then the next generation will have j type 1 (and $m - j$ type 2) genes with probability

$$\binom{m}{j} \left(\frac{i}{m}\right)^j \left(\frac{m-i}{m}\right)^{m-j}, \quad j = 0, 1, \ldots, m$$

Let X_n denote the number of type 1 genes in the nth generation, and assume that $X_0 = i$.
(a) Find $E[X_n]$.
(b) What is the probability that eventually all the genes will be type 1?

45. Consider an irreducible finite Markov chain with states $0, 1, \ldots, N$.
(a) Starting in state i, what is the probability the process will ever visit state j? Explain!
(b) Let $x_i = P\{\text{visit state } N \text{ before state } 0 | \text{start in } i\}$. Compute a set of linear equations that the x_i satisfy, $i = 0, 1, \ldots, N$.
(c) If $\sum_j j P_{ij} = i$ for $i = 1, \ldots, N - 1$, show that $x_i = i/N$ is a solution to the equations in part (b).

46. An individual possesses r umbrellas that he employs in going from his home to office, and vice versa. If he is at home (the office) at the beginning (end) of a day and it is raining, then he will take an umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (end) of a day with probability p.
   (a) Define a Markov chain with $r+1$ states, which will help us to determine the proportion of time that our man gets wet. (*Note*: He gets wet if it is raining, and all umbrellas are at his other location.)
   (b) Show that the limiting probabilities are given by

$$\pi_i = \begin{cases} \dfrac{q}{r+q}, & \text{if } i = 0 \\ \\ \dfrac{1}{r+q}, & \text{if } i = 1, \dots, r \end{cases} \qquad \text{where } q = 1 - p$$

   (c) What fraction of time does our man get wet?
   (d) When $r = 3$, what value of p maximizes the fraction of time he gets wet

*47. Let $\{X_n, n \geqslant 0\}$ denote an ergodic Markov chain with limiting probabilities π_i. Define the process $\{Y_n, n \geqslant 1\}$ by $Y_n = (X_{n-1}, X_n)$. That is, Y_n keeps track of the last two states of the original chain. Is $\{Y_n, n \geqslant 1\}$ a Markov chain? If so, determine its transition probabilities and find

$$\lim_{n\to\infty} P\{Y_n = (i, j)\}$$

48. Consider a Markov chain in steady state. Say that a k length run of zeroes ends at time m if

$$X_{m-k-1} \neq 0, \quad X_{m-k} = X_{m-k+1} = \ldots = X_{m-1} = 0, X_m \neq 0$$

Show that the probability of this event is $\pi_0(P_{0,0})^{k-1}(1 - P_{0,0})^2$, where π_0 is the limiting probability of state 0.

49. Let $P^{(1)}$ and $P^{(2)}$ denote transition probability matrices for ergodic Markov chains having the same state space. Let π^1 and π^2 denote the stationary (limiting) probability vectors for the two chains. Consider a process defined as follows:
   (a) $X_0 = 1$. A coin is then flipped and if it comes up heads, then the remaining states $X_1, \ldots$ are obtained from the transition probability matrix $P^{(1)}$ and if tails from the matrix $P^{(2)}$. Is $\{X_n, n \geqslant 0\}$ a Markov chain? If $p = P\{\text{coin comes up heads}\}$, what is $\lim_{n\to\infty} P(X_n = i)$?
   (b) $X_0 = 1$. At each stage the coin is flipped and if it comes up heads, then the next state is chosen according to $P^{(1)}$ and if tails comes up, then it is chosen according to $P^{(2)}$. In this case do the successive states constitute a Markov chain? If so, determine the transition probabilities. Show by a counterexample that the limiting probabilities are not the same as in part (a).

50. In Exercise 8, if today's flip lands heads, what is the expected number of additional flips needed until the pattern t, t, h, t, h, t, t occurs?

51. In Example 4.3, Gary is in a cheerful mood today. Find the expected number of days until he has been glum for three consecutive days.

52. A taxi driver provides service in two zones of a city. Fares picked up in zone A will have destinations in zone A with probability 0.6 or in zone B with probability 0.4. Fares picked up in zone B will have destinations in zone A with probability 0.3 or in zone B with probability 0.7. The driver's expected profit for a trip entirely in zone A is 6; for a trip entirely in zone B is 8; and for a trip that involves both zones is 12. Find the taxi driver's average profit per trip.

53. Find the average premium received per policyholder of the insurance company of Example 4.27 if $\lambda = 1/4$ for one-third of its clients, and $\lambda = 1/2$ for two-thirds of its clients.

54. Consider the Ehrenfest urn model in which M molecules are distributed between two urns, and at each time point one of the molecules is chosen at random and is then removed from its urn and placed in the other one. Let X_n denote the number of molecules in urn 1 after the nth switch and let $\mu_n = E[X_n]$. Show that
(a) $\mu_{n+1} = 1 + (1 - 2/M)\mu_n$.
(b) Use (a) to prove that

$$\mu_n = \frac{M}{2} + \left(\frac{M-2}{M}\right)^n \left(E[X_0] - \frac{M}{2}\right)$$

55. Consider a population of individuals each of whom possesses two genes that can be either type A or type a. Suppose that in outward appearance type A is dominant and type a is recessive. (That is, an individual will have only the outward characteristics of the recessive gene if its pair is aa.) Suppose that the population has stabilized, and the percentages of individuals having respective gene pairs AA, aa, and Aa are p, q, and r. Call an individual dominant or recessive depending on the outward characteristics it exhibits. Let S_{11} denote the probability that an offspring of two dominant parents will be recessive; and let S_{10} denote the probability that the offspring of one dominant and one recessive parent will be recessive. Compute S_{11} and S_{10} to show that $S_{11} = S_{10}^2$. (The quantities S_{10} and S_{11} are known in the genetics literature as *Snyder's ratios*.)

56. Suppose that on each play of the game a gambler either wins 1 with probability p or loses 1 with probability $1 - p$. The gambler continues betting until she or he is either up n or down m. What is the probability that the gambler quits a winner?

57. A particle moves among $n + 1$ vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability p or the counterclockwise direction with probability $q = 1 - p$. Starting at a specified state, call it state 0, let T be the time of the first return to state 0. Find the probability that all states have been visited by time T.

Hint: Condition on the initial transition and then use results from the gambler's ruin problem.

58. In the gambler's ruin problem of Section 4.5.1, suppose the gambler's fortune is presently i, and suppose that we know that the gambler's fortune will eventually reach N (before it goes to 0). Given this information, show that the probability he wins the next gamble is

$$\frac{p[1-(q/p)^{i+1}]}{1-(q/p)^i}, \quad \text{if } p \neq \tfrac{1}{2}$$

$$\frac{i+1}{2i}, \quad \text{if } p = \tfrac{1}{2}$$

Hint: The probability we want is

$$P\{X_{n+1} = i+1 | X_n = i, \lim_{m\to\infty} X_m = N\}$$
$$= \frac{P\{X_{n+1} = i+1, \lim_m X_m = N | X_n = i\}}{P\{\lim_m X_m = N | X_n = i\}}$$

59. For the gambler's ruin model of Section 4.5.1, let M_i denote the mean number of games that must be played until the gambler either goes broke or reaches a fortune of N, given that he starts with $i, i = 0, 1, \ldots, N$. Show that M_i satisfies

$$M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, \ldots, N-1$$

60. Solve the equations given in Exercise 59 to obtain

$$M_i = i(N-i), \quad \text{if } p = \tfrac{1}{2}$$
$$= \frac{i}{q-p} - \frac{N}{q-p}\frac{1-(q/p)^i}{1-(q/p)^N}, \quad \text{if } p \neq \tfrac{1}{2}$$

61. Suppose in the gambler's ruin problem that the probability of winning a bet depends on the gambler's present fortune. Specifically, suppose that α_i is the probability that the gambler wins a bet when his or her fortune is i. Given that the gambler's initial fortune is i, let $P(i)$ denote the probability that the gambler's fortune reaches N before 0.
   (a) Derive a formula that relates $P(i)$ to $P(i-1)$ and $P(i+1)$.
   (b) Using the same approach as in the gambler's ruin problem, solve the equation of part (a) for $P(i)$.
   (c) Suppose that i balls are initially in urn 1 and $N-i$ are in urn 2, and suppose that at each stage one of the N balls is randomly chosen, taken from whichever urn it is in, and placed in the other urn. Find the probability that the first urn becomes empty before the second.

*62. Consider the particle from Exercise 57. What is the expected number of steps the particle takes to return to the starting position? What is the probability that all other positions are visited before the particle returns to its starting state?

63. For the Markov chain with states 1, 2, 3, 4 whose transition probability matrix $\mathbf{P}$ is as specified below find f_{i3} and s_{i3} for $i = 1, 2, 3$.

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

64. Consider a branching process having $\mu < 1$. Show that if $X_0 = 1$, then the expected number of individuals that ever exist in this population is given by $1/(1-\mu)$. What if $X_0 = n$?

65. In a branching process having $X_0 = 1$ and $\mu > 1$, prove that π_0 is the *smallest* positive number satisfying Equation (4.20).

Hint: Let π be any solution of $\pi = \sum_{j=0}^{\infty} \pi^j P_j$. Show by mathematical induction that $\pi \geqslant P\{X_n = 0\}$ for all n, and let $n \to \infty$. In using the induction argue that

$$P\{X_n = 0\} = \sum_{j=0}^{\infty} (P\{X_{n-1} = 0\})^j P_j$$

66. For a branching process, calculate π_0 when
(a) $P_0 = \frac{1}{4}, P_2 = \frac{3}{4}$.
(b) $P_0 = \frac{1}{4}, P_1 = \frac{1}{2}, P_2 = \frac{1}{4}$.
(c) $P_0 = \frac{1}{6}, P_1 = \frac{1}{2}, P_3 = \frac{1}{3}$.

67. At all times, an urn contains N balls—some white balls and some black balls. At each stage, a coin having probability p, $0 < p < 1$, of landing heads is flipped. If heads appears, then a ball is chosen at random from the urn and is replaced by a white ball; if tails appears, then a ball is chosen from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the nth stage.
(a) Is $\{X_n, n \geqslant 0\}$ a Markov chain? If so, explain why.
(b) What are its classes? What are their periods? Are they transient or recurrent?
(c) Compute the transition probabilities P_{ij}.
(d) Let $N = 2$. Find the proportion of time in each state.
(e) Based on your answer in part (d) and your intuition, guess the answer for the limiting probability in the general case.
(f) Prove your guess in part (e) either by showing that Equation (4.7) is satisfied or by using the results of Example 4.35.
(g) If $p = 1$, what is the expected time until there are only white balls in the urn if initially there are i white and $N - i$ black?

*68. (a) Show that the limiting probabilities of the reversed Markov chain are the same as for the forward chain by showing that they satisfy the equations

$$\pi_j = \sum_i \pi_i Q_{ij}$$

(b) Give an intuitive explanation for the result of part (a).

69. M balls are initially distributed among m urns. At each stage one of the balls is selected at random, taken from whichever urn it is in, and then placed, at random, in one of the other $M - 1$ urns. Consider the Markov chain whose state at any time is the vector $(n_1, \ldots, n_m)$ where n_i denotes the number of balls in urn i. Guess at the limiting probabilities for this Markov chain and then verify your guess and show at the same time that the Markov chain is time reversible.

70. A total of m white and m black balls are distributed among two urns, with each urn containing m balls. At each stage, a ball is randomly selected from each urn and the two selected balls are interchanged. Let X_n denote the number of black balls in urn 1 after the nth interchange.
    (a) Give the transition probabilities of the Markov chain $X_n, n \geqslant 0$.
    (b) Without any computations, what do you think are the limiting probabilities of this chain?
    (c) Find the limiting probabilities and show that the stationary chain is time reversible.

71. It follows from Theorem 4.2 that for a time reversible Markov chain

$$P_{ij}P_{jk}P_{ki} = P_{ik}P_{kj}P_{ji}, \quad \text{for all } i, j, k$$

It turns out that if the state space is finite and $P_{ij} > 0$ for all i, j, then the preceding is also a sufficient condition for time reversibility. (That is, in this case, we need only check Equation (4.26) for paths from i to i that have only two intermediate states.) Prove this.

Hint: Fix i and show that the equations

$$\pi_j P_{jk} = \pi_k P_{kj}$$

are satisfied by $\pi_j = cP_{ij}/P_{ji}$, where c is chosen so that $\sum_j \pi_j = 1$.

72. For a time reversible Markov chain, argue that the rate at which transitions from i to j to k occur must equal the rate at which transitions from k to j to i occur.

73. Show that the Markov chain of Exercise 31 is time reversible.

74. A group of n processors is arranged in an ordered list. When a job arrives, the first processor in line attempts it; if it is unsuccessful, then the next in line tries it; if it too is unsuccessful, then the next in line tries it, and so on. When the job is successfully processed or after all processors have been unsuccessful, the job leaves the system. At this point we are allowed to reorder the processors, and a new job appears. Suppose that we use the one-closer reordering rule, which moves the processor that was successful one closer to the front of the line by interchanging its position with the one in front of it. If all processors were unsuccessful (or if the processor in the first position was successful), then the ordering remains the same. Suppose that each time processor i attempts a job then, independently of anything else, it is successful with probability p_i.
    (a) Define an appropriate Markov chain to analyze this model.
    (b) Show that this Markov chain is time reversible.
    (c) Find the long-run probabilities.

75. A Markov chain is said to be a tree process if
   (i) $P_{ij} > 0$ whenever $P_{ji} > 0$,
   (ii) for every pair of states i and j, $i \neq j$, there is a unique sequence of distinct states $i = i_0, i_1, \ldots, i_{n-1}, i_n = j$ such that

$$P_{i_k, i_{k+1}} > 0, \quad k = 0, 1, \ldots, n-1$$

In other words, a Markov chain is a tree process if for every pair of distinct states i and j there is a unique way for the process to go from i to j without reentering a state (and this path is the reverse of the unique path from j to i). Argue that an ergodic tree process is time reversible.

76. On a chessboard compute the expected number of plays it takes a knight, starting in one of the four corners of the chessboard, to return to its initial position if we assume that at each play it is equally likely to choose any of its legal moves. (No other pieces are on the board.)

Hint: Make use of Example 4.36.

77. In a Markov decision problem, another criterion often used, different than the expected average return per unit time, is that of the expected discounted return. In this criterion we choose a number $\alpha, 0 < \alpha < 1$, and try to choose a policy so as to maximize $E[\sum_{i=0}^{\infty} \alpha^i R(X_i, a_i)]$ (that is, rewards at time n are discounted at rate α^n). Suppose that the initial state is chosen according to the probabilities b_i. That is,

$$P\{X_0 = i\} = b_i, \quad i = 1, \ldots, n$$

For a given policy $\boldsymbol{\beta}$ let y_{ja} denote the expected discounted time that the process is in state j and action a is chosen. That is,

$$y_{ja} = E_{\boldsymbol{\beta}}\left[\sum_{n=0}^{\infty} \alpha^n I_{\{X_n = j, a_n = a\}}\right]$$

where for any event A the indicator variable I_A is defined by

$$I_A = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that

$$\sum_a y_{ja} = E\left[\sum_{n=0}^{\infty} \alpha^n I_{\{X_n = j\}}\right]$$

or, in other words, $\sum_a y_{ja}$ is the expected discounted time in state j under $\boldsymbol{\beta}$.

(b) Show that

$$\sum_j \sum_a y_{ja} = \frac{1}{1-\alpha},$$

$$\sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a)$$

Hint: For the second equation, use the identity

$$I_{\{X_{n+1}=j\}} = \sum_i \sum_a I_{\{X_{n=i}, a_{n=a}\}} I_{\{X_{n+1}=j\}}$$

Take expectations of the preceding to obtain

$$E\left[I_{X_{n+1}=j\}}\right] = \sum_i \sum_a E\left[I_{\{X_{n=i}, a_{n=a}\}}\right] P_{ij}(a)$$

(c) Let $\{y_{ja}\}$ be a set of numbers satisfying

$$\sum_j \sum_a y_{ja} = \frac{1}{1-\alpha},$$
$$\sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a) \tag{4.38}$$

Argue that y_{ja} can be interpreted as the expected discounted time that the process is in state j and action a is chosen when the initial state is chosen according to the probabilities b_j and the policy $\boldsymbol{\beta}$, given by

$$\beta_i(a) = \frac{y_{ia}}{\sum_a y_{ia}}$$

is employed.

Hint: Derive a set of equations for the expected discounted times when policy $\boldsymbol{\beta}$ is used and show that they are equivalent to Equation (4.38).

(d) Argue that an optimal policy with respect to the expected discounted return criterion can be obtained by first solving the linear program

$$\begin{aligned}
\text{maximize} \quad & \sum_j \sum_a y_{ja} R(j,a), \\
\text{such that} \quad & \sum_j \sum_a y_{ja} = \frac{1}{1-\alpha}, \\
& \sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a), \\
& y_{ja} \geqslant 0, \quad \text{all } j, a;
\end{aligned}$$

and then defining the policy β^* by

$$\beta_i^*(a) = \frac{y_{ia}^*}{\sum_a y_{ia}^*}$$

where the y_{ja}^* are the solutions of the linear program.

78. For the Markov chain of Exercise 5, suppose that $p(s|j)$ is the probability that signal s is emitted when the underlying Markov chain state is j, $j = 0, 1, 2$.

(a) What proportion of emissions are signal s?
(b) What proportion of those times in which signal s is emitted is 0 the underlying state?

79. In Example 4.43, what is the probability that the first 4 items produced are all acceptable?

Chapter 4

1. $P_{01} = 1, \quad P_{10} = \frac{1}{9}, \quad P_{21} = \frac{4}{9}, \quad P_{32} = 1$

$P_{11} = \frac{4}{9}, \quad P_{22} = \frac{4}{9}$

$P_{12} = \frac{4}{9}, \quad P_{23} = \frac{1}{9}$

2, 3.

$$P = \begin{array}{c} \\ (RRR) \\ (RRD) \\ (RDR) \\ (RDD) \\ (DRR) \\ (DRD) \\ (DDR) \\ (DDD) \end{array} \begin{array}{c} \begin{array}{cccccccc} (RRR) & (RRD) & (RDR) & (RDD) & (DRR) & (DRD) & (DDR) & (DDD) \end{array} \\ \left\| \begin{array}{cccccccc} .8 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & .4 & .6 & & & & \\ & & & & .6 & .4 & & \\ & & & & & & .4 & .6 \\ .6 & .4 & & & & & & \\ & & .4 & .6 & & & & \\ & & & & .6 & .4 & & \\ & & & & & & .2 & .8 \end{array} \right\| \end{array}$$

where D = dry and R = rain. For instance, (DDR) means that it is raining today, was dry yesterday, and was dry the day before yesterday.

4. Let the state space be $S = \{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$, where state $i(\bar{i})$ signifies that the present value is i, and the present day is even (odd).

5. Cubing the transition probability matrix, we obtain P^3:

$$\begin{bmatrix} 13/36 & 11/54 & 47/108 \\ 4/9 & 4/27 & 11/27 \\ 5/12 & 2/9 & 13/36 \end{bmatrix}$$

Thus,

$$\begin{aligned} E[X_3] &= P(X_3 = 1) + 2P(X_3 = 2) \\ &= \frac{1}{4}P^3_{01} + \frac{1}{4}P^3_{11} + \frac{1}{2}P^3_{21} \\ &\quad + 2\left[\frac{1}{4}P^3_{02} + \frac{1}{4}P^3_{12} + \frac{1}{2}P^3_{22}\right] \end{aligned}$$

6. It is immediate for $n = 1$, so assume for n. Now use induction.

7. $$\begin{aligned} P^2_{30} + P^2_{31} &= P_{31}P_{10} + P_{33}P_{11} + P_{33}P_{31} \\ &= (.2)(.5) + (.8)(0) + (.2)(0) + (.8)(.2) \\ &= .26 \end{aligned}$$

8. Let the state on any day be the number of the coin that is flipped on that day.

$$\underline{P} = \begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}$$

and so,

$$\underline{P}^2 = \begin{bmatrix} .67 & .33 \\ .66 & .34 \end{bmatrix}$$

and

$$\underline{P}^3 = \begin{bmatrix} .667 & .333 \\ .666 & .334 \end{bmatrix}$$

Hence,

$$\frac{1}{2}\left[P^3_{11} + P^3_{21}\right] \equiv .6665$$

If we let the state be 0 when the most recent flip lands heads and let it equal 1 when it lands tails, then the sequence of states is a Markov chain with transition probability matrix

$$\begin{bmatrix} .7 & .3 \\ .6 & .4 \end{bmatrix}$$

The desired probability is $P^4_{0,0} = .6667$

9. It is not a Markov chain because information about previous color selections would affect probabilities about the current makeup of the urn, which would affect the probability that the next selection is red.

10. The answer is $1 - P^3_{0,2}$ for the Markov chain with transition probability matrix

$$\begin{bmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ 0 & 0 & 1 \end{bmatrix}$$

11. The answer is $\dfrac{P^4_{2,2}}{1 - P^4_{2,0}}$ for the Markov chain with transition probability matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{bmatrix}$$

12. The result is not true. For instance, suppose that $P_{0,1} = P_{0,2} = 1/2, P_{1,0} = 1, P_{2,3} = 1$. Given $X_0 = 0$ and that state 3 has not been entered by time 2, the equality implies that X_1 is equally likely to be 1 or 2, which is not true because, given the information, X_1 is equal to 1 with certainty.

13. $P^n_{ij} = \sum_k P^{n-r}_{ik} P^r_{kj} > 0$

14. (i) $\{0, 1, 2\}$ recurrent.
  (ii) $\{0, 1, 2, 3\}$ recurrent.
  (iii) $\{0, 2\}$ recurrent, $\{1\}$ transient, $\{3, 4\}$ recurrent.
  (iv) $\{0, 1\}$ recurrent, $\{2\}$ recurrent, $\{3\}$ transient, $\{4\}$ transient.

15. Consider any path of states $i_0 = i, i_1, i_2, \ldots, i_n = j$ such that $P_{i_k i_{k+1}} > 0$. Call this a path from i to j. If j can be reached from i, then there must be a path from i to j. Let $i_0, \ldots, i_n$ be such a path. If all of the values $i_0, \ldots, i_n$ are not distinct, then there is a subpath from i to j having fewer elements (for instance, if $i, 1, 2, 4, 1, 3, j$ is a path, then so is $i, 1, 3, j$). Hence, if a path exists, there must be one with all distinct states.

16. If P_{ij} were (strictly) positive, then P^n_{ji} would be 0 for all n (otherwise, i and j would communicate). But then the process, starting in i, has a positive probability of at least P_{ij} of never returning to i. This contradicts the recurrence of i. Hence $P_{ij} = 0$.

17. $\sum_{i=1}^{n} Y_j/n \to E[Y]$ by the strong law of large numbers. Now $E[Y] = 2p - 1$. Hence, if $p > 1/2$, then $E[Y] > 0$, and so the average of the Y_is converges in this case to a positive number, which implies that $\sum_1^n Y_i \to \infty$ as $n \to \infty$. Hence, state 0 can be visited only a finite number of times and so must be transient. Similarly, if $p < 1/2$, then $E[Y] < 0$, and so $\lim \sum_1^n Y_i = -\infty$, and the argument is similar.

18. If the state at time n is the n^{th} coin to be flipped then a sequence of consecutive states constitutes a two-state Markov chain with transition probabilities

$$P_{1,1} = .6 = 1 - P_{1,2}, \quad P_{2,1} = .5 = P_{2,2}$$

(a) The stationary probabilities satisfy

$$\pi_1 = .6\pi_1 + .5\pi_2$$
$$\pi_1 + \pi_2 = 1$$

Solving yields that $\pi_1 = 5/9$, $\pi_2 = 4/9$. So the proportion of flips that use coin 1 is 5/9.

(b) $P^4_{1,2} = .44440$

19. The limiting probabilities are obtained from

$$r_0 = .7r_0 + .5r_1$$
$$r_1 = .4r_2 + .2r_3$$
$$r_2 = .3r_0 + .5r_1$$
$$r_0 + r_1 + r_2 + r_3 = 1$$

and the solution is

$$r_0 = \frac{1}{4}, \quad r_1 = \frac{3}{20}, \quad r_2 = \frac{3}{20}, \quad r_3 = \frac{9}{20}$$

The desired result is thus

$$r_0 + r_1 = \frac{2}{5}$$

20. If $\sum_{i=0}^{m} P_{ij} = 1$ for all j, then $r_j = 1/(M+1)$ satisfies

$$r_j = \sum_{i=0}^{m} r_i P_{ij}, \quad \sum_0^m r_j = 1$$

Hence, by uniqueness these are the limiting probabilities.

21. The transition probabilities are

$$P_{i,j} = \begin{cases} 1 - 3\alpha, & \text{if } j = i \\ \alpha, & \text{if } j \neq i \end{cases}$$

By symmetry,

$$P^n_{ij} = \frac{1}{3}(1 - P^n_{ii}), \quad j \neq i$$

So, let us prove by induction that

$$P^n_{i,j} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n, & \text{if } j = i \\ \frac{1}{4} - \frac{1}{4}(1 - 4\alpha)^n, & \text{if } j \neq i \end{cases}$$

As the preceding is true for $n = 1$, assume it for n. To complete the induction proof, we need to show that

$$P^{n+1}_{i,j} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^{n+1}, & \text{if } j = i \\ \frac{1}{4} - \frac{1}{4}(1 - 4\alpha)^{n+1}, & \text{if } j \neq i \end{cases}$$

Now,

$$P_{i,i}^{n+1} = P_{i,i}^{n} P_{i,i} + \sum_{j \neq i} P_{i,j}^{n} P_{j,i}$$
$$= \left(\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n\right)(1-3\alpha)$$
$$+ 3\left(\frac{1}{4} - \frac{1}{4}(1-4\alpha)^n\right)\alpha$$
$$= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n(1-3\alpha-\alpha)$$
$$= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n+1}$$

By symmetry, for $j \neq i$

$$P_{ij}^{n+1} = \frac{1}{3}\left(1 - P_{ii}^{n+1}\right) = \frac{1}{4} - \frac{1}{4}(1-4\alpha)^{n+1}$$

and the induction is complete.

By letting $n \to \infty$ in the preceding, or by using that the transition probability matrix is doubly stochastic, or by just using a symmetry argument, we obtain that $\pi_i = 1/4$.

22. Let X_n denote the value of Y_n modulo 13. That is, X_n is the remainder when Y_n is divided by 13. Now X_n is a Markov chain with states $0, 1, \ldots, 12$. It is easy to verify that $\sum_i P_{ij} = 1$ for all j. For instance, for $j = 3$:

$$\sum_i P_{ij} = P_{2,3} + P_{1,3} + P_{0,3} + P_{12,3} + P_{11,3} + P_{10,3}$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

Hence, from Problem 20, $r_i = \dfrac{1}{13}$.

23. (a) Letting 0 stand for a good year and 1 for a bad year, the successive states follow a Markov chain with transition probability matrix P:

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

Squaring this matrix gives P^2:

$$\begin{pmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{pmatrix}$$

Hence, if S_i is the number of storms in year i then

$$E[S_1] = E[S_1|X_1 = 0]P_{00} + E[S_1|X_1 = 1]P_{01}$$
$$= 1/2 + 3/2 = 2$$
$$E[S_2] = E[S_2|X_2 = 0]P_{00}^2 + E[S_2|X_2 = 1]P_{01}^2$$
$$= 5/12 + 21/12 = 26/12$$

Hence, $E[S_1 + S_2] = 25/6$.

(b) Multiplying the first row of P by the first column of P^2 gives

$$P_{00}^3 = 5/24 + 7/36 = 29/72$$

Hence, conditioning on the state at time 3 yields

$$P(S_3 = 0) = P(S_3 = 0|X_3 = 0)\frac{29}{72} + P(S_3 = 0|X_3 = 1)$$
$$\times \frac{43}{72} = \frac{29}{72}e^{-1} + \frac{43}{72}e^{-3}$$

(c) The stationary probabilities are the solution of

$$\pi_0 = \pi_0\frac{1}{2} + \pi_1\frac{1}{3}$$
$$\pi_0 + \pi_1 = 1$$

giving

$$\pi_0 = 2/5, \quad \pi_1 = 3/5.$$

Hence, the long-run average number of storms is $2/5 + 3(3/5) = 11/5$.

24. Let the state be the color of the last ball selected, call it 0 if that color was red, 1 if white, and 2 if blue. The transition probability matrix of this Markov chain is

$$P = \begin{bmatrix} 1/5 & 0 & 4/5 \\ 2/7 & 3/7 & 2/7 \\ 3/9 & 4/9 & 2/9 \end{bmatrix}$$

Solve for the stationary probabilities to obtain the solution.

25. Letting X_n denote the number of pairs of shoes at the door the runner departs from at the beginning of day n, then $\{X_n\}$ is a Markov chain with transition probabilities

$$\begin{aligned} P_{i,i} &= 1/4, \quad 0 < i < k \\ P_{i,i-1} &= 1/4, \quad 0 < i < k \\ P_{i,k-i} &= 1/4, \quad 0 < i < k \\ P_{i,k-i+1} &= 1/4, \quad 0 < i < k \end{aligned}$$

The first equation refers to the situation where the runner returns to the same door she left from and then chooses that door the next day; the second to the situation where the runner returns to the opposite door from which she left from and then chooses the original door the next day; and so on. (When some of the four cases above refer to the same transition probability, they should be added together. For instance, if $i = 4$, $k = 8$, then the preceding

states that $P_{i,\,i} = 1/4 = P_{i,\,k-i}$. Thus, in this case, $P_{4,\,4} = 1/2$.) Also,

$$
\begin{aligned}
P_{0,\,0} &= 1/2\\
P_{0,\,k} &= 1/2\\
P_{k,\,k} &= 1/4\\
P_{k,\,0} &= 1/4\\
P_{k,\,1} &= 1/4\\
P_{k,\,k-1} &= 1/4
\end{aligned}
$$

It is now easy to check that this Markov chain is doubly stochastic—that is, the column sums of the transition probability matrix are all 1—and so the long-run proportions are equal. Hence, the proportion of time the runner runs barefooted is $1/(k+1)$.

26. Let the state be the ordering, so there are $n!$ states. The transition probabilities are

$$P_{(i_1,\,\ldots,\,i_n),(i_j,\,i_1,\,\ldots,\,i_{j-1},\,i_{j+1},\,\ldots,\,i_n)} = \frac{1}{n}$$

It is now easy to check that this Markov chain is doubly stochastic and so, in the limit, all $n!$ possible states are equally likely.

27. The limiting probabilities are obtained from

$$
\begin{aligned}
r_0 &= \frac{1}{9}r_1\\
r_1 &= r_0 + \frac{4}{9}r_1 + \frac{4}{9}r_2\\
r_2 &= \frac{4}{9}r_1 + \frac{4}{9}r_2 + r_3\\
r_0 &+ r_1 + r_2 + r_3 = 1
\end{aligned}
$$

and the solution is $r_0 = r_3 = \dfrac{1}{20}$, $r_1 = r_2 = \dfrac{9}{20}$.

28. Letting π_w be the proportion of games the team wins then

$$\pi_w = \pi_w(.8) + (1 - \pi_w)(.3)$$

Hence, $\pi_w = 3/5$, yielding that the proportion of games that result in a team dinner is $3/5(.7) + 2/5(.2) = 1/2$. That is, fifty percent of the time the team has dinner.

29. Each employee moves according to a Markov chain whose limiting probabilities are the solution of

$$
\begin{aligned}
\Pi_1 &= .7\,\Pi_1 + .2\,\Pi_2 + .1\,\Pi_3\\
\Pi_2 &= .2\,\Pi_1 + .6\,\Pi_2 + .4\,\Pi_3\\
\Pi_1 &+ \Pi_2 + \Pi_3 = 1
\end{aligned}
$$

Solving yields $\Pi_1 = 6/17, \Pi_2 = 7/17, \Pi_3 = 4/17$. Hence, if N is large, it follows from the law of large numbers that approximately 6, 7, and 4 of each 17 employees are in categories 1, 2, and 3.

30. Letting X_n be 0 if the n^{th} vehicle is a car and letting it be 1 if the vehicle is a truck gives rise to a two-state Markov chain with transition probabilities

$$
\begin{aligned}
&P_{00} = 4/5, \quad P_{01} = 1/5\\
&P_{10} = 3/4, \quad P_{11} = 1/4
\end{aligned}
$$

The long-run proportions are the solutions of

$$
\begin{aligned}
r_0 &= \frac{4}{5}r_0 + \frac{3}{4}r_1\\
r_1 &= \frac{1}{5}r_0 + \frac{1}{4}r_1\\
r_0 &+ r_1 = 1
\end{aligned}
$$

Solving these gives the result

$$r_0 = \frac{15}{19}, \quad r_1 = \frac{4}{19}$$

That is, 4 out of every 19 cars is a truck.

31. Let the state on day n be 0 if sunny, 1 if cloudy, and 2 if rainy. This gives a three-state Markov chain with transition probability matrix

		0	1	2
	0	0	1/2	1/2
$P =$	1	1/4	1/2	1/4
	2	1/4	1/4	1/2

The equations for the long-run proportions are

$$
\begin{aligned}
r_0 &= \frac{1}{4}r_1 + \frac{1}{4}r_2\\
r_1 &= \frac{1}{2}r_0 + \frac{1}{2}r_1 + \frac{1}{4}r_2\\
r_2 &= \frac{1}{2}r_0 + \frac{1}{4}r_1 + \frac{1}{2}r_2\\
r_0 &+ r_1 + r_2 = 1
\end{aligned}
$$

By symmetry it is easy to see that $r_1 = r_2$. This makes it easy to solve and we obtain the result

$$r_0 = \frac{1}{5}, \quad r_1 = \frac{2}{5}, \quad r_2 = \frac{2}{5}$$

32. With the state being the number of off switches this is a three-state Markov chain. The equations for the long-run proportions are

$$r_0 = \frac{1}{16} r_0 + \frac{1}{4} r_1 + \frac{9}{16} r_2$$

$$r_1 = \frac{3}{8} r_0 + \frac{1}{2} r_1 + \frac{3}{8} r_2$$

$$r_0 + r_1 + r_2 = 1$$

This gives the solution

$$r_0 = 2/7, \quad r_1 = 3/7, \quad r_2 = 2/7$$

33. Consider the Markov chain whose state at time n is the type of exam number n. The transition probabilities of this Markov chain are obtained by conditioning on the performance of the class. This gives the following:

$$P_{11} = .3(1/3) + .7(1) = .8$$

$$P_{12} = P_{13} = .3(1/3) = .1$$

$$P_{21} = .6(1/3) + .4(1) = .6$$

$$P_{22} = P_{23} = .6(1/3) = .2$$

$$P_{31} = .9(1/3) + .1(1) = .4$$

$$P_{32} = P_{33} = .9(1/3) = .3$$

Let r_i denote the proportion of exams that are type $i, i = 1, 2, 3$. The r_i are the solutions of the following set of linear equations:

$$r_1 = .8\, r_1 + .6\, r_2 + .4\, r_3$$

$$r_2 = .1\, r_1 + .2\, r_2 + .3\, r_3$$

$$r_1 + r_2 + r_3 = 1$$

Since $P_{i2} = P_{i3}$ for all states i, it follows that $r_2 = r_3$. Solving the equations gives the solution

$$r_1 = 5/7, \quad r_2 = r_3 = 1/7$$

34. (a) π_i, $i = 1, 2, 3$, which are the unique solutions of the following equations:

$$\pi_1 = q_2\pi_2 + p_3\pi_3$$

$$\pi_2 = p_1\pi_1 + q_3\pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

(b) The proportion of time that there is a counterclockwise move from i that is followed by 5 clockwise moves is $\pi_i q_i p_{i-1} p_i p_{i+1} p_{i+2} p_{i+3}$, and so the answer to (b) is $\sum_{i=1}^{3} \pi_i q_i p_{i-1} p_i p_{i+1} p_{i+2} p_{i+3}$. In the preceding, $p_0 = p_3, p_4 = p_1, p_5 = p_2, p_6 = p_3$.

35. The equations are

$$r_0 = r_1 + \frac{1}{2} r_2 + \frac{1}{3} r_3 + \frac{1}{4} r_4$$

$$r_1 = \frac{1}{2} r_2 + \frac{1}{3} r_3 + \frac{1}{4} r_4$$

$$r_2 = \frac{1}{3} r_3 + \frac{1}{4} r_4$$

$$r_3 = \frac{1}{4} r_4$$

$$r_4 = r_0$$

$$r_0 + r_1 + r_2 + r_3 + r_4 = 1$$

The solution is

$$r_0 = r_4 = 12/37, \quad r_1 = 6/37, \quad r_2 = 4/37,$$
$$r_3 = 3/37$$

36. (a) $p_0 P_{0,0} + p_1 P_{0,1} = .4p_0 + .6p_1$

(b) $p_0 P^4_{0,0} + p_1 P^4_{0,1} = .2512p_0 + .7488p_1$

(c) $p_0\pi_0 + p_1\pi_1 = p_0/4 + 3p_1/4$

(d) Not a Markov chain.

37. Must show that

$$\pi_j = \sum_i \pi_i P^k_{i,j}$$

The preceding follows because the right-hand side is equal to the probability that the Markov chain with transition probabilities $P_{i,j}$ will be in state j at time k when its initial state is chosen according to its stationary probabilities, which is equal to its stationary probability of being in state j.

38. Because j is accessible from i, there is an n such that $P^n_{i,j} > 0$. Because $\pi_i P^n_{i,j}$ is the long-run proportion of time the chain is currently in state j and had been in state i exactly n time periods ago, the inequality follows.

39. Because recurrence is a class property it follows that state j, which communicates with the recurrent state i, is recurrent. But if j were positive recurrent, then by the previous exercise i would be as well. Because i is not, we can conclude that j is null recurrent.

40. (a) Follows by symmetry.

(b) If $\pi_i = a > 0$ then, for any n, the proportion of time the chain is in any of the states $1, \ldots, n$ is na. But this is impossible when $n > 1/a$.

41. (a) The number of transitions into state i by time n, the number of transitions originating from

state i by time n, and the number of time periods the chain is in state i by time n all differ by at most 1. Thus, their long-run proportions must be equal.

(b) $r_i P_{ij}$ is the long-run proportion of transitions that go from state i to state j.

(c) $\sum_j r_i P_{ij}$ is the long-run proportion of transitions that are into state j.

(d) Since r_j is also the long-run proportion of transitions that are into state j, it follows that

$$r_j = \sum_j r_i P_{ij}$$

42. (a) This is the long-run proportion of transitions that go from a state in A to one in A^c.

(b) This is the long-run proportion of transitions that go from a state in A^c to one in A.

(c) Between any two transitions from A to A^c there must be one from A^c to A. Similarly between any two transitions from A^c to A there must be one from A to A^c. Therefore, the long-run proportion of transitions that are from A to A^c must be equal to the long-run proportion of transitions that are from A^c to A.

43. Consider a typical state—say, 1 2 3. We must show

$$\prod_{123} = \prod_{123} P_{123,123} + \prod_{213} P_{213,123} + \prod_{231} P_{231,123}$$

Now $P_{123,123} = P_{213,123} = P_{231,123} = P_1$ and thus,

$$\prod_{123} = P_1\left[\prod_{123} + \prod_{213} + \prod_{231}\right]$$

We must show that

$$\prod_{123} = \frac{P_1 P_2}{1 - P_1}, \prod_{213} = \frac{P_2 P_1}{1 - P_2}, \prod_{231} = \frac{P_2 P_3}{1 - P_2}$$

satisfies the above, which is equivalent to

$$\begin{aligned} P_1 P_2 &= P_1\left[\frac{P_2 P_1}{1 - P_2} + \frac{P_2 P_3}{1 - P_2}\right] \\ &= \frac{P_1}{1 - P_2} P_2 (P_1 + P_3) \\ &= P_1 P_2 \quad \text{since } P_1 + P_3 = 1 - P_2 \end{aligned}$$

By symmetry all of the other stationary equations also follow.

44. Given X_n, $X_{n=1}$ is binomial with parameters m and $p = X_n/m$. Hence, $E[X_{n+1}|X_n] = m(X_n/m) = X_n$, and so $E[X_{n+1}] = E[X_n]$. So $E[X_n] = i$ for all n. To solve (b) note that as all states but 0 and m are transient, it follows that X_n will converge to either 0 or m. Hence, for n large

$$\begin{aligned} E[X_n] &= mP\{\text{hits } m\} + 0\, P\{\text{hits } 0\} \\ &= mP\{\text{hits } m\} \end{aligned}$$

But $E[X_n] = i$ and thus $P\{\text{hits } m\} = i/m$.

45. (a) 1, since all states communicate and thus all are recurrent since state space is finite.

(b) Condition on the first state visited from i.

$$x_i = \sum_{j=1}^{N-1} P_{ij} x_j + P_{iN}, \quad i = 1, \ldots, N-1$$

$$x_0 = 0, \quad x_N = 1$$

(c) Must show

$$\begin{aligned} \frac{i}{N} &= \sum_{j=1}^{N-1} \frac{j}{N} P_{ij} + P_{iN} \\ &= \sum_{j=0}^{N} \frac{j}{N} P_{ij} \end{aligned}$$

and follows by hypothesis.

46. (a) Let the state be the number of umbrellas he has at his present location. The transition probabilities are

$$P_{0,r} = 1, P_{i,r-i} = 1 - p, P_{i,r-i+1} = p, \quad i = 1, \ldots, r$$

(b) We must show that $\pi_j = \sum_i \pi_j P_{ij}$ is satisfied by the given solution. These equations reduce to

$$\begin{aligned} \pi_r &= \pi_0 + \pi_1 p \\ \pi_j &= \pi_{r-j}(1 - p) + \pi_{r-j+1} p, \quad j = 1, \ldots, r-1 \\ \pi_0 &= \pi_r(1 - p) \end{aligned}$$

and it is easily verified that they are satisfied.

(c) $p\pi_0 = \dfrac{pq}{r + q}$

(d) $$\begin{aligned} \frac{d}{dp}\left[\frac{p(1-p)}{4-p}\right] &= \frac{(4-p)(1-2p) + p(1-p)}{(4-p)^2} \\ &= \frac{p^2 - 8p + 4}{(4-p)^2} \end{aligned}$$

$$p^2 - 8p + 4 = 0 \Rightarrow p = \frac{8 - \sqrt{48}}{2} = .55$$

47. $\{Y_n, n \geq 1\}$ is a Markov chain with states (i, j).

$$P_{(i,j),(k,\ell)} = \begin{cases} 0, & \text{if } j \neq k \\ P_{j\ell}, & \text{if } j = k \end{cases}$$

where $P_{j\ell}$ is the transition probability for $\{X_n\}$.

$$\begin{aligned} \lim_{n\to\infty} P\{Y_n = (i,j)\} &= \lim_n P\{X_n = i, X_{n+1} = j\} \\ &= \lim_n [P\{X_n = i\}P_{ij}] \\ &= r_i P_{ij} \end{aligned}$$

48. Letting P be the desired probability, we obtain upon conditioning on X_{m-k-1} that

$$\begin{aligned} P &= \sum_{i\neq 0} P(X_{m-k-1} \neq 0, X_{m-k} = X_{m-k+1} = \cdots = X_{m-1} \\ &= 0, X_m \neq 0 | X_{m-k-1} = i)\pi_i \\ &= \sum_{i\neq 0} P_{i,0}(P_{0,0})^{k-1}(1 - P_{0,0})\pi_i \\ &= (P_{0,0})^{k-1}(1 - P_{0,0}) \sum_{i\neq 0} \pi_i P_{i,0} \\ &= (P_{0,0})^{k-1}(1 - P_{0,0}) \left(\sum_i \pi_i P_{i,0} - \pi_0 P_{0,0} \right) \\ &= (P_{0,0})^{k-1}(1 - P_{0,0})(\pi_0 - \pi_0 P_{0,0}) \end{aligned}$$

49. (a) No.

$$\lim P\{X_n = i\} = pr^1(i) + (1-p)r^2(i)$$

(b) Yes.

$$P_{ij} = pP_{ij}^{(1)} + (1-p)P_{ij}^{(2)}$$

50. Using the Markov chain of Exercise 9, $\mu_{h,t} = 1/.3$, $\mu_{t,h} = 1/.6$. Also, the stationary probabilities of this chain are $\pi_h = 2/3$, $\pi_t = 1/3$. Therefore,

$$E[A(t,t)] = \frac{1}{(1/3)(.4)(.6)(.3)(.6)(.3)(.4)} = 578.7$$

giving

$$\begin{aligned} E[N(tththtt)|X_0 = h] = E[N(t,t)|X_0 = h] \\ + E(A(t,t)] \end{aligned}$$

Also,

$$\begin{aligned} E[N(t,t)|X_0 = h] &= E[N(t)|X_0 = h] + \frac{1}{(1/3)(.4)} \\ &= \frac{13}{1.2} = 10.8 \end{aligned}$$

Therefore, $E[N(tththtt)|X_0 = h] = 589.5$

52. Let the state be the successive zonal pickup locations. Then $P_{A,A} = .6$, $P_{B,A} = .3$. The long-run proportions of pickups that are from each zone are

$$\pi_A = .6\pi_A + .3\pi_B = .6\pi_A + .3(1 - \pi_A)$$

Therefore, $\pi_A = 3/7$, $\pi_B = 4/7$. Let X denote the profit in a trip. Conditioning on the location of the pickup gives

$$\begin{aligned} E[X] &= \frac{3}{7}E[X|A] + \frac{4}{7}E[X|B] \\ &= \frac{3}{7}[.6(6) + .4(12)] + \frac{4}{7}[.3(12) + .7(8)] \\ &= 62/7 \end{aligned}$$

53. With $\pi_i(1/4)$ equal to the proportion of time a policyholder whose yearly number of accidents is Poisson distributed with mean $1/4$ is in Bonus-Malus state i, we have that the average premium is

$$\frac{2}{3}(326.375) + \frac{1}{3}[200\pi_1(1/4) + 250\pi_2(1/4) \\ + 400\pi_3(1/4) + 600\pi_4(1/4)]$$

54. $E[X_{n+1}] = E[E[X_{n+1}|X_n]]$

Now given X_n,

$$X_{n+1} = \begin{cases} X_n + 1, & \text{with probability } \dfrac{M - X_n}{M} \\ X_n - 1, & \text{with probability } \dfrac{X_n}{M} \end{cases}$$

Hence,

$$\begin{aligned} E[X_{n+1}|X_n] &= X_n + \frac{M - X_n}{M} - \frac{X_n}{M} \\ &= X_n + 1 - \frac{2X_n}{M} \end{aligned}$$

and so $E[X_{n+1}] = \left[1 - \dfrac{2}{M}\right] E[X_n] + 1$.

It is now easy to verify by induction that the formula presented in (b) is correct.

55. $S_{11} = P\{\text{offspring is aa} \mid \text{both parents dominant}\}$

$$\begin{aligned} &= \frac{P\{\text{aa, both dominant}\}}{P\{\text{both dominant}\}} \\ &= \frac{r^2 \frac{1}{4}}{(1-q)^2} = \frac{r^2}{4(1-q)^2} \end{aligned}$$

$$S_{10} = \frac{P\{\text{aa, 1 dominant and 1 recessive parent}\}}{P\{\text{1 dominant and 1 recessive parent}\}}$$
$$= \frac{P\{\text{aa, 1 parent aA and 1 parent aa}\}}{2q(1-q)}$$
$$= \frac{2qr\,\frac{1}{2}}{2q(1-q)}$$
$$= \frac{r}{2(1-q)}$$

56. This is just the probability that a gambler starting with m reaches her goal of $n + m$ before going broke, and is thus equal to $\frac{1-(q/p)^m}{1-(q/p)^{n+m}}$, where $q = 1 - p$.

57. Let A be the event that all states have been visited by time T. Then, conditioning on the direction of the first step gives

$$P(A) = P(A|\text{clockwise})p$$
$$+ P(A|\text{counterclockwise})q$$
$$= p\frac{1-q/p}{1-(q/p)^n} + q\frac{1-p/q}{1-(p/q)^n}$$

The conditional probabilities in the preceding follow by noting that they are equal to the probability in the gambler's ruin problem that a gambler that starts with 1 will reach n before going broke when the gambler's win probabilities are p and q.

58. Using the hint, we see that the desired probability is

$$P\{X_{n+1} = i+1 | X_n = i\}$$
$$\frac{P\{\lim X_m = N | X_n = i, X_n + 1 = i+1\}}{P\{\lim X_m = N | X_n = 1\}}$$
$$= \frac{p^P i + 1}{P_i}$$

and the result follows from Equation (4.74).

59. Condition on the outcome of the initial play.

61. With $P_0 = 0$, $P_N = 1$

$$P_i = \alpha_i P_{i+1} + (1-\alpha_i)P_{i-1}, \quad i = 1, \ldots, N-1$$

These latter equations can be rewritten as

$$P_{i+1} - P_i = \beta_i(P_i - P_{i-1})$$

where $\beta_i = (1-\alpha_i)/\alpha_i$. These equations can now be solved exactly as in the original gambler's ruin problem. They give the solution

$$P_i = \frac{1 + \sum_{j=1}^{i-1} C_j}{1 + \sum_{j=1}^{N-1} C_j}, \quad i = 1, \ldots, N-1$$

where

$$C_j = \prod_{i=1}^{j} \beta_i$$

(c) P_{N-i}, where $\alpha_i = (N-i)/N$

62. (a) Since $r_i = 1/5$ is equal to the inverse of the expected number of transitions to return to state i, it follows that the expected number of steps to return to the original position is 5.

(b) Condition on the first transition. Suppose it is to the right. In this case the probability is just the probability that a gambler who always bets 1 and wins each bet with probability p will, when starting with 1, reach γ before going broke. By the gambler's ruin problem this probability is equal to

$$\frac{1-q/p}{1-(q/p)^\gamma}$$

Similarly, if the first move is to the left then the problem is again the same gambler's ruin problem but with p and q reversed. The desired probability is thus

$$\frac{p-q}{1-(q/p)^\gamma} = \frac{q-p}{1-(p/q)^\gamma}$$

64. (a) $E\left[\sum_{k=0}^{\infty} X_k | X_0 = 1\right] = \sum_{k=0}^{\infty} E[X_k|X_0 = 1]$
$$= \sum_{k=0}^{\infty} \mu^k = \frac{1}{1-\mu}$$

(b) $E\left[\sum_{k=0}^{\infty} X_k | X_0 = n\right] = \frac{n}{1-\mu}$

65. $r \geq 0 = P\{X_0 = 0\}$. Assume that
$r \geq P\{X_{n-1} = 0\}$

$$P\{X_n = 0 = \sum_j P\{X_n = 0 | X_1 = j\} P_j$$
$$= \sum_j [P\{X_{n-1} = \}]^j P_j$$
$$\leq \sum_j r^j P_j$$
$$= r$$

66. (a) $r_0 = \frac{1}{3}$

(b) $r_0 = 1$

(c) $r_0 = \left(\sqrt{3}-1\right)\big/2$

67. (a) Yes, the next state depends only on the present and not on the past.

(b) One class, period is 1, recurrent.

(c) $P_{i,i+1} = P\dfrac{N-i}{N}, \quad i = 0, 1, \ldots, N-1$

$P_{i,i-1} = (1-P)\dfrac{i}{N}, \quad i = 1, 2, \ldots, N$

$P_{i,i} = P\dfrac{i}{N} + (1-p)\dfrac{(N-i)}{N}, \quad i = 0, 1, \ldots, N$

(d) See (e).

(e) $r_i = \begin{bmatrix} N \\ i \end{bmatrix} p^i(1-p)^{N-i}, \quad i = 0, 1, \ldots, N$

(f) Direct substitution or use Example 7a.

(g) Time $= \sum_{j=i}^{N-1} T_j$, where T_j is the number of flips to go from j to $j+1$ heads. T_j is geometric with $E[T_j] = N/j$. Thus, $E[\text{time}] = \sum_{j=i}^{N-1} N/j$.

68. (a) $\sum_i r_i Q_{ij} = \sum_i r_j P_{ji} = r_j \sum_i P_{ji} = r_j$

(b) Whether perusing the sequence of states in the forward direction of time or in the reverse direction the proportion of time the state is i will be the same.

69. $r(n_1, \ldots, n_m) = \dfrac{M!}{n_1, \ldots, n_m!} \left[\dfrac{1}{m}\right]^M$

We must now show that

$$r(n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots)\frac{n_j+1}{M}\frac{1}{M-1}$$
$$= r(n_1, \ldots, n_i, \ldots, n_j, \ldots)\frac{i}{M}\frac{1}{M-1}$$

or $\dfrac{n_j+1}{(n_i-1)!(n_j+1)!} = \dfrac{n_i}{n_i!n_j!}$, which follows.

70. (a) $P_{i,i+1} = \dfrac{(m-i)^2}{m^2}, \quad P_{i,i-1} = \dfrac{i^2}{m^2}$

$P_{i,i} = \dfrac{2i(m-i)}{m^2}$

(b) Since, in the limit, the set of m balls in urn 1 is equally likely to be any subset of m balls, it is intuitively clear that

$$\pi_i = \frac{\binom{m}{i}\binom{m}{m-i}}{\binom{2m}{m}} = \frac{\binom{m}{i}^2}{\binom{2m}{m}}$$

(c) We must verify that, with the π_i given in (b),

$\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$

That is, we must verify that

$(m-i)\binom{m}{i} = (i+1)\binom{m}{i+1}$

which is immediate.

71. If $r_j = c\dfrac{P_{ij}}{P_{ji}}$, then

$r_j P_{jk} = c\dfrac{P_{ij}P_{jk}}{P_{ji}}$

$r_k P_{kj} = c\dfrac{P_{jk}P_{kj}}{P_{ki}}$

and are thus equal by hypothesis.

72. Rate at which transitions from i to j to k occur $= r_i P_{ij} P_{jk}$, whereas the rate in the reverse order is $r_k P_{kj} P_{ji}$. So, we must show

$$r_i P_{ij} P_{jk} = r_k P_{kj} P_{ji}$$

Now, $r_i P_{ij} P_{jk} = r_j P_{ji} P_{jk}$ by reversibility

$= r_j P_{jk} P_{ji}$

$= r_k P_{kj} P_{ji}$ by reversibility

73. It is straightforward to check that $r_i P_{ij} = r_j P_{ji}$. For instance, consider states 0 and 1. Then

$r_0 p_{01} = (1/5)(1/2) = 1/10$

whereas

$r_1 p_{10} = (2/5)(1/4) = 1/10$

74. (a) The state would be the present ordering of the n processors. Thus, there are $n!$ states.

(b) Consider states $x = (x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ and $x^1 = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i, \ldots, x_n)$. With q_t equal to $1 - p_t$ the time reversible equations are

$$r(x) q_{x_i} p_{x_{i+1}} \prod_{k=1}^{i-1} q_{x_k} = r(x^1) q_{x_{i+1}} p_{x_i} \prod_{k=1}^{i-1} q_{x_k}$$

or

$$r(x) = \left(q_{x_{i+1}}/p_{x_{i+1}}\right)\left(q_{x_i}/p_{x_i}\right)^{-1} r\left(x^1\right)$$

Suppose now that we successively utilize the above identity until we reach the state $(1, 2, \ldots, n)$. Note that each time j is moved to the left we multiply by q_j/p_j and each time it moves to the right we multiply by $(q_j/p_j)^{-1}$. Since x_j, which is initially in position j, is to have a net move of $j - x_j$ positions to the left (so it will end up in position $j - (j - x_j) = x_j$) it follows from the above that

$$r(x) = C\prod_j \left(q_{x_j}/p_{x_j}\right)^{j-x} j$$

The value of C, which is equal to $r(1, 2, \ldots, n)$, can be obtained by summing over all states x and equating to 1. Since the solution given by the above value of $r(x)$ satisfies the time reversibility equations it follows that the chain is time reversible and these are the limiting probabilities.

75. The number of transitions from i to j in any interval must equal (to within 1) the number from j to i since each time the process goes from i to j in order to get back to i, it must enter from j.

76. We can view this problem as a graph with 64 nodes where there is an arc between 2 nodes if a knight can go from one node to another in a *single move*. The weights on each are equal to 1. It is easy to check that $\sum_i \sum_j w_{ij} = 336$, and for a corner node $i, \sum_j w_{ij} = 2$. Hence, from Example 7b, for one of the 4 corner nodes $i, \prod_i = 2/336$, and thus the mean time to return, which equals $1/r_i$, is $336/2 = 168$.

77. (a) $$\sum_a y_{ja} = \sum_a E_\beta\left[\sum_n a^n I_{\{X_n=j, a_n=a\}}\right]$$
$$= E_\beta\left[\sum_n a^n \sum_a I_{\{X_n=j, a_n=a\}}\right]$$
$$= E_\beta\left[\sum_n a^n I_{\{X_n=j\}}\right]$$

(b) $$\sum_j\sum_a y_{ja} = E_\beta\left[\sum_n a^n \sum_j I_{\{X_n=j\}}\right]$$
$$= E_\beta\left[\sum a^n\right] = \frac{1}{1-\alpha}$$

$$\sum_a y_{ja}$$
$$= b_j + E_\beta\left[\sum_{n=1}^{\infty} = a^n I_{\{X_n=j\}}\right]$$
$$= b_j + E_\beta\left[\sum_{n=0}^{\infty} a^{n+1} I_{\{X_{n+1}=j\}}\right]$$
$$= b_j + E_\beta\left[\sum_{n=0}^{\infty} = a^{n+1}\sum_{i,a} I_{\{X_n=i, a_n=a\}} I_{(X_{n+1}=j\}}\right]$$
$$= b_j + \sum_{n=0}^{\infty} a^{n+1}\sum_{i,a} E_\beta\left[I_{\{X_n=i, a_n=a\}}\right] P_{ij}(a)$$
$$= b_j + a\sum_{i,a}\sum_n a^n E_\beta\left[I_{(X_n=i, a_n=a\}}\right] P_{ij}(a)$$
$$= b_j + a\sum_{i,a} y_{ia}P_{ij}(a)$$

(c) Let $d_{j,a}$ denote the expected discounted time the process is in j, and a is chosen when policy β is employed. Then by the same argument as in (b):

$$\sum_a d_{ja}$$
$$= b_j + a\sum_{i,a}\sum_n a^n E_\beta[I\{X_n = i, a_n = a\}] P_{ij}(a)$$
$$= b_j + a\sum_{i,a}\sum_n a^n E_\beta\left[I_{\{X_n=i\}}\right]\frac{y_{ia}}{\sum_a y_{ia}} P_{ij}(a)$$
$$= b_j + a\sum_{i,a}\sum_a d_{ia,}\frac{y_{ia}}{\sum_a y_{ia}} P_{ij}(a)$$

and we see from Equation (9.1) that the above is satisfied upon substitution of $d_{ia} = y_{ia}$. As it is easy to see that $\sum_{i,a} d_{ia} = \frac{1}{1-a}$, the result follows since it can be shown that these linear equations have a unique solution.

(d) Follows immediately from previous parts. It is a well-know result in analysis (and easily proven) that if $\lim_{n\to\infty} a_n/n = a$ then $\lim_{n\to\infty}\sum_i^n a_i/n$ also equals a. The result follows from this since

$$
\begin{aligned}
E[R(X_n)] &= \sum_j R(j)P\{X_n = j\} \\
&= \sum_i R(j)r_j
\end{aligned}
$$

78. Let $\pi_j, j \geq 0$, be the stationary probabilities of the underlying chain.

(a) $\sum_j \pi_j p(s|j)$

(b) $p(j|s) = \dfrac{\pi_j p(s|j)}{\sum_j \pi_j p(s|j)}$

The Exponential Distribution and the Poisson Process

Exercises

1. The time T required to repair a machine is an exponentially distributed random variable with mean $\frac{1}{2}$ (hours).
   (a) What is the probability that a repair time exceeds $\frac{1}{2}$ hour?
   (b) What is the probability that a repair takes at least $12\frac{1}{2}$ hours given that its duration exceeds 12 hours?
2. Suppose that you arrive at a single-teller bank to find five other customers in the bank, one being served and the other four waiting in line. You join the end of the line. If the service times are all exponential with rate μ, what is the expected amount of time you will spend in the bank?
3. Let X be an exponential random variable. Without any computations, tell which one of the following is correct. Explain your answer.
   (a) $E[X^2|X > 1] = E[(X + 1)^2]$
   (b) $E[X^2|X > 1] = E[X^2] + 1$
   (c) $E[X^2|X > 1] = (1 + E[X])^2$
4. Consider a post office with two clerks. Three people, A, B, and C, enter simultaneously. A and B go directly to the clerks, and C waits until either A or B leaves before he begins service. What is the probability that A is still in the post office after the other two have left when
   (a) the service time for each clerk is exactly (nonrandom) ten minutes?
   (b) the service times are i with probability $\frac{1}{3}$, $i = 1, 2, 3$?
   (c) the service times are exponential with mean $1/\mu$?
5. The lifetime of a radio is exponentially distributed with a mean of ten years. If Jones buys a ten-year-old radio, what is the probability that it will be working after an additional ten years?

6. In Example 5.3 if server i serves at an exponential rate λ_i, $i = 1, 2$, show that

$$P\{\text{Smith is not last}\} = \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^2$$

*7. If X_1 and X_2 are independent nonnegative continuous random variables, show that

$$P\{X_1 < X_2 | \min(X_1, X_2) = t\} = \frac{r_1(t)}{r_1(t) + r_2(t)}$$

where $r_i(t)$ is the failure rate function of X_i.

8. Let $X_i, i = 1, \ldots, n$ be independent continuous random variables, with X_i having failure rate function $r_i(t)$. Let T be independent of this sequence, and suppose that $\sum_{i=1}^{n} P\{T = i\} = 1$. Show that the failure rate function $r(t)$ of X_T is given by

$$r(t) = \sum_{i=1}^{n} r_i(t) P\{T = i | X > t\}$$

9. Machine 1 is currently working. Machine 2 will be put in use at a time t from now. If the lifetime of machine i is exponential with rate λ_i, $i = 1, 2$, what is the probability that machine 1 is the first machine to fail?

*10. Let X and Y be independent exponential random variables with respective rates λ and μ. Let $M = \min(X, Y)$. Find
(a) $E[MX | M = X]$,
(b) $E[MX | M = Y]$,
(c) $\text{Cov}(X, M)$.

11. Let X, $Y_1, \ldots, Y_n$ be independent exponential random variables; X having rate λ, and Y_i having rate μ. Let A_j be the event that the jth smallest of these $n + 1$ random variables is one of the Y_i. Find $p = P\{X > \max_i Y_i\}$, by using the identity

$$p = P(A_1 \cdots A_n) = P(A_1)P(A_2|A_1) \cdots P(A_n|A_1 \cdots A_{n-1})$$

Verify your answer when $n = 2$ by conditioning on X to obtain p.

12. If X_i, $i = 1, 2, 3$, are independent exponential random variables with rates λ_i, $i = 1, 2, 3$, find
(a) $P\{X_1 < X_2 < X_3\}$,
(b) $P\{X_1 < X_2 | \max(X_1, X_2, X_3) = X_3\}$,
(c) $E[\max X_i | X_1 < X_2 < X_3]$,
(d) $E[\max X_i]$.

13. Find, in Example 5.10, the expected time until the nth person on line leaves the line (either by entering service or departing without service).

14. Let X be an exponential random variable with rate λ.
(a) Use the definition of conditional expectation to determine $E[X | X < c]$.
(b) Now determine $E[X | X < c]$ by using the following identity:

$$E[X] = E[X | X < c]P\{X < c\} + E[X | X > c]P\{X > c\}$$

15. One hundred items are simultaneously put on a life test. Suppose the lifetimes of the individual items are independent exponential random variables with mean 200 hours. The test will end when there have been a total of 5 failures. If T is the time at which the test ends, find $E[T]$ and $\text{Var}(T)$.

16. There are three jobs that need to be processed, with the processing time of job i being exponential with rate μ_i. There are two processors available, so processing on two of the jobs can immediately start, with processing on the final job to start when one of the initial ones is finished.
   (a) Let T_i denote the time at which the processing of job i is completed. If the objective is to minimize $E[T_1 + T_2 + T_3]$, which jobs should be initially processed if $\mu_1 < \mu_2 < \mu_3$?
   (b) Let M, called the *makespan*, be the time until all three jobs have been processed. With S equal to the time that there is only a single processor working, show that

$$2E[M] = E[S] + \sum_{i=1}^{3} 1/\mu_i$$

   For the rest of this problem, suppose that $\mu_1 = \mu_2 = \mu, \quad \mu_3 = \lambda$. Also, let $P(\mu)$ be the probability that the last job to finish is either job 1 or job 2, and let $P(\lambda) = 1 - P(\mu)$ be the probability that the last job to finish is job 3.
   (c) Express $E[S]$ in terms of $P(\mu)$ and $P(\lambda)$.
   Let $P_{i,j}(\mu)$ be the value of $P(\mu)$ when i and j are the jobs that are initially started.
   (d) Show that $P_{1,2}(\mu) \leqslant P_{1,3}(\mu)$.
   (e) If $\mu > \lambda$ show that $E[M]$ is minimized when job 3 is one of the jobs that is initially started.
   (f) If $\mu < \lambda$ show that $E[M]$ is minimized when processing is initially started on jobs 1 and 2.

17. A set of n cities is to be connected via communication links. The cost to construct a link between cities i and j is C_{ij}, $i \neq j$. Enough links should be constructed so that for each pair of cities there is a path of links that connects them. As a result, only $n - 1$ links need be constructed. A minimal cost algorithm for solving this problem (known as the minimal spanning tree problem) first constructs the cheapest of all the $\binom{n}{2}$ links. Then, at each additional stage it chooses the cheapest link that connects a city without any links to one with links. That is, if the first link is between cities 1 and 2, then the second link will either be between 1 and one of the links $3, \ldots, n$ or between 2 and one of the links $3, \ldots, n$. Suppose that all of the $\binom{n}{2}$ costs C_{ij} are independent exponential random variables with mean 1. Find the expected cost of the preceding algorithm if
   (a) $n = 3$,
   (b) $n = 4$.

*18. Let X_1 and X_2 be independent exponential random variables, each having rate μ. Let

$$X_{(1)} = \text{minimum}(X_1, X_2) \quad \text{and} \quad X_{(2)} = \text{maximum}(X_1, X_2)$$

Find
(a) $E[X_{(1)}]$,
(b) $\text{Var}[X_{(1)}]$,
(c) $E[X_{(2)}]$,
(d) $\text{Var}[X_{(2)}]$.

19. Repeat Exercise 18, but this time suppose that the X_i are independent exponentials with respective rates μ_i, $i = 1, 2$.

20. Consider a two-server system in which a customer is served first by server 1, then by server 2, and then departs. The service times at server i are exponential random variables with rates μ_i, $i = 1, 2$. When you arrive, you find server 1 free and two customers at server 2—customer A in service and customer B waiting in line.
(a) Find P_A, the probability that A is still in service when you move over to server 2.
(b) Find P_B, the probability that B is still in the system when you move over to server 2.
(c) Find $E[T]$, where T is the time that you spend in the system.

Hint: Write

$$T = S_1 + S_2 + W_A + W_B$$

where S_i is your service time at server i, W_A is the amount of time you wait in queue while A is being served, and W_B is the amount of time you wait in queue while B is being served.

21. In a certain system, a customer must first be served by server 1 and then by server 2. The service times at server i are exponential with rate μ_i, $i = 1, 2$. An arrival finding server 1 busy waits in line for that server. Upon completion of service at server 1, a customer either enters service with server 2 if that server is free or else remains with server 1 (blocking any other customer from entering service) until server 2 is free. Customers depart the system after being served by server 2. Suppose that when you arrive there is one customer in the system and that customer is being served by server 1. What is the expected total time you spend in the system?

22. Suppose in Exercise 21 you arrive to find two others in the system, one being served by server 1 and one by server 2. What is the expected time you spend in the system? Recall that if server 1 finishes before server 2, then server 1's customer will remain with him (thus blocking your entrance) until server 2 becomes free.

*23. A flashlight needs two batteries to be operational. Consider such a flashlight along with a set of n functional batteries—battery 1, battery $2, \ldots,$ battery n. Initially, battery 1 and 2 are installed. Whenever a battery fails, it is immediately replaced by the lowest numbered functional battery that has not yet been put in use. Suppose that the lifetimes of the different batteries are independent exponential random variables each having rate μ. At a random time, call it T, a battery will fail and our stockpile will be empty. At that moment exactly one of the batteries—which we call battery X—will not yet have failed.
(a) What is $P\{X = n\}$?
(b) What is $P\{X = 1\}$?
(c) What is $P\{X = i\}$?

(d) Find $E[T]$.
(e) What is the distribution of T?

24. There are two servers available to process n jobs. Initially, each server begins work on a job. Whenever a server completes work on a job, that job leaves the system and the server begins processing a new job (provided there are still jobs waiting to be processed). Let T denote the time until all jobs have been processed. If the time that it takes server i to process a job is exponentially distributed with rate μ_i, $i = 1, 2$, find $E[T]$ and $\text{Var}(T)$.

25. Customers can be served by any of three servers, where the service times of server i are exponentially distributed with rate μ_i, $i = 1, 2, 3$. Whenever a server becomes free, the customer who has been waiting the longest begins service with that server.
(a) If you arrive to find all three servers busy and no one waiting, find the expected time until you depart the system.
(b) If you arrive to find all three servers busy and one person waiting, find the expected time until you depart the system.

26. Each entering customer must be served first by server 1, then by server 2, and finally by server 3. The amount of time it takes to be served by server i is an exponential random variable with rate μ_i, $i = 1, 2, 3$. Suppose you enter the system when it contains a single customer who is being served by server 3.
(a) Find the probability that server 3 will still be busy when you move over to server 2.
(b) Find the probability that server 3 will still be busy when you move over to server 3.
(c) Find the expected amount of time that you spend in the system. (Whenever you encounter a busy server, you must wait for the service in progress to end before you can enter service.)
(d) Suppose that you enter the system when it contains a single customer who is being served by server 2. Find the expected amount of time that you spend in the system.

27. Show, in Example 5.7, that the distributions of the total cost are the same for the two algorithms.

28. Consider n components with independent lifetimes, which are such that component i functions for an exponential time with rate λ_i. Suppose that all components are initially in use and remain so until they fail.
(a) Find the probability that component 1 is the second component to fail.
(b) Find the expected time of the second failure.

Hint: Do not make use of part (a).

29. Let X and Y be independent exponential random variables with respective rates λ and μ, where $\lambda > \mu$. Let $c > 0$.
(a) Show that the conditional density function of X, given that $X + Y = c$, is

$$f_{X|X+Y}(x|c) = \frac{(\lambda - \mu)e^{-(\lambda-\mu)x}}{1 - e^{-(\lambda-\mu)c}}, \qquad 0 < x < c$$

(b) Use part (a) to find $E[X|X + Y = c]$.
(c) Find $E[Y|X + Y = c]$.

30. The lifetimes of A's dog and cat are independent exponential random variables with respective rates λ_d and λ_c. One of them has just died. Find the expected additional lifetime of the other pet.

31. A doctor has scheduled two appointments, one at 1 P.M. and the other at 1:30 P.M. The amounts of time that appointments last are independent exponential random variables with mean 30 minutes. Assuming that both patients are on time, find the expected amount of time that the 1:30 appointment spends at the doctor's office.

32. Let X be a uniform random variable on (0, 1), and consider a counting process where events occur at times $X + i$, for $i = 0, 1, 2, \ldots$.
 (a) Does this counting process have independent increments?
 (b) Does this counting process have stationary increments?

33. Let X and Y be independent exponential random variables with respective rates λ and μ.
 (a) Argue that, conditional on $X > Y$, the random variables $\min(X, Y)$ and $X - Y$ are independent.
 (b) Use part (a) to conclude that for any positive constant c

$$
\begin{aligned}
E[\min(X, Y)|X > Y + c] &= E[\min(X, Y)|X > Y] \\
&= E[\min(X, Y)] = \frac{1}{\lambda + \mu}
\end{aligned}
$$

 (c) Give a verbal explanation of why $\min(X, Y)$ and $X - Y$ are (unconditionally) independent.

34. Two individuals, A and B, both require kidney transplants. If she does not receive a new kidney, then A will die after an exponential time with rate μ_A, and B after an exponential time with rate μ_B. New kidneys arrive in accordance with a Poisson process having rate λ. It has been decided that the first kidney will go to A (or to B if B is alive and A is not at that time) and the next one to B (if still living).
 (a) What is the probability that A obtains a new kidney?
 (b) What is the probability that B obtains a new kidney?

35. Show that Definition 5.1 of a Poisson process implies Definition 5.3.

*36. Let $S(t)$ denote the price of a security at time t. A popular model for the process $\{S(t), t \geqslant 0\}$ supposes that the price remains unchanged until a "shock" occurs, at which time the price is multiplied by a random factor. If we let $N(t)$ denote the number of shocks by time t, and let X_i denote the ith multiplicative factor, then this model supposes that

$$
S(t) = S(0) \prod_{i=1}^{N(t)} X_i
$$

where $\prod_{i=1}^{N(t)} X_i$ is equal to 1 when $N(t) = 0$. Suppose that the X_i are independent exponential random variables with rate μ; that $\{N(t), t \geqslant 0\}$ is a Poisson process with rate λ; that $\{N(t), t \geqslant 0\}$ is independent of the X_i; and that $S(0) = s$.
 (a) Find $E[S(t)]$.
 (b) Find $E[S^2(t)]$.

37. A machine works for an exponentially distributed time with rate μ and then fails. A repair crew checks the machine at times distributed according to a Poisson process with rate λ; if the machine is found to have failed then it is immediately replaced. Find the expected time between replacements of machines.

38. Let $\{M_i(t), t \geqslant 0\}, i = 1, 2, 3$ be independent Poisson processes with respective rates $\lambda_i, i = 1, 2$, and set

$$N_1(t) = M_1(t) + M_2(t), \quad N_2(t) = M_2(t) + M_3(t)$$

The stochastic process $\{(N_1(t), N_2(t)), t \geqslant 0\}$ is called a bivariate Poisson process.
(a) Find $P\{N_1(t) = n, N_2(t) = m\}$.
(b) Find $\text{Cov}\big(N_1(t), N_2(t)\big)$.

39. A certain scientific theory supposes that mistakes in cell division occur according to a Poisson process with rate 2.5 per year, and that an individual dies when 196 such mistakes have occurred. Assuming this theory, find
(a) the mean lifetime of an individual,
(b) the variance of the lifetime of an individual.
Also approximate
(c) the probability that an individual dies before age 67.2,
(d) the probability that an individual reaches age 90,
(e) the probability that an individual reaches age 100.

*40. Show that if $\{N_i(t), t \geqslant 0\}$ are independent Poisson processes with rate $\lambda_i, i = 1, 2$, then $\{N(t), t \geqslant 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$ where $N(t) = N_1(t) + N_2(t)$.

41. In Exercise 40 what is the probability that the first event of the combined process is from the N_1 process?

42. Let $\{N(t), t \geqslant 0\}$ be a Poisson process with rate λ. Let S_n denote the time of the nth event. Find
(a) $E[S_4]$,
(b) $E[S_4|N(1) = 2]$,
(c) $E[N(4) - N(2)|N(1) = 3]$.

43. Customers arrive at a two-server service station according to a Poisson process with rate λ. Whenever a new customer arrives, any customer that is in the system immediately departs. A new arrival enters service first with server 1 and then with server 2. If the service times at the servers are independent exponentials with respective rates μ_1 and μ_2, what proportion of entering customers completes their service with server 2?

44. Cars pass a certain street location according to a Poisson process with rate λ. A woman who wants to cross the street at that location waits until she can see that no cars will come by in the next T time units.
(a) Find the probability that her waiting time is 0.
(b) Find her expected waiting time.

Hint: Condition on the time of the first car.

45. Let $\{N(t), t \geqslant 0\}$ be a Poisson process with rate λ that is independent of the nonnegative random variable T with mean μ and variance σ^2. Find
(a) $\text{Cov}(T, N(T))$,
(b) $\text{Var}(N(T))$.

46. Let $\{N(t),\ t \geqslant 0\}$ be a Poisson process with rate λ that is independent of the sequence $X_1, X_2, \ldots$ of independent and identically distributed random variables with mean μ and variance σ^2. Find

$$\mathrm{Cov}\left(N(t), \sum_{i=1}^{N(t)} X_i\right)$$

47. Consider a two-server parallel queuing system where customers arrive according to a Poisson process with rate λ, and where the service times are exponential with rate μ. Moreover, suppose that arrivals finding both servers busy immediately depart without receiving any service (such a customer is said to be lost), whereas those finding at least one free server immediately enter service and then depart when their service is completed.
    (a) If both servers are presently busy, find the expected time until the next customer enters the system.
    (b) Starting empty, find the expected time until both servers are busy.
    (c) Find the expected time between two successive lost customers.
48. Consider an n-server parallel queuing system where customers arrive according to a Poisson process with rate λ, where the service times are exponential random variables with rate μ, and where any arrival finding all servers busy immediately departs without receiving any service. If an arrival finds all servers busy, find
    (a) the expected number of busy servers found by the next arrival,
    (b) the probability that the next arrival finds all servers free,
    (c) the probability that the next arrival finds exactly i of the servers free.
49. Events occur according to a Poisson process with rate λ. Each time an event occurs, we must decide whether or not to stop, with our objective being to stop at the last event to occur prior to some specified time T, where $T > 1/\lambda$. That is, if an event occurs at time t, $0 \leqslant t \leqslant T$, and we decide to stop, then we win if there are no additional events by time T, and we lose otherwise. If we do not stop when an event occurs and no additional events occur by time T, then we lose. Also, if no events occur by time T, then we lose. Consider the strategy that stops at the first event to occur after some fixed time s, $0 \leqslant s \leqslant T$.
    (a) Using this strategy, what is the probability of winning?
    (b) What value of s maximizes the probability of winning?
    (c) Show that one's probability of winning when using the preceding strategy with the value of s specified in part (b) is $1/e$.
50. The number of hours between successive train arrivals at the station is uniformly distributed on $(0, 1)$. Passengers arrive according to a Poisson process with rate 7 per hour. Suppose a train has just left the station. Let X denote the number of people who get on the next train. Find
    (a) $E[X]$,
    (b) $\mathrm{Var}(X)$.
51. If an individual has never had a previous automobile accident, then the probability he or she has an accident in the next h time units is $\beta h + o(h)$; on the other hand, if he or she has ever had a previous accident, then the probability is $\alpha h + o(h)$. Find the expected number of accidents an individual has by time t.

52. Teams 1 and 2 are playing a match. The teams score points according to independent Poisson processes with respective rates λ_1 and λ_2. If the match ends when one of the teams has scored k more points than the other, find the probability that team 1 wins.

Hint: Relate this to the gambler's ruin problem.

53. The water level of a certain reservoir is depleted at a constant rate of 1000 units daily. The reservoir is refilled by randomly occurring rainfalls. Rainfalls occur according to a Poisson process with rate 0.2 per day. The amount of water added to the reservoir by a rainfall is 5000 units with probability 0.8 or 8000 units with probability 0.2. The present water level is just slightly below 5000 units.
(a) What is the probability the reservoir will be empty after five days?
(b) What is the probability the reservoir will be empty sometime within the next ten days?

54. A viral linear DNA molecule of length, say, 1 is often known to contain a certain "marked position," with the exact location of this mark being unknown. One approach to locating the marked position is to cut the molecule by agents that break it at points chosen according to a Poisson process with rate λ. It is then possible to determine the fragment that contains the marked position. For instance, letting m denote the location on the line of the marked position, then if L_1 denotes the last Poisson event time before m (or 0 if there are no Poisson events in $[0, m]$), and R_1 denotes the first Poisson event time after m (or 1 if there are no Poisson events in $[m, 1]$), then it would be learned that the marked position lies between L_1 and R_1. Find
(a) $P\{L_1 = 0\}$,
(b) $P\{L_1 < x\}$, $0 < x < m$,
(c) $P\{R_1 = 1\}$,
(d) $P\{R_1 > x\}$, $m < x < 1$.
By repeating the preceding process on identical copies of the DNA molecule, we are able to zero in on the location of the marked position. If the cutting procedure is utilized on n identical copies of the molecule, yielding the data L_i, R_i, $i = 1, \ldots, n$, then it follows that the marked position lies between L and R, where

$$L = \max_i L_i, \quad R = \min_i R_i$$

(e) Find $E[R - L]$, and in doing so, show that $E[R - L] \sim \frac{2}{n\lambda}$.

55. Consider a single server queuing system where customers arrive according to a Poisson process with rate λ, service times are exponential with rate μ, and customers are served in the order of their arrival. Suppose that a customer arrives and finds $n - 1$ others in the system. Let X denote the number in the system at the moment that customer departs. Find the probability mass function of X.

Hint: Relate this to a negative binomial random variable.

56. An event independently occurs on each day with probability p. Let $N(n)$ denote the total number of events that occur on the first n days, and let T_r denote the day on which the rth event occurs.
(a) What is the distribution of $N(n)$?
(b) What is the distribution of T_1?

(c) What is the distribution of T_r?

(d) Given that $N(n) = r$, show that the set of r days on which events occurred has the same distribution as a random selection (without replacement) of r of the values $1, 2, \ldots, n$.

*57. Events occur according to a Poisson process with rate $\lambda = 2$ per hour.

(a) What is the probability that no event occurs between 8 P.M. and 9 P.M.?

(b) Starting at noon, what is the expected time at which the fourth event occurs?

(c) What is the probability that two or more events occur between 6 P.M. and 8 P.M.?

58. Consider the coupon collecting problem where there are m distinct types of coupons, and each new coupon collected is type j with probability p_j, $\sum_{j=1}^{m} p_j = 1$. Suppose you stop collecting when you have a complete set of at least one of each type. Show that

$$P\{i \text{ is the last type collected}\} = E\left[\prod_{j \neq i}(1 - U^{\lambda_j/\lambda_i})\right]$$

where U is a uniform random variable on $(0, 1)$.

59. There are two types of claims that are made to an insurance company. Let $N_i(t)$ denote the number of type i claims made by time t, and suppose that $\{N_1(t), t \geqslant 0\}$ and $\{N_2(t), t \geqslant 0\}$ are independent Poisson processes with rates $\lambda_1 = 10$ and $\lambda_2 = 1$. The amounts of successive type 1 claims are independent exponential random variables with mean \$1000 whereas the amounts from type 2 claims are independent exponential random variables with mean \$5000. A claim for \$4000 has just been received; what is the probability it is a type 1 claim?

*60. Customers arrive at a bank at a Poisson rate λ. Suppose two customers arrived during the first hour. What is the probability that

(a) both arrived during the first 20 minutes?

(b) at least one arrived during the first 20 minutes?

61. A system has a random number of flaws that we will suppose is Poisson distributed with mean c. Each of these flaws will, independently, cause the system to fail at a random time having distribution G. When a system failure occurs, suppose that the flaw causing the failure is immediately located and fixed.

(a) What is the distribution of the number of failures by time t?

(b) What is the distribution of the number of flaws that remain in the system at time t?

(c) Are the random variables in parts (a) and (b) dependent or independent?

62. Suppose that the number of typographical errors in a new text is Poisson distributed with mean λ. Two proofreaders independently read the text. Suppose that each error is independently found by proofreader i with probability p_i, $i = 1, 2$. Let X_1 denote the number of errors that are found by proofreader 1 but not by proofreader 2. Let X_2 denote the number of errors that are found by proofreader 2 but not by proofreader 1. Let X_3 denote the number of errors that are found by both proofreaders. Finally, let X_4 denote the number of errors found by neither proofreader.

(a) Describe the joint probability distribution of X_1, X_2, X_3, X_4.
(b) Show that

$$\frac{E[X_1]}{E[X_3]} = \frac{1-p_2}{p_2} \quad \text{and} \quad \frac{E[X_2]}{E[X_3]} = \frac{1-p_1}{p_1}$$

Suppose now that λ, p_1, and p_2 are all unknown.
(c) By using X_i as an estimator of $E[X_i]$, $i = 1, 2, 3$, present estimators of p_1, p_2, and λ.
(d) Give an estimator of X_4, the number of errors not found by either proofreader.

63. Consider an infinite server queuing system in which customers arrive in accordance with a Poisson process with rate λ, and where the service distribution is exponential with rate μ. Let $X(t)$ denote the number of customers in the system at time t. Find
(a) $E[X(t+s)|X(s) = n]$;
(b) $\text{Var}[X(t+s)|X(s) = n]$.

Hint: Divide the customers in the system at time $t + s$ into two groups, one consisting of "old" customers and the other of "new" customers.

(c) Consider an infinite server queuing system in which customers arrive according to a Poisson process with rate λ, and where the service times are all exponential random variables with rate μ. If there is currently a single customer in the system, find the probability that the system becomes empty when that customer departs.

*64. Suppose that people arrive at a bus stop in accordance with a Poisson process with rate λ. The bus departs at time t. Let X denote the total amount of waiting time of all those who get on the bus at time t. We want to determine $\text{Var}(X)$. Let $N(t)$ denote the number of arrivals by time t.
(a) What is $E[X|N(t)]$?
(b) Argue that $\text{Var}[X|N(t)] = N(t)t^2/12$.
(c) What is $\text{Var}(X)$?

65. An average of 500 people pass the California bar exam each year. A California lawyer practices law, on average, for 30 years. Assuming these numbers remain steady, how many lawyers would you expect California to have in 2050?

66. Policyholders of a certain insurance company have accidents at times distributed according to a Poisson process with rate λ. The amount of time from when the accident occurs until a claim is made has distribution G.
(a) Find the probability there are exactly n incurred but as yet unreported claims at time t.
(b) Suppose that each claim amount has distribution F, and that the claim amount is independent of the time that it takes to report the claim. Find the expected value of the sum of all incurred but as yet unreported claims at time t.

67. Satellites are launched into space at times distributed according to a Poisson process with rate λ. Each satellite independently spends a random time (having distribution G) in space before falling to the ground. Find the probability that none of the satellites in the air at time t was launched before time s, where $s < t$.

68. Suppose that electrical shocks having random amplitudes occur at times distributed according to a Poisson process $\{N(t), t \geqslant 0\}$ with rate λ. Suppose that the

amplitudes of the successive shocks are independent both of other amplitudes and of the arrival times of shocks, and also that the amplitudes have distribution F with mean μ. Suppose also that the amplitude of a shock decreases with time at an exponential rate α, meaning that an initial amplitude A will have value $Ae^{-\alpha x}$ after an additional time x has elapsed. Let $A(t)$ denote the sum of all amplitudes at time t. That is,

$$A(t) = \sum_{i=1}^{N(t)} A_i e^{-\alpha(t-S_i)}$$

where A_i and S_i are the initial amplitude and the arrival time of shock i.

(a) Find $E[A(t)]$ by conditioning on $N(t)$.

(b) Without any computations, explain why $A(t)$ has the same distribution as does $D(t)$ of Example 5.21.

69. Let $\{N(t), t \geqslant 0\}$ be a Poisson process with rate λ. For $s < t$, find

(a) $P(N(t) > N(s))$;

(b) $P(N(s) = 0, N(t) = 3)$;

(c) $E[N(t)|N(s) = 4]$;

(d) $E[N(s)|N(t) = 4]$.

70. For the infinite server queue with Poisson arrivals and general service distribution G, find the probability that

(a) the first customer to arrive is also the first to depart.

Let $S(t)$ equal the sum of the remaining service times of all customers in the system at time t.

(b) Argue that $S(t)$ is a compound Poisson random variable.

(c) Find $E[S(t)]$.

(d) Find $\mathrm{Var}(S(t))$.

71. Let S_n denote the time of the nth event of the Poisson process $\{N(t),\ t \geqslant 0\}$ having rate λ. Show, for an arbitrary function g, that the random variable $\sum_{i=1}^{N(t)} g(S_i)$ has the same distribution as the compound Poisson random variable $\sum_{i=1}^{N(t)} g(U_i)$, where $U_1, U_2, \ldots$ is a sequence of independent and identically distributed uniform $(0,\ t)$ random variables that is independent of N, a Poisson random variable with mean λt. Consequently, conclude that

$$E\left[\sum_{i=1}^{N(t)} g(S_i)\right] = \lambda \int_0^t g(x)\,dx \quad \mathrm{Var}\left(\sum_{i=1}^{N(t)} g(S_i)\right) = \lambda \int_0^t g^2(x)\,dx$$

72. A cable car starts off with n riders. The times between successive stops of the car are independent exponential random variables with rate λ. At each stop one rider gets off. This takes no time, and no additional riders get on. After a rider gets off the car, he or she walks home. Independently of all else, the walk takes an exponential time with rate μ.

(a) What is the distribution of the time at which the last rider departs the car?

(b) Suppose the last rider departs the car at time t. What is the probability that all the other riders are home at that time?

73. Shocks occur according to a Poisson process with rate λ, and each shock independently causes a certain system to fail with probability p. Let T denote the time at which the system fails and let N denote the number of shocks that it takes.
   (a) Find the conditional distribution of T given that $N = n$.
   (b) Calculate the conditional distribution of N, given that $T = t$, and notice that it is distributed as 1 plus a Poisson random variable with mean $\lambda(1-p)t$.
   (c) Explain how the result in part (b) could have been obtained without any calculations.

74. The number of missing items in a certain location, call it X, is a Poisson random variable with mean λ. When searching the location, each item will independently be found after an exponentially distributed time with rate μ. A reward of R is received for each item found, and a searching cost of C per unit of search time is incurred. Suppose that you search for a fixed time t and then stop.
   (a) Find your total expected return.
   (b) Find the value of t that maximizes the total expected return.
   (c) The policy of searching for a fixed time is a static policy. Would a dynamic policy, which allows the decision as to whether to stop at each time t, depend on the number already found by t be beneficial?

   Hint: How does the distribution of the number of items not yet found by time t depend on the number already found by that time?

75. Suppose that the times between successive arrivals of customers at a single-server station are independent random variables having a common distribution F. Suppose that when a customer arrives, he or she either immediately enters service if the server is free or else joins the end of the waiting line if the server is busy with another customer. When the server completes work on a customer, that customer leaves the system and the next waiting customer, if there are any, enters service. Let X_n denote the number of customers in the system immediately before the nth arrival, and let Y_n denote the number of customers that remain in the system when the nth customer departs. The successive service times of customers are independent random variables (which are also independent of the interarrival times) having a common distribution G.
   (a) If F is the exponential distribution with rate λ, which, if any, of the processes $\{X_n\}$, $\{Y_n\}$ is a Markov chain?
   (b) If G is the exponential distribution with rate μ, which, if any, of the processes $\{X_n\}$, $\{Y_n\}$ is a Markov chain?
   (c) Give the transition probabilities of any Markov chains in parts (a) and (b).

76. For the model of Example 5.27, find the mean and variance of the number of customers served in a busy period.

77. Suppose that customers arrive to a system according to a Poisson process with rate λ. There are an infinite number of servers in this system so a customer begins service upon arrival. The service times of the arrivals are independent exponential random variables with rate μ, and are independent of the arrival process. Customers depart the system when their service ends. Let N be the number of arrivals before the first departure.
   (a) Find $P(N = 1)$.
   (b) Find $P(N = 2)$.

(c) Find $P(N = j)$.
(d) Find the probability that the first to arrive is the first to depart.
(e) Find the expected time of the first departure.

78. A store opens at 8 A.M. From 8 until 10 A.M. customers arrive at a Poisson rate of four an hour. Between 10 A.M. and 12 P.M. they arrive at a Poisson rate of eight an hour. From 12 P.M. to 2 P.M. the arrival rate increases steadily from eight per hour at 12 P.M. to ten per hour at 2 P.M.; and from 2 to 5 P.M. the arrival rate drops steadily from ten per hour at 2 P.M. to four per hour at 5 P.M.. Determine the probability distribution of the number of customers that enter the store on a given day.

79. Consider a nonhomogeneous Poisson process whose intensity function $\lambda(t)$ is bounded and continuous. Show that such a process is equivalent to a process of counted events from a (homogeneous) Poisson process having rate λ, where an event at time t is counted (independent of the past) with probability $\lambda(t)/\lambda$; and where λ is chosen so that $\lambda(s) < \lambda$ for all s.

80. Let $T_1, T_2, \ldots$ denote the interarrival times of events of a nonhomogeneous Poisson process having intensity function $\lambda(t)$.
(a) Are the T_i independent?
(b) Are the T_i identically distributed?
(c) Find the distribution of T_1.

81. (a) Let $\{N(t), t \geqslant 0\}$ be a nonhomogeneous Poisson process with mean value function $m(t)$. Given $N(t) = n$, show that the unordered set of arrival times has the same distribution as n independent and identically distributed random variables having distribution function

$$F(x) = \begin{cases} \dfrac{m(x)}{m(t)}, & x \leqslant t \\ 1, & x \geqslant t \end{cases}$$

(b) Suppose that workmen incur accidents in accordance with a nonhomogeneous Poisson process with mean value function $m(t)$. Suppose further that each injured man is out of work for a random amount of time having distribution F. Let $X(t)$ be the number of workers who are out of work at time t. By using part (a), find $E[X(t)]$.

82. Let $X_1, X_2, \ldots$ be independent positive continuous random variables with a common density function f, and suppose this sequence is independent of N, a Poisson random variable with mean λ. Define

$$N(t) = \text{number of } i \leqslant N : X_i \leqslant t$$

Show that $\{N(t), t \geqslant 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t) = \lambda f(t)$.

83. Suppose that $\{N_0(t), t \geqslant 0\}$ is a Poisson process with rate $\lambda = 1$. Let $\lambda(t)$ denote a nonnegative function of t, and let

$$m(t) = \int_0^t \lambda(s)\, ds$$

Define $N(t)$ by

$$N(t) = N_0(m(t))$$

Argue that $\{N(t), t \geqslant 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda(t), t \geqslant 0$.

Hint: Make use of the identity

$$m(t+h) - m(t) = m'(t)h + o(h)$$

*84. Let $X_1, X_2, \ldots$ be independent and identically distributed nonnegative continuous random variables having density function $f(x)$. We say that a record occurs at time n if X_n is larger than each of the previous values $X_1, \ldots, X_{n-1}$. (A record automatically occurs at time 1.) If a record occurs at time n, then X_n is called a *record value*. In other words, a record occurs whenever a new high is reached, and that new high is called the record value. Let $N(t)$ denote the number of record values that are less than or equal to t. Characterize the process $\{N(t),\ t \geqslant 0\}$ when
(a) f is an arbitrary continuous density function.
(b) $f(x) = \lambda e^{-\lambda x}$.

Hint: Finish the following sentence: There will be a record whose value is between t and $t + dt$ if the first X_i that is greater than t lies between ...

85. An insurance company pays out claims on its life insurance policies in accordance with a Poisson process having rate $\lambda = 5$ per week. If the amount of money paid on each policy is exponentially distributed with mean \$2000, what is the mean and variance of the amount of money paid by the insurance company in a four-week span?

86. In good years, storms occur according to a Poisson process with rate 3 per unit time, while in other years they occur according to a Poisson process with rate 5 per unit time. Suppose next year will be a good year with probability 0.3. Let $N(t)$ denote the number of storms during the first t time units of next year.
(a) Find $P\{N(t) = n\}$.
(b) Is $\{N(t)\}$ a Poisson process?
(c) Does $\{N(t)\}$ have stationary increments? Why or why not?
(d) Does it have independent increments? Why or why not?
(e) If next year starts off with three storms by time $t = 1$, what is the conditional probability it is a good year?

87. Determine

$$\text{Cov}[X(t), X(t+s)]$$

when $\{X(t), t \geqslant 0\}$ is a compound Poisson process.

88. Customers arrive at the automatic teller machine in accordance with a Poisson process with rate 12 per hour. The amount of money withdrawn on each transaction is a random variable with mean \$30 and standard deviation \$50. (A negative withdrawal means that money was deposited.) The machine is in use for 15 hours daily. Approximate the probability that the total daily withdrawal is less than \$6000.

89. Some components of a two-component system fail after receiving a shock. Shocks of three types arrive independently and in accordance with Poisson processes. Shocks of the first type arrive at a Poisson rate λ_1 and cause the first component to fail. Those of the second type arrive at a Poisson rate λ_2 and cause the second component to fail. The third type of shock arrives at a Poisson rate λ_3 and causes both components to fail. Let X_1 and X_2 denote the survival times for the two components. Show that the joint distribution of X_1 and X_2 is given by

$$P\{X_1 > s, X_1 > t\} = \exp\{-\lambda_1 s - \lambda_2 t - \lambda_3 \max(s,t)\}$$

This distribution is known as the *bivariate exponential distribution.*

90. In Exercise 89 show that X_1 and X_2 both have exponential distributions.

*91. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables. Show that the probability that the largest of them is greater than the sum of the others is $n/2^{n-1}$. That is, if

$$M = \max_j X_j$$

then show

$$P\left\{M > \sum_{i=1}^{n} X_i - M\right\} = \frac{n}{2^{n-1}}$$

Hint: What is $P\{X_1 > \sum_{i=2}^{n} X_i\}$?

92. Prove Equation (5.22).

93. Prove that

(a) $\max(X_1, X_2) = X_1 + X_2 - \min(X_1, X_2)$ and, in general,

(b)
$$\begin{aligned}\max(X_1, \ldots, X_n) = \sum_{1}^{n} X_i - \sum_{i<j}\sum \min(X_i, X_j) \\ + \sum_{i<j<k}\sum\sum \min(X_i, X_j, X_k) + \cdots \\ + (-1)^{n-1} \min(X_i, X_j, \ldots, X_n)\end{aligned}$$

(c) Show by defining appropriate random variables X_i, $i = 1, \ldots, n$, and by taking expectations in part (b) how to obtain the well-known formula

$$P\left(\bigcup_1^n A_i\right) = \sum_i P(A_i) - \sum_{i<j}\sum P(A_i A_j) + \cdots + (-1)^{n-1} P(A_1 \cdots A_n)$$

(d) Consider n independent Poisson processes—the ith having rate λ_i. Derive an expression for the expected time until an event has occurred in all n processes.

94. A two-dimensional Poisson process is a process of randomly occurring events in the plane such that

(i) for any region of area A the number of events in that region has a Poisson distribution with mean λA, and

(ii) the number of events in nonoverlapping regions are independent.

For such a process, consider an arbitrary point in the plane and let X denote its distance from its nearest event (where distance is measured in the usual Euclidean manner). Show that

(a) $P\{X > t\} = e^{-\lambda\pi t^2}$,

(b) $E[X] = \frac{1}{2\sqrt{\lambda}}$.

95. Let $\{N(t), t \geqslant 0\}$ be a conditional Poisson process with a random rate L.
    (a) Derive an expression for $E[L|N(t) = n]$.
    (b) Find, for $s > t$, $E[N(s)|N(t) = n]$.
    (c) Find, for $s < t$, $E[N(s)|N(t) = n]$.

96. For the conditional Poisson process, let $m_1 = E[L],\ m_2 = E[L^2]$. In terms of m_1 and m_2, find $\text{Cov}(N(s), N(t))$ for $s \leqslant t$.

97. Consider a conditional Poisson process in which the rate L is, as in Example 5.29, gamma distributed with parameters m and p. Find the conditional density function of L given that $N(t) = n$.

Chapter 5

1. (a) e^{-1} (b) e^{-1}

2. Let T be the time you spend in the system; let S_i be the service time of person i in the queue: let R be the remaining service time of the person in service; let S be your service time. Then,

$$E[T] = E[R + S_1 + S_2 + S_3 + S_4 + S]$$
$$= E[R] + \sum_{i=1}^{4} E[S_i] + E[S] = 6/\mu$$

where we have used the lack of memory property to conclude that R is also exponential with rate μ.

3. The conditional distribution of X, given that $X > 1$, is the same as the unconditional distribution of $1 + X$. Hence, (a) is correct.

4. (a) 0 (b) $\frac{1}{27}$ (c) $\frac{1}{4}$

5. e^{-1} by lack of memory.

6. Condition on which server initially finishes first. Now,

$$P\{\text{Smith is last}|\text{server 1 finishes first}\}$$
$$= P\{\text{server 1 finishes before server 2}\} \quad \text{by lack of memory}$$
$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Similarly,

$$P\{\text{Smith is last}|\text{server 2 finished first}\} = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

and thus

$$P\{\text{Smith is last}\} = \left[\frac{\lambda_1}{\lambda_1 + \lambda_2}\right]^2 + \left[\frac{\lambda_2}{\lambda_1 + \lambda_2}\right]^2$$

7. $P\{X_1 < X_2 | \min(X_1, X_2) = t\}$

$$= \frac{P\{X_1 < X_2, \min(X_1, X_2) = t\}}{P\{\min(X_1, X_2) = t\}}$$
$$= \frac{P\{X_1 = t, X_2 > t\}}{P\{X_1 = t, X_2 > t\} + P\{X_2 = t, X_1 > t\}}$$
$$= \frac{f_1(t)\bar{F}_2(t)}{f_1(t)\bar{F}_2(t) + f_2(t)\bar{F}_1(t)}$$

Dividing though by $\bar{F}_1(t)\bar{F}_2(t)$ yields the result. (For a more rigorous argument, replace "$= t$" by "$\in (t, t+\epsilon)$" throughout, and then let $\epsilon \to 0$.)

8. Let X_i have density f_i and tail distribution $\bar{F}_i$.

$$r(t) = \frac{\sum_{i=1}^{n} P\{T = i\} f_i(t)}{\sum_{j=1}^{n} P\{T = j\} \bar{F}_j(t)}$$
$$= \frac{\sum_{i=1}^{n} P\{T = i\} r_i(t) \bar{F}_i(t)}{\sum_{j=1}^{n} P\{T = j\} \bar{F}_j(t)}$$

The result now follows from

$$P\{T = i | X > t\} = \frac{P\{T = i\} \bar{F}_i(t)}{\sum_{j=1}^{n} P\{T = j\} \bar{F}_j(t)}$$

9. Condition on whether machine 1 is still working at time t, to obtain the answer,

$$1 - e^{-\lambda_1 t} + e^{-\lambda_1 t} \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

11. (a) Using Equation (5.5), the lack of memory property of the exponential, as well as the fact that the minimum of independent exponentials is exponential with a rate equal to the sum of their individual rates, it follows that

$$P(A_1) = \frac{n\mu}{\lambda + n\mu}$$

and, for $j > 1$,

$$P(A_j|A_1 \cdots A_{j-1}) = \frac{(n-j+1)\mu}{\lambda + (n-j+1)\mu}$$

Hence,

$$p = \prod_{j=1}^{n} \frac{(n-j+1)\mu}{\lambda + (n-j+1)\mu}$$

(b) When $n = 2$,
$P\{\max Y_i < X\}$

$$\begin{aligned}
&= \int_0^\infty P\{\max Y_i < X|X = x\}\lambda e^{-\lambda x}dx \\
&= \int_0^\infty P\{\max Y_i < x\}\lambda e^{-\lambda x}dx \\
&= \int_0^\infty (1 - e^{-\mu x})^2 \lambda e^{-\lambda x}dx \\
&= \int_0^\infty (1 - 2e^{-\mu x} + e^{-2\mu x})^2 \lambda e^{-\lambda x}dx \\
&= 1 - \frac{2\lambda}{\lambda + \mu} + \frac{\lambda}{2\mu + \lambda} \\
&= \frac{2\mu^2}{(\lambda + \mu)(\lambda + 2\mu)}
\end{aligned}$$

12. (a) $P\{X_1 < X_2 < X_3\}$

$$\begin{aligned}
&= P\{X_1 = \min(X_1, X_2, X_3)\} \\
&\quad P\{X_2 < X_3|X_1 = \min(X_1, X_2, X_3)\} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} P\{X_2 < X_3|X_1 \\
&\qquad\qquad = \min(X_1, X_2, X_3)\} \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_2 + \lambda_3}
\end{aligned}$$

where the final equality follows by the lack of memory property.

(b) $P\{X_2 < X_3|X_1 = \max(X_1, X_2, X_3)\}$

$$\begin{aligned}
&= \frac{P\{X_2 < X_3 < X_1\}}{P\{X_2 < X_3 < X_1\} + P\{X_3 < X_2 < X_1\}} \\
&= \frac{\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_3}{\lambda_1 + \lambda_3}}{\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_3}{\lambda_1 + \lambda_3} + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_2}{\lambda_1 + \lambda_2}} \\
&= \frac{1/(\lambda_1 + \lambda_3)}{1/(\lambda_1 + \lambda_3) + 1/(\lambda_1 + \lambda_2)}
\end{aligned}$$

(c) $\dfrac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \dfrac{1}{+ \lambda_2 + \lambda_3} + \dfrac{1}{\lambda_3}$

(d) $$\sum_{i \neq j \neq k} \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_j}{\lambda_j + \lambda_k} \left[\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k}\right]$$

where the sum is over all 6 permutations of 1, 2, 3.

13. Let T_n denote the time until the n^{th} person in line departs the line. Also, let D be the time until the first departure from the line, and let X be the additional time after D until T_n. Then,

$$\begin{aligned}
E[T_n] &= E[D] + E[X] \\
&= \frac{1}{n\theta + \mu} + \frac{(n-1)\theta + \mu}{n\theta + \mu} E[T_{n-1}]
\end{aligned}$$

where $E[X]$ was computed by conditioning on whether the first departure was the person in line. Hence,

$$E[T_n] = A_n + B_n E[T_{n-1}]$$

where

$$A_n = \frac{1}{n\theta + \mu}, \qquad B_n = \frac{(n-1)\theta + \mu}{n\theta + \mu}$$

Solving gives the solution

$$\begin{aligned}
E[T_n] &= A_n + \sum_{i=1}^{n-1} A_{n-i} \prod_{j=n-i+1}^{n} B_j \\
&= A_n + \sum_{i=1}^{n-1} 1/(n\theta + \mu) \\
&= \frac{n}{n\theta + \mu}
\end{aligned}$$

Another way to solve the preceding is to let I_j equal 1 if customer n is still in line at the time of the $(j-1)^{st}$ departure from the line, and let X_j denote the time between the $(j-1)^{st}$ and j^{th} departure from line. (Of course, these departures only refer to the first n people in line.) Then

$$T_n = \sum_{j=1}^{n} I_j X_j$$

The independence of I_j and X_j gives

$$E[T_n] = \sum_{j=1}^{n} E[I_j]E[X_j]$$

But,

$$\begin{aligned}
E[I_j] &= \frac{(n-1)\theta + \mu}{n\theta + \mu} \cdots \frac{(n-j+1)\theta + \mu}{(n-j+2)\theta + \mu} \\
&= \frac{(n-j+1)\theta + \mu}{n\theta + \mu}
\end{aligned}$$

and

$$E[X_j] = \frac{1}{(n-j+1)\theta + \mu}$$

which gives the result.

14. (a) The conditional density of X gives that $X < c$ is

$$f(x|X < c) = \frac{f(x)}{P\{x < c\}} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}, \ 0 < x < c$$

Hence,

$$E[X|X < c] = \int_0^c x\lambda e^{-\lambda x} dx / (1 - e^{-\lambda c})$$

Integration by parts yields

$$\int_0^c x\lambda e^{-\lambda x}\, dx = -xe^{-\lambda x}\Big|_0^c + \int_0^c e^{-\lambda x} dx$$
$$= -ce^{-\lambda c} + (1 - e^{-\lambda c})/\lambda$$

Hence,

$$E[X|X < c] = 1/\lambda - ce^{-\lambda c}/(1 - e^{-\lambda c})$$

(b) $1/\lambda = E[X|X < c](1 - e^{-\lambda c}) + (c + 1/\lambda)e^{-\lambda c}$
This simplifies to the same answer as given in part (a).

15. Let T_i denote the time between the $(i-1)^{th}$ and the i^{th} failure. Then the T_i are independent with T_i being exponential with rate $(101 - i)/200$. Thus,

$$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{101 - i}$$
$$Var(T) = \sum_{i=1}^{5} Var(T_i) = \sum_{i=1}^{5} \frac{(200)^2}{(101 - i)^2}$$

16. (a) Suppose i and j are initially begun, with k waiting for one of them to be completed. Then

$$E[T_i] + E[T_j] + E[T_k] = \tfrac{1}{\mu_i} + \tfrac{1}{\mu_j} + \tfrac{1}{\mu_i+\mu_j} + \tfrac{1}{\mu_k}$$
$$= \textstyle\sum_{i=1}^{3} \tfrac{1}{\mu_i} + \tfrac{1}{\mu_i+\mu_j}$$

Hence, the preceding is minimized when $\mu_i + \mu_j$ is as large as possible, showing that it is optimal to begin processing on jobs 2 and 3. Consequently, to minimize the expected sum of the completion times the jobs having largest rates should be initiated first.

(b) Letting X_i be the processing time of job i, this follows from the identity

$$2(M - S) + S = \sum_{i=1}^{3} X_i$$

which follows because if we interpret X_i as the work of job i then the total amount of work is $\sum_{i=1}^{3} X_i$, whereas work is processed at rate 2 per unit time when both servers are busy and at rate 1 per unit time when only a single processor is working.

(c) $E[S] = \dfrac{1}{\mu}P(\mu) + \dfrac{1}{\lambda}P(\lambda)$

(d) $P_{1,2}(\mu) = \dfrac{\lambda}{\mu + \lambda} < \dfrac{\lambda}{\mu + \lambda} + \dfrac{\mu}{\mu + \lambda}\dfrac{\lambda}{\mu + \lambda}$
$= P_{1,3}(\mu)$

(e) If $\mu > \lambda$ then $E[S]$ is minimized when $P(\mu)$ is as large as possible. Hence, because minimizing $E[S]$ is equivalent to minimizing $E[M]$, it follows that $E[M]$ is minimized when jobs 1 and 3 are initially processed.

(f) In this case $E[M]$ is minimized when jobs 1 and 2 are initially processed. In all cases $E[M]$ is minimized when the jobs having smallest rates are initiated first.

17. Let C_i denote the cost of the i^{th} link to be constructed, $i = 1, \ldots, n-1$. Note that the first link can be any of the $\binom{n}{2}$ possible links. Given the first one, the second link must connect one of the 2 cities joined by the first link with one of the $n-2$ cities without any links. Thus, given the first constructed link, the next link constructed will be one of $2(n-2)$ possible links. Similarly, given the first two links that are constructed, the next one to be constructed will be one of $3(n-3)$ possible links, and so on. Since the cost of the first link to be built is the minimum of $\binom{n}{2}$ exponentials with rate 1, it follows that

$$E[C_1] = 1\Big/\binom{n}{2}$$

By the lack of memory property of the exponential it follows that the amounts by which the costs of the other links exceed C_1 are independent exponentials with rate 1. Therefore, C_2 is equal to C_1 plus the minimum of $2(n-2)$ independent exponentials with rate 1, and so

$$E[C_2] = E[C_1] + \frac{1}{2(n-2)}$$

Similar reasoning then gives

$$E[C_3] = E[C_2] + \frac{1}{3(n-3)}$$

and so on.

19. (c) Letting $A = X_{(2)} - X_{(1)}$ we have

$E[X_{(2)}]$

$$= E[X_{(1)}] + E[A]$$

$$= \frac{1}{\mu_1 + \mu_2} + \frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_1}\frac{\mu_2}{\mu_1 + \mu_2}$$

The formula for $E[A]$ being obtained by conditioning on which X_i is largest.

(d) Let I equal 1 if $X_1 < X_2$ and let it be 2 otherwise. Since the conditional distribution of A (either exponential with rate μ_1 *or* μ_2) is determined by I, which is independent of $X_{(1)}$, it follows that A is independent of $X_{(1)}$.

Therefore,

$$Var(X_{(2)}) = Var(X_{(1)}) + Var(A)$$

With $p = \mu_1/(\mu_1 + \mu_2)$ we obtain, upon conditioning on I,

$$E[A] = p/\mu_2 + (1-p)/\mu_1,$$

$$E[A^2] = 2p/\mu_2^2 + 2(1-p)/\mu_1^2$$

Therefore,

$$Var(A) = 2p/\mu_2^2 + 2(1-p)/\mu_1^2 - (p/\mu_2 + (1-p)/\mu_1)^2$$

Thus,

$Var(X_{(2)})$

$$= 1/(\mu_1 + \mu_2)^2 + 2[p/\mu_2^2 + (1-p)/\mu_1^2] - (p/\mu_2 + (1-p)/\mu_1)^2$$

20. (a) $P_A = \dfrac{\mu_1}{\mu_1 + \mu_2}$

(b) $P_B = 1 - \left(\dfrac{\mu_2}{\mu_1 + \mu_2}\right)^2$

(c) $E[T] = 1/\mu_1 + 1/\mu_2 + P_A/\mu_2 + P_B/\mu_2$

21. $E[\text{time}] = E[\text{time waiting at 1}] + 1/\mu_1 + E[\text{time waiting at 2}] + 1/\mu_2$

Now,

$E[\text{time waiting at 1}] = 1/\mu_1$,

$E[\text{time waiting at 2}] = (1/\mu_2)\dfrac{\mu_1}{\mu_1 + \mu_2}$

The last equation follows by conditioning on whether or not the customer waits for server 2. Therefore,

$$E[\text{time}] = 2/\mu_1 + (1/\mu_2)[1 + \mu_1/(\mu_1 + \mu_2)]$$

22. $E[\text{time}] = E[\text{time waiting for server 1}] + 1/\mu_1 + E[\text{time waiting for server 2}] + 1/\mu_2$

Now, the time spent waiting for server 1 is the remaining service time of the customer with server 1 plus any additional time due to that customer blocking your entrance. If server 1 finishes before server 2 this additional time will equal the additional service time of the customer with server 2. Therefore,

$E[\text{time waiting for server 1}]$

$$= 1/\mu_1 + E[\text{Additional}]$$

$$= 1/\mu_1 + (1/\mu_2[\mu_1/(\mu_1 + \mu_2)]$$

Since when you enter service with server 1 the customer preceding you will be entering service with server 2, it follows that you will have to wait for server 2 if you finish service first. Therefore, conditioning on whether or not you finish first

$E[\text{time waiting for server 2}]$

$$= (1/\mu_2)[\mu_1/(\mu_1 + \mu_2)]$$

Thus,

$$E[\text{time}] = 2/\mu_1 + (2/\mu_2)[\mu_1/(\mu_1 + \mu_2)] + 1/\mu_2$$

23. (a) 1/2.

(b) $(1/2)^{n-1}$: whenever battery 1 is in use and a failure occurs the probability is 1/2 that it is not battery 1 that has failed.

(c) $(1/2)^{n-i+1}$, $\quad i > 1$.

(d) T is the sum of $n-1$ independent exponentials with rate 2μ (since each time a failure occurs the time until the next failure is exponential with rate 2μ).

(e) Gamma with parameters $n-1$ and 2μ.

24. Let T_i denote the time between the $(i-1)^{th}$ and the i^{th} job completion. Then the T_i are independent, with $T_i, i = 1, \ldots, n-1$ being exponential with rate $\mu_1 + \mu_2$. With probability $\dfrac{\mu_1}{\mu_1 + \mu_2}$, T_n is exponential with rate μ_2, and with probability $\dfrac{\mu_2}{\mu_1 + \mu_2}$ it is exponential with rate μ_1. Therefore,

$$E[T] = \sum_{i=1}^{n-1} E[T_i] + E[T_n] = (n-1)\frac{1}{\mu_1 + \mu_2} + \frac{\mu_1}{\mu_1 + \mu_2}\frac{1}{\mu_2} + \frac{\mu_2}{\mu_1 + \mu_2}\frac{1}{\mu_1}$$

$$Var(T) = \sum_{i=1}^{n-1} Var(T_i) + Var(T_n)$$
$$= (n-1)\frac{1}{(\mu_1+\mu_2)^2} + Var(T_n)$$

Now use

$$Var(T_n) = E[T_n^2] - (E[T_n])^2$$
$$= \frac{\mu_1}{\mu_1+\mu_2}\frac{2}{\mu_2^2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{2}{\mu_1^2} - \left(\frac{\mu_1}{\mu_1+\mu_2}\frac{1}{\mu_2} + \frac{\mu_2}{\mu_1+\mu_2}\frac{1}{\mu_1}\right)^2$$

25. Parts (a) and (b) follow upon integration. For part (c), condition on which of X or Y is larger and use the lack of memory property to conclude that the amount by which it is larger is exponential rate λ. For instance, for $x < 0$,

$$fx - y(x)dx$$
$$= P\{X < Y\}P\{-x < Y - X < -x + dx | Y > X\}$$
$$= \frac{1}{2}\lambda e^{\lambda x}dx$$

For (d) and (e), condition on I.

26. (a) $\displaystyle \frac{1}{\mu_1+\mu_2+\mu_3} + \sum_{i=1}^{3}\frac{\mu_i}{\mu_1+\mu_2+\mu_3}\frac{1}{\mu_i} = \frac{4}{\mu_1+mu_2+\mu_3}$

(b) $\displaystyle \frac{1}{\mu_1+\mu_2+\mu_3} + \text{(a)} = \frac{5}{\mu_1+\mu_2+\mu_3}$

27. (a) $\displaystyle \frac{\mu_1}{\mu_1+\mu_3}$

(b) $\displaystyle \frac{\mu_1}{\mu_1+\mu_3}\frac{\mu_2}{\mu_2+\mu_3}$

(c) $\displaystyle \sum_i \frac{1}{\mu_i} + \frac{\mu_1}{\mu_1+\mu_3}\frac{\mu_2}{\mu_2+\mu_3}\frac{1}{\mu_3}$

(d) $\displaystyle \sum_i \frac{1}{\mu_i} + \frac{\mu_1}{\mu_1+\mu_2}\left[\frac{1}{\mu_2} + \frac{\mu_2}{\mu_2+\mu_3}\frac{1}{\mu_3}\right] + \frac{\mu_2}{\mu_1+\mu_2}\frac{\mu_1}{\mu_1+\mu_3}\frac{\mu_2}{\mu_2+\mu_3}\frac{1}{\mu_3}$

28. For both parts, condition on which item fails first.

(a) $\displaystyle \sum_{i\neq 1}\frac{\lambda_i}{\sum_{j=1}^{n}\lambda_j}\frac{\lambda_1}{\sum_{j\neq i}\lambda_j}$

(b) $\displaystyle \frac{1}{\sum_{i=1}^{n}\lambda_j} + \sum_{i=1}^{n}\frac{\lambda_i}{\sum_{j=1}^{n}\lambda_j}\frac{1}{\sum_{j\neq i}\lambda_j}$

29. (a) $f_{X|X+Y(x|c)} = Cf_{X.\,X+Y(x,\,c)}$
$$= C_1 f_{X_Y(x,c-x)}$$
$$= f_X(x)f_Y(c-x)$$
$$= C_2e^{-\lambda x}e^{-\mu(c-x)}, \quad 0 < x < c$$
$$= C_3e^{-(\lambda-\mu)x}, \quad 0 < x < c$$

where none of the C_i depend on x. Hence, we can conclude that the conditional distribution is that of an exponential random variable conditioned to be less than c.

(b) $\displaystyle E[X|X+Y=c] = \frac{1-e^{-(\lambda-\mu)c}(1+(\lambda-\mu)c)}{\lambda(1-e^{-(\lambda-\mu)c})}$

(c) $c = E[X+Y|X+Y=c] = E[X|X+Y=c] + E[Y|X+Y=c]$

implying that

$$E[Y|X+Y=c]$$
$$= c - \frac{1-e^{-(\lambda-\mu)c}(1+(\lambda-\mu)c)}{\lambda(1-e^{-(\lambda-\mu)c})}$$

30. Condition on which animal died to obtain

E[additional life]

$= E$[additional life | dog died]

$$\frac{\lambda_d}{\lambda_c+\lambda_d} + E[\text{additional life} \mid \text{cat died}]\frac{\lambda_c}{\lambda_c+\lambda_d}$$
$$= \frac{1}{\lambda_c}\frac{\lambda_d}{\lambda_c+\lambda_d} + \frac{1}{\lambda_d}\frac{\lambda_c}{\lambda_c+\lambda_d}$$

31. Condition on whether the 1 PM appointment is still with the doctor at 1:30, and use the fact that if she or he is then the remaining time spent is exponential with mean 30. This gives

E[time spent in office]

$$= 30(1-e^{-30/30}) + (30+30)e^{-30/30}$$
$$= 30 + 30e^{-1}$$

32. (a) no; (b) yes

33. (a) By the lack of memory property, no matter when Y fails the remaining life of X is exponential with rate λ.

(b) $E[\min(X,Y)|X > Y + c]$

$$= E[\min(X,Y)|X > Y, X - Y > c]$$

$$= E[\min(X,Y)|X > Y]$$

where the final equality follows from (a).

34. (a) $\dfrac{\lambda}{\lambda + \mu_A}$

(b) $\dfrac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B} \cdot \dfrac{\lambda}{\lambda + \mu_B}$

37. $\dfrac{1}{\mu} + \dfrac{1}{\lambda}$

38. Let $k = \min(n, m)$, and condition on $M_2(t)$.

$$P\{N_1(t) = n, N_2(t) = m\}$$

$$= \sum_{j=0}^{k} P\{N_1(t) = n, N_2(t) = m | M_2(t) = j\} \times e^{-\lambda_2 t}\frac{(\lambda_2 t)^j}{j!}$$

$$= \sum_{j=0}^{k} e^{-\lambda_1 t}\frac{(\lambda_1 t)^{n-j}}{(n-j)!} e^{-\lambda_3 t}\frac{(\lambda_3 t)^{m-j}}{(m-j)!} e^{-\lambda_2 t}\frac{(\lambda_2 t)^j}{j!}$$

39. (a) $196/2.5 = 78.4$

(b) $196/(2.5)^2 = 31.36$

We use the central limit theorem to justify approximating the life distribution by a normal distribution with mean 78.4 and standard deviation $\sqrt{31.36} = 5.6$. In the following, Z is a standard normal random variable.

(c) $P\{L < 67.2\} \approx P\left\{Z < \dfrac{67.2 - 78.4}{5.6}\right\}$

$$= P\{Z < -2\} = .0227$$

(d) $P\{L > 90\} \approx P\left\{Z > \dfrac{90 - 78.4}{5.6}\right\}$

$$= P\{Z > 2.07\} = .0192$$

(e) $P\{L > 100\} \approx P\left\{Z > \dfrac{100 - 78.4}{5.6}\right\}$

$$= P\{Z > 3.857\} = .00006$$

40. The easiest way is to use Definition 5.1. It is easy to see that $\{N(t), t \geq 0\}$ will also possess stationary and independent increments. Since the sum of two independent Poisson random variables is also Poisson, it follows that $N(t)$ is a Poisson random variable with mean $(\lambda_1 + \lambda_2)t$.

41. $\lambda_1/(\lambda_1 + \lambda_2)$

42. (a) $E[S_4] = 4/\lambda$

(b) $E[S_4|N(1) = 2]$

$$= 1 + E[\text{time for 2 more events}] = 1 + 2/\lambda$$

(c) $E[N(4) - N(2)|N(1) = 3] = E[N(4) - N(2)]$

$$= 2\lambda$$

The first equality used the independent increments property.

43. Let S_i denote the service time at server i, $i = 1, 2$ and let X denote the time until the next arrival. Then, with p denoting the proportion of customers that are served by both servers, we have

$$p = P\{X > S_1 + S_2\}$$

$$= P\{X > S_1\}PX > S_1 + S_2|X > S_1\}$$

$$= \frac{\mu_1}{\mu_1 + \lambda}\,\frac{\mu_2}{\mu_2 + \lambda}$$

44. (a) $e^{-\lambda T}$

(b) Let W denote the waiting time and let X denote the time until the first car. Then

$$E[W] = \int_0^\infty E[W|X = x]\lambda e^{-\lambda x}dx$$

$$= \int_0^T E[W|X = x]\lambda e^{-\lambda x}dx + \int_T^\infty E[W|X = x]\lambda e^{-\lambda x}dx$$

$$= \int_0^T (x + E[W])\lambda e^{-\lambda x}dx + Te^{-\lambda T}$$

Hence,

$$E[W] = T + e^{\lambda T}\int_0^T x\lambda e^{-\lambda x}dx$$

45. $E[N(T)] = E[E[N(T)|T]] = E[\lambda T] = \lambda E[T]$

$$E[TN(T)] = E[E[TN(T)|T]] = E[T\lambda T] = \lambda E[T^2]$$

$$E[N^2(T)] = E\left[E[N^2(T)|T]\right] = E[\lambda T + (\lambda T)^2]$$

$$= \lambda E[T] + \lambda^2 E[T^2]$$

Hence,

$$Cov(T, N(T)) = \lambda E[T^2] - E[T]\lambda E[T] = \lambda\sigma^2$$

and

$$Var(N(T)) = \lambda E[T] + \lambda^2 E[T^2] - (\lambda E[T])^2$$
$$= \lambda\mu + \lambda^2\sigma^2$$

46. $$E[\sum_{i=1}^{N(t)} X_i] = E\left[E[\sum_{i=1}^{N(t)} X_i | N(t)]\right]$$
$$= E[\mu N(t)] = \mu\lambda t$$
$$E[N(t)\sum_{i=1}^{N(t)} X_i] = E\left[E[N(t)\sum_{i=1}^{N(t)} X_i | N(t)]\right]$$
$$= E[\mu N^2(t)] = \mu(\lambda t + \lambda^2 t^2)$$

Therefore,

$$Cov(N(t), \sum_{i=1}^{N(t)} X_i) = \mu(\lambda t + \lambda^2 t^2) - \lambda t(\mu\lambda t) = \mu\lambda t$$

47. (a) $1/(2\mu) + 1/\lambda$

(b) Let T_i denote the time until both servers are busy when you start with i busy servers $i = 0, 1$. Then,

$$E[T_0] = 1/\lambda + E[T_1]$$

Now, starting with 1 server busy, let T be the time until the first event (arrival or departure); let $X = 1$ if the first event is an arrival and let it be 0 if it is a departure; let Y be the additional time after the first event until both servers are busy.

$$E[T_1] = E[T] + E[Y]$$
$$= \frac{1}{\lambda+\mu} + E[Y|X=1]\frac{\lambda}{\lambda+\mu}$$
$$+ E[Y|X=0]\frac{\mu}{\lambda+\mu}$$
$$= \frac{1}{\lambda+\mu} + E[T_0]\frac{\mu}{\lambda+\mu}$$

Thus,

$$E[T_0] - \frac{1}{\lambda} = \frac{1}{\lambda+\mu} + E[T_0]\frac{\mu}{\lambda+\mu}$$

or

$$E[T_0] = \frac{2\lambda+\mu}{\lambda^2}$$

Also,

$$E[T_1] = \frac{\lambda+\mu}{\lambda^2}$$

(c) Let L_i denote the time until a customer is lost when you start with i busy servers. Then, reasoning as in part (b) gives that

$$E[L_2] = \frac{1}{\lambda+\mu} + E[L_1]\frac{\mu}{\lambda+\mu}$$
$$= \frac{1}{\lambda+\mu} + (E[T_1] + E[L_2])\frac{\mu}{\lambda+\mu}$$
$$= \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda^2} + E[L_2]\frac{\mu}{\lambda+\mu}$$

Thus,

$$E[L_2] = \frac{1}{\lambda} + \frac{\mu(\lambda+\mu)}{\lambda^3}$$

48. Given T, the time until the next arrival, N, the number of busy servers found by the next arrival, is a binomial random variable with parameters n and $p = e^{-\mu T}$.

(a) $$E[N] = \int E[N|T=t]\lambda e^{-\lambda t}dt$$
$$= \int ne^{-\mu t}\lambda e^{-\lambda t}dt = \frac{n\lambda}{\lambda+\mu}$$

For (b) and (c), you can either condition on T, or use the approach of part (a) of Exercise 11 to obtain

$$P\{N=0\} = \prod_{j=1}^{n} \frac{(n-j+1)\mu}{\lambda + (n-j+1)\mu}$$

$$P\{N = n-i\}$$
$$= \frac{\lambda}{\lambda + (n-i)\mu}\prod_{j=1}^{i} \frac{(n-j+1)\mu}{\lambda + (n-j+1)\mu}$$

49. (a) $P\{N(T) - N(s) = 1\} = \lambda(T-s)e^{-\lambda(T-s)}$

(b) Differentiating the expression in part (a) and then setting it equal to 0 gives

$$e^{-\lambda(T-s)} = \lambda(T-s)e^{-\lambda(T-s)}$$

implying that the maximizing value is

$$s = T - 1/\lambda$$

(c) For $s = T - 1/\lambda$, we have that $\lambda(T-s) = 1$ and thus,

$$P\{N(T) - N(s) = 1\} = e^{-1}$$

50. Let T denote the time until the next train arrives; and so T is uniform on $(0, 1)$. Note that, conditional on T, X is Poisson with mean $7T$.

(a) $E[X] = E[E[X|T]] = E[7T] = 7/2$

(b) $E[X|T] = 7T$, $Var(X|T) = 7T$. By the conditional variance formula
$Var(X) = 7E[T] + 49Var[T] = 7/2 + 49/12 = 91/12$.

51. Condition on X, the time of the first accident, to obtain

$$E[N(t)] = \int_0^\infty E[N(t)|X = s]\beta e^{-\beta s}ds = \int_0^t (1 + \alpha(t-s))\beta e^{-\beta s}ds$$

52. This is the gambler's ruin probability that, starting with k, the gambler's fortune reaches $2k$ before 0 when her probability of winning each bet is $p = \lambda_1/(\lambda_1 + \lambda_2)$. The desired probability is
$\dfrac{1-(\lambda_2/\lambda_1)^k}{1-(\lambda_2/\lambda_1)^{2k}}$.

53. (a) e^{-1}
(b) $e^{-1} + e^{-1}(.8)e^{-1}$

54. (a) $P\{L_1 = 0\} = e^{-\lambda m}$
(b) $P\{L_1 < x\} = e^{-\lambda(m-x)}$
(c) $P\{R_1 = 1\} = e^{-\lambda(1-m)}$
(d) $P\{R_1 > x\} = e^{-\lambda(x-m)}$
(e)

$$E[R] = \int_0^1 P\{R > x\}dx = m + \int_m^1 P\{R > x\}dx = m + \int_m^1 e^{-n\lambda(x-m)}dx = m + \frac{1-e^{-n\lambda(1-m)}}{n\lambda}$$

Now, using that

$$P\{L > x\} = 1 - P\{L \le x\} = 1 - e^{-n\lambda(m-x)}, \quad 0 < x < m$$

gives

$$E\{L\} = \int_0^m (1 - e^{-n\lambda(m-x)})dx = m - \frac{1-e^{-n\lambda m}}{n\lambda}$$

Hence,

$$E[R-L] = \frac{1-e^{-n\lambda(1-m)}}{n\lambda} + \frac{1-e^{-n\lambda m}}{n\lambda} \approx \frac{2}{n\lambda} \quad \text{when } n \text{ is large}$$

55. As long as customers are present to be served, every event (arrival or departure) will, independently of other events, be a departure with probability $p = \mu/(\lambda + \mu)$. Thus $P\{X = m\}$ is the probability that there have been a total of m tails at the moment that the n^{th} head occurs, when independent flips of a coin having probability p of coming up heads are made: that is, it is the probability that the n^{th} head occurs on trial number $n + m$. Hence,

$$p\{X = m\} = \binom{n+m-1}{n-1} p^n(1-p)^m$$

56. (a) It is a binomial (n, p) random variable.
(b) It is geometric with parameter p.
(c) It is a negative binomial with parameters r, p.
(d) Let $0 < i_1 < i_2, \cdots < i_r < n$. Then,

$$P\{\text{events at } i_1, \ldots, i_r | N(n) = r\} = \frac{P\{\text{events at } i_1, \ldots, i_r, N(n) = r\}}{P\{N(n) = r\}} = \frac{P^r(1-p)^{n-r}}{\binom{n}{r} P^r(1-p)^{n-r}} = \frac{1}{\binom{n}{r}}$$

57. (a) e^{-2}
(b) 2 p.m.

58. Let $L_i = P\{i \text{ is the last type collected}\}$.

$$L_i = P\{X_i = \max_{j=1,\ldots,n} X_j\} = \int_0^\infty p_i e^{-p_i x} \prod_{j \ne i} (1 - e^{-p_j x})dx = \int_0^1 \prod_{j \ne i} (1 - y^{p_j/p_i})dy \quad (y = e^{-p_i x}) = E\left[\prod_{j \ne i} (1 - U^{p_j/p_i})\right]$$

59. The unconditional probability that the claim is type 1 is 10/11. Therefore,

$$P(1|4000) = \frac{P(4000|1)P(1)}{P(4000|1)P(1) + P(4000|2)P(2)} = \frac{e^{-4}10/11}{e^{-4}10/11 + .2e^{-.8}1/11}$$

61. (a) Poisson with mean $cG(t)$.

(b) Poisson with mean $c[1 - G(t)]$.

(c) Independent.

62. Each of a Poisson number of events is classified as either being of type 1 (if found by proofreader 1 but not by 2) or type 2 (if found by 2 but not by 1) or type 3 (if found by both) or type 4 (if found by neither).

(a) The X_i are independent Poisson random variables with means

$$E[X_1] = \lambda p_1(1 - p_2),$$
$$E[X_2] = \lambda(1 - p_1)p_2,$$
$$E[X_3] = \lambda p_1 p_2,$$
$$E[X_4] = \lambda(1 - p_1)(1 - p_2).$$

(b) Follows from the above.

(c) Using that $(1 - p_1)/p_1 = E[X_2]/E[X_3] = X_2/X_3$ we can approximate p_1 by $X_3/(X_2 + X_3)$. Thus p_1 is estimated by the fraction of the errors found by proofreader 2 that are also found by proofreader 1. Similarly, we can estimate p_2 by $X_3/(X_1 + X_3)$.

The total number of errors found, $X_1 + X_2 + X_3$, has mean

$$E[X_1 + X_2 + X_3] = \lambda[1 - (1 - p_1)(1 - p_2)]$$
$$= \lambda\left[1 - \frac{X_2 X_1}{(X_2 + X_3)(X_1 + X_3)}\right]$$

Hence, we can estimate λ by

$$(X_1 + X_2 + X_3)/\left[1 - \frac{X_2 X_1}{(X_2 + X_3)(X_1 + X_3)}\right]$$

For instance, suppose that proofreader 1 finds 10 errors, and proofreader 2 finds 7 errors, including 4 found by proofreader 1. Then $X_1 = 6, X_2 = 3, X_3 = 4$. The estimate of p_1 is $4/7$, and that of p_2 is $4/10$. The estimate of λ is $13/(1 - 18/70) = 17.5$.

(d) Since λ is the expected total number of errors, we can use the estimator of λ to estimate this total. Since 13 errors were discovered we would estimate X_4 to equal 4.5.

63. Let X and Y be respectively the number of customers in the system at time $t + s$ that were present at time s, and the number in the system at $t + s$ that were not in the system at time s. Since there are an infinite number of servers, it follows that X and Y are independent (even if given the number is the system at time s). Since the service distribution is exponential with rate μ, it follows that given that $X(s) = n$, X will be binomial with parameters n and $p = e^{-\mu t}$. Also Y, which is independent of $X(s)$, will have the same distribution as $X(t)$.

Therefore, Y is Poisson with mean $\lambda \int_0^t e^{-\mu y} dy$

$$= \lambda(1 - e^{-\mu t})/\mu$$

(a) $E[X(t + s)|X(s) = n]$

$$= E[X|X(s) = n] + E[Y|X(s) = n].$$
$$= ne^{-\mu t} + \lambda(1 - e^{-\mu t})/\mu$$

(b) $Var(X(t + s)|X(s) = n)$

$$= Var(X + Y|X(s) = n)$$
$$= Var(X|X(s) = n) + Var(Y)$$
$$= ne^{-\mu t}(1 - e^{-\mu t}) + \lambda(1 - e^{-\mu t})/\mu$$

The above equation uses the formulas for the variances of a binomial and a Poisson random variable.

(c) Consider an infinite server queuing system in which customers arrive according to a Poisson process with rate λ, and where the service times are all exponential random variables with rate μ. If there is currently a single customer in the system, find the probability that the system becomes empty when that customer departs.

Condition on R, the remaining service time:

$P\{\text{empty}\}$

$$= \int_0^\infty P\{\text{empty}|R = t\}\mu e^{-\mu t} dt$$
$$= \int_0^\infty \exp\left\{-\lambda \int_0^t e^{-\mu y} dy\right\} \mu e^{-\mu t} dt$$
$$= \int_0^\infty \exp\left\{-\frac{\lambda}{\mu}(1 - e^{-\mu t})\right\} \mu e^{-\mu t} dt$$
$$= \int_0^1 e^{-\lambda x/\mu} dx$$
$$= \frac{\mu}{\lambda}(1 - e^{-\lambda/\mu})$$

where the preceding used that $P\{\text{empty}|R = t\}$ is equal to the probability that an $M/M/\infty$ queue is empty at time t.

64. (a) Since, given $N(t)$, each arrival is uniformly distributed on $(0, t)$ it follows that

$$E[X|N(t)] = N(t)\int_0^t (t-s)ds/t = N(t)\ t/2$$

(b) Let $U_1, U_2, \ldots$ be independent uniform (0, t) random variables.

Then

$$Var(X|N(t)=n) = Var\left[\sum_{i=1}^{n}(t-U_i)\right]$$
$$= nVar(U_i) = nt^2/12$$

(c) By (a), (b), and the conditional variance formula,

$$Var(X) = Var(N(t)t/2) + E[N(t)t^2/12]$$
$$= \lambda t t^2/4 + \lambda t t^2/12 = \lambda t^3/3$$

65. This is an application of the infinite server Poisson queue model. An arrival corresponds to a new lawyer passing the bar exam, the service time is the time the lawyer practices law. The number in the system at time t is, for large t, approximately a Poisson random variable with mean $\lambda\mu$ where λ is the arrival rate and μ the mean service time. This latter statement follows from

$$\int_0^n [1-G(y)]dy = \mu$$

where μ is the mean of the distribution G. Thus, we would expect $500 \cdot 30 = 15,000$ lawyers.

66. The number of unreported claims is distributed as the number of customers in the system for the infinite server Poisson queue.

(a) $e^{-a(t)}(a(t))^n/n!$, where $a(t) = \lambda\int_0^t \bar{G}(y)dy$

(b) $a(t)\mu_F$, where μ_F is the mean of the distribution F.

67. If we count a satellite if it is launched before time s but remains in operation at time t, then the number of items counted is Poisson with mean $m(t) = \int_0^s \bar{G}(t-y)dy$. The answer is $e^{-m(t)}$.

68. $E[A(t)|N(t)=n]$

$$= E[A]e^{-\alpha t}E\left[\sum_{i=1}^{n} e^{\alpha s_i}|N(t)=n\right]$$
$$= E[A]e^{-\alpha t}E\left[\sum_{i=1}^{n} e^{\alpha U_{(i)}}\right]$$
$$= E[A]e^{-\alpha t}E\left[\sum_{i=1}^{n} e^{\alpha U_i}\right]$$
$$= nE[A]e^{-\alpha t}E\left[e^{\alpha U}\right]$$
$$= nE[A]e^{-\alpha t}\int_0^t e^{\alpha x}\frac{1}{t}dx$$
$$= nE[A]\frac{1-e^{-\alpha t}}{\alpha t}$$

Therefore,

$$E[A(t)] = E\left[N(t)E[A]\frac{1-e^{-\alpha t}}{\alpha t}\right] = \lambda E[A]\frac{1-e^{-\alpha t}}{\alpha t}$$

Going backwards from t to 0, events occur according to a Poisson process and an event occurring a time s (from the starting time t) has value $Ae^{-\alpha s}$ attached to it.

69. (a) $1-e^{-\lambda(t-s)}$

(b) $e^{-\lambda s}e^{-\lambda(t-s)}[\lambda(t-s)]^3/3!$

(c) $4+\lambda(t-s)$

(d) $4s/t$

70. (a) Let A be the event that the first to arrive is the first to depart, let S be the first service time, and let $X(t)$ denote the number of departures by time t.

$$P(A) = \int P(A|S=t)g(t)dt$$
$$= \int P\{X(t)=0\}g(t)dt$$
$$= \int e^{-\lambda\int_0^t G(y)dy}g(t)dt$$

(b) Given $N(t)$, the number of arrivals by t, the arrival times are iid uniform $(0, t)$. Thus, given $N(t)$, the contribution of each arrival to the total remaining service times are independent with the same distribution, which does not depend on $N(t)$.

(c) and (d) If, conditional on $N(t)$, X is the contribution of an arrival, then

$$E[X] = \frac{1}{t}\int_0^t\int_{t-s}^{\infty}(s+y-t)g(y)dyds$$

$$E[X^2] = \frac{1}{t}\int_0^t\int_{t-s}^{\infty}(s+y-t)^2 g(y)dyds$$

$$E[S(t)] = \lambda t E[X] \quad Var(S(t)) = \lambda t E[X^2]$$

71. Let $U_1, \ldots$ be independent uniform $(0, t)$ random variables that are independent of $N(t)$, and let $U_{(i,n)}$ be the i^{th} smallest of the first n of them.

$$P\left\{\sum_{i=1}^{N(t)} g(S_i) < x\right\}$$

$$= \sum_n P\left\{\sum_{i=1}^{N(t)} g(S_i) < x | N(t) = n\right\} P\{N(t) = n\}$$

$$= \sum_n P\left\{\sum_{i=1}^{n} g(S_i) < x | N(t) = n\right\} P\{N(t) = n\}$$

$$= \sum_n P\left\{\sum_{i=1}^{n} g(U_{(i,n)}) < x\right\} P\{N(t) = n\}$$

(Theorem 5.2)

$$= \sum_n P\left\{\sum_{i=1}^{n} g(U_i) < x\right\} P\{N(t) = n\}$$

$$\left(\sum_{i=1}^{n} g(U_{(i,n)}) = \sum_{i=1}^{n} g(U_i)\right)$$

$$= \sum_n P\left\{\sum_{i=1}^{n} g(U_i) < x | N(t) = n\right\} P\{N(t) = n\}$$

$$= \sum_n P\left\{\sum_{i=1}^{N(t)} g(U_i) < x | N(t) = n\right\} P\{N(t) = n\}$$

$$= P\left\{\sum_{i=1}^{N(t)} g(U_i) < x\right\}$$

72. (a) Call the random variable S_n. Since it is the sum of n independent exponentials with rate λ, it has a graze distribution with parameters n and λ.

(b) Use the result that given $S_n = t$ the set of times at which the first $n-1$ riders departed are independent uniform $(0, t)$ random variables. Therefore, each of these riders will still be walking at time t with probability

$$p = \int_0^t e^{-\mu(t-s)} ds/t = \frac{1 - e^{-\mu t}}{\mu t}$$

Hence, the probability that none of the riders are walking at time t is $(1-p)^{n-1}$.

73. (a) It is the gamma distribution with parameters n and λ.

(b) For $n \geq 1$,

$P\{N = n | T = t\}$

$$= \frac{P\{T = t | N = n\} p(1-p)^{n-1}}{f_T(t)}$$

$$= C\frac{(\lambda t)^{n-1}}{(n-1)!}(1-p)^{n-1}$$

$$= C\frac{(\lambda(1-p)t)^{n-1}}{(n-1)!}$$

$$= e^{-\lambda(1-p)t}\frac{(\lambda(1-p)t)^{n-1}}{(n-1)!}$$

where the last equality follows since the probabilities must sum to 1.

(c) The Poisson events are broken into two classes, those that cause failure and those that do not. By Proposition 5.2, this results in two independent Poisson processes with respective rates λp and $\lambda(1-p)$. By independence it follows that given that the first event of the first process occurred at time t the number of events of the second process by this time is Poisson with mean $\lambda(1-p)t$.

74. (a) Since each item will, independently, be found with probability $1 - e^{-\mu t}$ it follows that the number found will be Poisson distribution with mean $\lambda(1 - e^{-\mu t})$. Hence, the total expected return is $R\lambda(1 - e^{-\mu t}) - Ct$.

(b) Calculus now yields that the maximizing value of t is given by

$$t = \frac{1}{\mu}\log\left(\frac{R\lambda\mu}{C}\right)$$

provided that $R\lambda\mu > C$; if the inequality is reversed then $t = 0$ is best.

(c) Since the number of items not found by any time t is independent of the number found (since each of the Poisson number of items will independently either be counted with probability $1 - e^{-\mu t}$ or uncounted with probability $e^{-\mu t}$) there is no added gain in letting the decision on whether to stop at time t depend on the number already found.

75. (a) $\{Y_n\}$ is a Markov chain with transition probabilities given by

$$P_{0j} = a_j, \quad P_{i,i-1+j} = a_j, \quad j \geq 0$$

where

$$a_j = \int \frac{e^{-\lambda t}(\lambda t)^j}{j!} dG(t)$$

(b) $\{X_n\}$ is a Markov chain with transition probabilities

$$P_{i,i+1-j} = \beta_j,\ j = 0, 1, \ldots, i, P_{i,0} = \sum_{k=i+1}^{\infty} \beta_j$$

where

$$\beta_j = \int \frac{e^{-\mu t}(\mu t)^j}{j!} dF(t)$$

76. Let Y denote the number of customers served in a busy period. Note that given S, the service time of the initial customer in the busy period, it follows by the argument presented in the text that the conditional distribution of $Y - 1$ is that of the compound Poisson random variable $\sum_{i=1}^{N(S)} Y_i$, where the Y_i have the same distribution as does Y. Hence,

$$E[Y|S] = 1 + \lambda S E[Y]$$

$$Var(Y|S) = \lambda S E[Y^2]$$

Therefore,

$$E[Y] = \frac{1}{1 - \lambda E[S]}$$

Also, by the conditional variance formula

$$\begin{aligned} Var(Y) &= \lambda E[S]E[Y^2] + (\lambda E[Y])^2 Var(S) \\ &= \lambda E[S]Var(Y) + \lambda E[S](E[Y])^2 \\ &\quad + (\lambda E[Y])^2 Var(S) \end{aligned}$$

implying that

$$Var(Y) = \frac{\lambda E[S](E[Y])^2 + (\lambda E[Y])^2 Var(S)}{1 - \lambda E[S]}$$

77. (a) $\dfrac{\mu}{\lambda + \mu}$

(b) $\dfrac{\lambda}{\lambda + \mu} \dfrac{2\mu}{\lambda + 2\mu}$

(c) $\displaystyle\prod_{i=1}^{j-1} \frac{\lambda}{\lambda + i\mu} \frac{j\mu}{\lambda + j\mu}$, $j > 1$

(d) Conditioning on N yields the solution; namely $\displaystyle\sum_{j=1}^{\infty} \frac{1}{j} P(N = j)$

(e) $\displaystyle\sum_{j=1}^{\infty} P(N = j) \sum_{i=0}^{j} \frac{1}{\lambda + i\mu}$

78. Poisson with mean 63.

79. Consider a Poisson process with rate λ in which an event at time t is counted with probability $\lambda(t)/\lambda$ independently of the past. Clearly such a process will have independent increments. In addition,

$$\begin{aligned} &P\{2 \text{ or more counted events in}(t, t + h)\} \\ &\quad \le P\{2 \text{ or more events in}(t, t + h)\} \\ &\quad = o(h) \end{aligned}$$

and

$$\begin{aligned} &P\{1 \text{ counted event in } (t, t + h)\} \\ &\quad = P\{1 \text{ counted} \mid 1 \text{ event}\} P(1 \text{ event}) \\ &\qquad + P\{1 \text{ counted} \mid \ge 2 \text{ events}\} P\{\ge 2\} \\ &\quad = \int_t^{t+h} \frac{\lambda(s)}{\lambda} \frac{ds}{h} (\lambda h + o(h)) + o(h) \\ &\quad = \frac{\lambda(t)}{\lambda} \lambda h + o(h) \\ &\quad = \lambda(t) h + o(h) \end{aligned}$$

80. (a) No.

(b) No.

(c) $P\{T_1 > t\} = P\{N(t) = 0\} = e^{-m(t)}$ where

$$m(t) = \int_0^t \lambda(s) ds$$

81. (a) Let S_i denote the time of the ith event, $i \ge 1$. Let $t_i + h_i < t_{i+1}$, $t_n + h_n \le t$.

$P\{t_i < S_i < t_i + h_i,\ i = 1, \ldots, n | N(t) = n\}$

$$\begin{aligned} &P\{1 \text{ event in } (t_i, t_i + h_i),\quad i = 1, \ldots, n, \\ &= \frac{\text{no events elsewhere in } (0, t)}{P\{N(t) = n\}} \\ &\quad \left[\prod_{i=1}^{n} e^{-(m(t_i + h_i) - m(t_i))} [m(t_i + h_i) - m(t_i)]\right] \\ &= \frac{e^{-[m(t) - \sum_i m(t_i + h_i) - m(t_i)]}}{e^{-m(t)}[m(t)]^n / n!} \\ &= \frac{n \prod_i^n [m(t_i + h_i) - m(t_i)]}{[m(t)]^n} \end{aligned}$$

Dividing both sides by $h_1 \cdots h_n$ and using the fact that $m(t_i + h_i) - m(t_i) = \int_{t_i}^{t_i + h} \lambda(s)\ ds = \lambda(t_i) h + o(h)$ yields upon letting the $h_i \to 0$:

$$f_{S_1 \cdots S_2}(t_1, \ldots, t_n | N(t) = n)$$

$$= n! \prod_{i=1}^{n} [\lambda(t_i)/m(t)]$$

and the right-hand side is seen to be the joint density function of the order statistics from a set of n independent random variables from the distribution with density function $f(x) = m(x)/m(t), x \leq t$.

(b) Let $N(t)$ denote the number of injuries by time t. Now given $N(t) = n$, it follows from part (b) that the n injury instances are independent and identically distributed. The probability (density) that an arbitrary one of those injuries was at s is $\lambda(s)/m(t)$, and so the probability that the injured party will still be out of work at time t is

$$p = \int_0^t P\{\text{out of work at } t | \text{injured at } s\} \frac{\lambda(s)}{m(t)} d\zeta$$

$$= \int_0^t [1 - F(t-s)] \frac{\lambda(s)}{m(t)} d\zeta$$

Hence, as each of the $N(t)$ injured parties have the same probability p of being out of work at t, we see that

$$E[X(t)] | N(t)] = N(t)p$$

and thus,

$$E[X(t)] = pE[N(t)]$$

$$= pm(t)$$

$$= \int_0^t [1 - F(t-s)] \lambda(s)\, ds$$

82. Interpret N as a number of events, and correspond X_i to the i^{th} event. Let $I_1, I_2, \ldots, I_k$ be k nonoverlaping intervals. Say that an event from N is a type j event if its corresponding X lies in I_j, $j = 1, 2, \ldots, k$. Say that an event from N is a type $k+1$ event otherwise. It then follows that the numbers of type $j, j = 1, \ldots, k$, events—call these numbers $N(I_j), j = 1, \ldots, k$—are independent Poisson random variables with respective means

$$E[N(I_j)] = \lambda P\{X_i \in I_j\} = \lambda \int_{I_j} f(s) ds$$

The independence of the $N(I_j)$ establishes that the process $\{N(t)\}$ has independent increments. Because $N(t+h) - N(t)$ is Poisson distributed with mean

$$E[N(t+h) - N(t)] = \lambda \int_t^{t+h} f(s) ds$$
$$= \lambda h f(t) + o(h)$$

it follows that

$$P\{N(t+h) - N(t) = 0\} = e^{-(\lambda h f(t) + o(h))}$$
$$= 1 - \lambda h f(t) + o(h)$$

$$P\{N(t+h) - N(t) = 1\}$$
$$= (\lambda h f(t) + o(h)) e^{-(\lambda h f(t) + o(h))}$$
$$= (\lambda h f(t) + o(h)$$

As the preceding also implies that

$$P\{N(t+h) - N(t) \geq 2\} = o(h)$$

the verification is complete.

83. Since $m(t)$ is increasing it follows that nonoverlapping time intervals of the $\{N(t)\}$ process will correspond to nonoverlapping intervals of the $\{N_o(t)\}$ process. As a result, the independent increment property will also hold for the $\{N(t)\}$ process. For the remainder we will use the identity

$$m(t+h) = m(t) + \lambda(t)h + o(h)$$

$$P\{N(t+h) - N(t) \geq 2\}$$
$$= P\{N_o[m(t+h)] - N_o[m(t)] \geq 2\}$$
$$= P\{N_o[m(t) + \lambda(t)h + o(h)] - N_o[m(t)] \geq 2\}$$
$$= o[\lambda(t)h + o(h)] = o(h)$$

$$P\{N(t+h) - N(t) = 1\}$$
$$= P\{N_o[m(t) + \lambda(t)h + o(h)] - N_o[m(t)] = 1\}$$
$$= P\{1 \text{ event of Poisson process in interval of length } \lambda(t)h + o(h)]\}$$
$$= \lambda(t)h + o(h)$$

84. There is a record whose value is between t and $t + dt$ if the first X larger than t lies between t and $t + dt$. From this we see that, independent of all record values less that t, there will be one between t and $t + dt$ with probability $\lambda(t)dt$ where $\lambda(t)$ is the failure rate function given by

$$\lambda(t) = f(t)/[1 - F(t)]$$

Since the counting process of record values has, by the above, independent increments we can conclude (since there cannot be multiple record values because the X_i are continuous) that it is a

nonhomogeneous Poisson process with intensity function $\lambda(t)$. When f is the exponential density, $\lambda(t) = \lambda$ and so the counting process of record values becomes an ordinary Poisson process with rate λ.

85. \$ 40,000 and \$$1.6 \times 10^8$.

86. (a) $P\{N(t) = n\} = .3\, e^{-3t}(3t)^n/n! + .7e^{-5t}(5t)^n/n!$

(b) No!

(c) Yes! The probability of n events in any interval of length t will, by conditioning on the type of year, be as given in (a).

(d) No! Knowing how many storms occur in an interval changes the probability that it is a good year and this affects the probability distribution of the number of storms in other intervals.

(e) $P\{\text{good}|N(1) = 3\}$

$$= \frac{P\{N(1) = 3|\text{good}\}\, P\{\text{good}\}}{P\{N(1) = 3|\text{good}\}P\{\text{good}\} + P\{N(1) = 3|\text{bad}\}P\{\text{bad}\}}$$

$$= \frac{(e^{-3}3^3/3!).3}{(e^{-3}3^3/3!).3 + (1e^{-5}5^3/3!).7}$$

87. $Cov[X(t), X(t+s)]$

$= Cov[X(t), X(t) + X(t+s) - X(t)]$

$= Cov[X(t), X(t)] + Cov[X(t), X(t+s) - X(t)]$

$= Cov[X(t), X(t)]$ by independent increments

$= Var[X(t)] = \lambda t E[Y^2]$

88. Let $X(15)$ denote the daily withdrawal. Its mean and variance are as follows:

$E[X(15)] = 12 \cdot 15 \cdot 30 = 5400$

$Var[X(15)] = 12 \cdot 15 \cdot [30 \cdot 30 + 50 \cdot 50] = 612,000$

Hence,

$P\{X(15) \le 6000\}$

$$= P\left\{\frac{X(15) - 5400}{\sqrt{612,000}} \le \frac{600}{\sqrt{612,000}}\right\}$$

$= P\{Z \le .767\}$ where Z is a standard normal

$= .78$ from Table 7.1 of Chapter 2.

89. Let T_i denote the arrival time of the first type i shock, $i = 1, 2, 3$.

$P\{X_1 > s, X_2 > t\}$

$= P\{T_1 > s, T_3 > s, T_2 > t, T_3 > t\}$

$= P\{T_1 > s, T_2 > t, T_3 > \max(s, t)\}$

$= e^{-\lambda_1 s}e^{-\lambda_2 t}e^{-\lambda_3 \max(s,t)}$

90. $P\{X_1 > s\} = P\{X_1 > s, X_2 > 0\}$

$= e^{-\lambda_1 s}e^{-\lambda_3 s}$

$= e^{-(\lambda_1+\lambda_3)s}$

91. To begin, note that

$$P\left[X_1 > \sum_2^n X_i\right]$$

$= P\{X_1 > X_2\}P\{X_1 - X_2 > X_3|X_1 > X_2\}$

$= P\{X_1 - X_2 - X_3 > X_4|X_1 > X_2 + X_3\}\ldots$

$= P\{X_1 - X_2 \cdots - X_{n-1} > X_n|X_1 > X_2 + \cdots + X_{n-1}\}$

$= (1/2)^{n-1}$

Hence,

$$P\left\{M > \sum_{i=1}^n X_i - M\right\} = \sum_{i-1}^n P\left\{X_1 > \sum_{j\ne i}^n X_i\right\}$$
$$= n/2^{n-1}$$

92. $M_2(t) = \sum_i J_i$

where $J_i = \begin{cases} 1, & \text{if bug } i \text{ contributes 2 errors by } t \\ 0, & \text{otherwise} \end{cases}$

and so

$E[M_2(t)] = \sum_i P\{N_i(t) = 2\} = \sum_i e^{-\lambda_i t}(\lambda_i t)^2/2$

93. (a) $\max(X_1, X_2) + \min(X_1, X_2) = X_1 + X_2$.

(b) This can be done by induction:

$\max\{(X_1, \ldots, X_n)$

$= \max(X_1, \max(X_2, \ldots, X_n))$

$= X_1 + \max(X_2, \ldots, X_n)$
$\quad - \min(X_1, \max(X_2, \ldots, X_n))$

$= X_1 + \max(X_2, \ldots, X_n)$
$\quad - \max(\min(X_1, X_2), \ldots, \min(X_1, X_n))$.

Now use the induction hypothesis.

A second method is as follows:

Suppose $X_1 \le X_2 \le \cdots \le X_n$. Then the coefficient of X_i on the right side is

$$1 - \begin{bmatrix} n-i \\ 1 \end{bmatrix} + \begin{bmatrix} n-i \\ 2 \end{bmatrix} - \begin{bmatrix} n-i \\ 3 \end{bmatrix} + \cdots$$
$$= (1-1)^{n-i}$$
$$= \begin{cases} 0, & i \ne n \\ 1, & i = n \end{cases}$$

and so both sides equal X_n. By symmetry the result follows for all other possible orderings of the X's.

(c) Taking expectations of (b) where X_i is the time of the first event of the i^{th} process yields

$$\sum_i \lambda_i^{-1} - \sum_i \sum_{<j} (\lambda_i + \lambda_j)^{-1}$$
$$+ \sum_i \sum_{<j} \sum_{<k} (\lambda_i + \lambda_j + \lambda_k)^{-1} - \cdots$$
$$+ (-1)^{n+1} \left[\sum_1^n \lambda_i \right]^{-1}$$

94. (i) $P\{X > t\}$
$$= P\{\text{no events in a circle of area } rt^2\}$$
$$= e^{-\lambda r t^2}$$

(ii) $$E[X] = \int_0^\infty P\{X > t\} dt$$
$$= \int_0^\infty e^{-\lambda r t^2} dt$$
$$= \frac{1}{\sqrt{2r\lambda}} \int_0^\infty e^{-x^2/2} dx \quad \text{by } x = t\sqrt{2\lambda r}$$
$$= \frac{1}{2\sqrt{\lambda}}$$

where the last equality follows since

$1/\sqrt{2r} \int_0^\infty e^{-x^2/2} dx = 1/2$ since it represents the probability that a standard normal random variable is greater than its mean.

95. $$E[L|N(t) = n] = \frac{\int x g(x) e^{-xt} (xt)^n dx}{\int g(x) e^{-xt} (xt)^n dx}$$

Conditioning on L yields

$$E[N(s)|N(t) = n]$$
$$= E[E[N(s)|N(t) = n, L]|N(t) = n]$$
$$= E[n + L(s-t)|N(t) = n]$$
$$= n + (s-t)E[L|N(t) = n]$$

For (c), use that for any value of L, given that there have been n events by time t, the set of n event times are distributed as the set of n independent uniform $(0, t)$ random variables. Thus, for $s < t$

$$E[N(s)|N(t) = n] = ns/t$$

96. $$E[N(s)N(t)|L] = E[E[N(s)N(t)|L, N(s)]|L]$$
$$= E[N(s)E[N(t)|L, N(s)]|L]$$
$$= E[N(s)[N(s) + L(t-s)]|L]$$
$$= E[N^2(s)|L] + L(t-s)E[N(s)|L]$$
$$= Ls + (Ls)^2 + (t-s)sL^2$$

Thus,

$$Cov(N(s), N(t)) = sm_1 + stm_2 - stm_1^2$$

97. With $C = 1/P(N(t) = n)$, we have

$$f_{L|N(t)}(\lambda|n) = Ce^{-\lambda t} \frac{(\lambda t)^n}{n!} pe^{-p\lambda} \frac{(p\lambda)^{m-1}}{(m-1)!}$$
$$= Ke^{-(p+t)\lambda} \lambda^{n+m-1}$$

where K does not depend on λ. But we recognize the preceding as the gamma density with parameters $n + m, p + t$, which is thus the conditional density.

Continuous-Time Markov Chains

Exercises

1. A population of organisms consists of both male and female members. In a small colony any particular male is likely to mate with any particular female in any time interval of length h, with probability $\lambda h + o(h)$. Each mating immediately produces one offspring, equally likely to be male or female. Let $N_1(t)$ and $N_2(t)$ denote the number of males and females in the population at t. Derive the parameters of the continuous-time Markov chain $\{N_1(t), N_2(t)\}$, i.e., the v_i, P_{ij} of Section 6.2.

*2. Suppose that a one-celled organism can be in one of two states—either A or B. An individual in state A will change to state B at an exponential rate α; an individual in state B divides into two new individuals of type A at an exponential rate β. Define an appropriate continuous-time Markov chain for a population of such organisms and determine the appropriate parameters for this model.

3. Consider two machines that are maintained by a single repairman. Machine i functions for an exponential time with rate μ_i before breaking down, $i = 1, 2$. The repair times (for either machine) are exponential with rate μ. Can we analyze this as a birth and death process? If so, what are the parameters? If not, how can we analyze it?

*4. Potential customers arrive at a single-server station in accordance with a Poisson process with rate λ. However, if the arrival finds n customers already in the station, then he will enter the system with probability α_n. Assuming an exponential service rate μ, set this up as a birth and death process and determine the birth and death rates.

5. There are N individuals in a population, some of whom have a certain infection that spreads as follows. Contacts between two members of this population occur in accordance with a Poisson process having rate λ. When a contact occurs, it is equally likely to involve any of the $\binom{N}{2}$ pairs of individuals in the population. If a contact involves an infected and a noninfected individual, then with probability p the noninfected individual becomes infected. Once infected, an individual remains infected throughout. Let $X(t)$ denote the number of infected members of the population at time t.

   (a) Is $\{X(t), t \geqslant 0\}$ a continuous-time Markov chain?
   (b) Specify its type.
   (c) Starting with a single infected individual, what is the expected time until all members are infected?

6. Consider a birth and death process with birth rates $\lambda_i = (i+1)\lambda, i \geqslant 0$, and death rates $\mu_i = i\mu, i \geqslant 0$.

   (a) Determine the expected time to go from state 0 to state 4.
   (b) Determine the expected time to go from state 2 to state 5.
   (c) Determine the variances in parts (a) and (b).

*7. Individuals join a club in accordance with a Poisson process with rate λ. Each new member must pass through k consecutive stages to become a full member of the club. The time it takes to pass through each stage is exponentially distributed with rate

μ. Let $N_i(t)$ denote the number of club members at time t who have passed through exactly i stages, $i = 1, \dots, k-1$. Also, let $\mathbf{N}(t) = (N_1(t), N_2(t), \dots, N_{k-1}(t))$.

(a) Is $\{\mathbf{N}(t), t \geqslant 0\}$ a continuous-time Markov chain?

(b) If so, give the infinitesimal transition rates. That is, for any state $\mathbf{n} = (n_1, \dots, n_{k-1})$ give the possible next states along with their infinitesimal rates.

8. Consider two machines, both of which have an exponential lifetime with mean $1/\lambda$. There is a single repairman that can service machines at an exponential rate μ. Set up the Kolmogorov backward equations; you need not solve them.

9. The birth and death process with parameters $\lambda_n = 0$ and $\mu_n = \mu, n > 0$ is called a pure death process. Find $P_{ij}(t)$.

10. Consider two machines. Machine i operates for an exponential time with rate λ_i and then fails; its repair time is exponential with rate $\mu_i, i = 1, 2$. The machines act independently of each other. Define a four-state continuous-time Markov chain that jointly describes the condition of the two machines. Use the assumed independence to compute the transition probabilities for this chain and then verify that these transition probabilities satisfy the forward and backward equations.

*11. Consider a Yule process starting with a single individual—that is, suppose $X(0) = 1$. Let T_i denote the time it takes the process to go from a population of size i to one of size $i+1$.

(a) Argue that $T_i, i = 1, \dots, j$, are independent exponentials with respective rates $i\lambda$.

(b) Let $X_1, \dots, X_j$ denote independent exponential random variables each having rate λ, and interpret X_i as the lifetime of component i. Argue that $\max(X_1, \dots, X_j)$ can be expressed as

$$\max(X_1, \dots, X_j) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_j$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_j$ are independent exponentials with respective rates $j\lambda$, $(j-1)\lambda, \dots, \lambda$.

Hint: Interpret ε_i as the time between the $i-1$ and the ith failure.

(c) Using (a) and (b) argue that

$$P\{T_1 + \cdots + T_j \leqslant t\} = (1 - e^{-\lambda t})^j$$

(d) Use (c) to obtain

$$P_{1j}(t) = (1 - e^{-\lambda t})^{j-1} - (1 - e^{-\lambda t})^j = e^{-\lambda t}(1 - e^{-\lambda t})^{j-1}$$

and hence, given $X(0) = 1$, $X(t)$ has a geometric distribution with parameter $p = e^{-\lambda t}$.

(e) Now conclude that

$$P_{ij}(t) = \binom{j-1}{i-1} e^{-\lambda t i}(1 - e^{-\lambda t})^{j-i}$$

12. Each individual in a biological population is assumed to give birth at an exponential rate λ, and to die at an exponential rate μ. In addition, there is an exponential rate

of increase θ due to immigration. However, immigration is not allowed when the population size is N or larger.

(a) Set this up as a birth and death model.

(b) If $N = 3, 1 = \theta = \lambda, \mu = 2$, determine the proportion of time that immigration is restricted.

13. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean $\frac{1}{4}$ hour.

(a) What is the average number of customers in the shop?

(b) What is the proportion of potential customers that enter the shop?

(c) If the barber could work twice as fast, how much more business would he do?

14. Potential customers arrive at a full-service, one-pump gas station at a Poisson rate of 20 cars per hour. However, customers will only enter the station for gas if there are no more than two cars (including the one currently being attended to) at the pump. Suppose the amount of time required to service a car is exponentially distributed with a mean of five minutes.

(a) What fraction of the attendant's time will be spent servicing cars?

(b) What fraction of potential customers are lost?

15. A service center consists of two servers, each working at an exponential rate of two services per hour. If customers arrive at a Poisson rate of three per hour, then, assuming a system capacity of at most three customers,

(a) what fraction of potential customers enter the system?

(b) what would the value of part (a) be if there was only a single server, and his rate was twice as fast (that is, $\mu = 4$)?

*16. The following problem arises in molecular biology. The surface of a bacterium consists of several sites at which foreign molecules—some acceptable and some not—become attached. We consider a particular site and assume that molecules arrive at the site according to a Poisson process with parameter λ. Among these molecules a proportion α is acceptable. Unacceptable molecules stay at the site for a length of time that is exponentially distributed with parameter μ_1, whereas an acceptable molecule remains at the site for an exponential time with rate μ_2. An arriving molecule will become attached only if the site is free of other molecules. What percentage of time is the site occupied with an acceptable (unacceptable) molecule?

17. Each time a machine is repaired it remains up for an exponentially distributed time with rate λ. It then fails, and its failure is either of two types. If it is a type 1 failure, then the time to repair the machine is exponential with rate μ_1; if it is a type 2 failure, then the repair time is exponential with rate μ_2. Each failure is, independently of the time it took the machine to fail, a type 1 failure with probability p and a type 2 failure with probability $1 - p$. What proportion of time is the machine down due to a type 1 failure? What proportion of time is it down due to a type 2 failure? What proportion of time is it up?

18. After being repaired, a machine functions for an exponential time with rate λ and then fails. Upon failure, a repair process begins. The repair process proceeds sequentially through k distinct phases. First a phase 1 repair must be performed, then a

phase 2, and so on. The times to complete these phases are independent, with phase i taking an exponential time with rate μ_i, $i = 1, \ldots, k$.

(a) What proportion of time is the machine undergoing a phase i repair?

(b) What proportion of time is the machine working?

*19. A single repairperson looks after both machines 1 and 2. Each time it is repaired, machine i stays up for an exponential time with rate λ_i, $i = 1, 2$. When machine i fails, it requires an exponentially distributed amount of work with rate μ_i to complete its repair. The repairperson will always service machine 1 when it is down. For instance, if machine 1 fails while 2 is being repaired, then the repairperson will immediately stop work on machine 2 and start on 1. What proportion of time is machine 2 down?

20. There are two machines, one of which is used as a spare. A working machine will function for an exponential time with rate λ and will then fail. Upon failure, it is immediately replaced by the other machine if that one is in working order, and it goes to the repair facility. The repair facility consists of a single person who takes an exponential time with rate μ to repair a failed machine. At the repair facility, the newly failed machine enters service if the repairperson is free. If the repairperson is busy, it waits until the other machine is fixed; at that time, the newly repaired machine is put in service and repair begins on the other one. Starting with both machines in working condition, find

(a) the expected value and

(b) the variance of the time until both are in the repair facility.

(c) In the long run, what proportion of time is there a working machine?

21. Suppose that when both machines are down in Exercise 20 a second repairperson is called in to work on the newly failed one. Suppose all repair times remain exponential with rate μ. Now find the proportion of time at least one machine is working, and compare your answer with the one obtained in Exercise 20.

22. Customers arrive at a single-server queue in accordance with a Poisson process having rate λ. However, an arrival that finds n customers already in the system will only join the system with probability $1/(n+1)$. That is, with probability $n/(n+1)$ such an arrival will not join the system. Show that the limiting distribution of the number of customers in the system is Poisson with mean λ/μ.

23. A job shop consists of three machines and two repairmen. The amount of time a machine works before breaking down is exponentially distributed with mean 10. If the amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8, then

(a) what is the average number of machines not in use?

(b) what proportion of time are both repairmen busy?

*24. Consider a taxi station where taxis and customers arrive in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, an arriving customer that does not find a taxi waiting leaves. Find

(a) the average number of taxis waiting, and

(b) the proportion of arriving customers that get taxis.

25. Customers arrive at a service station, manned by a single server who serves at an exponential rate μ_1, at a Poisson rate λ. After completion of service the customer

then joins a second system where the server serves at an exponential rate μ_2. Such a system is called a *tandem* or *sequential* queueing system. Assuming that $\lambda < \mu_i$, $i = 1, 2$, determine the limiting probabilities.

Hint: Try a solution of the form $P_{n,m} = C\alpha^n \beta^m$, and determine C, α, β.

26. Consider an ergodic $M/M/s$ queue in steady state (that is, after a long time) and argue that the number presently in the system is independent of the sequence of past departure times. That is, for instance, knowing that there have been departures 2, 3, 5, and 10 time units ago does not affect the distribution of the number presently in the system.

27. In the $M/M/s$ queue if you allow the service rate to depend on the number in the system (but in such a way so that it is ergodic), what can you say about the output process? What can you say when the service rate μ remains unchanged but $\lambda > s\mu$?

28. If $\{X(t)\}$ and $\{Y(t)\}$ are independent continuous-time Markov chains, both of which are time reversible, show that the process $\{X(t), Y(t)\}$ is also a time reversible Markov chain.

29. Consider a set of n machines and a single repair facility to service these machines. Suppose that when machine i, $i = 1, \ldots, n$, fails it requires an exponentially distributed amount of work with rate μ_i to repair it. The repair facility divides its efforts equally among all failed machines in the sense that whenever there are k failed machines each one receives work at a rate of $1/k$ per unit time. If there are a total of r working machines, including machine i, then i fails at an instantaneous rate λ_i/r.
    (a) Define an appropriate state space so as to be able to analyze the preceding system as a continuous-time Markov chain.
    (b) Give the instantaneous transition rates (that is, give the q_{ij}).
    (c) Write the time reversibility equations.
    (d) Find the limiting probabilities and show that the process is time reversible.

30. Consider a graph with nodes $1, 2, \ldots, n$ and the $\binom{n}{2}$ arcs (i, j), $i \neq j$, $i, j, = 1, \ldots, n$. (See Section 3.6.2 for appropriate definitions.) Suppose that a particle moves along this graph as follows: Events occur along the arcs (i, j) according to independent Poisson processes with rates λ_{ij}. An event along arc (i, j) causes that arc to become excited. If the particle is at node i at the moment that (i, j) becomes excited, it instantaneously moves to node j, $i, j = 1, \ldots, n$. Let P_j denote the proportion of time that the particle is at node j. Show that

$$P_j = \frac{1}{n}$$

Hint: Use time reversibility.

31. A total of N customers move about among r servers in the following manner. When a customer is served by server i, he then goes over to server j, $j \neq i$, with probability $1/(r-1)$. If the server he goes to is free, then the customer enters service; otherwise he joins the queue. The service times are all independent, with the service times at server i being exponential with rate μ, $i = 1, \ldots, r$. Let the state at any time be the vector $(n_1, \ldots, n_r)$, where n_i is the number of customers presently at server i, $i = 1, \ldots, r$, $\sum_i n_i = N$.

(a) Argue that if $X(t)$ is the state at time t, then $\{X(t), t \geqslant 0\}$ is a continuous-time Markov chain.
(b) Give the infinitesimal rates of this chain.
(c) Show that this chain is time reversible, and find the limiting probabilities.

32. Customers arrive at a two-server station in accordance with a Poisson process having rate λ. Upon arriving, they join a single queue. Whenever a server completes a service, the person first in line enters service. The service times of server i are exponential with rate μ_i, $i = 1, 2$, where $\mu_1 + \mu_2 > \lambda$. An arrival finding both servers free is equally likely to go to either one. Define an appropriate continuous-time Markov chain for this model, show it is time reversible, and find the limiting probabilities.

*33. Consider two $M/M/1$ queues with respective parameters λ_i, μ_i, $i = 1, 2$. Suppose they share a common waiting room that can hold at most three customers. That is, whenever an arrival finds her server busy and three customers in the waiting room, she goes away. Find the limiting probability that there will be n queue 1 customers and m queue 2 customers in the system.

Hint: Use the results of Exercise 28 together with the concept of truncation.

34. Four workers share an office that contains four telephones. At any time, each worker is either "working" or "on the phone." Each "working" period of worker i lasts for an exponentially distributed time with rate λ_i, and each "on the phone" period lasts for an exponentially distributed time with rate μ_i, $i = 1, 2, 3, 4$.
(a) What proportion of time are all workers "working"?
Let $X_i(t)$ equal 1 if worker i is working at time t, and let it be 0 otherwise.
Let $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$.
(b) Argue that $\{\mathbf{X}(t), t \geqslant 0\}$ is a continuous-time Markov chain and give its infinitesimal rates.
(c) Is $\{\mathbf{X}(t)\}$ time reversible? Why or why not?
Suppose now that one of the phones has broken down. Suppose that a worker who is about to use a phone but finds them all being used begins a new "working" period.
(d) What proportion of time are all workers "working"?

35. Consider a time reversible continuous-time Markov chain having infinitesimal transition rates q_{ij} and limiting probabilities $\{P_i\}$. Let A denote a set of states for this chain, and consider a new continuous-time Markov chain with transition rates q^*_{ij} given by

$$q^*_{ij} = \begin{cases} cq_{ij}, & \text{if } i \in A,\ j \notin A \\ q_{ij}, & \text{otherwise} \end{cases}$$

where c is an arbitrary positive number. Show that this chain remains time reversible, and find its limiting probabilities.

36. Consider a system of n components such that the working times of component i, $i = 1, \ldots, n$, are exponentially distributed with rate λ_i. When a component fails, however, the repair rate of component i depends on how many other components are down. Specifically, suppose that the instantaneous repair rate of component i, $i = 1, \ldots, n$, when there are a total of k failed components, is $\alpha^k \mu_i$.

(a) Explain how we can analyze the preceding as a continuous-time Markov chain. Define the states and give the parameters of the chain.

(b) Show that, in steady state, the chain is time reversible and compute the limiting probabilities.

37. For the continuous-time Markov chain of Exercise 3 present a uniformized version.

38. In Example 6.20, we computed $m(t) = E[O(t)]$, the expected occupation time in state 0 by time t for the two-state continuous-time Markov chain starting in state 0. Another way of obtaining this quantity is by deriving a differential equation for it.

(a) Show that

$$m(t+h) = m(t) + P_{00}(t)h + o(h)$$

(b) Show that

$$m'(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

(c) Solve for $m(t)$.

39. Let $O(t)$ be the occupation time for state 0 in the two-state continuous-time Markov chain. Find $E[O(t)|X(0) = 1]$.

40. Consider the two-state continuous-time Markov chain. Starting in state 0, find $\text{Cov}[X(s), X(t)]$.

41. Let Y denote an exponential random variable with rate λ that is independent of the continuous-time Markov chain $\{X(t)\}$ and let

$$\bar{P}_{ij} = P\{X(Y) = j|X(0) = i\}$$

(a) Show that

$$\bar{P}_{ij} = \frac{1}{v_i + \lambda} \sum_k q_{ik} \bar{P}_{kj} + \frac{\lambda}{v_i + \lambda} \delta_{ij}$$

where δ_{ij} is 1 when $i = j$ and 0 when $i \neq j$.

(b) Show that the solution of the preceding set of equations is given by

$$\bar{\mathbf{P}} = (\mathbf{I} - \mathbf{R}/\lambda)^{-1}$$

where $\bar{\mathbf{P}}$ is the matrix of elements $\bar{P}_{ij}$, $\mathbf{I}$ is the identity matrix, and $\mathbf{R}$ the matrix specified in Section 6.8.

(c) Suppose now that $Y_1, \ldots, Y_n$ are independent exponentials with rate λ that are independent of $\{X(t)\}$. Show that

$$P\{X(Y_1 + \cdots + Y_n) = j|X(0) = i\}$$

is equal to the element in row i, column j of the matrix $\bar{\mathbf{P}}^n$.

(d) Explain the relationship of the preceding to Approximation 2 of Section 6.8.

*42. (a) Show that Approximation 1 of Section 6.8 is equivalent to uniformizing the continuous-time Markov chain with a value v such that $vt = n$ and then approximating $P_{ij}(t)$ by P_{ij}^{*n}.

(b) Explain why the preceding should make a good approximation.

Hint: What is the standard deviation of a Poisson random variable with mean n?

Chapter 6

1. Let us assume that the state is (n, m). Male i mates at a rate λ with female j, and therefore it mates at a rate λm. Since there are n males, matings occur at a rate λnm. Therefore,

$$v_{(n,m)} = \lambda nm$$

Since any mating is equally likely to result in a female as in a male, we have

$$P_{(n,m);(n+1,m)} = P_{(n,m)(n,m+1)} = \frac{1}{2}$$

2. Let $N_A(t)$ be the number of organisms in state A and let $N_B(t)$ be the number of organisms in state B. Then clearly $\{N_A(t); N_B(t)\}$ is a continuous Markov chain with

$$v_{\{n,m\}} = \alpha n + \beta m$$

$$P_{\{n,m\};\{n-1;m+1\}} = \frac{\alpha n}{\alpha n + \beta m}$$

$$P_{\{n,m\};\{n+2;m-1\}} = \frac{\beta m}{\alpha n + \beta m}$$

3. This is not a birth and death process since we need more information than just the number working. We also must know which machine is working. We can analyze it by letting the states be

b : both machines are working
1 : 1 is working, 2 is down
2 : 2 is working, 1 is down
0_1: both are down, 1 is being serviced
0_2: both are down, 2 is being serviced

$$v_b = \mu_1 + \mu_2,\ v_1 = \mu_1 + \mu,\ v_2 = \mu_2 + \mu,$$

$$v_{0_1} = v_{0_2} = \mu$$

$$P_{b,1} = \frac{\mu_2}{\mu_2 + \mu_1} = 1 - P_{b,2}, \quad P_{1,b} = \frac{\mu}{\mu + \mu_1} = 1 - P_{1,0_2}$$

$$P_{2,b} = \frac{\mu}{\mu + \mu_2} = 1 - P_{2,0_1}, \quad P_{0_1,1} = P_{0_2,2} = 1$$

4. Let $N(t)$ denote the number of customers in the station at time t. Then $\{N(t)\}$ is a birth and death process with

$$\lambda_n = \lambda\alpha_n, \quad \mu_n = \mu$$

5. (a) Yes.

(b) It is a pure birth process.

(c) If there are i infected individuals then since a contact will involve an infected and an uninfected individual with probability $i(n-i)/\binom{n}{2}$, it follows that the birth rates are $\lambda_i = \lambda i(n-i)/\binom{n}{2}$, $i = 1, \ldots, n$. Hence,

$$E[\text{time all infected}] = \frac{n(n-1)}{2\lambda} \sum_{i=1}^{n} 1/[i(n-i)]$$

6. Starting with $E[T_0] = \frac{1}{\lambda_0} = \frac{1}{\lambda}$, employ the identity

$$E[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[T_{i-1}]$$

to successively compute $E[T_i]$ for $i = 1, 2, 3, 4$.

(a) $E[T_0] + \cdots + E[T_3]$

(b) $E[T_2] + E[T_3] + E[T_4]$

7. (a) Yes!

(b) For $n = (n_1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1})$ let

$$S_i(n) = (n_1, \ldots, n_{i-1}, n_{i+1} + 1, \ldots, n_{k-1}),$$
$$i = 1, \ldots, k-2$$
$$S_{k-1}(n) = (n_1, \ldots, n_i, n_{i+1}, \ldots n_{k-1} - 1),$$
$$S_0(n) = (n_1 + 1, \ldots, n_i, n_{i+1}, \ldots, n_{k-1})$$

Then

$$q_{n, S_1(n)} = n_i\mu, \qquad i = 1, \ldots, k-1$$
$$q_{n, S_0(n)} = \lambda$$

8. The number of failed machines is a birth and death process with

$$\lambda_0 = 2\lambda \qquad \mu_1 = \mu_2 = \mu$$
$$\lambda_1 = \lambda \qquad \mu_n = 0, n \neq 1, 2$$

$\lambda_n = 0, n > 1.$

Now substitute into the backward equations.

9. Since the death rate is constant, it follows that as long as the system is nonempty, the number of deaths in any interval of length t will be a Poisson random variable with mean μt. Hence,

$$P_{ij}(t) = e^{-\mu t}(\mu t)^{i-j}/(i-j)!, \quad 0 < j \le i$$

$$P_{i,0}(t) = \sum_{k=i}^{\infty} e^{-\mu t}(\mu t)^k/k!$$

10. Let $I_j(t) = \begin{cases} 0, & \text{if machine } j \text{ is working at time } t \\ 1, & \text{otherwise} \end{cases}$

Also, let the state be $(I_1(t), I_2(t))$.

This is clearly a continuous-time Markov chain with

$$v_{(0,0)} = \lambda_1 + \lambda_2 \quad \lambda_{(0,0);\,(0,1)} = \lambda_2 \quad \lambda_{(0,0);\,(1,0)} = \lambda_1$$

$$v_{(0,1)} = \lambda_1 + \mu_2 \quad \lambda_{(0,1);\,(0,0)} = \mu_2 \quad \lambda_{(0,1);\,(1,1)} = \lambda_1$$

$$v_{(1,0)} = \mu_1 + \lambda_2 \quad \lambda_{(1,0);\,(0,0)} = \mu_1 \quad \lambda_{(1,0);\,(1,1)} = \lambda_2$$

$$v_{(1,1)} = \mu_1 + \mu_2 \quad \lambda_{(1,1);\,(0,1)} = \mu_1 \quad \lambda_{(1,1);\,(1,0)} = \lambda_2$$

By the independence assumption, we have

(a) $P_{(i,j)(k,\ell)}(t) = P_{(i,k)}(t)Q_{(j,\ell)}(t)$

where $P_{i,k}(t) =$ probability that the first machine be in state k at time t given that it was at state i at time 0.

$Q_{j,\ell}(t)$ is defined similarly for the second machine. By Example 4(c) we have

$$P_{00}(t) = [\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1]/(\lambda_1 + \mu_1)$$

$$P_{10}(t) = [\mu_1 - \mu_1 e^{-(\mu_1+\lambda_1)t}]/(\lambda_1 + \mu_1)$$

And by the same argument,

$$P_{11}(t) = [\mu_1 e^{-(\mu_1+\lambda_1)t} + \lambda_1]/(\lambda_1 + \mu_1)$$

$$P_{01}(t) = [\lambda_1 - \lambda_1 e^{-(\mu_1+\lambda_1)t}]/(\lambda_1 + \mu_1)$$

Of course, the similar expressions for the second machine are obtained by replacing (λ_1, μ_1) by (λ_2, μ_2). We get $P_{(i,j)(k,\ell)}(t)$ by formula (a). For instance,

$$\begin{aligned} P_{(0,0)(0,0)}(t) &= P_{(0,0)}(t)Q_{(0,0)}(t) \\ &= \frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]}{(\lambda_1 + \mu_1)} \times \frac{\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_2 + \mu_2)} \end{aligned}$$

Let us check the forward and backward equations for the state $\{(0,0);\ (0,0)\}$.

Backward equation

We should have

$$P'_{(0,0),(0,0)}(t) = (\lambda_1 + \lambda_2)\left[\frac{\lambda_2}{\lambda_1+\lambda_2}P_{(0,1)(0,0)}(t) + \frac{\lambda_1}{\lambda_1+\lambda_2}P_{(1,0)(0,0)}(t) - P_{(0,0)(0,0)}(t)\right]$$

or

$$P'_{(0,0)(0,0)}(t) = \lambda_2 P_{(0,1)(0,0)}(t) + \lambda_1 P_{(1,0)(0,0)}(t) - (\lambda_1 + \lambda_2)P_{(0,0)(0,0)}(t)$$

Let us compute the right-hand side ($r.h.s.$) of this expression:

$r.h.s.$

$$\begin{aligned}
&= \lambda_2 \frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]\left[\mu_2 - \mu_2 e^{-(\mu_2+\lambda_2)t}\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&\quad + \frac{\left[\mu_1 - \mu_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&\quad - (\lambda_1+\lambda_2) \\
&\quad \frac{\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&= \frac{\lambda_2\left[\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&\quad \times \left[\mu_2 - \mu_2 e^{-(\mu_2+\lambda_2)t} - \lambda_2 e^{-(\lambda_2+\mu_2)t} - \mu_2\right] \\
&\quad + \frac{\lambda_1\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)} \\
&\quad \times \left[\mu_1 - \mu_1 e^{-(\mu_1+\lambda_1)t} - \mu_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right] \\
&= \left[-\lambda_2 e^{-(\lambda_2+\mu_2)t}\right]\left[\frac{\lambda_1 e^{-(\lambda_1+\mu_1)t} + \mu_1}{\lambda_1+\mu_1}\right] \\
&\quad + \left[-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\frac{\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2}{\lambda_2+\mu_2}\right] \\
&= Q'_{00}(t)P_{00}(t) + P'_{00}(t)Q_{00}(t) = [P_{00}(t)Q_{00}(t)]' \\
&= [P_{(0,0)(0,0)}(t)]'
\end{aligned}$$

So, for this state, the backward equation is satisfied.

Forward equation

According to the forward equation, we should now have

$$P'_{(0,\,0)(0,\,0)}(t) = \mu_2 P_{(0,\,0)(0,\,1)}(t) + \mu_1 P_{(0,\,0)(1,\,0)}(t) - (\lambda_1 + \lambda_1) P_{(0,\,0)(0,\,0)}(t)$$

Let us compute the right-hand side:

r.h.s.

$$= \mu_2 \frac{\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]\left[\lambda_2 - \lambda_2 e^{-(\lambda_2+\mu_2)t}\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$+ \mu_1 \frac{\left[\lambda_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right]\left[\lambda_2 e^{-(\lambda_2+\mu_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda 2+\mu_2)}$$

$$-(\lambda_1+\lambda_2) \frac{\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}$$

$$= \frac{\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]}{(\lambda_1+\mu_1)}$$

$$\times \frac{\left[\mu_2\lambda_2 - \lambda_2 e^{-(\lambda_2+\mu_2)t} - \lambda_2\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]\right]}{\lambda_2+\mu_2}$$

$$+ \frac{\left[\lambda_2 e^{-(\mu_2+\lambda_2)t} + \mu_2\right]}{(\lambda_2+\mu_2)}$$

$$\times \frac{\left[\mu_1\left[\lambda_1 - \lambda_1 e^{-(\lambda_1+\mu_1)t}\right] - \lambda_1\left[\lambda_1 e^{-(\mu_1+\lambda_1)t} + \mu_1\right]\right]}{(\lambda_1+\mu_1)}$$

$$= P_{00}(t)\left[-\lambda_2 e^{-(\mu_2+\lambda_2)t}\right] + Q_{00}(t)\left[-\lambda_1 e^{-(\lambda_1+\mu_1)t}\right]$$

$$= P_{00}(t)Q'_{00}(t) + Q_{00}(t)P'00(t) = \left[P_{(0,0)(0,0)}(t)\right]$$

In the same way, we can verify Kolmogorov's equations for all the other states.

11. (b) Follows from the hint upon using the lack of memory property and the fact that ϵ_i, the minimum of $j-(i-1)$ independent exponentials with rate λ, is exponential with rate $(j-i+1)\lambda$.

(c) From (a) and (b)

$$P\{T_1 + \cdots + T_j \le t\} = P\left\{\max_{1\le i\le j} X_i \le t\right\} = (1-e^{-\lambda t})^j$$

(d) With all probabilities conditional on $X(0) = 1$

$$\begin{aligned} P_{1j}(t) &= P\{X(t) = j\} \\ &= P\{X(t) \ge j\} - P\{X(t) \ge j+1\} \\ &= P\{T_1 + \cdots + T_j \le t\} \\ &\quad - P\{T_1 + \cdots + T_{j+1} \le t\} \end{aligned}$$

(e) The sum of independent geometrics, each having parameter $p = e^{-\lambda t}$, is negative binomial with parameters i, p. The result follows since starting with an initial population of i is equivalent to having i independent Yule processes, each starting with a single individual.

12. (a) If the state is the number of individuals at time t, we get a birth and death process with

$$\lambda_n = n\lambda + \theta, \qquad n < N$$
$$\lambda_n = n\lambda, \qquad n \ge N$$
$$\mu_n = n\mu$$

(b) Let P_i be the long-run probability that the system is in state i. Since this is also the proportion of time the system is in state i, we are looking for $\sum_{i=3}^{\infty} P_i$.

We have $\lambda_k P_k = \mu_{k+1} P_{k+1}$.

This yields

$$P_1 = \frac{\theta}{\mu} P_0$$

$$P_2 = \frac{\lambda+\theta}{2\mu} P_1 = \frac{\theta(\lambda+\theta)}{2\mu^2} P_0$$

$$P_3 = \frac{2\lambda+\theta}{2\mu} P_2 = \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3} P_0$$

For $k \ge 4$, we get

$$P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1}$$

which implies

$$P_k = \frac{(k-1)(k-2)\cdots(3)}{(k)(k-1)\cdots(4)} \left[\frac{\lambda}{\mu}\right]^{k-3}$$

$$P_k = \frac{3}{k}\left[\frac{\lambda}{\mu}\right]^{k-3} P_3$$

therefore $\displaystyle\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k,$

but $\sum_{k=1}^{\infty} \frac{1}{k}\left[\frac{\lambda}{\mu}\right]^k = \log\left[\frac{1}{1-\frac{\lambda}{\mu}}\right]$

$= \log\left[\frac{\mu}{\mu-\lambda}\right]$ if $\frac{\lambda}{\mu} < 1$

So $\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3 P_3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]$

$$\sum_{k=3}^{\infty} P_k = 3\left[\frac{\mu}{\lambda}\right]^3\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right] \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{6\mu^3}P_0$$

Now $\sum_{0}^{\infty} P_i = 1$ implies

$$P_0 = \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{1}{2\lambda^3}\theta(\lambda+\theta)(2\lambda+\theta) \times \left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right]^{-1}$$

And finally,

$$\sum_{k=3}^{\infty} P_k = \left[\left[\frac{1}{2\lambda^3}\right]\left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right] \theta(\lambda+\theta)(2\lambda+\theta)\right] \Big/ \left[1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{2\lambda^3} \times \left[\log\left[\frac{\mu}{\mu-\lambda}\right] - \frac{\lambda}{\mu} - \frac{1}{2}\left[\frac{\lambda}{\mu}\right]^2\right]\right]$$

13. With the number of customers in the shop as the state, we get a birth and death process with

$\lambda_0 = \lambda_1 = 3, \quad \mu_1 = \mu_2 = 4$

Therefore

$P_1 = \frac{3}{4}P_0, \quad P_2 = \frac{3}{4}, \quad P_1 = \left[\frac{3}{4}\right]^2 P_0$

And since $\sum_{0}^{2} P_i = 1$, we get

$$P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{16}{37}$$

(a) The average number of customers in the shop is

$$P_1 + 2P_2 = \left[\frac{3}{4} + 2\left[\frac{3}{4}\right]^2\right]P_0$$
$$= \frac{30}{16}\left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2\right]^{-1} = \frac{30}{37}$$

(b) The proportion of customers that enter the shop is

$$\frac{\lambda(1-P_2)}{\lambda} = 1 - P_2 = 1 - \frac{9}{16}\cdot\frac{16}{37} = \frac{28}{37}$$

(c) Now $\mu = 8$, and so

$$P_0 = \left[1 + \frac{3}{8} + \left[\frac{3}{8}\right]^2\right]^{-1} = \frac{64}{97}$$

So the proportion of customers who now enter the shop is

$$1 - P_2 = 1 - \left[\frac{3}{8}\right]^2\frac{264}{97} = 1 - \frac{9}{97} = \frac{88}{97}$$

The rate of added customers is therefore

$$\lambda\left[\frac{88}{97}\right] - \lambda\left[\frac{28}{37}\right] = 3\left[\frac{88}{97} - \frac{28}{37}\right] = 0.45$$

The business he does would improve by 0.45 customers per hour.

14. Letting the number of cars in the station be the state, we have a birth and death process with

$\lambda_0 = \lambda_1 = \lambda_2 = 20, \quad \lambda_i = 0, \ i > 2$

$\mu_1 = \mu_2 = 12$

Hence,

$P_1 = \frac{5}{3}P_0, \ P_2 = \frac{5}{3}P_1 = \left[\frac{5}{3}\right]^2 P_0$

$P_3 = \frac{5}{3}P_2 = \left[\frac{5}{3}\right]^3 P_0$

and as $\sum_{0}^{3} P_i = 1$, we have

$$P_0 = \left[1 + \frac{5}{3} + \left[\frac{5}{3}\right]^2 + \left[\frac{5}{3}\right]^2\right]^{-1} = \frac{27}{272}$$

(a) The fraction of the attendant's time spent servicing cars is equal to the fraction of time there are cars in the system and is therefore $1 - P_0 = 245/272$.

(b) The fraction of potential customers that are lost is equal to the fraction of customers that arrive when there are three cars in the station and is therefore

$$P_3 = \left[\frac{5}{3}\right]^3 P_0 = 125/272$$

15. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \ \lambda_i = 0, \quad i \geq 4$$
$$\mu_1 = 2, \ \mu_2 = \mu_3 = 4$$

Therefore, the balance equations reduce to

$$P_1 = \frac{3}{2}P_0, \ P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0, \ P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0$$

And therefore,

$$P_0 = \left[1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right]^{-1} = \frac{32}{143}$$

(a) The fraction of potential customers that enter the system is

$$\frac{\lambda(1 - P_3)}{\lambda} = 1 - P_3 = 1 - \frac{27}{32} \times \frac{32}{143} = \frac{116}{143}$$

(b) With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0 \ P_2 = \frac{3}{4}P_1 = \left[\frac{3}{4}\right]^2 P_0 \ P_3 = \left[\frac{3}{4}\right]^3 P_0$$

and $\quad P_0 = \left[1 + \frac{3}{4} + \left[\frac{3}{4}\right]^2 + \left[\frac{3}{4}\right]^3\right]^{-1} = \frac{64}{175}$

So that now

$$1 - P_3 = 1 - \frac{27}{64} = 1 - \frac{64}{175} = \frac{148}{175}$$

16. Let the state be

0: an acceptable molecule is attached

1: no molecule attached

2: an unacceptable molecule is attached.

Then this is a birth and death process with balance equations

$$P_{12} = \frac{\mu}{\lambda}P_0$$
$$P_2 = \frac{\lambda(1-\alpha)}{\mu_1}P_1 = \frac{(1-\alpha)}{\alpha}\frac{\mu_2}{\mu_1}P_0$$

Since $\sum_0^2 P_i = 1$, we get

$$P_0 = \left[1 + \frac{\mu_2}{\lambda\alpha} + \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}\right]^{-1}$$
$$= \frac{\lambda\alpha\mu_1}{\lambda\alpha\mu_1 + \mu_1\mu_2 + \lambda(1-\alpha)\mu_2}$$

P_0 is the percentage of time the site is occupied by an acceptable molecule.

The percentage of time the site is occupied by an unacceptable molecule is

$$P_2 = \frac{1-\alpha}{\alpha}\frac{\mu_2}{\mu_1}P_0 = \frac{\lambda(1-\alpha)\mu_2}{\lambda\alpha\mu_1 + \mu_1 + \lambda(1-\alpha)\mu_2}$$

17. Say the state is 0 if the machine is up, say it is i when it is down due to a type i failure, $i = 1, 2$. The balance equations for the limiting probabilities are as follows.

$$\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$\mu_1 P_1 = \lambda p P_0$$
$$\mu_2 P_2 = \lambda(1-p)P_0$$
$$P_0 + P_1 + P_2 = 1$$

These equations are easily solved to give the results

$$P_0 = (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1}$$
$$P_1 = \lambda p P_0/\mu_1, \qquad P_2 = \lambda(1-p)P_0/\mu_2$$

18. There are $k + 1$ states; state 0 means the machine is working, state i means that it is in repair phase i, $i = 1, \ldots, k$. The balance equations for the limiting probabilities are

$$\lambda P_0 = \mu_k P_k$$
$$\mu_1 P_1 = \lambda P_0$$
$$\mu_i P_i = \mu_{i-1}P_{i-1}, \quad i = 2, \ldots, k$$
$$P_0 + \cdots + P_k = 1$$

To solve, note that

$$\mu_i P_i = \mu_{i-1}P_{i-1} = \mu_{i-2}P_{i-2} = \cdots = \lambda P_0$$

Hence,

$$P_i = (\lambda/\mu_i)P_0$$

and, upon summing,

$$1 = P_0\left[1 + \sum_{i=1}^{k}(\lambda/\mu_i)\right]$$

Therefore,

$$P_0 = \left[1 + \sum_{i=1}^{k}(\lambda/\mu_i)\right]^{-1}, \quad P_i = (\lambda/\mu_i)P_0,$$

$i = 1, \ldots, k$

The answer to part (a) is P_i and to part (b) is P_0.

19. There are 4 states. Let state 0 mean that no machines are down, state 1 that machine 1 is down and 2 is up, state 2 that machine 1 is up and 2 is down, and 3 that both machines are down. The balance equations are as follows:

$$(\lambda_1 + \lambda_2)P_0 = \mu_1 P_1 + \mu_2 P_2$$
$$(\mu_1 + \lambda_2)P_1 = \lambda_1 P_0 + \mu_1 P_3$$
$$(\lambda_1 + \mu_2)P_2 = \lambda_2 P_0$$
$$\mu_1 P_3 = \mu_2 P_1 + \mu_1 P_2$$
$$P_0 + P_1 + P_2 + P_3 = 1$$

These equations are easily solved and the proportion of time machine 2 is down is $P_2 + P_3$.

20. Letting the state be the number of down machines, this is a birth and death process with parameters

$$\lambda_i = \lambda, \quad i = 0, 1$$
$$\mu_i = \mu, \quad i = 1, 2$$

By the results of Example 3g, we have

$$E[\text{time to go from 0 to 2}] = 2/\lambda + \mu/\lambda^2$$

Using the formula at the end of Section 3, we have

Var(time to go from 0 to 2)

$$= Var(T_0) + Var(T_1)$$
$$= \frac{1}{\lambda^2} + \frac{1}{\lambda(\lambda+\mu)} + \frac{\mu}{\lambda^3} + \frac{\mu}{\mu+\lambda}(2/\lambda + \mu/\lambda^2)^2$$

Using Equation (5.3) for the limiting probabilities of a birth and death process, we have

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2}$$

21. How we have a birth and death process with parameters

$$\lambda_i = \lambda, \quad i = 1, 2$$
$$\mu_i = i\mu, \quad i = 1, 2$$

Therefore,

$$P_0 + P_1 = \frac{1 + \lambda/\mu}{1 + \lambda/\mu + (\lambda/\mu)^2/2}$$

and so the probability that at least one machine is up is higher in this case.

22. The number in the system is a birth and death process with parameters

$$\lambda_n = \lambda/(n+1), \quad n \geq 0$$
$$\mu_n = \mu, \quad n \geq 1$$

From Equation (5.3),

$$1/P_0 = 1 + \sum_{n=1}^{\infty}(\lambda/\mu)^n/n! = e^{\lambda/\mu}$$

and

$$P_n = P_0(\lambda/\mu)^n/n! = e^{-\lambda/\mu}(\lambda/\mu)^n/n!, \quad n \geq 0$$

23. Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10}, \ \lambda_1 = \frac{2}{10}, \ \lambda_2 = \frac{1}{10}, \ \lambda_i = 0, \quad i \geq 3$$
$$\mu_1 = \frac{1}{8}, \ \mu_2 = \frac{2}{8}, \ \mu_3 = \frac{2}{8}$$

The balance equations reduce to

$$P_1 = \frac{3/10}{1/8}P_0 = \frac{12}{5}P_0$$
$$P_2 = \frac{2/10}{2/8}P_1 = \frac{4}{5}P_1 = \frac{48}{25}P_0$$
$$P_3 = \frac{1/10}{2/8}P_2 = \frac{4}{10}P_3 = \frac{192}{250}P_0$$

Hence, using $\sum_{0}^{3} P_i = 1$ yields

$$P_0 = \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right]^{-1} = \frac{250}{1522}$$

(a) Average number not in use

$$= P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}$$

(b) Proportion of time both repairmen are busy

$$= P_2 + P_3 = \frac{672}{1522} = \frac{336}{761}$$

24. We will let the state be the number of taxis waiting. Then, we get a birth and death process with $\lambda_n = 1\mu_n = 2$. This is a $M/M/1$, and therefore,

(a) Average number of taxis waiting $= \dfrac{1}{\mu - \lambda}$

$$= \frac{1}{2-1} = 1$$

(b) The proportion of arriving customers that get taxis is the proportion of arriving customers that find at least one taxi waiting. The rate of arrival of such customers is $2(1 - P_0)$. The proportion of such arrivals is therefore

$$\frac{2(1-P_0)}{2} = 1 - P_0 = 1 - \left[1 - \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu} = \frac{1}{2}$$

25. If $N_i(t)$ is the number of customers in the ith system $(i = 1, 2)$, then let us take $\{N_1(t), N_2(t)\}$ as the state. The balance equation are with $n \geq 1, m \geq 1$.

(a) $\lambda P_{0,\,0} = \mu_2 P_{0,\,1}$

(b) $P_{n,\,0}(\lambda + \mu_1) = \lambda P_{n-1,\,0} + \mu_2 P_{n,\,1}$

(c) $P_{0,\,m}(\lambda + \mu_2) = \mu_1 P_{1,\,m-1} + \mu_2 P_{0,\,m+1}$

(d) $P_{n,\,m}(\lambda + \mu_1 + \mu_2) = \lambda P_{n-1,\,m} + \mu_1 P_{n+1,\,m-1} + \mu_2 P_{n,\,m+1}$

We will try a solution of the form $C\alpha^n \beta^m = P_{n,\,m}$. From (a), we get

$$\lambda C = \mu_2 C\beta = \beta = \frac{\lambda}{\mu_2}$$

From (b),

$$(\lambda + \mu_1)C\alpha^n = \lambda C\alpha^{n-1} + \mu_2 C\alpha^n \beta$$

or

$$(\lambda + \mu_1)\alpha = \lambda + \mu_2\alpha\beta = \lambda + \mu_2\alpha\frac{\lambda}{\mu} = \lambda + \lambda\alpha$$

and $\quad \mu_1\alpha = \lambda \Rightarrow \alpha = \dfrac{\lambda}{\mu_1}$

To get C, we observe that $\sum\limits_{n,\,m} P_{n,\,m} = 1$

but

$$\sum_{n,\,m} P_{n,\,m} = C\sum_n \alpha^n \sum_m \beta^m = C\left[\frac{1}{1-\alpha}\right]\left[\frac{1}{1-\beta}\right]$$

and $\quad C = \left[1 - \dfrac{\lambda}{\mu_1}\right]\left[1 - \dfrac{\lambda}{\mu_2}\right]$

Therefore a solution of the form $C\alpha^n\beta^n$ must be given by

$$P_{n,\,m} = \left[1 - \frac{\lambda}{\mu_1}\right]\left[\frac{\lambda}{\mu_1}\right]^n \left[1 - \frac{\lambda}{\mu_2}\right]\left[\frac{\lambda}{\mu_2}\right]^m$$

It is easy to verify that this also satisfies (c) and (d) and is therefore the solution of the balance equations.

26. Since the arrival process is Poisson, it follows that the sequence of future arrivals is independent of the number presently in the system. Hence, by time reversibility the number presently in the system must also be independent of the sequence of past departures (since looking backwards in time departures are seen as arrivals).

27. It is a Poisson process by time reversibility. If $\lambda > \delta\mu$, the departure process will (in the limit) be a Poisson process with rate $\delta\mu$ since the servers will always be busy and thus the time between departures will be independent random variables each with rate $\delta\mu$.

28. Let P_{ij}^x, V_i^x denote the parameters of the $X(t)$ and P_{ij}^y, V_i^y of the $Y(t)$ process; and let the limiting probabilities be P_i^x, P_i^y, respectively. By independence we have that for the Markov chain $\{X(t), Y(t)\}$ its parameters are

$$V_{(i,\,\ell)} = V_i^x + V_\ell^y$$

$$P_{(i,\,\ell),(j,\,\ell)} = \frac{V_i^x}{V_i^x + V_\ell^y} P_{ij}^x$$

$$P_{(i,\,\ell),\,(i,\,k)} = \frac{V_\ell^y}{V_i^x + V_\ell^y} P_{\ell k}^y$$

and

$$\lim_{t\to\infty} P\{(X(t), Y(t)) = (i, j)\} = P_i^x P_j^y$$

Hence, we need show that

$$P_i^x P_\ell^y V_i^x P_{ij}^x = P_j^x P_\ell^y V_j^x P_{ji}^x$$

(That is, rate from (i, ℓ) to (j, ℓ) equals the rate from (j, ℓ) to (i, ℓ)). But this follows from the fact that the rate from i to j in $X(t)$ equals the rate from j to i; that is,

$$P_i^x V_i^x P_{ij}^x = P_j^x V_j^x P_{ji}^x$$

The analysis is similar in looking at pairs (i, ℓ) and (i, k).

29. (a) Let the state be S, the set of failed machines.

(b) For $i \in S, j \in S^c$,

$$q_{S,\, S-i} = \mu_i/|S|, q_{S,\, S+j} = \lambda_j$$

where $S - i$ is the set S with i deleted and $S + j$ is similarly S with j added. In addition, $|S|$ denotes the number of elements in S.

(c) $P_S q_{S,\, S-i} = P_{S-i} q_{S-i,\, S}$

(d) The equation in (c) is equivalent to

$$P_S \mu_i/|S| = P_{S-i}\lambda_i$$

or

$$P_S = P_{S-i}|S|\lambda_i/\mu_i$$

Iterating this recursion gives

$$P_S = P_0(|S|)! \prod_{i\in S}(\lambda_i/\mu_i)$$

where 0 is the empty set. Summing over all S gives

$$1 = P_0 \sum_S (|S|)! \prod_{i\in S}(\lambda_i/\mu_i)$$

and so

$$P_S = \frac{(|S|)! \prod_{i\in S}(\lambda_i/\mu_i)}{\sum_S (|S|)! \prod_{i\in S}(\lambda_i/\mu_i)}$$

As this solution satisfies the time reversibility equations, it follows that, in the steady state, the chain is time reversible with these limiting probabilities.

30. Since λ_{ij} is the rate it enters j when in state i, all we need do to prove both time reversibility and that P_j is as given is to verify that

$$\lambda_{kj}P_k = \lambda_{jk}P_j \sum_1^n P_j = 1$$

Since $\lambda_{kj} = \lambda_{jk}$, we see that $P_j \equiv 1/n$ satisfies the above.

31. (a) This follows because of the fact that all of the service times are exponentially distributed and thus memoryless.

(b) Let $n = (n_1, \ldots, n_i, \ldots, n_j, \ldots, n_r)$, where $n_i > 0$ and let $n' = (n_1, \ldots, n_i - 1, \ldots, n_j - 1, \ldots, n_r)$. Then $q_{n,\, n'} = \mu_i/(r-1)$.

(c) The process is time reversible if we can find probabilities $P(n)$ that satisfy the equations

$$P(n)\mu_i/(r-1) = P(n')\mu_j/(r-1)$$

where n and n' are as given in part (b). The above equations are equivalent to

$$\mu_i P(n) = \mu_j/P(n')$$

Since $n_i = n'_i + 1$ and $n'_j = n_j + 1$ (where n_k refers to the k^{th} component of the vector n), the above equation suggests the solution

$$P(n) = C\prod_{k=1}^{r}(1/\mu_k)^n k$$

where C is chosen to make the probabilities sum to 1. As $P(n)$ satisfies all the time reversibility equations it follows that the chain is time reversible and the $P(n)$ given above are the limiting probabilities.

32. The states are 0, 1, 1′, $n, n \geq 2$. State 0 means the system is empty, state 1 (1′) means that there is one in the system and that one is with server 1 (2); state $n, n \geq 2$, means that there are n customers in the system. The time reversibility equations are as follows:

$$(\lambda/2)P_0 = \mu_1 P_1$$

$$(\lambda/2)P_0 = \mu_2 P_{1'}$$

$$\lambda P_1 = \mu_2 P_2$$

$$\lambda P_{1'} = \mu_1 P_2$$

$$\lambda P_n = \mu P_{n+1}, n \geq 2$$

where $\mu = \mu_1 + \mu_2$. Solving the last set of equations (with $n \geq 2$) in terms of P_2 gives

$$P_{n+1} = (\lambda/\mu)P_n$$

$$= (\lambda/\mu)^2 P_{n-1} = \cdots = (\lambda/\mu)^{n-1}P_2$$

That is,

$$P_{n+2} = (\lambda/\mu)^n P_2, \quad n \geq 0$$

The third and fourth equations above yield

$$P_1 = (\mu_2/\lambda)P_2$$

$$P_{1'} = (\mu_1/\lambda)P_2$$

The second equation yields

$$P_0 = (2\mu_2/\lambda)P_{1'} = (2\mu_1\mu_2/\lambda^2)P_2$$

Thus all the other probabilities are determined in terms of P_0. However, we must now verify that the

top equation holds for this solution. This is shown as follows:

$$P_0 = (2\mu_1/\lambda)P_1 = (2\mu_1\mu_2/\lambda^2)P_2$$

Thus all the time reversible equations hold when the probabilities are given (in terms of P_2) as shown above. The value of P_2 is now obtained by requiring all the probabilities to sum to 1. The fact that this sum will be finite follows from the assumption that $\lambda/\mu < 1$.

33. Suppose first that the waiting room is of infinite size. Let $X_i(t)$ denote the number of customers at server $i, i = 1, 2$. Then since each of the $M/M/1$ processes $\{X_i(t)\}$ is time-reversible, it follows by Problem 28 that the vector process $\{(X_1(t), X_2(t)), t \geq 0\}$ is a time-reversible Markov chain. Now the process of interest is just the truncation of this vector process to the set of states A where

$$A = \{(0, m) : m \leq 4\} \cup \{(n, 0) : n \leq 4\}$$
$$\cup \{(n, m) : nm > 0, n + m \leq 5\}$$

Hence, the probability that there are n with server 1 and n with server 2 is

$$P_{n,m} = k(\lambda_1/\mu_1)^n(1 - \lambda_1/\mu_1)(\lambda_2/\mu_2)^m(1 - \lambda_2/\mu_2),$$
$$= C(\lambda_1/\mu_1)^n(\lambda_2/\mu_2)^m, \quad (n, m) \in A$$

The constant C is determined from

$$\sum P_{n,n} = 1$$

where the sum is over all (n, m) in A.

34. The process $\{X_i(t)\}$ is a two state continuous-time Markov chain and its limiting probability is

$$\lim_{t\to\infty} P\{X_i(t) = 1\} = \mu_i/(\mu_i + \lambda_i), \quad i = 1, \ldots, 4$$

(a) By independence,
proportion of time all working
$$= \prod_{i=1}^{4} \mu_i/(\mu_i + \lambda_i)$$

(b) It is a continuous-time Markov chain since the processes $\{X_i(t)\}$ are independent with each being a continuous-time Markov chain.

(c) Yes, by Problem 28 since each of the processes $\{X_i(t)\}$ is time reversible.

(d) The model that supposes that one of the phones is down is just a truncation of the process $\{X(t)\}$ to the set of states A, where A includes all 16 states except $(0, 0, 0, 0)$. Hence, for the truncated model

$$P\{\text{all working/truncated}\}$$
$$= P\{\text{all working}\}/(1 - P(0,0,0,0)$$
$$= \frac{\prod_{i=1}^{4}(\mu_i/(\mu_i + \lambda_i)}{1 - \prod_{i=1}^{4}(\lambda_i/(\lambda_i + \mu_i)}$$

35. We must find probabilities P_i^n such that

$$P_i^n q_{ij}^n = P_j^n q_{ji}^n$$

or

$$\begin{aligned} cP_i^n q_{ij} &= P_j^n q_{ji}, && \text{if } i \in A, j \notin A \\ P_i q_{ij} &= cP_j^n q_{ji}, && \text{if } i \notin A, j \in A \\ P_i q_{ij} &= P_j q_{ji}, && \text{otherwise} \end{aligned}$$

Now, $P_i q_{ij} = P_j q_{ji}$ and so if we let

$$P_i^n = \begin{matrix} kP_i/c & \text{if } i \in A \\ kP_i & \text{if } i \notin A \end{matrix}$$

then we have a solution to the above equations. By choosing k to make the sum of the P_j^n equal to 1, we have the desired result. That is,

$$k = \left(\sum_{i \in A} P_i/c - \sum_{i \notin A} P_i\right)^{-1}$$

36. In Problem 3, with the state being the number of machines down, we have

$$v_0 = 2\lambda\, P_{0,1} = 1$$
$$v_1 = \lambda + \mu\ P_{1,0} = \frac{\mu}{(\lambda + \mu)} P_{1,2} = \frac{1}{(\lambda + \mu)}$$
$$v_2 = \mu\ P_{2,1} = 1$$

We will choose $v = 2\lambda = 2\mu$, then the uniformized version is given by

$$v_i^n = 2(\lambda + \mu) \quad \text{for } i = 0, 1, 2$$
$$P_{00}^n = 1 - \frac{2\lambda}{2(\lambda + \mu)} = \frac{\lambda}{(\lambda + \mu)}$$
$$P_{01}^n = \frac{2\lambda}{2(\lambda + \mu)} \cdot 1 = \frac{\lambda}{(\lambda + \mu)}$$
$$P_{10}^n = \frac{\lambda + \mu}{2(\lambda + \mu)} \cdot \frac{\mu}{(\lambda + \mu)} = \frac{\mu}{2(\lambda + \mu)}$$

$$P_{11}^n = 1 - \frac{\lambda+\mu}{2(\lambda+\mu)} = \frac{1}{2}$$

$$P_{12}^n = \frac{\lambda+\mu}{2(\lambda+\mu)}\frac{\lambda}{(\lambda+\mu)} = \frac{\lambda}{2(\lambda+\mu)}$$

$$P_{21}^n = \frac{\mu}{2(\lambda+\mu)}$$

$$P_{22}^n = 1 - \frac{\mu}{2(\lambda+\mu)} = \frac{2\lambda+\mu}{2(\lambda+\mu)}$$

37. The state of any time is the set of down components at that time. For $S \subset \{1, 2, \ldots, n\}$, $i \notin S, j \in S$

$$q(S, S+i) = \lambda_i$$

$$q(S, S-j) = \mu_j \alpha^{|S|}$$

where $S + i = S \cup \{i\}, S - j = S \cap \{j\}^c, |S| =$ number of elements in S.

The time reversible equations are

$$P(S)\mu_i \alpha^{|S|} = P(S-i)\lambda_i, \quad i \in S$$

The above is satisfied when, for $S = \{i_1, i_2, \ldots, i_k\}$

$$P(S) = \frac{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}}{\mu_{i_1}\mu_{i_2}\cdots\mu_{i_k}\alpha^{k(k+1)/2}}P(\phi)$$

where $P(\phi)$ is determined so that

$$\sum P(S) = 1$$

where the sum is over all the 2^n subsets of $\{1, 2, \ldots, n\}$.

38. Say that the process is "on" when in state 0.

(a) $E[0(t+h)] = E[0(t) + \text{ on time in } (t, t+h)]$
$= n(t) + E[\text{on time in } (t, t+h)]$

Now

$E[\text{on time in } (t, t+h)|X(t) = 0] = h + o(h)$

$E[\text{on time in } (t, t+h)|X(t) = 1] = o(h)$

So, by the above

$$n(t+h) = n(t) + P_{00}(t)h + o(h)$$

(b) From (a) we see that

$$\frac{n(t+h) - n(t)}{h} = P_{00}(t) + o(h)/h$$

Let $h = 0$ to obtain

$$n'(t) = P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}$$

Integrating gives

$$n(t) = \frac{\mu t}{\lambda+\mu} - \frac{\lambda}{(\lambda+\mu)^2}e^{-(\lambda+\mu)t} + C$$

Since $m(0) = 0$ it follows that $C = \lambda/(\lambda+\mu)^2$.

39. $E[0(t)|x(0) = 1] = t - E[\text{time in } 1|X(0) = 1]$

$$= t - \frac{\lambda t}{\lambda+\mu} - \frac{\mu}{(\lambda+\mu)^2}[1 - e^{-(\lambda+\mu)t}]$$

The final equality is obtained from Example 7b (or Problem 38) by interchanging λ and μ.

40. $Cov[X(s), X(t)] = E[X(s)X(t)] - E[X(s)]EX(t)]$

Now,

$$X(s)X(t) = \begin{cases} 1 & \text{if } X(s) = X(t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, for $s \le t$

$E[X(s)X(t)]$

$= P\{X(s) = X(t) = 1|X(0) = 0\}$

$= P_{00}(s)P_{00}(t-s)$ by the Markovian property

$$= \frac{1}{(\lambda+\mu)^2}[\mu + \lambda e^{-(\lambda+\mu)s}][\mu + \lambda e^{-(\lambda+\mu)(t-s)}]$$

Also,

$E[X(s)]E[X(t)]$

$$= \frac{1}{(\lambda+\mu)^2}[\mu + \lambda e^{-(\lambda+\mu)s}][\mu + \lambda e^{-(\lambda+\mu)t}]$$

Hence,

$Cov[X(s), X(t)]$

$$= \frac{1}{(\lambda+\mu)^2}[\mu + \lambda e^{-(\lambda+\mu)s}]\lambda e^{-(\lambda+\mu)t}[e^{(\lambda+\mu)s} - 1]$$

41. (a) Letting T_i denote the time until a transition out of i occurs, we have

$$P_{ij} = P\{X(Y) = j\} = P\{X(Y) = j \mid T_i < Y\}$$
$$\times \frac{v_i}{v_i + \lambda} + P\{X(Y) = j | Y \le T_i\}\frac{\lambda}{\lambda + v_i}$$
$$= \sum_k P_{ik}P_{kj}\frac{v_i}{v_i + \lambda} + \frac{\delta_{ij}\lambda}{\lambda + v_i}$$

The first term on the right follows upon conditioning on the state visited from i (which is k with probability P_{ik}) and then using the lack of memory property of the exponential to assert that given a transition into k occurs before time Y then the state at Y is probabilistically the

same as if the process had started in state k and we were interested in the state after an exponential time with rate λ. As $q_{ik} = v_i P_{ik}$, the result follows.

(b) From (a)

$$(\lambda + v_i)\bar{P}_{ij} = \sum_k q_{ik}\bar{P}_{kj} + \lambda\delta_{ij}$$

or

$$-\lambda\delta_{ij} = \sum_k r_{ik}\bar{P}_{kj} - \lambda\bar{P}_{ij}$$

or, in matrix terminology,

$$-\lambda I = R\bar{P} - \lambda I\bar{P}$$
$$= (R - \lambda I)\bar{P}$$

implying that

$$\bar{P} = -\lambda I(R - \lambda I)^{-1} = -(R/\lambda - I)^{-1}$$
$$= (I - R/\lambda)^{-1}$$

(c) Consider, for instance,

$$P\{X(Y_1 + Y_2) = j|X(0) = i\}$$
$$= \sum_k P\{X(Y_1 + Y_2) = j|X(Y_1) = k, X(0) = i)$$
$$P\{X(Y_1) = k|X(0) = i\}$$
$$= \sum_k P\{X(Y_1 + Y_2) = j|X(Y_1) = k\}\bar{P}_{ik}$$
$$= \sum_k P\{X(Y_2) = j|X(0) = k\}\bar{P}_{ik}$$
$$= \sum_k \bar{P}_{kj}\bar{P}_{ik}$$

and thus the state at time $Y_1 + Y_2$ is just the 2-stage transition probabilities of $\bar{P}_{ij}$. The general case can be established by induction.

(d) The above results in exactly the same approximation as Approximation 2 in Section 6.8.

42. (a) The matrix P^* can be written as

$$P^* = I + R/v$$

and so P_{ij}^{*n} can be obtained by taking the i, j element of $(I + R/v)^n$, which gives the result when $v = n/t$.

(b) Uniformization shows that $P_{ij}(t) = E\left[P_{ij}^{*N}\right]$, where N is independent of the Markov chain with transition probabilities P_{ij}^* and is Poisson distributed with mean vt. Since a Poisson random variable with mean vt has standard deviation $(vt)^{1/2}$, it follows that for large values of vt it should be near vt. (For instance, a Poisson random variable with mean 10^6 has standard deviation 10^3 and thus will, with high probability, be within 3000 of 10^6.) Hence, since for fixed i and j, P_{ij}^{*m} should not vary much for values of m about vt when vt is large, it follows that, for large vt

$$E\left[P_{ij}^{*N}\right] \approx P_{ij}^{*n}, \quad \text{where } n = vt$$

Renewal Theory and Its Applications

Exercises

1. Is it true that
   (a) $N(t) < n$ if and only if $S_n > t$?
   (b) $N(t) \leqslant n$ if and only if $S_n \geqslant t$?
   (c) $N(t) > n$ if and only if $S_n < t$?

2. Suppose that the interarrival distribution for a renewal process is Poisson distributed with mean μ. That is, suppose

$$P\{X_n = k\} = e^{-\mu}\frac{\mu^k}{k!}, \quad k = 0, 1, \ldots$$

   (a) Find the distribution of S_n.
   (b) Calculate $P\{N(t) = n\}$.

*3. If the mean-value function of the renewal process $\{N(t), t \geqslant 0\}$ is given by $m(t) = t/2, t \geqslant 0$, what is $P\{N(5) = 0\}$?

4. Let $\{N_1(t),\ t \geqslant 0\}$ and $\{N_2(t),\ t \geqslant 0\}$ be independent renewal processes. Let $N(t) = N_1(t) + N_2(t)$.
   (a) Are the interarrival times of $\{N(t),\ t \geqslant 0\}$ independent?
   (b) Are they identically distributed?
   (c) Is $\{N(t),\ t \geqslant 0\}$ a renewal process?
5. Let $U_1, U_2, \dots$ be independent uniform (0, 1) random variables, and define N by

$$N = \min\{n : U_1 + U_2 + \cdots + U_n > 1\}$$

   What is $E[N]$?

*6. Consider a renewal process $\{N(t),\ t \geqslant 0\}$ having a gamma (r, λ) interarrival distribution. That is, the interarrival density is

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{r-1}}{(r-1)!}, \quad x > 0$$

   (a) Show that

$$P\{N(t) \geqslant n\} = \sum_{i=nr}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

   (b) Show that

$$m(t) = \sum_{i=r}^{\infty} \left[\frac{i}{r}\right] \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

   where $[i/r]$ is the largest integer less than or equal to i/r.

   Hint: Use the relationship between the gamma (r, λ) distribution and the sum of r independent exponentials with rate λ to define $N(t)$ in terms of a Poisson process with rate λ.

7. Mr. Smith works on a temporary basis. The mean length of each job he gets is three months. If the amount of time he spends between jobs is exponentially distributed with mean 2, then at what rate does Mr. Smith get new jobs?

*8. A machine in use is replaced by a new machine either when it fails or when it reaches the age of T years. If the lifetimes of successive machines are independent with a common distribution F having density f, show that
   (a) the long-run rate at which machines are replaced equals

$$\left[\int_0^T x f(x)\,dx + T(1 - F(T))\right]^{-1}$$

   (b) the long-run rate at which machines in use fail equals

$$\frac{F(T)}{\int_0^T x f(x)\,dx + T[1 - F(T)]}$$

9. A worker sequentially works on jobs. Each time a job is completed, a new one is begun. Each job, independently, takes a random amount of time having distribution F to complete. However, independently of this, shocks occur according to a Poisson process with rate λ. Whenever a shock occurs, the worker discontinues working on the present job and starts a new one. In the long run, at what rate are jobs completed?

10. Consider a renewal process with mean interarrival time μ. Suppose that each event of this process is independently "counted" with probability p. Let $N_C(t)$ denote the number of counted events by time t, $t > 0$.
    (a) Is $N_C(t)$, $t \geqslant 0$ a renewal process?
    (b) What is $\lim_{t\to\infty} N_C(t)/t$?

11. A renewal process for which the time until the initial renewal has a different distribution than the remaining interarrival times is called a *delayed* (or a *general*) *renewal process*. Prove that Proposition 7.1 remains valid for a delayed renewal process. (In general, it can be shown that all of the limit theorems for a renewal process remain valid for a delayed renewal process provided that the time until the first renewal has a finite mean.)

12. Events occur according to a Poisson process with rate λ. Any event that occurs within a time d of the event that immediately preceded it is called a d-event. For instance, if $d = 1$ and events occur at times $2, 2.8, 4, 6, 6.6, \ldots$, then the events at times 2.8 and 6.6 would be d-events.
    (a) At what rate do d-events occur?
    (b) What proportion of all events are d-events?

13. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. The nonnegative integer valued random variable N is said to be a *stopping time* for the sequence if the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$. The idea being that the X_i are observed one at a time—first X_1, then X_2, and so on—and N represents the number observed when we stop. Hence, the event $\{N = n\}$ corresponds to stopping after having observed $X_1, \ldots, X_n$ and thus must be independent of the values of random variables yet to come, namely, $X_{n+1}, X_{n+2}, \ldots$.
    (a) Let $X_1, X_2, \ldots$ be independent with

$$P\{X_i = 1\} = p = 1 - P\{X_i = 0\}, \quad i \geqslant 1$$

Define

$$N_1 = \min\{n : X_1 + \cdots + X_n = 5\}$$

$$N_2 = \begin{cases} 3, & \text{if } X_1 = 0 \\ 5, & \text{if } X_1 = 1 \end{cases}$$

$$N_3 = \begin{cases} 3, & \text{if } X_4 = 0 \\ 2, & \text{if } X_4 = 1 \end{cases}$$

Which of the N_i are stopping times for the sequence $X_1, \ldots$? An important result, known as *Wald's equation* states that if $X_1, X_2, \ldots$ are independent and identically distributed and have a finite mean $E(X)$, and if N is a stopping time

for this sequence having a finite mean, then

$$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X]$$

To prove Wald's equation, let us define the indicator variables $I_i, i \geqslant 1$ by

$$I_i = \begin{cases} 1, & \text{if } i \leqslant N \\ 0, & \text{if } i > N \end{cases}$$

(b) Show that

$$\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i I_i$$

From part (b) we see that

$$E\left[\sum_{i=1}^{N} X_i\right] = E\left[\sum_{i=1}^{\infty} X_i I_i\right]$$
$$= \sum_{i=1}^{\infty} E[X_i I_i]$$

where the last equality assumes that the expectation can be brought inside the summation (as indeed can be rigorously proven in this case).

(c) Argue that X_i and I_i are independent.

Hint: I_i equals 0 or 1 depending on whether or not we have yet stopped after observing which random variables?

(d) From part (c) we have

$$E\left[\sum_{i=1}^{N} X_i\right] = \sum_{i=1}^{\infty} E[X]E[I_i]$$

Complete the proof of Wald's equation.

(e) What does Wald's equation tell us about the stopping times in part (a)?

14. Wald's equation can be used as the basis of a proof of the elementary renewal theorem. Let $X_1, X_2, \ldots$ denote the interarrival times of a renewal process and let $N(t)$ be the number of renewals by time t.

(a) Show that whereas $N(t)$ is *not* a stopping time, $N(t) + 1$ is.

Hint: Note that

$$N(t) = n \Leftrightarrow X_1 + \cdots + X_n \leqslant t \quad \text{and} \quad X_1 + \cdots + X_{n+1} > t$$

(b) Argue that

$$E\left[\sum_{i=1}^{N(t)+1} X_i\right] = \mu[m(t) + 1]$$

(c) Suppose that the X_i are bounded random variables. That is, suppose there is a constant M such that $P\{X_i < M\} = 1$. Argue that

$$t < \sum_{i=1}^{N(t)+1} X_i < t + M$$

(d) Use the previous parts to prove the elementary renewal theorem when the interarrival times are bounded.

15. Consider a miner trapped in a room that contains three doors. Door 1 leads him to freedom after two days of travel; door 2 returns him to his room after a four-day journey; and door 3 returns him to his room after a six-day journey. Suppose at all times he is equally likely to choose any of the three doors, and let T denote the time it takes the miner to become free.

(a) Define a sequence of independent and identically distributed random variables $X_1, X_2 \ldots$ and a stopping time N such that

$$T = \sum_{i=1}^{N} X_i$$

Note: You may have to imagine that the miner continues to randomly choose doors even after he reaches safety.

(b) Use Wald's equation to find $E[T]$.

(c) Compute $E\left[\sum_{i=1}^{N} X_i | N = n\right]$ and note that it is not equal to $E[\sum_{i=1}^{n} X_i]$.

(d) Use part (c) for a second derivation of $E[T]$.

16. A deck of 52 playing cards is shuffled and the cards are then turned face up one at a time. Let X_i equal 1 if the ith card turned over is an ace, and let it be 0 otherwise, $i = 1, \ldots, 52$. Also, let N denote the number of cards that need be turned over until all four aces appear. That is, the final ace appears on the Nth card to be turned over. Is the equation

$$E\left[\sum_{i=1}^{N} X_i\right] = E[N]E[X_i]$$

valid? If not, why is Wald's equation not applicable?

17. In Example 7.6, suppose that potential customers arrive in accordance with a renewal process having interarrival distribution F. Would the number of events by time t constitute a (possibly delayed) renewal process if an event corresponds to a customer

(a) entering the bank?

(b) leaving the bank?

What if F were exponential?

*18. Compute the renewal function when the interarrival distribution F is such that

$$1 - F(t) = pe^{-\mu_1 t} + (1 - p)e^{-\mu_2 t}$$

19. For the renewal process whose interarrival times are uniformly distributed over $(0, 1)$, determine the expected time from $t = 1$ until the next renewal.

20. For a renewal reward process consider

$$W_n = \frac{R_1 + R_2 + \cdots + R_n}{X_1 + X_2 + \cdots + X_n}$$

where W_n represents the average reward earned during the first n cycles. Show that $W_n \to E[R]/E[X]$ as $n \to \infty$.

21. Consider a single-server bank for which customers arrive in accordance with a Poisson process with rate λ. If a customer will enter the bank only if the server is free when he arrives, and if the service time of a customer has the distribution G, then what proportion of time is the server busy?

*22. The lifetime of a car has a distribution H and probability density h. Ms. Jones buys a new car as soon as her old car either breaks down or reaches the age of T years. A new car costs C_1 dollars and an additional cost of C_2 dollars is incurred whenever a car breaks down. Assuming that a T-year-old car in working order has an expected resale value $R(T)$, what is Ms. Jones' long-run average cost?

23. Consider the gambler's ruin problem where on each bet the gambler either wins 1 with probability p or loses 1 with probability $1 - p$. The gambler will continue to play until his winnings are either $N - i$ or $-i$. (That is, starting with i the gambler will quit when his fortune reaches either N or 0.) Let T denote the number of bets made before the gambler stops. Use Wald's equation, along with the known probability that the gambler's final winnings are $N - i$, to find $E[T]$.

Hint: Let X_j be the gambler's winnings on bet $j, j \geqslant 1$. What are the possible values of $\sum_{j=1}^{T} X_j$? What is $E\left[\sum_{j=1}^{T} X_j\right]$?

24. Wald's equation can also be proved by using renewal reward processes. Let N be a stopping time for the sequence of independent and identically distributed random variables $X_i, i \geqslant 1$.
    (a) Let $N_1 = N$. Argue that the sequence of random variables $X_{N_1+1}, X_{N_1+2}, \ldots$ is independent of $X_1, \ldots, X_N$ and has the same distribution as the original sequence $X_i, i \geqslant 1$.

    Now treat $X_{N_1+1}, X_{N_1+2}, \ldots$ as a new sequence, and define a stopping time N_2 for this sequence that is defined exactly as N_1 is on the original sequence. (For instance, if $N_1 = \min\{n: X_n > 0\}$, then $N_2 = \min\{n: X_{N_1+n} > 0\}$.) Similarly, define a stopping time N_3 on the sequence $X_{N_1+N_2+1}, X_{N_1+N_2+2}, \ldots$ that is identically defined on this sequence as N_1 is on the original sequence, and so on.
    (b) Is the reward process in which X_i is the reward earned during period i a renewal reward process? If so, what is the length of the successive cycles?
    (c) Derive an expression for the average reward per unit time.
    (d) Use the strong law of large numbers to derive a second expression for the average reward per unit time.
    (e) Conclude Wald's equation.

25. Suppose in Example 7.13 that the arrival process is a Poisson process and suppose that the policy employed is to dispatch the train every t time units.

(a) Determine the average cost per unit time.
(b) Show that the minimal average cost per unit time for such a policy is approximately $c/2$ plus the average cost per unit time for the best policy of the type considered in that example.

26. Consider a train station to which customers arrive in accordance with a Poisson process having rate λ. A train is summoned whenever there are N customers waiting in the station, but it takes K units of time for the train to arrive at the station. When it arrives, it picks up all waiting customers. Assuming that the train station incurs a cost at a rate of nc per unit time whenever there are n customers present, find the long-run average cost.

27. A machine consists of two independent components, the ith of which functions for an exponential time with rate λ_i. The machine functions as long as at least one of these components function. (That is, it fails when both components have failed.) When a machine fails, a new machine having both its components working is put into use. A cost K is incurred whenever a machine failure occurs; operating costs at rate c_i per unit time are incurred whenever the machine in use has i working components, $i = 1, 2$. Find the long-run average cost per unit time.

28. In Example 7.15, what proportion of the defective items produced is discovered?

29. Consider a single-server queueing system in which customers arrive in accordance with a renewal process. Each customer brings in a random amount of work, chosen independently according to the distribution G. The server serves one customer at a time. However, the server processes work at rate i per unit time whenever there are i customers in the system. For instance, if a customer with workload 8 enters service when there are three other customers waiting in line, then if no one else arrives that customer will spend 2 units of time in service. If another customer arrives after 1 unit of time, then our customer will spend a total of 1.8 units of time in service provided no one else arrives.

    Let W_i denote the amount of time customer i spends in the system. Also, define $E[W]$ by

$$E[W] = \lim_{n\to\infty} (W_1 + \cdots + W_n)/n$$

and so $E[W]$ is the average amount of time a customer spends in the system.

    Let N denote the number of customers that arrive in a busy period.

(a) Argue that

$$E[W] = E[W_1 + \cdots + W_N]/E[N]$$

Let L_i denote the amount of work customer i brings into the system; and so the L_i, $i \geqslant 1$, are independent random variables having distribution G.

(b) Argue that at any time t, the sum of the times spent in the system by all arrivals prior to t is equal to the total amount of work processed by time t.

Hint: Consider the rate at which the server processes work.

(c) Argue that

$$\sum_{i=1}^{N} W_i = \sum_{i=1}^{N} L_i$$

(d) Use Wald's equation (see Exercise 13) to conclude that

$$E[W] = \mu$$

where μ is the mean of the distribution G. That is, the average time that customers spend in the system is equal to the average work they bring to the system.

*30. For a renewal process, let $A(t)$ be the age at time t. Prove that if $\mu < \infty$, then with probability 1

$$\frac{A(t)}{t} \to 0 \quad \text{as } t \to \infty$$

31. If $A(t)$ and $Y(t)$ are, respectively, the age and the excess at time t of a renewal process having an interarrival distribution F, calculate

$$P\{Y(t) > x | A(t) = s\}$$

32. Determine the long-run proportion of time that $X_{N(t)+1} < c$.

33. In Example 7.14, find the long-run proportion of time that the server is busy.

34. An $M/G/\infty$ queueing system is cleaned at the fixed times $T, 2T, 3T, \ldots$. All customers in service when a cleaning begins are forced to leave early and a cost C_1 is incurred for each customer. Suppose that a cleaning takes time $T/4$, and that all customers who arrive while the system is being cleaned are lost, and a cost C_2 is incurred for each one.
(a) Find the long-run average cost per unit time.
(b) Find the long-run proportion of time the system is being cleaned.

*35. Satellites are launched according to a Poisson process with rate λ. Each satellite will, independently, orbit the earth for a random time having distribution F. Let $X(t)$ denote the number of satellites orbiting at time t.
(a) Determine $P\{X(t) = k\}$.

Hint: Relate this to the $M/G/\infty$ queue.

(b) If at least one satellite is orbiting, then messages can be transmitted and we say that the system is functional. If the first satellite is orbited at time $t = 0$, determine the expected time that the system remains functional.

Hint: Make use of part (a) when $k = 0$.

36. Each of n skiers continually, and independently, climbs up and then skis down a particular slope. The time it takes skier i to climb up has distribution F_i, and it is independent of her time to ski down, which has distribution H_i, $i = 1, \ldots, n$. Let $N(t)$ denote the total number of times members of this group have skied down the slope by time t. Also, let $U(t)$ denote the number of skiers climbing up the hill at time t.
(a) What is $\lim_{t\to\infty} N(t)/t$?
(b) Find $\lim_{t\to\infty} E[U(t)]$.
(c) If all F_i are exponential with rate λ and all G_i are exponential with rate μ, what is $P\{U(t) = k\}$?

37. There are three machines, all of which are needed for a system to work. Machine i functions for an exponential time with rate λ_i before it fails, $i = 1, 2, 3$. When a machine fails, the system is shut down and repair begins on the failed machine. The time to fix machine 1 is exponential with rate 5; the time to fix machine 2 is uniform on (0, 4); and the time to fix machine 3 is a gamma random variable with parameters $n = 3$ and $\lambda = 2$. Once a failed machine is repaired, it is as good as new and all machines are restarted.
   (a) What proportion of time is the system working?
   (b) What proportion of time is machine 1 being repaired?
   (c) What proportion of time is machine 2 in a state of suspended animation (that is, neither working nor being repaired)?

38. A truck driver regularly drives round trips from A to B and then back to A. Each time he drives from A to B, he drives at a fixed speed that (in miles per hour) is uniformly distributed between 40 and 60; each time he drives from B to A, he drives at a fixed speed that is equally likely to be either 40 or 60.
   (a) In the long run, what proportion of his driving time is spent going to B?
   (b) In the long run, for what proportion of his driving time is he driving at a speed of 40 miles per hour?

39. A system consists of two independent machines that each function for an exponential time with rate λ. There is a single repairperson. If the repairperson is idle when a machine fails, then repair immediately begins on that machine; if the repairperson is busy when a machine fails, then that machine must wait until the other machine has been repaired. All repair times are independent with distribution function G and, once repaired, a machine is as good as new. What proportion of time is the repairperson idle?

40. Three marksmen take turns shooting at a target. Marksman 1 shoots until he misses, then marksman 2 begins shooting until he misses, then marksman 3 until he misses, and then back to marksman 1, and so on. Each time marksman i fires he hits the target, independently of the past, with probability P_i, $i = 1, 2, 3$. Determine the proportion of time, in the long run, that each marksman shoots.

41. Each time a certain machine breaks down it is replaced by a new one of the same type. In the long run, what percentage of time is the machine in use less than one year old if the life distribution of a machine is
   (a) uniformly distributed over (0, 2)?
   (b) exponentially distributed with mean 1?

*42. For an interarrival distribution F having mean μ, we defined the equilibrium distribution of F, denoted F_e, by

$$F_e(x) = \frac{1}{\mu} \int_0^x [1 - F(y)]\, dy$$

   (a) Show that if F is an exponential distribution, then $F = F_e$.
   (b) If for some constant c,

$$F(x) = \begin{cases} 0, & x < c \\ 1, & x \geqslant c \end{cases}$$

show that F_e is the uniform distribution on $(0, c)$. That is, if interarrival times are identically equal to c, then the equilibrium distribution is the uniform distribution on the interval $(0, c)$.

(c) The city of Berkeley, California, allows for two hours parking at all non-metered locations within one mile of the University of California. Parking officials regularly tour around, passing the same point every two hours. When an official encounters a car he or she marks it with chalk. If the same car is there on the official's return two hours later, then a parking ticket is written. If you park your car in Berkeley and return after three hours, what is the probability you will have received a ticket?

43. Consider a renewal process having interarrival distribution F such that

$$\bar{F}(x) = \tfrac{1}{2}e^{-x} + \tfrac{1}{2}e^{-x/2}, \quad x > 0$$

That is, interarrivals are equally likely to be exponential with mean 1 or exponential with mean 2.

(a) Without any calculations, guess the equilibrium distribution F_e.
(b) Verify your guess in part (a).

44. An airport shuttle bus picks up all passengers waiting at a bus stop and drops them off at the airport terminal; it then returns to the stop and repeats the process. The times between returns to the stop are independent random variables with distribution F, mean μ, and variance σ^2. Passengers arrive at the bus stop in accordance with a Poisson process with rate λ. Suppose the bus has just left the stop, and let X denote the number of passengers it picks up when it returns.

(a) Find $E[X]$.
(b) Find $\text{Var}(X)$.
(c) At what rate does the shuttle bus arrive at the terminal without any passengers?

Suppose that each passenger that has to wait at the bus stop more than c time units writes an angry letter to the shuttle bus manager.

(d) What proportion of passengers write angry letters?
(e) How does your answer in part (d) relate to $F_e(x)$?

45. Consider a system that can be in either state 1 or 2 or 3. Each time the system enters state i it remains there for a random amount of time having mean μ_i and then makes a transition into state j with probability P_{ij}. Suppose

$$P_{12} = 1, \quad P_{21} = P_{23} = \tfrac{1}{2}, \quad P_{31} = 1$$

(a) What proportion of transitions takes the system into state 1?
(b) If $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$, then what proportion of time does the system spend in each state?

46. Consider a semi-Markov process in which the amount of time that the process spends in each state before making a transition into a different state is exponentially distributed. What kind of process is this?

47. In a semi-Markov process, let t_{ij} denote the conditional expected time that the process spends in state i given that the next state is j.

(a) Present an equation relating μ_i to the t_{ij}.
(b) Show that the proportion of time the process is in i and will next enter j is equal to $P_i P_{ij} t_{ij}/\mu_i$.

Hint: Say that a cycle begins each time state i is entered. Imagine that you receive a reward at a rate of 1 per unit time whenever the process is in i and heading for j. What is the average reward per unit time?

48. A taxi alternates between three different locations. Whenever it reaches location i, it stops and spends a random time having mean t_i before obtaining another passenger, $i = 1, 2, 3$. A passenger entering the cab at location i will want to go to location j with probability P_{ij}. The time to travel from i to j is a random variable with mean m_{ij}. Suppose that $t_1 = 1$, $t_2 = 2$, $t_3 = 4$, $P_{12} = 1$, $P_{23} = 1$, $P_{31} = \frac{2}{3} = 1 - P_{32}$, $m_{12} = 10$, $m_{23} = 20$, $m_{31} = 15$, $m_{32} = 25$. Define an appropriate semi-Markov process and determine
(a) the proportion of time the taxi is waiting at location i, and
(b) the proportion of time the taxi is on the road from i to j, $i, j = 1, 2, 3$.

*49. Consider a renewal process having the gamma (n, λ) interarrival distribution, and let $Y(t)$ denote the time from t until the next renewal. Use the theory of semi-Markov processes to show that

$$\lim_{t \to \infty} P\{Y(t) < x\} = \frac{1}{n} \sum_{i=1}^{n} G_{i,\lambda}(x)$$

where $G_{i,\lambda}(x)$ is the gamma (i, λ) distribution function.

50. To prove Equation (7.24), define the following notation:

$$X_i^j \equiv \text{time spent in state } i \text{ on the } j\text{th visit to this state;}$$
$$N_i(m) \equiv \text{number of visits to state } i \text{ in the first } m \text{ transitions}$$

In terms of this notation, write expressions for
(a) the amount of time during the first m transitions that the process is in state i;
(b) the proportion of time during the first m transitions that the process is in state i.

Argue that, with probability 1,
(c) $\displaystyle\sum_{j=1}^{N_i(m)} \frac{X_i^j}{N_i(m)} \to \mu_i \qquad \text{as } m \to \infty$
(d) $N_i(m)/m \to \pi_i \qquad \text{as } m \to \infty$.
(e) Combine parts (a), (b), (c), and (d) to prove Equation (7.24).

51. In 1984 the country of Morocco in an attempt to determine the average amount of time that tourists spend in that country on a visit tried two different sampling procedures. In one, they questioned randomly chosen tourists as they were leaving the country; in the other, they questioned randomly chosen guests at hotels. (Each tourist stayed at a hotel.) The average visiting time of the 3000 tourists chosen from hotels was 17.8, whereas the average visiting time of the 12,321 tourists questioned at departure was 9.0. Can you explain this discrepancy? Does it necessarily imply a mistake?

52. Let X_i, $i = 1, 2, \ldots$, be the interarrival times of the renewal process $\{N(t)\}$, and let Y, independent of the X_i, be exponential with rate λ.

(a) Use the lack of memory property of the exponential to argue that

$$P\{X_1 + \cdots + X_n < Y\} = (P\{X < Y\})^n$$

(b) Use part (a) to show that

$$E[N(Y)] = \frac{E[e^{-\lambda X}]}{1 - E[e^{-\lambda X}]}$$

where X has the interarrival distribution.

53. Write a program to approximate $m(t)$ for the interarrival distribution $F * G$, where F is exponential with mean 1 and G is exponential with mean 3.

54. Let X_i, $i \geqslant 1$, be independent random variables with $p_j = P\{X = j\}$, $j \geqslant 1$. If $p_j = j/10$, $j = 1, 2, 3, 4$, find the expected time and the variance of the number of variables that need be observed until the pattern 1, 2, 3, 1, 2 occurs.

55. A coin that comes up heads with probability 0.6 is continually flipped. Find the expected number of flips until either the sequence *thht* or the sequence *ttt* occurs, and find the probability that *ttt* occurs first.

56. Random digits, each of which is equally likely to be any of the digits 0 through 9, are observed in sequence.
(a) Find the expected time until a run of 10 distinct values occurs.
(b) Find the expected time until a run of 5 distinct values occurs.

57. Let $h(x) = P\{\sum_{i=1}^{T} X_i > x\}$ where $X_1, X_2, \ldots$ are independent random variables having distribution function F_e and T is independent of the X_i and has probability mass function $P\{T = n\} = \rho^n(1 - \rho)$, $n \geqslant 0$. Show that $h(x)$ satisfies Equation (7.53).

Hint: Start by conditioning on whether $T = 0$ or $T > 0$.

Chapter 7

1. (a) Yes, (b) no, (c) no.

2. (a) S_n is Poisson with mean $n\mu$.

   (b) $P\{N(t) = n\}$

   $$= P\{N(t) \geq n\} - P\{N(t) \geq n+1\}$$
   $$= P\{S_n \leq t\} - P\{S_{n+1} \leq t\}$$
   $$= \sum_{k=0}^{[t]} e^{-n\mu}(n\mu)^k/k!$$
   $$- \sum_{k=0}^{[t]} e^{-(n+1)\mu}[(n+1)\mu]^k/k!$$

   where $[t]$ is the largest integer not exceeding t.

3. By the one-to-one correspondence of $m(t)$ and F, it follows that $\{N(t), t \geq 0\}$ is a Poisson process with rate 1/2. Hence,

   $P\{N(5) = 0) = e^{-5/2}$

4. (a) No! Suppose, for instance, that the interarrival times of the first renewal process are identically equal to 1. Let the second be a Poisson process. If the first interarrival time of the process $\{N(t), t \geq 0\}$ is equal to 3/4, then we can be certain that the next one is less than or equal to 1/4.

   (b) No! Use the same processes as in (a) for a counter example. For instance, the first interarrival will equal 1 with probability $e^{-\lambda}$, where λ is the rate of the Poisson process. The probability will be different for the next interarrival.

   (c) No, because of (a) or (b).

5. The random variable N is equal to $N(I) + 1$ where $\{N(t)\}$ is the renewal process whose interarrival distribution is uniform on (0, 1). By the results of Example 2c,

   $E[N] = a(1) + 1 = e$

6. (a) Consider a Poisson process having rate λ and say that an event of the renewal process occurs whenever one of the events numbered $r, 2r, 3r, \ldots$ of the Poisson process occur. Then

   $P\{N(t) \geq n\}$

   $$= P\{nr \text{ or more Poisson events by } t\}$$
   $$= \sum_{i=nr}^{\infty} e^{-\lambda t}(\lambda t)^i/i!$$

   (b) $E[N(t)]$

   $$= \sum_{n=1}^{\infty} P\{N(t) \geq n\} = \sum_{n=1}^{\infty}\sum_{i=nr}^{\infty} e^{-\lambda t}(\lambda t)^i/i!$$
   $$= \sum_{i=r}^{\infty}\sum_{n=1}^{[i/r]} e^{-\lambda t}(\lambda t)^i/i! = \sum_{i=r}^{\infty} [i/r] e^{-\lambda t}(\lambda t)^i/i!$$

7. Once every five months.

8. (a) The number of replaced machines by time t constitutes a renewal process. The time between replacements equals

   T, if lifetime of new machine is $\geq T$

   x, if lifetime of new machine is $x, x < T$.

   Hence,

   E[time between replacements]

   $$= \int_0^T x f(x)dx + T[1 - F(T)]$$

   and the result follows by Proposition 3.1.

   (b) The number of machines that have failed in use by time t constitutes a renewal process. The mean time between in-use failures, $E[F]$, can be calculated by conditioning on the lifetime of the initial machine as

   $E[F] = E[E[F|\text{lifetime of initial machine}]]$

   Now

   $E[F|\text{lifetime of machine is } x]$

   $$= \begin{cases} x, & \text{if } x \leq T \\ T + E[F], & \text{if } x > T \end{cases}$$

Hence,

$$E[F] = \int_0^T xf(x)dx + (T + E[F])[1 - F(T)]$$

or

$$E[F] = \frac{\int_0^T xf(x)dx + T[1 - F(T)]}{F(T)}$$

and the result follows from Proposition 3.1.

9. A job completion constitutes a reneval. Let T denote the time between renewals. To compute $E[T]$ start by conditioning on W, the time it takes to finish the next job:

$$E[T] = E[E[T|W]]$$

Now, to determine $E[T|W = w]$ condition on S, the time of the next shock. This gives

$$E[T|W = w] = \int_0^\infty E[T|W = w, S = x]\lambda e^{-\lambda x}dx$$

Now, if the time to finish is less than the time of the shock then the job is completed at the finish time; otherwise everything starts over when the shock occurs. This gives

$$E[T|W = w, S = x] = \begin{cases} x + E[T], & \text{if} \quad x < w \\ w, & \text{if} \quad x \geq w \end{cases}$$

Hence,

$$E[T|W = w]$$

$$= \int_0^w (x + E[T])\lambda e^{-\lambda x}dx + w\int_w^\infty \lambda e^{-\lambda x}dx$$

$$= E[T][1 - e^{-\lambda w}] + 1/\lambda - we^{-\lambda w} - \frac{1}{\lambda}e^{-\lambda w} - we^{-\lambda w}$$

Thus,

$$E[T|W] = (E[T] + 1/\lambda)(1 - e^{-\lambda W})$$

Taking expectations gives

$$E[T] = (E[T] + 1/\lambda)(1 - E[e^{-\lambda W}])$$

and so

$$E[T] = \frac{1 - E[e^{-\lambda W}]}{\lambda E[e^{-\lambda W}]}$$

In the above, W is a random variable having distribution F and so

$$E[e^{-\lambda W}] = \int_0^\infty e^{-\lambda w}f(w)dw$$

10. Yes, ρ/μ

11. $$\frac{N(t)}{t} = \frac{1}{t} + \frac{\text{number of renewals in } (X_1, t)}{t}$$

Since $X_1 < \infty$, Proposition 3.1 implies that

$$\frac{\text{number of renewals in } (X_1, t)}{t} - \frac{1}{\mu} \text{ as } t - \infty.$$

12. Let X be the time between successive d-events. Conditioning on T, the time until the next event following a d-event, gives

$$E[X] = \int_0^d x\lambda e^{-\lambda x}dx + \int_d^\infty (x + E[X]\lambda e^{-\lambda x}dx$$
$$= 1/\lambda + E[X]e^{-\lambda d}$$

Therefore, $E[X] = \dfrac{1}{\lambda(1 - e^{-\lambda d})}$

(a) $\dfrac{1}{E[X]} = \lambda(1 - e^{-\lambda d})$

(b) $1 - e^{-\lambda d}$

13. (a) N_1 and N_2 are stopping times. N_3 is not.

(b) Follows immediately from the definition of I_i.

(c) The value of I_i is completely determined from $X_1, \ldots, X_{i-1}$ (e.g., $I_i = 0$ or 1 depending upon whether or not we have stopped after observing $X_1, \ldots, X_{i-1}$). Hence, I_i is independent of X_i.

(d) $\sum_{i=1}^\infty E[I_i] = \sum_{i=1}^\infty P\{N \geq i\} = E[N]$

(e) $E\left[X_1 + \cdots + X_{N_1}\right] = E[N_1]E[X]$

But $X_1 + \cdots + X_{N_1} = 5, E[X] = p$ and so

$E[N_1] = 5/p$

$E\left[X_1 + \cdots + X_{N_2}\right] = E[N_2]E[X]$

$E[X] = p, E[N_2] = 5p + 3(1 - p) = 3 + 2p$

$E\left[X_1 + \cdots + X_{N_2}\right] = (3 + 2p)p$

14. (a) It follows from the hint that $N(t)$ is not a stopping time since $N(t) = n$ depends on X_{n+1}.

Now $N(t) + 1 = n(\Leftrightarrow)N(t) = n - 1$

$(\Leftrightarrow)X_1 + \cdots + X_{n-1} \leq t,$

$X_1 + \cdots + X_n > t,$

and so $N(t) + 1 = n$ depends only on $X_1, \ldots, X_n$. Thus $N(t) + 1$ is a stopping time.

(b) Follows upon application of Wald's equation—using $N(t)+1$ as the stopping time.

(c) $\sum_{i=1}^{N(t)+1} X_i$ is the time of the first renewal after t. The inequality follows directly from this interpretation since there must be at least one renewal in the interval between t and $t+m$.

(e) $t < \sum_{i=1}^{N(t)+1} X_i < t+M$

Taking expectations and using (b) yields

$t < \mu(m(t)+1) < t+M$

or

$t-\mu < \mu m(t) < t+M-\mu$

or

$\frac{1}{\mu} - \frac{1}{t} < \frac{m(t)}{t} < \frac{1}{\mu} + \frac{M-\mu}{\mu t}$

Let $t \to \infty$ to see that $\frac{m(t)}{t} \to \frac{1}{\mu}$.

15. (a) X_i = amount of time he has to travel after his ith choice (we will assume that he keeps on making choices even after becoming free). N is the number of choices he makes until becoming free.

(b) $E[T] = E\left[\sum_1^N X_i\right] = E[N]E[X]$

N is a geometric random variable with $P = 1/3$, so

$E[N] = 3, E[X] = \frac{1}{3}(2+4+6) = 4$

Hence, $E[T] = 12$.

(c) $E\left[\sum_1^N X_i | N=n\right] = (n-1)\frac{1}{2}(4+6) + 2 = 5n - 3$, since given $N=n, X_1, \ldots, X_{n-1}$ are equally likely to be either 4 or 6, $X_n = 2$, $E\left(\sum_1^n X_i\right) = 4n$.

(d) From (c),

$E\left[\sum_1^N X_i\right] = E[5N-3] = 15-3 = 12$

16. No, since $\sum_{1=i}^{N} X_i = 4$ and $E[X_i] = 1/13$, which would imply that $E[N] = 52$, which is clearly incorrect. Wald's equation is not applicable since the X_i are not independent.

17. (i) Yes. (ii) No—Yes, if F exponential.

18. We can imagine that a renewal corresponds to a machine failure, and each time a new machine is put in use its life distribution will be exponential with rate μ_1 with probability p, and exponential with rate μ_2 otherwise. Hence, if our state is the index of the exponential life distribution of the machine presently in use, then this is a 2-state continuous-time Markov chain with intensity rates

$q_{1,2} = \mu_1(1-p), q_{2,1} = \mu_2 p$

Hence,

$$P_{11}(t) = \frac{\mu_1(1-p)}{\mu_1(1-p)+\mu_2 p} \exp\left\{-\left[\mu_1(1-p)+\mu_2 p\right]t\right\} + \frac{\mu_2 p}{\mu_1(1-p)+\mu_2 p}$$

with similar expressions for the other transition probabilities ($P_{12}(t) = 1 - P_{11}(t)$, and $P_{22}(t)$ is the same with $\mu_2 p$ and $\mu_1(1-p)$ switching places). Conditioning on the initial machine now gives

$$E[Y(t)] = pE[Y(t)|X(0)=1] + (1-p)E[Y(t)|X(0)=2]$$

$$= p\left[\frac{P_{11}(t)}{\mu_1} + \frac{P_{12}(t)}{\mu_2}\right] + (1-p)\left[\frac{P_{21}(t)}{\mu_1} + \frac{P_{22}(t)}{\mu_2}\right]$$

Finally, we can obtain $m(t)$ from

$\mu[m(t)+1] = t + E[Y(t)]$

where

$\mu = p/\mu_1 + (1-p)/\mu_2$

is the mean interarrival time.

19. Since, from Example 2c, $m(t) = e^t - 1, 0 < t \le 1$, we obtain upon using the identity $t + E[Y(t)] = \mu[m(t)+1]$ that $E[Y(1)] = e/2 - 1$.

20. $W_n = \frac{(R_1 + \cdots + R_n)}{(X_1 + \cdots + X_n)/n} \to \frac{ER}{EX}$

by the strong law of large numbers.

21. $\frac{\mu_G}{\mu + 1/\lambda}$, where μ_G is the mean of G.

22. Cost of a cycle $= C_1 + C_2 I - R(T)(1-I)$.

$I = \begin{cases} 1, & \text{if } X < T \\ 0, & \text{if } X \ge T \end{cases}$ where X = life of car.

Hence,

E[cost of a cycle]

$$= C_1 + C_2H(T) - R(T)[1 - H(T)]$$

Also,

$$E[\text{time of cycle} = \int E[time|X = x]h(x)dx$$
$$= \int_0^t xh(x)dx + T[1 - H(T)]$$

Thus the average cost per unit time is given by

$$\frac{C_1 + C_2H(T) - R(T)[1 - H(T)]}{\int_0^t xh(x)dx + T[1 - H(T)]}$$

23. Using that $E[X] = 2p - 1$, we obtain from Wald's equation when $p \neq 1/2$ that

$$E[T](2p-1) = E\left[\sum_{j=1}^{T} X_j\right]$$
$$= (N-i)\frac{1-(q/p)^i}{1-(q/p)^N} - i\left[1 - \frac{1-(q/p)^i}{1-(q/p)^N}\right]$$
$$= N\frac{1-(q/p)^i}{1-(q/p)^N} - i$$

yielding the result:

$$E[T] = \frac{N\dfrac{1-(q/p)^i}{1-(q/p)^N} - i}{2p-1}, \quad p \neq 1/2$$

When $p = 1/2$, we can easily show by a conditioning argument that $E[T] = i(N - i)$

24. Let $N_1 = N$ denote the stopping time. Because X_i, $i \geq 1$, are independent and identically distributed, it follows by the definition of a stopping time that the event $\{N_1 = n\}$ is independent of the values X_{n+i}, $i \geq 1$. But this implies that the sequence of random variables $X_{N_1+1}, X_{N_1+2}, \ldots$ is independent of $X_1, \ldots, X_N$ and has the same distribution as the original sequence X_i, $i \geq 1$. Thus if we let N_2 be a stopping time on $X_{N_1+1}, X_{N_1+2}, \ldots$ that is defined exactly as is N_1 is on the original sequence, then $X_{N_1+1}, X_{N_1+2}, \ldots, X_{N_1+N_2}$ is independent of and has the same distribution as does $X_1, \ldots, X_{N_1}$. Similarly, we can define a stopping time N_3 on the sequence $X_{N_1+N_2+1}, X_{N_1+N_2+2}, \ldots$ that is identically defined on this sequence as is N_1 on the original sequence, and so on. If we now consider a reward process for which X_i is the reward earned during period i, then this reward process is a renewal reward process whose cycle lengths are $N_1, N_2, \ldots$. By the renewal reward theorem,

$$\text{average reward per unit time} = \frac{E[X_1 + \cdots + X_N]}{E[N]}$$

But the average reward per unit time is $\lim_{n\to\infty} \sum_{i=1}^{n} X_i/n$, which, by the strong law of large numbers, is equal to $E[X]$. Thus,

$$E[X] = \frac{E[X_1 + \ldots X_N]}{E[N]}$$

25. Say that a new cycle begins each time a train is dispatched. Then, with C being the cost of a cycle, we obtain, upon conditioning on $N(t)$, the number of arrivals during a cycle, that

$$E[C] = E[E[C|N(t)]] = E[K + N(t)ct/2]$$
$$= k + \lambda ct^2/2$$

Hence,

$$\text{average cost per unit time} = \frac{E[C]}{t} = \frac{K}{t} + \lambda ct/2$$

Calculus shows that the preceding is minimized when $t = \sqrt{2K/(\lambda c)}$, with the average cost equal to $\sqrt{2\lambda Kc}$.

On the other hand, the average cost for the N policy of Example 7.12 is $c(N-1)/2 + \lambda K/N$. Treating N as a continuous variable yields that its minimum occurs at $N = \sqrt{2\lambda K/c}$, with a resulting minimal average cost of $\sqrt{2\lambda Kc} - c/2$.

26. $$\frac{[c + 2c + \cdots + (N-1)c]/\lambda + KNc + \lambda K^2c/2}{N/\lambda + K}$$
$$= \frac{c(N-1)N/2\lambda + KNc + \lambda K^2c/2}{N/\lambda + K}$$

27. Say that a new cycle begins when a machine fails; let C be the cost per cycle; let T be the time of a cycle.

$$E[C] = K + \frac{c_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{c_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{c_1}{\lambda_1}$$
$$E[T] = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{1}{\lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{1}{\lambda_1}$$

T the long-run average cost per unit time is $E[C]/E[T]$.

28. For N large, out of the first N items produced there will be roughly Nq defective items. Also, there will be roughly NP_I inspected items, and as each

inspected item will still be, independently, defective with probability q, it follows that there will be roughly $NP_I q$ defective items discovered. Hence, the proportion of defective items that are discovered is, in the limit,

$$NP_1 q/Nq = P_I = \frac{(1/p)^k}{(1/p)^k - 1 + 1/\alpha}$$

29. (a) Imagine that you are paid a reward equal to W_i on day i. Since everything starts over when a busy period ends, it follows that the reward process constitutes a renewal reward process with cycle time equal to N and with the reward during a cycle equal to $W_1 + \cdots + W_N$. Thus $E[W]$, the average reward per unit time, is $E[W_1 + \cdots + W_N]/E[N]$.

(b) The sum of the times in the system of all customers and the total amount of work that has been processed both start equal to 0 and both increase at the same rate. Hence, they are always equal.

(c) This follows from (b) by looking at the value of the two totals at the end of the first busy period.

(d) It is easy to see that N is a stopping time for the $L_i, i \geq 1$, and so, by Wald's Equation, $E\left[\sum_{i=1}^{N} L_i\right] = E[L]E[N]$. Thus, from (a) and (c), we obtain that $E[W] = E[L]$.

30. $$\frac{A(t)}{t} = \frac{t - S_{N(t)}}{t} = 1 - \frac{S_{N(t)}}{t} = 1 - \frac{S_{N(t)}}{N(t)}\frac{N(t)}{t}$$

The result follows since $S_{N(t)}/N(t) — \mu$ (by the strong law of large numbers) and $N(t)/t — 1/\mu$.

31. $$\begin{aligned} P\{E(t) > x | A(t) = s\} &= P\{0 \text{ renewals in } (t, t+x] | A(t) = s\} \\ &= P\{\text{interarrival} > x + s | A(t) = s\} \\ &= P\{\text{interarrival} > x + s | \text{interarrival} > s\} \\ &= \frac{1 - F(x+s)}{1 - F(s)} \end{aligned}$$

32. Say that the system is off at t if the excess at t is less than c. Hence, the system is off the last c time units of a renewal interval. Hence,

proportion of time excess is less than c

$$\begin{aligned} &= E[\text{off time in a renewal cycle}]/[X] \\ &= E[\min(X, c)]/E[X] \\ &= \int_0^c (1 - F(x))dx/E[X] \end{aligned}$$

33. Let B be the amount of time the server is busy in a cycle; let X be the remaining service time of the person in service at the beginning of a cycle.

$$\begin{aligned} E[B] &= E[B|X < t](1 - e^{-\lambda t}) + E[B|X > t]e^{-\lambda t} \\ &= E[X|X < t](1 - e^{-\lambda t}) + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\lambda t} \\ &= E[X] - E[X|X > t]e^{-\lambda t} + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\lambda t} \\ &= \tfrac{1}{\mu} - \left(t + \frac{1}{\mu}\right)e^{-\lambda t} + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\lambda t} \\ &= \frac{1}{\mu}\left[1 - \frac{\lambda}{\lambda + \mu}e^{-\lambda t}\right] \end{aligned}$$

More intuitively, writing $X = B + (X - B)$, and noting that $X - B$ is the additional amount of service time remaining when the cycle ends, gives

$$\begin{aligned} E[B] &= E[X] - E[X - B] \\ &= \frac{1}{\mu} - \frac{1}{\mu}P(X > B) \\ &= \frac{1}{\mu} - \frac{1}{\mu}e^{-\lambda t}\frac{\lambda}{\lambda + \mu} \end{aligned}$$

The long-run proportion of time that the server is busy is $\dfrac{E[B]}{t + 1/\lambda}$.

34. A cycle begins immediately after a cleaning starts. Let C be the cost of a cycle.

$$E[C] = \lambda C_2 T/4 + C_1 \lambda \int_0^{3T/4} \bar{G}(y)dy$$

where the preceding uses that the number of customers in an $M/G/\infty$ system at time t is Poisson distributed with mean $\lambda \int_0^t \bar{G}(y)dy$. The long-run average cost is $E[C]/T$. The long-run proportion of time that the system is being cleaned is $\dfrac{T/4}{T} = 1/4$.

35. (a) We can view this as an $M/G/\infty$ system where a satellite launching corresponds to an arrival and F is the service distribution. Hence,

$$P\{X(t) = k\} = e^{-\lambda(t)}[\lambda(t)]^k/k!$$

where $\lambda(t) = \lambda \int_0^t (1 - F(s))ds$.

(b) By viewing the system as an alternating renewal process that is on when there is at least one satellite orbiting, we obtain

$$\lim P\{X(t) = 0\} = \frac{1/\lambda}{1/\lambda + E[T]}$$

where T, the on time in a cycle, is the quantity of interest. From part (a)

$$\lim P\{X(t) = 0\} = e^{-\lambda\mu}$$

where $\mu = \int_0^\infty (1 - F(s))ds$ is the mean time that a satellite orbits. Hence,

$$e^{-\lambda\mu} = \frac{1/\lambda}{1/\lambda + E[T]}$$

and so

$$E[T] = \frac{1 - e^{-\lambda\mu}}{\lambda e^{-\lambda\mu}}$$

36. (a) If we let $N_i(t)$ denote the number of times person i has skied down by time t, then $\{N_i(t)\}$ is a (delayed) renewal process. As $N(t) = \sum N_i(t)$, we have

$$\lim \frac{N(t)}{t} = \sum_i \lim \frac{N_i(t)}{t} = \sum_i \frac{1}{\mu_i + \theta_i}$$

where μ_i and θ_i are respectively the mean of the distributions F_i and G_i.

(b) For each skier, whether they are climbing up or skiing down constitutes an alternating renewal process, and so the limiting probability that skier i is climbing up is $p_i = \mu_i/(\mu_i + \theta_i)$. From this we obtain

$$\lim P\{U(t) = k\} = \sum_S \left\{ \prod_{i \in S} p_i \prod_{i \in S^c} (1 - p_i) \right\}$$

where the above sum is over all of the $\begin{bmatrix} n \\ k \end{bmatrix}$ subsets S of size k.

(c) In this case the location of skier i, whether going up or down, is a 2-state continuous-time Markov chain. Letting state 0 correspond to going up, then since each skier acts independently according to the same probability, we have

$$P\{U(t) = k\} = \begin{bmatrix} n \\ k \end{bmatrix} [P_{00}(t)]^k [1 - P_{00}(t)]^{n-k}$$

where $P_{00}(t) = (\lambda e^{-(\lambda+\mu)t} + \mu)/(\lambda + \mu)$.

37. (a) This is an alternating renewal process, with the mean off time obtained by conditioning on which machine fails to cause the off period.

$$E[\text{off}] = \sum_{i=1}^{3} E[\text{off}|i \text{ fails}]P\{i \text{ fails}\}$$

$$= (1/5)\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + (2)\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$+ (3/2)\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

As the on time in a cycle is exponential with rate equal to $\lambda_1 + \lambda_2 + \lambda_3$, we obtain that p, the proportion of time that the system is working is

$$p = \frac{1/(\lambda_1 + \lambda_2 + \lambda_3)}{E[C]}$$

where

$$E[C] = E[\text{cycle time}]$$

$$= 1/(\lambda_1 + \lambda_2 + \lambda_3) + E[\text{off}]$$

(b) Think of the system as a renewal reward process by supposing that we earn 1 per unit time that machine 1 is being repaired. Then, r_1, the proportion of time that machine 1 is being repaired is

$$r_1 = \frac{(1/5)\dfrac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}$$

(c) By assuming that we earn 1 per unit time when machine 2 is in a state of suspended animation, shows that, with s_2 being the proportion of time that 2 is in a state of suspended animation,

$$s_2 = \frac{(1/5)\dfrac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + (3/2)\dfrac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}$$

38. Let $T_{e,f}$ denote the time it takes to go from e to f, and let d be the distance between A to B. Then, with S being the driver's speed

$$E[T_{A,B}] = \frac{1}{20} \int_{40}^{60} E[T_{A,B}|S = s]ds$$

$$= \frac{1}{20} \int_{40}^{60} \frac{d}{s} ds$$

$$= \frac{d}{20} \log(3/2)$$

Also,

$$E[T_{B,A}] = E[T_{B,A}|S = 40](1/2) + E[T_{B,A}|S$$
$$= 60](1/2) = \frac{1}{2}(d/40 + d/60)$$
$$= d/48$$

(a) $$\frac{E[T_{A,B}]}{E[T_{A,B}] + E[T_{B,A}]} = \frac{\frac{1}{20}\log(3/2)}{\frac{1}{20}\log(3/2) + 1/48}$$

(b) By assuming that a reward is earned at a rate of 1 per unit time whenever he is driving at a speed of 40 miles per hour, we see that p, the proportion of time this is the case, is

$$p = \frac{(1/2)d/40}{E[T_{A,B}] + E[T_{B,A}]} = \frac{\frac{1}{80}}{\frac{1}{20}\log(3/2) + 1/48}$$

39. Let B be the length of a busy period. With S equal to the service time of the machine whose failure initiated the busy period, and T equal to the remaining life of the other machine at that moment, we obtain

$$E[B] = \int E[B|S = s]g(s)ds$$

Now,

$$E[B|S = s] = E[B|S = s,\ T \leq s](1 - e^{-\lambda s})$$
$$+ E[B|S = s, T > s]e^{-\lambda s}$$
$$= (s + E[B])(1 - e^{-\lambda s}) + se^{-\lambda s}$$
$$= s + E[B](1 - e^{-\lambda s})$$

Substituting back gives

$$E[B] = E[S] + E[B]E[1 - e^{-\lambda S}]$$

or

$$E[B] = \frac{E[S]}{E[e^{-\lambda S}]}$$

Hence,

$$E[\text{idle}] = \frac{1/(2\lambda)}{1/(2\lambda) + E[B]}$$

40. Proportion of time 1 shoots $= \dfrac{1/(1 - P_1)}{\sum_{j=1}^{3} 1/(1 - P_j)}$ by alternating renewal process (or by semi-Markov process) since $1/(1 - P_j)$ is the mean time marksman j shoots. Similarly, proportion of time i shoots $= \dfrac{1/(1 - P_i)}{\sum 1/(1 - P_j)}$.

41. $$\int_0^1 \frac{(1 - F(x)dx}{\mu}$$
$$= \begin{cases} \displaystyle\int_0^1 \frac{2 - x}{2}\, dx = \frac{3}{4} \text{ in part (i)} \\ \displaystyle\int_0^1 e^{-x} \quad dx = 1 - e^{-1} \text{ in part (ii)} \end{cases}$$

42. (a) $$F_e(x) = \frac{1}{\mu}\int_0^x e^{-y/\mu}dy = 1 - e^{-x/\mu}$$

(b) $$F_e(x) = \frac{1}{c}\int_0^x dy = x/c, \quad 0 \leq x \leq c$$

(c) You will receive a ticket if, starting when you park, an official appears within 1 hour. From Example 5.1c the time until the official appears has the distribution F_e, which, by part (a), is the uniform distribution on (0, 2). Thus, the probability is equal to 1/2.

43. Since half the interarrival times will be exponential with mean 1 and half will be exponential with mean 2, it would seem that because the exponentials with mean 2 will last, on average, twice as long, that

$$\bar{F}_e(x) = \frac{2}{3}e^{-x/2} + \frac{1}{3}e^{-x}$$

With $\mu = (1)1/2\ +\ (2)1/2 = 3/2$ equal to the mean interarrival time

$$\bar{F}_e(x) = \int_x^\infty \frac{\bar{F}(y)}{\mu}dy$$

and the earlier formula is seen to be valid.

44. Let T be the time it takes the shuttle to return. Now, given T, X is Poisson with mean λT. Thus,

$$E[X|T] = \lambda T, \quad Var(X|T) = \lambda T$$

Consequently,

(a) $E[X] = E[E[X|T]] = \lambda E[T]$

(b) $$Var(X) = E[Var(X|T)] + Var(E[X|T])$$
$$= \lambda E[T] + \lambda^2 Var(T)$$

(c) Assume that a reward of 1 is earned each time the shuttle returns empty. Then, from renewal

reward theory, r, the rate at which the shuttle returns empty, is

$$r = \frac{P\{\text{empty}\}}{E[T]}$$

$$= \frac{\int P\{\text{empty}|T = t\}f(t)dt}{E[T]}$$

$$= \frac{\int e^{-\lambda t} f(t)dt}{E[T]}$$

$$= \frac{E[e^{-\lambda T}]}{E[T]}$$

(d) Assume that a reward of 1 is earned each time that a customer writes an angry letter. Then, with N_a equal to the number of angry letters written in a cycle, it follows that r_a, the rate at which angry letters are written, is

$$r_a = E[N_a]/E[T]$$

$$= \int E[N_a|T = t]f(t)dt/E[T]$$

$$= \int_c^\infty \lambda(t - c)f(t)dt/E[T]$$

$$= \lambda E[(T - c)^+]/E[T]$$

Since passengers arrive at rate λ, this implies that the proportion of passengers that write angry letters is r_a/λ.

(e) Because passengers arrive at a constant rate, the proportion of them that have to wait more than c will equal the proportion of time that the age of the renewal process (whose event times are the return times of the shuttle) is greater than c. It is thus equal to $\bar{F}_e(c)$.

45. The limiting probabilities for the Markov chain are given as the solution of

$$r_1 = r_2\frac{1}{2} + r_3$$

$$r_2 = r_1$$

$$r_1 + r_2 + r_3 = 1$$

or

$$r_1 = r_2 = \frac{2}{5}, \quad r_3 = \frac{1}{5}$$

(a) $r_1 = \frac{2}{5}$

(b) $P_i = \frac{r_i\mu_i}{\sum_i r_i\mu_i}$ and so,

$$P_1 = \frac{2}{9}, \; P_2 = \frac{4}{9}, \; P_3 = \frac{3}{9}.$$

46. Continuous-time Markov chain.

47. (a) By conditioning on the next state, we obtain the following:

$$\mu_j = E[\text{time in } i]$$

$$= \sum E[\text{time in } i|\text{next state is } j]P_{ij}$$

$$= \sum_i t_{ij}P_{ij}$$

(b) Use the hint. Then,

$$E[\text{reward per cycle}]$$

$$= E[\text{reward per cycle}|\text{next state is } j]P_{ij}$$

$$= t_{ij}P_{ij}$$

Also,

$E[\text{time of cycle}] = E[\text{time between visits to } i]$

Now, if we had supposed a reward of 1 per unit time whenever the process was in state i and 0 otherwise then using the same cycle times as above we have that

$$P_i = \frac{E[\text{reward is cycle}]}{E[\text{time of cycle}]} = \frac{\mu_i}{E[\text{time of cycle}]}$$

Hence,

$$E[\text{time of cycle}] = \mu_i/P_i$$

and so

$$\text{average reward per unit time} = t_{ij}P_{ij}P_i/\mu_i$$

The above establishes the result since the average reward per unit time is equal to the proportion of time the process is in i and will next enter j.

48. Let the state be the present location if the taxi is waiting or let it be the most recent location if it is on the road. The limiting probabilities of the embedded Markov chain satisfy

$$r_1 = \frac{2}{3}r_3$$

$$r_2 = r_1 + \frac{1}{3}r_3$$

$$r_1 + r_2 + r_3 = 1$$

Solving yields

$$r_1 = \frac{1}{4}, \quad r_2 = r_3 = \frac{3}{8}$$

The mean time spent in state i before entering another state is

$$\mu_1 = 1 + 10 = 11, \quad \mu_2 = 2 + 20 = 22,$$

$$\mu_3 = 4 + \left[\frac{2}{3}\right]15 + \left[\frac{1}{3}\right]25 = \frac{67}{3},$$

and so the limiting probabilities are

$$P_1 = \frac{66}{465}, P_2 = \frac{198}{465}, P_3 = \frac{201}{465}.$$

The time the state is i is broken into 2 parts—the time t_i waiting at i, and the time traveling. Hence, the proportion of time the taxi is waiting at state i is $P_i t_i/(t_i/\mu_i)$. The proportion of time it is traveling from i to j is $P_i m_{ij}/(t_i + \mu_i)$.

49. Think of each interarrival time as consisting of n independent phases—each of which is exponentially distributed with rate λ—and consider the semi–Markov process whose state at any time is the phase of the present interarrival time. Hence, this semi-Markov process goes from state 1 to 2 to 3 … to n to 1, and so on. Also the time spent in each state has the same distribution. Thus, clearly the limiting probabilities of this semi-Markov chain are $P_i = 1/n, i = 1, \ldots, n$. To compute $\lim P\{Y(t) < x\}$, we condition on the phase at time t and note that if it is $n - i + 1$, which will be the case with probability $1/n$, then the time until a renewal occurs will be the sum of i exponential phases, which will thus have a gamma distribution with parameters i and λ.

50. (a) $\sum_{j=1}^{N_i(m)} X_i^j$

(b) $$\frac{\sum_{j=1}^{N_i(m)} X_i^j}{\sum_i \sum_{j=1}^{N_i(m)} X_i^j}$$

(c) Follows from the strong law of large numbers since the X_i^j are independent and identically distributed and have mean μ_i.

(d) This is most easily proven by first considering the model under the assumption that each transition takes one unit of time. Then $N_i(m)/m$ is the rate at which visits to i occur, which, as such visits can be thought of as being renewals, converges to

(E[number of transitions between visits]$)^{-1}$

by Proposition 3.1. But, by Markov-chain theory, this must equal x_i. As the quantity in (d) is clearly unaffected by the actual times between transition, the result follows.

Equation (6.2) now follows by dividing numerator and denominator of (b) by m; by writing

$$\frac{X_i^j}{m} = \frac{X_i^j}{N_i(m)} \frac{N_i(m)}{(m)}$$

and by using (c) and (d).

51. It is an example of the inspection paradox. Because every tourist spends the same time in departing the country, those questioned at departure constitute a random sample of all visiting tourists. On the other hand, if the questioning is of randomly chosen hotel guests then, because longer staying guests are more likely to be selected, it follows that the average time of the ones selected will be larger than the average of all tourists. The data that the average of those selected from hotels was approximately twice as large as from those selected at departure are consistent with the possibility that the time spent in the country by a tourist is exponential with a mean approximately equal to 9.

52. (a) $P\{X_1 + \cdots + X_n < Y\}$

$= P\{X_1 + \cdots + X_n < Y | X_n < Y\} P\{X_n < Y\}$

$= P\{X_1 + \cdots + X_{n-1} < Y\} P\{X < Y\}$

where the above follows because given that $Y > X_n$ the amount by which it is greater is, by the lack of memory property, also exponential with rate λ. Repeating this argument yields the result.

(b) $$E[N(Y)] = \sum_{n=1}^{\infty} P\{N(Y) \geq n\}$$

$$= \sum_{n=1}^{\infty} P\{X_1 + \cdots + X_n \leq Y\}$$

$$= \sum_{n=1}^{\infty} P\{X < Y\}^n = \frac{P}{1-P}$$

where

$$P = P\{X < Y\} = \int P\{X < Y | X = x\} f(x)dx$$

$$= \int e^{-\lambda x} f(x)dx = E[e^{-\lambda x}]$$

54. Let T denote the number of variables that need be observed until the pattern first appears. Also, let T^{∞} denote the number that need be observed once the pattern appears until it next appears. Let $p = p_1^2 p_2^2 p_3$

$$p^{-1} = E[T^{\infty}]$$
$$= E[T] - E[T_{1,2}]$$
$$= E[T] - (p_1 p_2)^{-1}$$

Hence, $E[T] = 8383.333$. Now, since $E[I(5)I(8)] = (.1)^3(.2)^3(.3)^2$, we obtain from Equation (7.45) that

$$Var(T^{\infty}) = (1/p)^2 - 9/p + 2(1/p)^3(.1)^3(.2)^3(.3)^2$$
$$= 6.961943 \times 10^7$$

Also,

$$Var(T_{1,2}) = (.02)^{-2} - 3(.02)^{-1} = 2350$$

and so

$$Var(T) = Var(T_{1,2}) + Var(T^{\infty}) \approx 6.96 \times 10^7$$

55. $E[T(1)] = (.24)^{-2} + (.4)^{-1} = 19.8611$,

$E[T(2)] = 24.375$, $E[T_{12}] = 21.875$,

$E[T_{2,1}] = 17.3611$. The solution of the equations

$$19.861 = E[M] + 17.361P(2)$$
$$24.375 = E[M] + 21.875P(1)$$
$$1 = P(1) + P(2)$$

gives the results

$$P(2) \approx .4425, \quad E[M] \approx 12.18$$

56. (a) $\dfrac{(10)^{10}}{10!} \sum_{i=0}^{9} i!/(10)^i$

(b) Define a renewal process by saying that a renewal occurs the first time that a run of 5 consecutive distinct values occur. Also, let a reward of 1 be earned whenever the previous 5 data values are distinct. Then, letting R denote the reward earned between renewal epochs, we have that

$$E[R] = 1 + \sum_{i=1}^{4} E\big[\text{reward earned a time } i \text{ after a renewal}\big]$$
$$= 1 + \sum_{i=1}^{4} \binom{5+i}{i} \Big/ \binom{10}{i}$$
$$= 1 + 6/10 + 7/15 + 7/15 + 6/10$$
$$= 47/15$$

If R_i is the reward earned at time i then for $i \geq 5$

$$E[R_i] = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6/(10)^{10} = 189/625$$

Hence,

$$E[T] = (47/15)(625/189) \approx 10.362$$

57. $$P\{\sum_{i=1}^{T} X_i > x\} = P\{\sum_{i=1}^{T} X_i > x | T = 0\}(1-\rho) + P\{\sum_{i=1}^{T} X_i > x | T > 0\}\rho$$

$$= P\{\sum_{i=1}^{T} X_i > x | T > 0\}\rho$$

$$= \rho \int_0^{\infty} P\{\sum_{i=1}^{T} X_i > x | T > 0, X_1 = y\} \frac{\bar{F}(y)}{\mu} dy$$

$$= \frac{\rho}{\mu} \int_0^{x} P\{\sum_{i=1}^{T} X_i > x | T > 0, X_1 = y\} \bar{F}(y) dy + \frac{\rho}{\mu} \int_x^{\infty} \bar{F}(y) dy$$

$$= \frac{\rho}{\mu} \int_0^{x} h(x-y)\bar{F}(y) dy + \frac{\rho}{\mu} \int_x^{\infty} \bar{F}(y) dy$$

$$= h(0) + \frac{\rho}{\mu} \int_0^{x} h(x-y)\bar{F}(y) dy - \frac{\rho}{\mu} \int_0^{x} \bar{F}(y) dy$$

where the final equality used that

$$h(0) = \rho = \frac{\rho}{\mu} \int_0^{\infty} \bar{F}(y) dy$$

Queueing Theory

Exercises

1. For the $M/M/1$ queue, compute
   (a) the expected number of arrivals during a service period and
   (b) the probability that no customers arrive during a service period.

   Hint: "Condition."

*2. Machines in a factory break down at an exponential rate of six per hour. There is a single repairman who fixes machines at an exponential rate of eight per hour. The cost incurred in lost production when machines are out of service is \$10 per hour per machine. What is the average cost rate incurred due to failed machines?

3. The manager of a market can hire either Mary or Alice. Mary, who gives service at an exponential rate of 20 customers per hour, can be hired at a rate of \$3 per hour. Alice, who gives service at an exponential rate of 30 customers per hour, can be hired at a rate of \$$C$ per hour. The manager estimates that, on the average, each customer's time is worth \$1 per hour and should be accounted for in the model. Assume customers arrive at a Poisson rate of 10 per hour
   (a) What is the average cost per hour if Mary is hired? If Alice is hired?
   (b) Find C if the average cost per hour is the same for Mary and Alice.

4. Suppose that a customer of the $M/M/1$ system spends the amount of time $x > 0$ waiting in queue before entering service.
   (a) Show that, conditional on the preceding, the number of other customers that were in the system when the customer arrived is distributed as $1 + P$, where P is a Poisson random variable with mean λ.
   (b) Let W_Q^* denote the amount of time that an $M/M/1$ customer spends in queue. As a by-product of your analysis in part (a), show that

$$P\{W_Q^* \leqslant x\} = \begin{cases} 1 - \frac{\lambda}{\mu} & \text{if } x = 0 \\ 1 - \frac{\lambda}{\mu} + \frac{\lambda}{\mu}(1 - e^{-(\mu-\lambda)x}) & \text{if } x > 0 \end{cases}$$

5. It follows from Exercise 4 that if, in the $M/M/1$ model, W_Q^* is the amount of time that a customer spends waiting in queue, then

$$W_Q^* = \begin{cases} 0, & \text{with probability } 1 - \lambda/\mu \\ \text{Exp}(\mu - \lambda), & \text{with probability } \lambda/\mu \end{cases}$$

where $\mathrm{Exp}(\mu - \lambda)$ is an exponential random variable with rate $\mu - \lambda$. Using this, find $\mathrm{Var}(W_Q^*)$.

6. Suppose we want to find the covariance between the times spent in the system by the first two customers in an $M/M/1$ queueing system. To obtain this covariance, let S_i be the service time of customer i, $i = 1, 2$, and let Y be the time between the two arrivals.
   (a) Argue that $(S_1 - Y)^+ + S_2$ is the amount of time that customer 2 spends in the system, where $x^+ = \max(x, 0)$.
   (b) Find $\mathrm{Cov}(S_1, (S_1 - Y)^+ + S_2)$.

   Hint: Compute both $E[(S - Y)^+]$ and $E[S_1(S_1 - Y)^+]$ by conditioning on whether $S_1 > Y$.

*7. Show that W is smaller in an $M/M/1$ model having arrivals at rate λ and service at rate 2μ than it is in a two-server $M/M/2$ model with arrivals at rate λ and with each server at rate μ. Can you give an intuitive explanation for this result? Would it also be true for W_Q?

8. A facility produces items according to a Poisson process with rate λ. However, it has shelf space for only k items and so it shuts down production whenever k items are present. Customers arrive at the facility according to a Poisson process with rate μ. Each customer wants one item and will immediately depart either with the item or empty handed if there is no item available.
   (a) Find the proportion of customers that go away empty handed.
   (b) Find the average time that an item is on the shelf.
   (c) Find the average number of items on the shelf.

9. A group of n customers moves around among two servers. Upon completion of service, the served customer then joins the queue (or enters service if the server is free) at the other server. All service times are exponential with rate μ. Find the proportion of time that there are j customers at server 1, $j = 0, \ldots, n$.

10. A group of m customers frequents a single-server station in the following manner. When a customer arrives, he or she either enters service if the server is free or joins the queue otherwise. Upon completing service the customer departs the system, but then returns after an exponential time with rate θ. All service times are exponentially distributed with rate μ.
    (a) Find the average rate at which customers enter the station.
    (b) Find the average time that a customer spends in the station per visit.

11. Consider a single-server queue with Poisson arrivals and exponential service times having the following variation: Whenever a service is completed a departure occurs only with probability α. With probability $1 - \alpha$ the customer, instead of leaving, joins the end of the queue. Note that a customer may be serviced more than once.
    (a) Set up the balance equations and solve for the steady-state probabilities, stating conditions for it to exist.
    (b) Find the expected waiting time of a customer from the time he arrives until he enters service for the first time.
    (c) What is the probability that a customer enters service exactly n times, $n = 1, 2, \ldots$?
    (d) What is the expected amount of time that a customer spends in service (which does not include the time he spends waiting in line)?

Hint: Use part (c).

(e) What is the distribution of the total length of time a customer spends being served?

Hint: Is it memoryless?

*12. A supermarket has two exponential checkout counters, each operating at rate μ. Arrivals are Poisson at rate λ. The counters operate in the following way:

(i) One queue feeds both counters.

(ii) One counter is operated by a permanent checker and the other by a stock clerk who instantaneously begins checking whenever there are two or more customers in the system. The clerk returns to stocking whenever he completes a service, and there are fewer than two customers in the system.

(a) Find P_n, proportion of time there are n in the system.

(b) At what rate does the number in the system go from 0 to 1? From 2 to 1?

(c) What proportion of time is the stock clerk checking?

Hint: Be a little careful when there is one in the system.

13. Two customers move about among three servers. Upon completion of service at server i, the customer leaves that server and enters service at whichever of the other two servers is free. (Therefore, there are always two busy servers.) If the service times at server i are exponential with rate μ_i, $i = 1, 2, 3$, what proportion of time is server i idle?

14. Consider a queueing system having two servers and no queue. There are two types of customers. Type 1 customers arrive according to a Poisson process having rate λ_1, and will enter the system if either server is free. The service time of a type 1 customer is exponential with rate μ_1. Type 2 customers arrive according to a Poisson process having rate λ_2. A type 2 customer requires the simultaneous use of both servers; hence, a type 2 arrival will only enter the system if both servers are free. The time that it takes (the two servers) to serve a type 2 customer is exponential with rate μ_2. Once a service is completed on a customer, that customer departs the system.

(a) Define states to analyze the preceding model.

(b) Give the balance equations.

In terms of the solution of the balance equations, find

(c) the average amount of time an entering customer spends in the system;

(d) the fraction of served customers that are type 1.

15. Consider a sequential-service system consisting of two servers, A and B. Arriving customers will enter this system only if server A is free. If a customer does enter, then he is immediately served by server A. When his service by A is completed, he then goes to B if B is free, or if B is busy, he leaves the system. Upon completion of service at server B, the customer departs. Assume that the (Poisson) arrival rate is two customers an hour, and that A and B serve at respective (exponential) rates of four and two customers an hour.

(a) What proportion of customers enter the system?

(b) What proportion of entering customers receive service from B?

(c) What is the average number of customers in the system?

(d) What is the average amount of time that an entering customer spends in the system?

16. Customers arrive at a two-server system according to a Poisson process having rate $\lambda = 5$. An arrival finding server 1 free will begin service with that server. An arrival finding server 1 busy and server 2 free will enter service with server 2. An arrival finding both servers busy goes away. Once a customer is served by either server, he departs the system. The service times at server i are exponential with rates μ_i, where $\mu_1 = 4,\ \mu_2 = 2$.
   (a) What is the average time an entering customer spends in the system?
   (b) What proportion of time is server 2 busy?
17. Customers arrive at a two-server station in accordance with a Poisson process with a rate of two per hour. Arrivals finding server 1 free begin service with that server. Arrivals finding server 1 busy and server 2 free begin service with server 2. Arrivals finding both servers busy are lost. When a customer is served by server 1, she then either enters service with server 2 if 2 is free or departs the system if 2 is busy. A customer completing service at server 2 departs the system. The service times at server 1 and server 2 are exponential random variables with respective rates of four and six per hour.
   (a) What fraction of customers do not enter the system?
   (b) What is the average amount of time that an entering customer spends in the system?
   (c) What fraction of entering customers receives service from server 1?
18. Arrivals to a three-server system are according to a Poisson process with rate λ. Arrivals finding server 1 free enter service with 1. Arrivals finding 1 busy but 2 free enter service with 2. Arrivals finding both 1 and 2 busy do not join the system. After completion of service at either 1 or 2 the customer will then either go to server 3 if 3 is free or depart the system if 3 is busy. After service at 3 customers depart the system. The service times at i are exponential with rate μ_i, $i = 1, 2, 3$.
   (a) Define states to analyze the above system.
   (b) Give the balance equations.
   (c) In terms of the solution of the balance equations, what is the average time that an entering customer spends in the system?
   (d) Find the probability that a customer who arrives when the system is empty is served by server 3.
19. The economy alternates between good and bad periods. During good times customers arrive at a certain single-server queueing system in accordance with a Poisson process with rate λ_1, and during bad times they arrive in accordance with a Poisson process with rate λ_2. A good time period lasts for an exponentially distributed time with rate α_1, and a bad time period lasts for an exponential time with rate α_2. An arriving customer will only enter the queueing system if the server is free; an arrival finding the server busy goes away. All service times are exponential with rate μ.
   (a) Define states so as to be able to analyze this system.
   (b) Give a set of linear equations whose solution will yield the long-run proportion of time the system is in each state.

   In terms of the solutions of the equations in part (b),
   (c) what proportion of time is the system empty?
   (d) what is the average rate at which customers enter the system?

20. There are two types of customers. Type 1 and 2 customers arrive in accordance with independent Poisson processes with respective rate λ_1 and λ_2. There are two servers. A type 1 arrival will enter service with server 1 if that server is free; if server 1 is busy and server 2 is free, then the type 1 arrival will enter service with server 2. If both servers are busy, then the type 1 arrival will go away. A type 2 customer can only be served by server 2; if server 2 is free when a type 2 customer arrives, then the customer enters service with that server. If server 2 is busy when a type 2 arrives, then that customer goes away. Once a customer is served by either server, he departs the system. Service times at server i are exponential with rate μ_i, $i = 1, 2$.

Suppose we want to find the average number of customers in the system.

(a) Define states.

(b) Give the balance equations. Do not attempt to solve them.

In terms of the long-run probabilities, what is

(c) the average number of customers in the system?

(d) the average time a customer spends in the system?

*21. Suppose in Exercise 20 we want to find out the proportion of time there is a type 1 customer with server 2. In terms of the long-run probabilities given in Exercise 20, what is

(a) the rate at which a type 1 customer enters service with server 2?

(b) the rate at which a type 2 customer enters service with server 2?

(c) the fraction of server 2's customers that are type 1?

(d) the proportion of time that a type 1 customer is with server 2?

22. Customers arrive at a single-server station in accordance with a Poisson process with rate λ. All arrivals that find the server free immediately enter service. All service times are exponentially distributed with rate μ. An arrival that finds the server busy will leave the system and roam around "in orbit" for an exponential time with rate θ at which time it will then return. If the server is busy when an orbiting customer returns, then that customer returns to orbit for another exponential time with rate θ before returning again. An arrival that finds the server busy and N other customers in orbit will depart and not return. That is, N is the maximum number of customers in orbit.

(a) Define states.

(b) Give the balance equations.

In terms of the solution of the balance equations, find

(c) the proportion of all customers that are eventually served;

(d) the average time that a served customer spends waiting in orbit.

23. Consider the $M/M/1$ system in which customers arrive at rate λ and the server serves at rate μ. However, suppose that in any interval of length h in which the server is busy there is a probability $\alpha h + o(h)$ that the server will experience a breakdown, which causes the system to shut down. All customers that are in the system depart, and no additional arrivals are allowed to enter until the breakdown is fixed. The time to fix a breakdown is exponentially distributed with rate β.

(a) Define appropriate states.

(b) Give the balance equations.

In terms of the long-run probabilities,

(c) what is the average amount of time that an entering customer spends in the system?

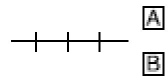


Figure 8.4

(d) what proportion of entering customers complete their service?
(e) what proportion of customers arrive during a breakdown?

*24. Reconsider Exercise 23, but this time suppose that a customer that is in the system when a breakdown occurs remains there while the server is being fixed. In addition, suppose that new arrivals during a breakdown period are allowed to enter the system. What is the average time a customer spends in the system?

25. Poisson (λ) arrivals join a queue in front of two parallel servers A and B, having exponential service rates μ_A and μ_B (see Figure 8.4). When the system is empty, arrivals go into server A with probability α and into B with probability $1 - \alpha$. Otherwise, the head of the queue takes the first free server.
(a) Define states and set up the balance equations. Do not solve.
(b) In terms of the probabilities in part (a), what is the average number in the system? Average number of servers idle?
(c) In terms of the probabilities in part (a), what is the probability that an arbitrary arrival will get serviced in A?

26. In a queue with unlimited waiting space, arrivals are Poisson (parameter λ) and service times are exponentially distributed (parameter μ). However, the server waits until K people are present before beginning service on the first customer; thereafter, he services one at a time until all K units, and all subsequent arrivals, are serviced. The server is then "idle" until K new arrivals have occurred.
(a) Define an appropriate state space, draw the transition diagram, and set up the balance equations.
(b) In terms of the limiting probabilities, what is the average time a customer spends in queue?
(c) What conditions on λ and μ are necessary?

27. Consider a single-server exponential system in which ordinary customers arrive at a rate λ and have service rate μ. In addition, there is a special customer who has a service rate μ_1. Whenever this special customer arrives, she goes directly into service (if anyone else is in service, then this person is bumped back into queue). When the special customer is not being serviced, she spends an exponential amount of time (with mean $1/\theta$) out of the system.
(a) What is the average arrival rate of the special customer?
(b) Define an appropriate state space and set up balance equations.
(c) Find the probability that an ordinary customer is bumped n times.

*28. Let D denote the time between successive departures in a stationary $M/M/1$ queue with $\lambda < \mu$. Show, by conditioning on whether or not a departure has left the system empty, that D is exponential with rate λ.

Hint: By conditioning on whether or not the departure has left the system empty we see that

$$D = \begin{cases} \text{Exponential}(\mu), & \text{with probability } \lambda/\mu \\ \text{Exponential}(\lambda) * \text{Exponential}(\mu), & \text{with probability } 1 - \lambda/\mu \end{cases}$$

where Exponential(λ) $*$ Exponential(μ) represents the sum of two independent exponential random variables having rates μ and λ. Now use moment-generating functions to show that D has the required distribution.

Note that the preceding does not prove that the departure process is Poisson. To prove this we need show not only that the interdeparture times are all exponential with rate λ, but also that they are independent.

29. Potential customers arrive to a single-server hair salon according to a Poisson process with rate λ. A potential customer who finds the server free enters the system; a potential customer who finds the server busy goes away. Each potential customer is type i with probability p_i, where $p_1 + p_2 + p_3 = 1$. Type 1 customers have their hair washed by the server; type 2 customers have their hair cut by the server; and type 3 customers have their hair first washed and then cut by the server. The time that it takes the server to wash hair is exponentially distributed with rate μ_1, and the time that it takes the server to cut hair is exponentially distributed with rate μ_2.
    (a) Explain how this system can be analyzed with four states.
    (b) Give the equations whose solution yields the proportion of time the system is in each state.
    In terms of the solution of the equations of (b), find
    (c) the proportion of time the server is cutting hair;
    (d) the average arrival rate of entering customers.
30. For the tandem queue model verify that
$$P_{n,m} = (\lambda/\mu_1)^n(1-\lambda/\mu_1)(\lambda/\mu_2)^m(1-\lambda/\mu_2)$$
    satisfies the balance equation (8.15).
31. Consider a network of three stations with a single server at each station. Customers arrive at stations 1, 2, 3 in accordance with Poisson processes having respective rates 5, 10, and 15. The service times at the three stations are exponential with respective rates 10, 50, and 100. A customer completing service at station 1 is equally likely to (i) go to station 2, (ii) go to station 3, or (iii) leave the system. A customer departing service at station 2 always goes to station 3. A departure from service at station 3 is equally likely to either go to station 2 or leave the system.
    (a) What is the average number of customers in the system (consisting of all three stations)?
    (b) What is the average time a customer spends in the system?
32. Consider a closed queueing network consisting of two customers moving among two servers, and suppose that after each service completion the customer is equally likely to go to either server—that is, $P_{1,2} = P_{2,1} = \frac{1}{2}$. Let μ_i denote the exponential service rate at server i, $i = 1, 2$.
    (a) Determine the average number of customers at each server.
    (b) Determine the service completion rate for each server.
33. Explain how a Markov chain Monte Carlo simulation using the Gibbs sampler can be utilized to estimate
    (a) the distribution of the amount of time spent at server j on a visit.

    Hint: Use the arrival theorem.

(b) the proportion of time a customer is with server j (i.e., either in server j's queue or in service with j).

34. For open queueing networks
   (a) state and prove the equivalent of the arrival theorem;
   (b) derive an expression for the average amount of time a customer spends waiting in queues.

35. Customers arrive at a single-server station in accordance with a Poisson process having rate λ. Each customer has a value. The successive values of customers are independent and come from a uniform distribution on $(0, 1)$. The service time of a customer having value x is a random variable with mean $3 + 4x$ and variance 5.
   (a) What is the average time a customer spends in the system?
   (b) What is the average time a customer having value x spends in the system?

*36. Compare the $M/G/1$ system for first-come, first-served queue discipline with one of last-come, first-served (for instance, in which units for service are taken from the top of a stack). Would you think that the queue size, waiting time, and busy-period distribution differ? What about their means? What if the queue discipline was always to choose at random among those waiting? Intuitively, which discipline would result in the smallest variance in the waiting time distribution?

37. In an $M/G/1$ queue,
   (a) what proportion of departures leave behind 0 work?
   (b) what is the average work in the system as seen by a departure?

38. For the $M/G/1$ queue, let X_n denote the number in the system left behind by the nth departure.
   (a) If

$$X_{n+1} = \begin{cases} X_n - 1 + Y_n, & \text{if } X_n \geqslant 1 \\ Y_n, & \text{if } X_n = 0 \end{cases}$$

   what does Y_n represent?
   (b) Rewrite the preceding as

$$X_{n+1} = X_n - 1 + Y_n + \delta_n \tag{8.64}$$

   where

$$\delta_n = \begin{cases} 1, & \text{if } X_n = 0 \\ 0, & \text{if } X_n \geqslant 1 \end{cases}$$

   Take expectations and let $n \to \infty$ in Equation (8.64) to obtain

$$E[\delta_\infty] = 1 - \lambda E[S]$$

   (c) Square both sides of Equation (8.64), take expectations, and then let $n \to \infty$ to obtain

$$E[X_\infty] = \frac{\lambda^2 E[S^2]}{2(1 - \lambda E[S])} + \lambda E[S]$$

   (d) Argue that $E[X_\infty]$, the average number as seen by a departure, is equal to L.

*39. Consider an $M/G/1$ system in which the first customer in a busy period has the service distribution G_1 and all others have distribution G_2. Let C denote the number of customers in a busy period, and let S denote the service time of a customer chosen at random.

Argue that

(a) $a_0 = P_0 = 1 - \lambda E[S]$.

(b) $E[S] = a_0 E[S_1] + (1 - a_0)E[S_2]$ where S_i has distribution G_i.

(c) Use (a) and (b) to show that $E[B]$, the expected length of a busy period, is given by

$$E[B] = \frac{E[S_1]}{1 - \lambda E[S_2]}$$

(d) Find $E[C]$.

40. Consider a $M/G/1$ system with $\lambda E[S] < 1$.

(a) Suppose that service is about to begin at a moment when there are n customers in the system.

  (i) Argue that the additional time until there are only $n - 1$ customers in the system has the same distribution as a busy period.

  (ii) What is the expected additional time until the system is empty?

(b) Suppose that the work in the system at some moment is A. We are interested in the expected additional time until the system is empty—call it $E[T]$. Let N denote the number of arrivals during the first A units of time.

  (i) Compute $E[T|N]$.

  (ii) Compute $E[T]$.

41. Carloads of customers arrive at a single-server station in accordance with a Poisson process with rate 4 per hour. The service times are exponentially distributed with rate 20 per hour. If each carload contains either 1, 2, or 3 customers with respective probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, compute the average customer delay in queue.

42. In the two-class priority queueing model of Section 8.6.2, what is W_Q? Show that W_Q is less than it would be under FIFO if $E[S_1] < E[S_2]$ and greater than under FIFO if $E[S_1] > E[S_2]$.

43. In a two-class priority queueing model suppose that a cost of C_i per unit time is incurred for each type i customer that waits in queue, $i = 1, 2$. Show that type 1 customers should be given priority over type 2 (as opposed to the reverse) if

$$\frac{E[S_1]}{C_1} < \frac{E[S_2]}{C_2}$$

44. Consider the priority queueing model of Section 8.6.2 but now suppose that if a type 2 customer is being served when a type 1 arrives then the type 2 customer is bumped out of service. This is called the preemptive case. Suppose that when a bumped type 2 customer goes back in service his service begins at the point where it left off when he was bumped.

(a) Argue that the work in the system at any time is the same as in the non-preemptive case.

(b) Derive W_Q^1.

Hint: How do type 2 customers affect type 1s?

(c) Why is it not true that

$$V_Q^2 = \lambda_2 E[S_2] W_Q^2$$

(d) Argue that the work seen by a type 2 arrival is the same as in the nonpreemptive case, and so

$$W_Q^2 = W_Q^2(\text{nonpreemptive}) + E[\text{extra time}]$$

where the extra time is due to the fact that he may be bumped.

(e) Let N denote the number of times a type 2 customer is bumped. Why is

$$E[\text{extra time}|N] = \frac{NE[S_1]}{1 - \lambda_1 E[S_1]}$$

Hint: When a type 2 is bumped, relate the time until he gets back in service to a "busy period."

(f) Let S_2 denote the service time of a type 2. What is $E[N|S_2]$?

(g) Combine the preceding to obtain

$$W_Q^2 = W_Q^2(\text{nonpreemptive}) + \frac{\lambda_1 E[S_1] E[S_2]}{1 - \lambda_1 E[S_1]}$$

*45. Calculate explicitly (not in terms of limiting probabilities) the average time a customer spends in the system in Exercise 24.

46. In the $G/M/1$ model if G is exponential with rate λ show that $\beta = \lambda/\mu$.

47. Verify Erlang's loss formula, Equation (8.60), when $k = 1$.

48. Verify the formula given for the P_i of the $M/M/k$.

49. In the Erlang loss system suppose the Poisson arrival rate is $\lambda = 2$, and suppose there are three servers, each of whom has a service distribution that is uniformly distributed over $(0, 2)$. What proportion of potential customers is lost?

50. In the $M/M/k$ system,
(a) what is the probability that a customer will have to wait in queue?
(b) determine L and W.

51. Verify the formula for the distribution of W_Q^* given for the $G/M/k$ model.

*52. Consider a system where the interarrival times have an arbitrary distribution F, and there is a single server whose service distribution is G. Let D_n denote the amount of time the nth customer spends waiting in queue. Interpret S_n, T_n so that

$$D_{n+1} = \begin{cases} D_n + S_n - T_n, & \text{if } D_n + S_n - T_n \geqslant 0 \\ 0, & \text{if } D_n + S_n - T_n < 0 \end{cases}$$

53. Consider a model in which the interarrival times have an arbitrary distribution F, and there are k servers each having service distribution G. What condition on F and G do you think would be necessary for there to exist limiting probabilities?

Chapter 8

1. (a) E[number of arrivals]

$$\begin{aligned}
&= E[E\{\text{number of arrivals}|\text{service period is } S\}]\\
&= E[\lambda S]\\
&= \lambda/\mu
\end{aligned}$$

(b) $P\{0 \text{ arrivals}\}$

$$\begin{aligned}
&= E[P\{0 \text{ arrivals}|\text{service period is } S\}]\\
&= E[P\{N(S) = 0\}]\\
&= E[e^{-\lambda S}]\\
&= \int_0^x e^{-\lambda s}\mu e^{-\mu s} ds\\
&= \frac{\mu}{\lambda + \mu}
\end{aligned}$$

2. This problem can be modeled by an $M/M/1$ queue in which $\lambda = 6$, $\mu = 8$. The average cost rate will be

$10 per hour per machine $\times$ average number of broken machines.

The average number of broken machines is just L, which can be computed from Equation (3.2):

$$\begin{aligned}
L &= \lambda/(\mu - \lambda)\\
&= \frac{6}{2} = 3
\end{aligned}$$

Hence, the average cost rate = $30/hour.

3. Let $C_M =$ Mary's average cost/hour and $C_A =$ Alice's average cost/hour.

Then, $C_M = \$3 + \$1 \times$ (Average number of customers in queue when Mary works),

and $C_A = \$C + \$1 \times$ (Average number of customers in queue when Alice works).

The arrival stream has parameter $\lambda = 10$, and there are two service parameters—one for Mary and one for Alice:

$\mu_M = 20, \quad \mu_A = 30.$

Set $L_M =$ average number of customers in queue when Mary works and
$L_A =$ average number of customers in queue when Alice works.

Then using Equation (3.2), $L_M = \dfrac{10}{(20 - 10)} = 1$

$$L_A = \frac{10}{(20 - 10)} = \frac{1}{2}$$

So
$$\begin{aligned}
C_M &= \$3 + \$1/\text{customer} \times L_M \text{ customers}\\
&= \$3 + \$1\\
&= \$4/\text{hour}
\end{aligned}$$

Also,
$$\begin{aligned}
C_A &= \$C + \$1/\text{customer} \times L_A \text{ customers}\\
&= \$C + \$1 \times \frac{1}{2}\\
&= \$C + \frac{1}{2}\ /\ \text{hour}
\end{aligned}$$

(b) We can restate the problem this way: If $C_A = C_M$, solve for C.

$$4 = C + \frac{1}{2} \Rightarrow C = \$3.50/\text{hour}$$

i.e., $3.50/hour is the most the employer should be willing to pay Alice to work. At a higher wage his average cost is lower with Mary working.

4. Let N be the number of other customers that were in the system when the customer arrived, and let $C = 1/f_{W_Q^*}(x)$. Then

$$\begin{aligned}
f_{N|W_Q^*}(n|x) &= C f_{W_Q^*|N}(x|n)P\{N = n\}\\
&= C\mu e^{-\mu x}\frac{(\mu x)^{n-1}}{(n-1)!}(\lambda/\mu)^n(1 - \lambda/\mu)\\
&= K\frac{(\lambda x)^{n-1}}{(n-1)!}
\end{aligned}$$

where

$$K = \frac{1}{f_{W_Q^*}(x)} \mu e^{-\mu x}(\lambda/\mu)(1-\lambda/\mu)$$

Using

$$1 = \sum_{n=1}^{\infty} f_{N|W_Q^*}(n|x) = K \sum_{n=1}^{\infty} \frac{(\lambda x)^{n-1}}{(n-1)!} = Ke^{\lambda x}$$

shows that

$$f_{N|W_Q^*}(n|x) = e^{-\lambda x}\frac{(\lambda x)^{n-1}}{(n-1)!}, \qquad n > 0$$

Thus, $N-1$ is Poisson with mean λx.

The preceding also yields that for $x > 0$

$$\begin{aligned} f_{W_Q^*}(x) &= e^{\lambda x}\mu e^{-\mu x}(\lambda/\mu)(1-\lambda/\mu) \\ &= \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)x} \end{aligned}$$

Hence, for $x > 0$

$$\begin{aligned} P\{W_Q^* \le x\} &= P\{W_Q^* = 0\} + \int_0^x f_{W_Q^*}(y)dy \\ &= 1 - \frac{\lambda}{\mu} + \frac{\lambda}{\mu}(1-e^{-(\mu-\lambda)x}) \end{aligned}$$

5. Let I equal 0 if $W_Q^* = 0$ and let it equal 1 otherwise. Then,

$$E[W_Q^*|I=0] = 0$$

$$E[W_Q^*|I=1] = (\mu-\lambda)^{-1}$$

$$Var(W_Q^*|I=0) = 0$$

$$Var(W_Q^*|I=1) = (\mu-\lambda)^{-2}$$

Hence,

$$E[Var(W_Q^*|I] = (\mu-\lambda)^{-2}\lambda/\mu$$

$$Var(E[W_Q^*|I]) = (\mu-\lambda)^{-2}\lambda/\mu(1-\lambda/\mu)$$

Consequently, by the conditional variance formula,

$$Var(W_Q^*) = \frac{\lambda}{\mu(\mu-\lambda)^2} + \frac{\lambda}{\mu^2(\mu-\lambda)}$$

6. $$\begin{aligned} E[(S_1-Y)^+] &= E[(S_1-Y)^+|S_1>Y]\frac{\lambda}{\lambda+\mu} \\ &= \frac{\lambda}{\mu(\lambda+\mu)} \end{aligned}$$

Also,

$$\begin{aligned} &E[S_1(S_1-Y)^+] \\ &= E[S_1(S_1-Y)^+|S_1>Y]\frac{\lambda}{\lambda+\mu} \\ &= \frac{\lambda}{\lambda+\mu}(E[(S_1-Y)(S_1-Y)^+|S_1>Y] \\ &\quad + E[Y(S_1-Y)^+|S_1>Y]) \\ &= \frac{\lambda}{\lambda+\mu}(E[S_1^2] + E[Y|S_1>Y]E[(S_1-Y)^+|S_1>Y]) \\ &= \frac{\lambda}{\lambda+\mu}\left(\frac{2}{\mu^2} + \frac{1}{\lambda+\mu}\frac{1}{\mu}\right) \end{aligned}$$

Hence,

$$\begin{aligned} \text{Cov}(S_1,\ (S_1-Y)^+ + S_2) &= \frac{\lambda}{\lambda+\mu}\left(\frac{2}{\mu^2} + \frac{1}{\lambda+\mu}\frac{1}{\mu}\right) \\ &\quad - \frac{\lambda}{\mu^2(\lambda+\mu)} \\ &= \frac{\lambda}{\mu^2(\lambda+\mu)} + \frac{\lambda}{\mu(\lambda+\mu)^2} \end{aligned}$$

7. To compute W for the $M/M/2$, set up balance equations as

$$\lambda p_0 = \mu p_1 \qquad \text{(each server has rate } \mu)$$

$$(\lambda+\mu)p_1 = \lambda p_0 + 2\mu p_2$$

$$(\lambda+2\mu)p_n = \lambda p_{n-1} + 2\mu p_{n+1}, \qquad n \ge 2$$

These have solutions $P_n = \rho^n/2^{n-1}p_0$ where $\rho = \lambda/\mu$.

The boundary condition $\sum_{n=0}^{\infty} P_n = 1$ implies

$$P_0 = \frac{1-\rho/2}{1+\rho/2} = \frac{(2-\rho)}{(2+\rho)}$$

Now we have P_n, so we can compute L, and hence W from $L = \lambda W$:

$$\begin{aligned} L = \sum_{n=0}^{\infty} np_n &= \rho p_0 \sum_{n=0}^{\infty} n\left[\frac{\rho}{2}\right]^{n-1} \\ &= 2p_0 \sum_{n=0}^{\infty} n\left[\frac{\rho}{2}\right]^{n} \\ &= 2\frac{(2-\rho)}{(2+\rho)}\frac{(\rho/2)}{(1-\rho/2)^2} \\ &= \frac{4\rho}{(2+\rho)(2-\rho)} \\ &= \frac{4\mu\lambda}{(2\mu+\lambda)(2\mu-\lambda)} \end{aligned}$$

From $L = \lambda W$ we have

$$W = W_{m/m/2} = \frac{4\mu}{(2\mu+\lambda)(2\mu-\lambda)}$$

The $M/M/1$ queue with service rate 2μ has

$$Wm/m/1 = \frac{1}{2\mu - \lambda}$$

from Equation (3.3). We assume that in the $M/M/1$ queue, $2\mu > \lambda$ so that the queue is stable. But then $4\mu > 2\mu + \lambda$, or $\frac{4\mu}{2\mu + \lambda} > 1$, which implies $Wm/m/2 > Wm/m/1$.

The intuitive explanation is that if one finds the queue empty in the $M/M/2$ case, it would do no good to have two servers. One would be better off with one faster server.

Now let $W_Q^1 = W_Q(M/M/1)$

$$W_Q^2 = W_Q(M/M/2)$$

Then,

$$W_Q^1 = Wm/m/1 - 1/2\mu$$

$$W_Q^2 = Wm/m/2 - 1/\mu$$

So,

$$W_Q^1 = \frac{\lambda}{2\mu(2\mu - \lambda)} \qquad (3.3)$$

and

$$W_Q^2 = \frac{\lambda^2}{\mu(2\mu - \lambda)(2\mu + \lambda)}$$

Then,

$$W_Q^1 > W_Q^2 \Leftrightarrow \frac{1}{2} > \frac{\lambda}{(2\mu + \lambda)}$$
$$\lambda < 2\mu$$

Since we assume $\lambda < 2\mu$ for stability in the $M/M/1$, $W_Q^2 < W_Q^1$ whenever this comparison is possible, i.e., whenever $\lambda < 2\mu$.

8. This model is mathematically equivalent to the $M/M/1$ queue with finite capacity k. The produced items constitute the arrivals to the queue, and the arriving customers constitute the services. That is, if we take the state of the system to be the number of items presently available then we just have the model of Section 8.3.2.

   (a) The proportion of customers that go away empty-handed is equal to P_0, the proportion of time there are no items on the shelves. From Section 8.3.2,

   $$P_0 = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{k+1}}$$

   (b) $W = \dfrac{L}{\lambda(1 - P_k)}$ where L is given by Equation (8.12).

   (c) The average number of items in stock is L.

9. Take the state to be the number of customers at server 1. The balance equations are

$$\mu P_0 = \mu P_1$$
$$2\mu P_j = \mu P_{j+1} + \mu P_{j-1}, \quad 1 \le j < n$$
$$\mu P_n = \mu P_{n-1}$$
$$1 = \sum_{j=0}^{n} P_j$$

It is easy to check that the solution to these equations is that all the P_js are equal, so $P_j = 1/(n+1)$, $j = 0, \ldots, n$.

10. The state is the number of customers in the system, and the balance equations are

$$m\theta P_0 = \mu P_1$$
$$((m-j)\theta + \mu)P_j = (m - j + 1)\theta P_{j-1} + \mu P_{j+1}, \quad 0 < j < m$$
$$\mu P_m = \theta P_{m-1}$$
$$1 = \sum_{j=0}^{m} P_j$$

   (a) $\lambda_\alpha = \sum_{j=0}^{m} (m-j)\theta P_j$

   (b) $L/\lambda_\alpha = \sum_{j=0}^{m} jP_j / \sum_{j=0}^{m} (m-j)\theta P_j$

11. (a) $\lambda P_0 = \alpha\mu P_1$

   $$(\lambda + \alpha\mu)P_n = \lambda P_{n-1} + \alpha\mu P_{n+1}, \quad n \ge 1$$

   These are exactly the same equations as in the $M/M/1$ with $\alpha\mu$ replacing μ. Hence,

   $$P_n = \left[\frac{\lambda}{\alpha\mu}\right]^n \left[1 - \frac{\lambda}{\alpha\mu}\right], \quad n \ge 0$$

   and we need the condition $\lambda < \alpha\mu$.

   (b) If T is the waiting time until the customer first enters service, then conditioning on the number present when he arrives yields

   $$E[T] = \sum_n E[T|n \text{ present}]P_n = \sum_n \frac{n}{\mu} P_n = \frac{L}{\mu}$$

Since $L = \sum nP_n$, and the P_n are the same as in the $M/M/1$ with λ and $\alpha\mu$, we have that $L = \lambda/(\alpha\mu - \lambda)$ and so

$$E[T] = \frac{\lambda}{\mu(\alpha\mu - \lambda)}$$

(c) $P\{\text{enters service exactly } n \text{ times}\}$

$$= (1-\alpha)^{n-1}\alpha$$

(d) This is expected number of services $\times$ mean services time $= 1/\alpha\mu$

(e) The distribution is easily seen to be memoryless. Hence, it is exponential with rate $\alpha\mu$.

12. (a)

$$\lambda p_0 = \mu p_1$$

$$(\lambda + \mu)p_1 = \lambda p_0 + 2\mu p_2$$

$$(\lambda + 2\mu)p_n = \lambda p_{n-1} + 2\mu p_{n+1} \quad n \geq 2$$

These are the same balance equations as for the $M/M/2$ queue and have solution

$$p_0 = \left[\frac{2\mu - \lambda}{2\mu + \lambda}\right], \quad p_n = \frac{\lambda^n}{2^{n-1}\mu^n} p_0$$

(b) The system goes from 0 to 1 at rate $\lambda p_0 = \dfrac{\lambda(2\mu - \lambda)}{(2\mu + \lambda)}$. The system goes from 2 to 1 at rate

$$2\mu p_2 = \frac{\lambda^2(2\mu - \lambda)}{\mu(2\mu + \lambda)}.$$

(c) Introduce a new state cl to indicate that the stock clerk is checking by himself. The balance equation for P_{cl} is

$$(\lambda + \mu)p_{cl} = \mu p_2$$

The reason for p_2 is that it is only if the checker completes service first in p_2 that the system moves to state cl. Then

$$p_{cl} = \frac{\mu}{\lambda + \mu} p_2 = \frac{\lambda^2}{2\mu(\lambda + \mu)} \frac{(2\mu - \lambda)}{(2\mu + \lambda)}$$

Finally, the proportion of time the stock clerk is checking is

$$p_{cl} + \sum_{n=2}^{\infty} p_n = p_{cl} + \frac{2\lambda^2}{\mu(2\mu + \lambda)}$$

13. Let the state be the idle server. The balance equations are

Rate Leave = Rate Enter,

$$(\mu_2 + \mu_3)P_1 = \frac{\mu_1}{\mu_1 + \mu_2} P_3 + \frac{\mu_1}{\mu_1 + \mu_3} P_2,$$

$$(\mu_1 + \mu_3)P_2 = \frac{\mu_2}{\mu_2 + \mu_3} P_1 + \frac{\mu_2}{\mu_2 + \mu_1} P_3,$$

$$\mu_1 + \mu_2 + \mu_3 = 1.$$

These are to be solved and the quantity P_i represents the proportion of time that server i is idle.

14. There are 4 states, defined as follows: 0 means the system is empty, i that there are i type 1 customers in the system, $i = 1, 2$, and 1_2 that there is one type 2 customer in the system.

(b)

$$(\lambda_1 + \lambda_2)P_0 = \mu_1 P_1 + \mu_2 P_{1_2}$$

$$(\lambda_1 + \mu_1)P_1 = \lambda_1 P_0 + 2\mu_1 P_2$$

$$2\mu_1 P_2 = \lambda_1 P_1$$

$$\mu_2 P_{1_2} = \lambda_2 P_0$$

$$P_0 + P_1 + P_2 + P_{1_2} = 1$$

(c) $$W = \frac{L}{\lambda_a} = \frac{P_1 + 2P_2 + P_{1_2}}{(\lambda_1 + \lambda_2)P_0 + \lambda_1 P_1}$$

(d) Let F_1 be the fraction of served customers that are type 1. Then F_1

$$= \frac{\text{rate at which type 1 customers join the system}}{\text{rate at which customers join the system}}$$

$$= \frac{\lambda_1(P_0 + P_1)}{\lambda_1(P_0 + P_1) + \lambda_2 P_0}$$

15. There are four states $= 0, 1_A, 1_B, 2$. Balance equations are

$$2P_0 = 2P_{1_B}$$

$$4P_{1_A} = 2P_0 + 2P_2$$

$$4P_{1_B} = 4P_{1_A} + 4P_2$$

$$6P_2 = 2P_{1_B}$$

$$P_0 + P_{1_A} + P_{1_B} + P_2 = 1 \Rightarrow P_0 = \frac{3}{9}$$

$$P_{1_A} = \frac{2}{9}, P_{1_B} = \frac{3}{9}, P_2 = \frac{1}{9}$$

(a) $P_0 + P_{1_B} = \dfrac{2}{3}$

(b) By conditioning upon whether the state was 0 or 1_B when he entered we get that the desired probability is given by

$$\frac{1}{2} + \frac{1}{2}\frac{2}{6} = \frac{4}{6}$$

(c) $P_{1_A} + P_{1_B} + 2P_2 = \dfrac{7}{9}$

(d) Again, condition on the state when he enters to obtain

$$\frac{1}{2}\left[\frac{1}{4} + \frac{1}{2}\right] + \frac{1}{2}\left[\frac{1}{4} + \frac{2}{6}\frac{1}{2}\right] = \frac{7}{12}$$

This could also have been obtained from (a) and (c) by the formula $W = \frac{L}{\lambda a}$.

That is, $W = \dfrac{\frac{7}{9}}{2\left[\frac{2}{3}\right]} = \frac{7}{12}$.

16. Let the states be $(0,0), (1,0), (0,1)$, and $(1,1)$, where state (i,j) means that there are i customers with server 1 and j with server 2. The balance equations are as follows.

$$\lambda P_{00} = \mu_1 P_{10} + \mu_2 P_{01}$$
$$(\lambda + \mu_1)P_{10} = \lambda P_{00} + \mu_2 P_{11}$$
$$(\lambda + \mu_2)P_{01} = \mu_1 P_{11}$$
$$(\mu_1 + \mu_2)P_{11} = \lambda P_{01} + \lambda P_{10}$$
$$P_{00} + P_{01} + P_{10} + P_{11} = 1$$

Substituting the values $\lambda = 5, \mu_1 = 4, \mu_2 = 2$ and solving yields the solution

$P_{00} = 128/513$, $P_{10} = 110/513$, $P_{01} = 100/513$, $P_{11} = 175/513$

(a) $W = L/\lambda_a = [1(P_{01} + P_{10}) + 2P_{11}]/[\lambda(1 - P_{11})] = 56/119$

Another way is to condition on the state as seen by the arrival. Letting T denote the time spent, this gives

$$W = E[T|00]128/338 + E[T|01]100/338$$
$$+ E[T|10]110/338$$
$$= (1/4)(228/338) + (1/2)(110/338)$$
$$= 56/119$$

(b) $P_{01} + P_{11} = 275/513$

17. The state space can be taken to consist of states $(0,0), (0,1), (1,0), (1,1)$, where the i^{th} component of the state refers to the number of customers at server i, $i = 1, 2$. The balance equations are

$$2P_{0,0} = 6P_{0,1}$$
$$8P_{0,1} = 4P_{1,0} + 4P_{1,1}$$
$$6P_{1,0} = 2P_{0,0} + 6P_{1,1}$$
$$10P_{1,1} = 2P_{0,1} + 2P_{1,0}$$
$$1 = P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1}$$

Solving these equations gives $P_{0,0} = 1/2$, $P_{0,1} = 1/6, P_{1,0} = 1/4, P_{1,1} = 1/12$.

(a) $P_{1,1} = 1/12$

(b) $W = \dfrac{L}{\lambda_a} = \dfrac{P_{0,1} + P_{1,0} + 2P_{1,1}}{2(1 - P_{1,1})} = \dfrac{7}{22}$

(c) $\dfrac{P_{0,0} + P_{0,1}}{1 - P_{1,1}} = \dfrac{8}{11}$

18. (a) Let the state be (i,j,k) if there are i customers with server 1, j customers with server 2, and k customers with server 3.

(b)
$$\lambda P_{0,0,0} = \mu_3 P_{0,0,1}$$
$$(\lambda + \mu_1)P_{1,0,0} = \lambda P_{0,0,0} + \mu_3 P_{1,0,1}$$
$$(\lambda + \mu_2)P_{0,1,0} = \mu_3 P_{0,1,1}$$
$$(\lambda + \mu_3)P_{0,0,1} = \mu_1 P_{1,0,0} + \mu_2 P_{0,1,0}$$
$$(\mu_1 + \mu_2)P_{1,1,0} = \lambda P_{1,0,0} + \lambda P_{0,1,0} + \mu_3 P_{1,1,1}$$
$$(\lambda + \mu_1 + \mu_3)P_{1,0,1} = \lambda P_{0,0,1} + \mu_2 P_{1,1,1}$$
$$(\lambda + \mu_2 + \mu_3)P_{0,1,1} = \mu_1 P_{1,1,1}$$
$$(\mu_1 + \mu_2 + \mu_3)P_{1,1,1} = \lambda P_{0,1,1} + \lambda P_{1,0,1}$$
$$\sum_{i,j,k} P_{i,j,k} = 1$$

(c) $W = \dfrac{L}{\lambda_a}$
$$= \frac{P_{1,0,0} + P_{0,1,0} + P_{0,0,1} + 2(P_{1,1,0} + P_{1,0,1} + P_{0,1,1}) + 3P_{1,1,1}}{\lambda(1 - P_{1,1,0} - P_{1,1,1})}$$

(d) Let $Q_{1,j,k}$ be the probability that the person at server 1 will be eventually served by server 3 when there are j currently at server 2 and k at server 3. The desired probability is $Q_{1,0,0}$. Conditioning on the next event yields

$$Q_{1,0,0} = \frac{\mu_1}{\lambda + \mu_1} + \frac{\lambda}{\lambda + \mu_1} Q_{1,1,0}$$
$$Q_{1,1,0} = \frac{\mu_1}{\mu_1 + \mu_2} + \frac{\mu_2}{\mu_1 + \mu_2} Q_{1,0,1}$$
$$Q_{1,0,1} = \frac{\lambda}{\lambda + \mu_1 + \mu_3} Q_{1,1,1} + \frac{\mu_3}{\lambda + \mu_1 + \mu_3} Q_{1,0,0}$$
$$Q_{1,1,1} = \frac{\mu_2}{\mu_1 + \mu_2 + \mu_3} Q_{1,0,1} + \frac{\mu_3}{\mu_1 + \mu_2 + \mu_3} Q_{1,1,0}$$

Now solve for $Q_{1,0,0}$.

19. (a) Say that the state is $(n, 1)$ whenever it is a good period and there are n in the system, and say that it is $(n, 2)$ whenever it is a bad period and there are n in the system, $n = 0, 1$.

(b)
$$(\lambda_1 + \alpha_1)P_{0,1} = \mu P_{1,1} + \alpha_2 P_{0,2}$$
$$(\lambda_2 + \alpha_2)P_{0,2} = \mu P_{1,2} + \alpha_1 P_{0,1}$$
$$(\mu + \alpha_1)P_{1,1} = \lambda_1 P_{0,1} + \alpha_2 P_{1,2}$$
$$(\mu + \alpha_2)P_{1,2} = \lambda_2 P_{0,2} + \alpha_1 P_{1,1}$$

$P_{0,\,1} + P_{0,\,2} + P_{1,\,1} + P_{1,\,2} = 1$

(c) $P_{0,\,1} + P_{0,\,2}$

(d) $\lambda_1 P_{0,\,1} + \lambda_2 P_{0,\,2}$

20. (a) The states are $0, (1,\ 0), (0,\ 1)$ and $(1,\ 1)$, where 0 means that the system is empty, $(1,\ 0)$ that there is one customer with server 1 and none with server 2, and so on.

(b) $(\lambda_1 + \lambda_2)P_0 = \mu_1 P_{10} + \mu_2 P_{01}$

$$(\lambda_1 + \lambda_2 + \mu_1)P_{10} = \lambda_1 P_0 + \mu_2 P_{11}$$

$$(\lambda_1 + \mu_2)P_{01} = \lambda_2 P_0 + \mu_1 P_{11}$$

$$(\mu_1 + \mu_2)P_{11} = \lambda_1 P_{01} + (\lambda_1 + \lambda_2)P_{10}$$

$$P_0 + P_{10} + P_{01} + P_{11} = 1$$

(c) $L = P_{01} + P_{10} + 2P_{11}$

(d) $W = L/\lambda_a = L/[\lambda_1(1 - P_{11}) + \lambda_2(P_0 + P_{10})]$

21. (a) $\lambda_1 P_{10}$

(b) $\lambda_2(P_0 + P_{10})$

(c) $\lambda_1 P_{10}/[\lambda_1 P_{10} + \lambda_2(P_0 + P_{10})]$

(d) This is equal to the fraction of server 2's customers that are type 1 multiplied by the proportion of time server 2 is busy. (This is true since the amount of time server 2 spends with a customer does not depend on which type of customer it is.) By (c) the answer is thus

$$(P_{01} + P_{11})\lambda_1 P_{10}/[\lambda_1 P_{10} + \lambda_2(P_0 + P_{10})]$$

22. The state is the pair $(i, j), i = 0, 1, 0 \le j \le n$ where i signifies the number of customers in service and j the number in orbit. The balance equations are

$$(\lambda + j\theta)P_{0,j} = \mu P_{1,j}, \quad j = 0, \ldots, N$$

$$(\lambda + \mu)P_{1,j} = \lambda P_{0,j} + (j+1)\theta P_{0,j+1}, \quad j = 0, \ldots, N-1$$

$$\mu P_{1,N} = \lambda P_{0,N}$$

(c) $1 - P_{1,\,N}$

(d) The average number of customers in the system is

$$L = \sum_{i,\,j} (i + j)P_{i,\,j}$$

Hence, the average time that an entering customer spends in the system is $W = L/\lambda(1 - P_{1,N})$, and the average time that an entering customer spends in orbit is $W - 1/\mu$.

23. (a) The states are $n, n \ge 0$, and b. State n means there are n in the system and state b means that a breakdown is in progress.

(b) $\beta P_b = a(1 - P_0)$

$$\lambda P_0 = \mu P_1 + \beta P_b$$

$$(\lambda + \mu + a)P_n = \lambda P_{n-1} + \mu P_{n+1}, \quad n \ge 1$$

(c) $W = L/\lambda_n = \sum_{n=1}^{\infty} nP_a/[\lambda(1 - P_b)]$

(d) Since rate at which services are completed $= \mu(1 - P_0 - P_b)$ it follows that the proportion of customers that complete service is

$$\mu(1 - P_0 - P_b)/\lambda_a = \mu(1 - P_0 - P_b)/[\lambda(1 - P_b)]$$

An equivalent answer is obtained by conditioning on the state as seen by an arrival. This gives the solution

$$\sum_{n=0}^{\infty} P_n[\mu/(\mu + a)]^{n+1}$$

where the above uses that the probability that $n + 1$ services of present customers occur before a breakdown is $[\mu/(\mu + a)]^{n+1}$.

(e) P_b

24. The states are now $n, n \ge 0$, and $n', n \ge 1$ where the state is n when there are n in the system and no breakdown, and it is n' when there are n in the system and a breakdown is in progress. The balance equations are

$$\lambda P_0 = \mu P_1$$

$$(\lambda + \mu + \alpha)P_n = \lambda P_{n-1} + \mu P_{n+1} + \beta P_{n'}, \quad n \ge 1$$

$$(\beta + \lambda)P_{1'} = \alpha P_1$$

$$(\beta + \lambda)P_{n'} = \alpha P_n + \lambda P_{(n-1)'}, \quad n \ge 2$$

$$\sum_{n=0}^{\infty} P_n + \sum_{n=1}^{\infty} P_{n'} = 1.$$

In terms of the solution to the above,

$$L = \sum_{n=1}^{\infty} n(P_n + P_{n'})$$

and so

$$W = L/\lambda_\alpha = L/\lambda$$

25. (a)
$$\lambda P_0 = \mu_A P_A + \mu_B P_B$$
$$(\lambda + \mu_A)P_A = a\lambda P_0 + \mu_B P_2$$
$$(\lambda + \mu_B)P_B = (1-a)\lambda P_0 + \mu_A P_2$$
$$(\lambda + \mu_A + \mu_B)P_n = \lambda P_{n-1} + (\mu_A + \mu_B)P_{n+1'}$$
$n \geq 2$ where $P_1 = P_A + P_B$.

(b) $L = P_A + P_B + \sum_{n=2}^{\infty} nP_n$

Average number of idle servers $= 2P_0 + P_A + P_B$.

(c) $P_0 + P_B + \dfrac{\mu_A}{\mu_A + \mu_B} \sum_{n=2}^{\infty} P_n$

26. States are $0, 1, 1', \ldots, k-1 (k-1)', k, k+1, \ldots$ with the following interpretation

$0 =$ system is empty

$n = n$ in system and server is working

$n' = n$ in system and server is idle,

$n = 1, 2, \ldots, k-1$

(a)
$$\lambda P_0 = \mu P_1, (\lambda + \mu)P_1 = \mu P_2$$
$$\lambda P'_n = \lambda P_{(n-1)'} \; n = 1, \ldots, k-1$$
$$(\lambda + \mu)P_k = \lambda P_{(k-1)'} + \mu P_{k+1} + \lambda P_{k-1}$$
$$(\lambda + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1'} \; n > k$$

(b) $\dfrac{k-1}{\lambda}P_0 + \sum_{n=1}^{k-1}\left[\dfrac{k-1-n}{\lambda} + \dfrac{n}{\mu}\right]P_{n'} + \sum_{n-1}^{\infty} P_n \dfrac{n}{\mu}$

(c) $\lambda < \mu$

27. (a) The special customer's arrival rate is act θ because we must take into account his service time. In fact, the mean time between his arrivals will be $1/\theta + 1/\mu_1$. Hence, the arrival rate is $(1/\theta + 1/\mu_1)^{-1}$.

(b) Clearly we need to keep track of whether the special customer is in service. For $n \geq 1$, set

$P_n = Pr\{n$ customers in system regular customer in service$\}$,

$P_n^S = Pr\{n$ customers in system, special customer in service$\}$, and

$P_0 = Pr\{0$ customers in system$\}$.

$$(\lambda + \theta)P_0 = \mu P_1 + \mu_1 P_1^S$$
$$(\lambda + \theta + \mu)P_n = \lambda P_{n-1} + \mu P_{n+1} + \mu_1 P_{n+1}^S$$
$$(\lambda + \mu)P_n^S = \theta P_{n-1} + \lambda P_{n-1'}^S$$
$$n \geq 1 \; [P_0^S = P_0]$$

(c) Since service is memoryless, once a customer resumes service it is as if his service has started anew. Once he begins a particular service, he will complete it if and only if the next arrival of the special customer is after his service. The probability of this is Pr {Service < Arrival of special customer} $= \mu/(\mu + \theta)$, since service and special arrivals are independent exponential random variables. So,

$Pr\{$bumped exactly n times$\}$
$$= (1 - \mu/(\mu + \theta))^n(\mu/(\mu + \theta))$$
$$= (\theta/(\mu + \theta))^n(\mu/(\mu + \theta))$$

In essence, the number of times a customer is bumped in service is a geometric random variable with parameter $\mu/(\mu + \theta)$.

28. If a customer leaves the system busy, the time until the next departure is the time of a service. If a customer leaves the system empty, the time until the next departure is the time until an arrival plus the time of a service.

Using moment-generating functions we get

$$E\{e^{\delta D}\} = \frac{\lambda}{\mu}E\{e^{\delta D}|\text{system left busy}\}$$
$$+ \left[1 - \frac{\lambda}{\mu}\right] E\{e^{\delta D}|\text{system left empty}\}$$
$$= \left[\frac{\lambda}{\mu}\right]\left[\frac{\mu}{\mu - \delta}\right] + \left[1 - \frac{\lambda}{\mu}\right]\left[E\{e^{\delta(X+Y)}\}\right]$$

where X has the distribution of interarrival times, Y has the distribution of service times, and X and Y are independent.

Then
$$E\{e^{\delta(X+Y)}\} = E\{e^{\delta X}e^{\delta Y}\}$$
$$= E[e^{\delta X}]E[e^{\delta Y}] \text{ by independence}$$
$$= \left[\frac{\lambda}{\lambda - \delta}\right]\left[\frac{\mu}{\mu - \delta}\right]$$

So,

$$E\{e^{\delta D}\} = \left[\frac{\lambda}{\mu}\right]\left[\frac{\mu}{\mu-\delta}\right] + \left[1-\frac{\lambda}{\mu}\right]\left[\frac{\lambda}{\lambda-\delta}\right]\left[\frac{\mu}{\mu-\delta}\right]$$

$$= \frac{\lambda}{(\lambda-\delta)}.$$

By the uniqueness of generating functions, it follows that D has an exponential distribution with parameter λ.

29. (a) Let state 0 mean that the server is free; let state 1 mean that a type 1 customer is having a wash; let state 2 mean that the server is cutting hair; and let state 3 mean that a type 3 is getting a wash.

(b) $\lambda P_0 = \mu_1 P_1 + \mu_2 P_2$

$$\mu_1 P_1 = \lambda p_1 P_0$$

$$\mu_2 P_2 = \lambda p_2 P_0 + \mu_1 P_3$$

$$\mu_1 P_3 = \lambda p_3 P_0$$

$$P_0 + P_1 + P_2 + P_3 = 1$$

(c) P_2

(d) λP_0

Direct substitution now verifies the equation.

31. The total arrival rates satisfy

$$\lambda_1 = 5$$

$$\lambda_2 = 10 + \frac{1}{3}5 + \frac{1}{2}\lambda_3$$

$$\lambda_3 = 15 + \frac{1}{3}5 + \lambda_2$$

Solving yields that $\lambda_1 = 5$, $\lambda_2 = 40$, $\lambda_3 = 170/3$. Hence,

$$L = \sum_{i=1}^{3}\frac{\lambda_i}{\mu_i - \lambda_i} = \frac{82}{13}$$

$$W = \frac{L}{r_1 + r_2 + r_3} = \frac{41}{195}$$

32. Letting the state be the number of customers at server 1, the balance equations are

$$(\mu_2/2)P_0 = (\mu_1/2)P_1$$

$$(\mu_1/2 + \mu_2/2)P_1 = (\mu_2/2)P_0 + (\mu_1/2)P_2$$

$$(\mu_1/2)P_2 = (\mu_2/2)P_1$$

$$P_0 + P_1 + P_2 = 1$$

Solving yields that

$$P_1 = (1 + \mu_1/\mu_2 + \mu_2/\mu_1)^{-1}, \quad P_0 = \mu_1/\mu_2 P_1,$$

$$P_2 = \mu_2/\mu_1 P_1$$

Hence, letting L_i be the average number of customers at server i, then

$$L_1 = P_1 + 2P_2, \qquad L_2 = 2 - L_1$$

The service completion rate for server 1 is $\mu_1(1 - P_0)$, and for server 2 it is $\mu_2(1 - P_2)$.

33. (a) Use the Gibbs sampler to simulate a Markov chain whose stationary distribution is that of the queuing network system with $m-1$ customers. Use this simulated chain to estimate $P_{i,\,m-1}$, the steady state probability that there are i customers at server j for this system. Since, by the arrival theorem, the distribution function of the time spent at server j in the m customer system is $\sum_{i=0}^{m-1} P_{i,\,m-1}G_{i+1}(x)$, where $G_k(x)$ is the probability that a gamma (k, μ) random variable is less than or equal to x, this enables us to estimate the distribution function.

(b) This quantity is equal to the average number of customers at server j divided by m.

34. $W_Q = L_Q/\lambda_\alpha = \dfrac{\sum_j \dfrac{\lambda_j^2}{\mu_j(\mu_j - \lambda_j)}}{\sum_j r_j}$

35. Let S and U denote, respectively, the service time and value of a customer. Then U is uniform on (0, 1) and

$$E[S|U] = 3 + 4U, \quad Var(S|U) = 5$$

Hence,

$$E[S] = E\{E[S|U]\} = 3 + 4E[U] = 5$$

$$Var(S) = E[Var(S|U)] + Var(E[S|U])$$

$$= 5 + 16Var(U) = 19/3$$

Therefore,

$$E[S^2] = 19/3 + 25 = 94/3$$

(a) $W = W_Q + E[S] = \dfrac{94\lambda/3}{1 - \delta\lambda} + 5$

(b) $W_Q + E[S|U = x] = \dfrac{94\lambda/3}{1 - \delta\lambda} + 3 + 4x$

36. The distributions of the queue size and busy period are the same for all three disciplines; that of the waiting time is different. However, the means are identical. This can be seen by using $W = L/\lambda$, since L is the same for all. The smallest variance in the waiting time occurs under first-come, first-served and the largest under last-come, first-served.

37. (a) The proportion of departures leaving behind 0 work

$=$ proportion of departures leaving an empty system

$=$ proportion of arrivals finding an empty system

$=$ proportion of time the system is empty (by Poisson arrivals)

$= P_0$

(b) The average amount of work as seen by a departure is equal to the average number it sees multiplied by the mean service time (since no customers seen by a departure have yet started service). Hence,

Average work as seen by a departure

$=$ average number it sees $\times E[S]$

$=$ average number an arrival sees $\times E[S]$

$= LE[S]$ by Poisson arrivals

$$= \lambda(W_Q + E[S])E[S]$$

$$= \frac{\lambda^2 E[S]E[S^2]}{\lambda - \lambda E[S]} + \lambda(E[S])^2$$

38. (a) $Y_n =$ number of arrivals during the $(n+1)$st service.

(b) Taking expectations we get

$$EX_{n+1} = EX_n - 1 + EY_n + E\delta_n$$

Letting $n \to \infty$, EX_{n+1} and EX_n cancel, and $EY_\infty = EY_1$. Therefore,

$$E\delta_\infty = 1 - EY_1$$

To compute EY_1, condition on the length of service S; $E[Y_1|S = t] = \lambda t$ by Poisson arrivals. But $E[\lambda S]$ is just λES. Hence,

$$E\delta_\infty = 1 - \lambda ES$$

(c) Squaring Equation (8.1) we get

$$(^*) X_{n+1}^2 = X_n^2 + 1 + Y_n^2 + 2(X_nY_n - X_n) - 2Y_n + \delta_n(2Y_n + 2X_n - 1)$$

But taking expectations, there are a few facts to notice:

$E\delta_n S_n = 0 \quad \text{since} \quad \delta_n S_n \equiv 0$

Y_n and X_n are independent random variables because $Y_n =$ number of arrivals during the $(n+1)^{st}$ service. Hence,

$EX_nY_n = EX_nEY_n$

For the same reason, Y_n and δ_n are independent random variables, so $E\delta_nY_n = E\delta_nEY_n$.

$EY_n^2 = \lambda ES + \lambda^2 ES^2$ by the same conditioning argument of part (b).

Finally also note $\delta_n^2 \equiv \delta_n$.

Taking expectations of (*) gives

$$EX_{n+1}^2 = EX_n^2 + 1 + \lambda E(S) + \lambda^2 E(S^2)$$
$$+ 2EX_n(\lambda E(S) - 1)$$
$$- 2\lambda E(S) + 2\lambda E(S)E\delta_n - E\delta_n$$

Letting $n \to \infty$ cancels EX_n^2 and EX_{n+1}^2, and $E\delta_n \to E\delta_\infty = 1 - \lambda E(S)$. This leaves

$$0 = \lambda^2 E(S^2) + 2EX_\infty(\lambda E(S) - 1) + 2\lambda E(S)$$
$$[1 - \lambda E(S)]$$

which gives the result upon solving for EX_∞.

(d) If customer n spends time W_n in system, then by Poisson arrivals $E[X_n|W_n] = \lambda W_n$. Hence, $EX_n = \lambda EW_n$ and letting $n \to \infty$ yields $EX_\infty = \lambda W = L$. It also follows since the average number as seen by a departure is always equal to the average number as seen by an arrival, which in this case equals L by Poisson arrivals.

39. (a) $a_0 = P_0$ due to Poisson arrivals. Assuming that each customer pays 1 per unit time while in service the cost identity (2.1) states that

Average number in service $= \lambda E[S]$

or

$$1 - P_0 = \lambda E[S]$$

(b) Since a_0 is the proportion of arrivals that have service distribution G_1 and $1 - a_0$ the proportion having service distribution G_2, the result follows.

(c) We have

$$P_0 = \frac{E[I]}{E[I] + E[B]}$$

and $E[I] = 1/\lambda$ and thus,

$$E[B] = \frac{1-P_0}{\lambda P_0} = \frac{E[S]}{1-\lambda E[S]}$$

Now from (a) and (b) we have

$$E[S] = (1-\lambda E[S])E[S_1] + \lambda E[S]E[S_2]$$

or

$$E[S] = \frac{E[S_1]}{1+\lambda E[S_1] + \lambda E[S_2]}$$

Substitution into $E[B] = E[S]/(1-\lambda E[S])$ now yields the result.

40. (a) (i) A little thought reveals that time to go from n to $n-1$ is independent of n.

(ii) $nE[B] = \dfrac{nE[S]}{1-\lambda E[S]}$

(b) (i) $E[T|N] = A + NE[B]$

(ii) $E[T] = A + E[N]E[B]$

$$= A + \frac{\lambda A E[S]}{1-\lambda E[S]} = \frac{A}{1-\lambda E[S]}$$

41. $E[N] = 2, E[N^2] = 9/2, E[S^2] = 2E^2[S] = 1/200$

$$W = \frac{\frac{1}{20}\frac{5}{2}/4 + 4 \cdot 2/400}{1-8/20} = \frac{41}{480}$$

$$W_Q = \frac{41}{480} - \frac{1}{20} = \frac{17}{480}$$

42. For notational ease, set $\alpha = \lambda_1/(\lambda_1+\lambda_2)$ = proportion of customers that are type I.

$\rho_1 = \lambda_1 E(S_1),\ \rho_2 E(S_2)$

Since the priority rule does not affect the amount of work in system compared to *FIFO* and $W^Q_{FIFO} = V$, we can use Equation (6.5) for W^Q_{FIFO}. Now $W_Q = \alpha W^1_Q + (1-\alpha)W^2_Q$ by averaging over both classes of customers. It is easy to check that W_Q then becomes

$$W_Q = \frac{\left[\lambda_1 ES_1^2 + \lambda_2 ES_2^2\right][\alpha(1-\rho_1-\rho_2)+(1-\alpha)]}{2(1-\rho_1-\rho_2)(1-\rho_1)}$$

which we wish to compare to

$$W^Q_{FIFO} = \frac{\left[\lambda_1 ES_1^2 + \lambda_2 ES_2^2\right]}{2(1-\rho_1-\rho_2)} \cdot \frac{(1-\rho_1)}{(1-\rho_1)}$$

Then $W_Q < W^Q_{FIFO} \Leftrightarrow \alpha(-\rho_1-\rho_2) \le -\rho_1$

$$\Leftrightarrow \alpha\rho_2 > (1-\alpha)\rho_1$$

$$\Leftrightarrow \frac{\lambda_1}{\lambda_1+\lambda_2} \cdot \lambda_2 E(S_2) > \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \lambda_1 ES_1$$

$$\Leftrightarrow E(S_2) > E(S_1)$$

43. Problem 42 shows that if $\mu_1 > \mu_2$, then serving 1's first minimizes average wait. But the same argument works if $c_1\mu_1 > c_2\mu_2$, i.e.,

$$\frac{E(S_1)}{c_1} < \frac{E(S_2)}{\mu_1}$$

44. (a) As long as the server is busy, work decreases by 1 per unit time and jumps by the service of an arrival even though the arrival may go directly into service. Since the bumped customer's remaining service does not change by being bumped, the total work in system remains the same as for nonpreemptive, which is the same as *FIFO*.

(b) As far as type I customers are concerned, the type II customers do not exist. A type I customer's delay only depends on other type I customers in system when he arrives. Therefore, $W^1_Q = V^1$ = amount of type I work in system.

By part (a), this is the same V^1 as for the nonpreemptive case (6.6). Therefore,

$$W^1_Q = \lambda_1 E(S_1)W^1_Q + \frac{\lambda_1 E\left[S_1^2\right]}{2}$$

or

$$W^1_Q = \frac{\lambda_1 E\left[S_1^2\right]}{2(1-\lambda_1 E(S_1))}$$

Note that this is the same as for an $M/G/1$ queue that has only type I customers.

(c) This does not account for the fact that some type II work in queue may result from customers that have been bumped from service, and so their average work would not be $E[S]$.

(d) If a type II arrival finds a bumped type II in queue, then a type I is in service. But in the nonpreemptive case, the only difference is that the type II bumped customer is served ahead of the type I, both of whom still go before the arrival. So the total amount of work found facing the arrival is the same in both cases. Hence,

$$W_Q^2 = \underbrace{V_Q^2 \text{ (nonpreemptive)}}_{\substack{\text{total work found} \\ \text{by type II}}} + \underbrace{E \text{ (extra time)}}_{\substack{\text{extra time due} \\ \text{to being bumped}}}$$

(e) As soon as a type II is bumped, he will not return to service until all type I's arriving during the first type I's service have departed, all further type I's who arrived during the additional type I services have departed, and so on. That is, each time a type II customer is bumped, he waits back in queue for one type I busy period. Because the type I customers do not see the type IIs at all, their busy period is just an $M/G_1/1$ busy period with mean

$$\frac{E(S_1)}{1-\lambda_1 E(S_1)}$$

So given that a customer is bumped N times, we have

$$E\{\text{extra time}|N\} = \frac{NE(S_1)}{1-\lambda_1 E(S_1)}$$

(f) Since arrivals are Poisson, $E[N|S_2] = \lambda_1 S_2$, and so $EN = \lambda_1 ES_2$.

(g) From (e) and (f), $E(\text{extra time}) = \dfrac{\lambda_1 E(S_2)E(S_1)}{1-\lambda_1 E(S_1)}$. Combining this with (e) gives the result.

45. By regarding any breakdowns that occur during a service as being part of that service, we see that this is an $M/G/1$ model. We need to calculate the first two moments of a service time. Now the time of a service is the time T until something happens (either a service completion or a breakdown) plus any additional time A. Thus,

$$E[S] = E[T + A]$$
$$= E[T] + E[A]$$

To compute $E[A]$ we condition upon whether the happening is a service or a breakdown. This gives

$$E[A] = E[A|\text{service}]\frac{\mu}{\mu+\alpha} + E[A|\text{breakdown}]\frac{\alpha}{\mu+\alpha}$$
$$= E[A|\text{breakdown}]\frac{\alpha}{\mu+\alpha}$$
$$= (1/\beta + E[S])\frac{\alpha}{\mu+\alpha}$$

Since, $E[T] = 1/(\alpha+\mu)$ we obtain

$$E[S] = \frac{1}{\alpha+\mu} + (1/\beta + E[S])\frac{\alpha}{\mu+\alpha}$$

or

$$E[S] = 1/\mu + \alpha/(\mu\beta)$$

We also need $E[S^2]$, which is obtained as follows.

$$E[S^2] = E[(T+A)^2]$$
$$= E[T^2] + 2E[AT] + E[A^2]$$
$$= E[T^2] + 2E[A]E[T] + E[A^2]$$

The independence of A and T follows because the time of the first happening is independent of whether the happening was a service or a breakdown. Now,

$$E[A^2] = E[A^2|\text{breakdown}]\frac{\alpha}{\mu+\alpha}$$
$$= \frac{\alpha}{\mu+\alpha}E[(\text{down time} + S^\alpha)^2]$$
$$= \frac{\alpha}{\mu+\alpha}\left\{E[\text{down}^2] + 2E[\text{down}]E[S] + E[S^2]\right\}$$
$$= \frac{\alpha}{\mu+\alpha}\left\{\frac{2}{\beta^2} + \frac{2}{\beta}\left[\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right] + E[S^2]\right\}$$

Hence,

$$E[S^2] = \frac{2}{(\mu+\beta)^2} + 2\left[\frac{\alpha}{\beta(\mu+\alpha)} + \frac{\alpha}{\mu+\alpha}\left(\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right)\right] + \frac{\alpha}{\mu+\alpha}\left\{\frac{2}{\beta^2} + \frac{2}{\beta}\left[\frac{1}{\mu} + \frac{\alpha}{\mu\beta}\right] + E[S^2]\right\}$$

Now solve for $E[S^2]$. The desired answer is

$$W_Q = \frac{\lambda E[S^2]}{2(1-\lambda E[S])}$$

In the above, S^α is the additional service needed after the breakdown is over. S^α has the same distribution as S. The above also uses the fact that the expected square of an exponential is twice the square of its mean.

Another way of calculating the moments of S is to use the representation

$$S = \sum_{i=1}^{N}(T_i + B_i) + T_{N+1}$$

where N is the number of breakdowns while a customer is in service, T_i is the time starting when service commences for the i^{th} time until a happening

occurs, and B_i is the length of the i^{th} breakdown. We now use the fact that, given N, all of the random variables in the representation are independent exponentials with the T_i having rate $\mu + \alpha$ and the B_i having rate β. This yields

$$E[S|N] = (N+1)/(\mu+\alpha) + N/\beta$$

$$Var(S|N) = (N+1)/(\mu+\alpha)^2 + N/\beta^2$$

Therefore, since $1 + N$ is geometric with mean $(\mu+\alpha)/\mu$ (and variance $\alpha(\alpha+\mu)/\mu^2$) we obtain

$$E[S] = 1/\mu + \alpha/(\mu\beta)$$

and, using the conditional variance formula,

$$Var(S) = [1/(\mu+\alpha) + 1/\beta]^2\alpha(\alpha+\mu)/\mu^2 \\ + 1/[\mu(\mu+\alpha)] + \alpha/\mu\beta^2)$$

46. β is to be the solution of Equation (7.3):

$$\beta = \int_0^\infty e^{-\mu t(1-\beta)}dG(t)$$

If $G(t) = 1 - e^{-\lambda t}(\lambda < \mu)$ and $\beta = \lambda/\mu$

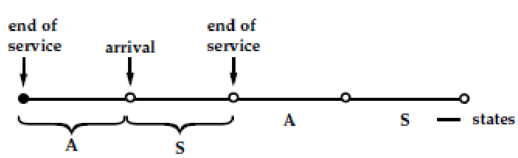
$$\int_0^\infty e^{-\mu t(1-\lambda/\mu)}dG(t) = \int_0^\infty e^{-\mu t(1-\lambda/\mu)}\lambda e^{-\lambda t}dt$$
$$= \int_0^\infty e^{-\mu t}dt$$
$$= \frac{\lambda}{\mu} = \beta$$

The equation checks out.

47. For $k = 1$, Equation (8.1) gives

$$P_0 = \frac{1}{1+\lambda E(S)} = \frac{(\lambda)}{(\lambda)+E(S)} \quad P_1 = \frac{\lambda(ES)}{1+\lambda E(S)}$$
$$= \frac{E(S)}{\lambda + E(S)}$$

One can think of the process as an *alteracting renewal process.* Since arrivals are Poisson, the time until the next arrival is still exponential with parameter λ.



The basic result of alternating renewal processes is that the limiting probabilities are given by

$$P\{\text{being in "state } S\text{"}\} = \frac{E(S)}{E(A)+E(S)} \text{ and}$$

$$P\{\text{being in "state } A\text{"}\} = \frac{E(A)}{E(A)+E(S)}$$

These are exactly the Erlang probabilities given above since $E[A] = 1/\lambda$. Note this uses Poisson arrivals in an essential way, viz., to know the distribution of time until the next arrival after a service is still exponential with parameter λ.

48. The easiest way to check that the P_i are correct is simply to check that they satisfy the balance equations:

$$\lambda p_0 = \mu p_1$$
$$(\lambda+\mu)p_1 = \lambda p_0 + 2\mu p_2$$
$$(\lambda+2\mu)p_2 = \lambda p_1 + 3\mu p_3$$
$$(\lambda+i\mu)p_i = \lambda p_{i-1} + (i+1)\mu p_{i+1}, \quad 0 < i \le k$$
$$(\lambda+k\mu)p_n = \lambda p_{n-1} + k\mu p_{n+1}, \quad n \ge k$$

or

$$p_1 = \frac{1}{\mu}P_0$$
$$p_2 = \frac{\lambda^2}{2\mu^2}P_0$$
$$p_i = \frac{\lambda^i}{\mu^1 i!}P_0, \quad 0 < i \le k$$
$$p_{k+n} = \frac{\lambda^{k+n}}{\mu^{k+n}k!k^n}P_0, \quad n \ge 1$$

In this form it is easy to check that the p_i of Equation (8.2) solves the balance equations.

49. $$P_3 = \frac{\dfrac{(\lambda E[S])^3}{3!}}{\sum_{j=0}^{3}\dfrac{(\lambda E[S])^j}{j!}}, \quad \lambda = 2, E[S] = 1$$
$$= \frac{8}{38}$$

50. (i) $P\{\text{arrival finds all servers busy}\}$

$$= \sum_{i=k}^{\infty} P_i = \frac{\left[\dfrac{\lambda}{\mu}\right]^k \dfrac{k\mu}{k\mu-\lambda}}{k!\sum_{i=0}^{k-1}\dfrac{\left[\dfrac{\lambda}{\mu}\right]^i}{1!} + \left[\dfrac{\lambda}{\mu}\right]^k \dfrac{k\mu}{k\mu-\lambda}}$$

(ii) $W = W_Q + 1/\mu$ where W_Q is as given by Equation (7.3), $L = \lambda W$.

51. Note that when all servers are busy, the departures are exponential with rate $k\mu$. Now see Problem 26.

52. S_n is the service time of the n^{th} customer. T_n is the time between the arrival of the n^{th} and $(n+1)^{\text{st}}$ customer.

53. $1/\mu_F < k/\mu_G$, where μ_F and μ_G are the respective means of F and G.

Reliability Theory

Exercises

1. Prove that, for any structure function ϕ,

$$\phi(\mathbf{x}) = x_i\phi(1_i, \mathbf{x}) + (1 - x_i)\phi(0_i, \mathbf{x})$$

where

$$(1_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n),$$
$$(0_i, \mathbf{x}) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

2. Show that
   (a) if $\phi(0, 0, \ldots, 0) = 0$ and $\phi(1, 1, \ldots, 1) = 1$, then

$$\min x_i \leqslant \phi(\mathbf{x}) \leqslant \max x_i$$

   (b) $\phi(\max(\mathbf{x}, \mathbf{y})) \geqslant \max(\phi(\mathbf{x}), \phi(\mathbf{y}))$
   (c) $\phi(\min(\mathbf{x}, \mathbf{y})) \leqslant \min(\phi(\mathbf{x}), \phi(\mathbf{y}))$
3. For any structure function, we define the dual structure ϕ^{D} by

$$\phi^{\mathrm{D}}(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x})$$

   (a) Show that the dual of a parallel (series) system is a series (parallel) system.
   (b) Show that the dual of a dual structure is the original structure.
   (c) What is the dual of a k-out-of-n structure?
   (d) Show that a minimal path (cut) set of the dual system is a minimal cut (path) set of the original structure.

*4. Write the structure function corresponding to the following:
   (a)

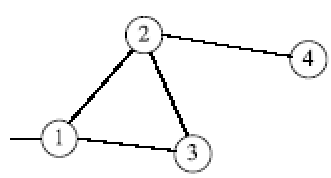


Figure 9.16

   (b)

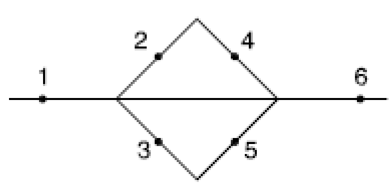


Figure 9.17

(c)

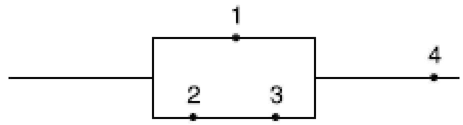


Figure 9.18

5. Find the minimal path and minimal cut sets for:
(a)

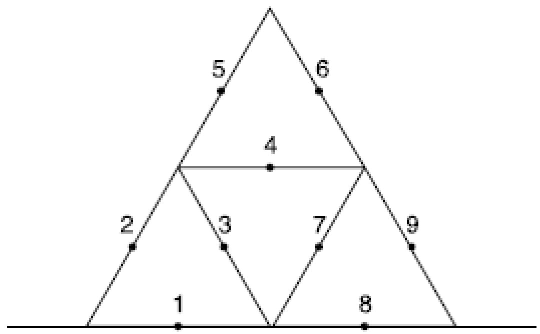


Figure 9.19

(b)

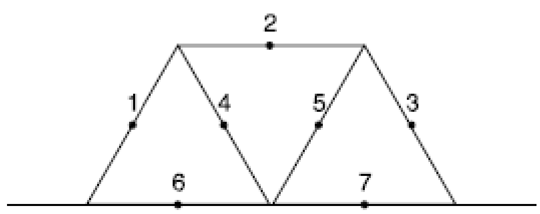


Figure 9.20

*6. The minimal path sets are {1, 2, 4}, {1, 3, 5}, and {5, 6}. Give the minimal cut sets.

7. The minimal cut sets are {1, 2, 3}, {2, 3, 4}, and {3, 5}. What are the minimal path sets?

8. Give the minimal path sets and the minimal cut sets for the structure given by Figure 9.21.

9. Component i is said to be *relevant* to the system if for some state vector $\mathbf{x}$,

$$\phi(1_i, \mathbf{x}) = 1, \qquad \phi(0_i, \mathbf{x}) = 0$$

Otherwise, it is said to be *irrelevant*.

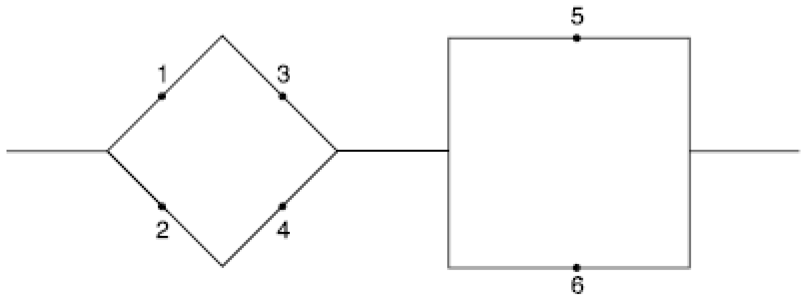


Figure 9.21

(a) Explain in words what it means for a component to be irrelevant.
(b) Let $A_1, \ldots, A_s$ be the minimal path sets of a system, and let S denote the set of components. Show that $S = \bigcup_{i=1}^{s} A_i$ if and only if all components are relevant.
(c) Let $C_1, \ldots, C_k$ denote the minimal cut sets. Show that $S = \bigcup_{i=1}^{k} C_i$ if and only if all components are relevant.

10. Let t_i denote the time of failure of the ith component; let $\tau_\phi(t)$ denote the time to failure of the system ϕ as a function of the vector $\mathbf{t} = (t_1, \ldots, t_n)$. Show that

$$\max_{1 \leqslant j \leqslant s} \min_{i \in A_j} t_i = \tau_\phi(\mathbf{t}) = \min_{1 \leqslant j \leqslant k} \max_{i \in C_j} t_i$$

where $C_1, \ldots, C_k$ are the minimal cut sets, and $A_1, \ldots, A_s$ the minimal path sets.

11. Give the reliability function of the structure of Exercise 8.

*12. Give the minimal path sets and the reliability function for the structure in Figure 9.22.

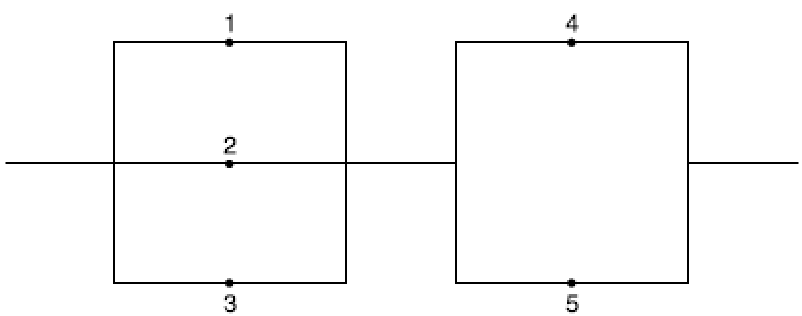


Figure 9.22

13. Let $r(\mathbf{p})$ be the reliability function. Show that

$$r(\mathbf{p}) = p_i r(1_i, \mathbf{p}) + (1 - p_i) r(0_i, \mathbf{p})$$

14. Compute the reliability function of the bridge system (see Figure 9.11) by conditioning upon whether or not component 3 is working.
15. Compute upper and lower bounds of the reliability function (using Method 2) for the systems given in Exercise 4, and compare them with the exact values when $p_i \equiv \frac{1}{2}$.
16. Compute the upper and lower bounds of $r(\mathbf{p})$ using both methods for the
    (a) two-out-of-three system and
    (b) two-out-of-four system.
    (c) Compare these bounds with the exact reliability when
        (i) $p_i \equiv 0.5$
        (ii) $p_i \equiv 0.8$
        (iii) $p_i \equiv 0.2$

*17. Let N be a nonnegative, integer-valued random variable. Show that

$$P\{N > 0\} \geqslant \frac{(E[N])^2}{E[N^2]}$$

and explain how this inequality can be used to derive additional bounds on a reliability function.

Hint:

$$\begin{aligned} E[N^2] &= E[N^2 \mid N > 0]P\{N > 0\} && \text{(Why?)} \\ &\geqslant (E[N \mid N > 0])^2 P\{N > 0\} && \text{(Why?)} \end{aligned}$$

Now multiply both sides by $P\{N > 0\}$.

18. Consider a structure in which the minimal path sets are $\{1, 2, 3\}$ and $\{3, 4, 5\}$.
    (a) What are the minimal cut sets?
    (b) If the component lifetimes are independent uniform $(0, 1)$ random variables, determine the probability that the system life will be less than $\frac{1}{2}$.
19. Let $X_1, X_2, \ldots, X_n$ denote independent and identically distributed random variables and define the order statistics $X_{(1)}, \ldots, X_{(n)}$ by

$$X_{(i)} \equiv i\text{th smallest of } X_1, \ldots, X_n$$

Show that if the distribution of X_j is IFR, then so is the distribution of $X_{(i)}$.

Hint: Relate this to one of the examples of this chapter.

20. Let F be a continuous distribution function. For some positive α, define the distribution function G by

$$\bar{G}(t) = (\bar{F}(t))^\alpha$$

Find the relationship between $\lambda_G(t)$ and $\lambda_F(t)$, the respective failure rate functions of G and F.

21. Consider the following four structures:

(i)

Figure 9.23

(ii)

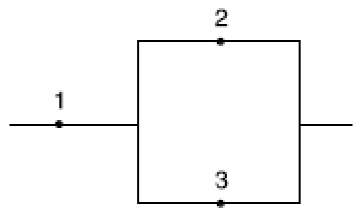


Figure 9.24

(iii)

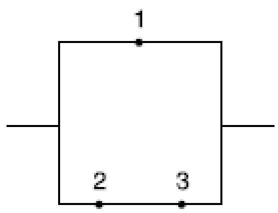


Figure 9.25

(iv)

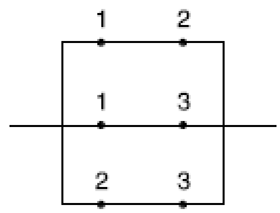


Figure 9.26

Let F_1, F_2, and F_3 be the corresponding component failure distributions; each of which is assumed to be IFR (increasing failure rate). Let F be the system failure distribution. All components are independent.

(a) For which structures is F necessarily IFR if $F_1 = F_2 = F_3$? Give reasons.
(b) For which structures is F necessarily IFR if $F_2 = F_3$? Give reasons.
(c) For which structures is F necessarily IFR if $F_1 \neq F_2 \neq F_3$? Give reasons.

*22. Let X denote the lifetime of an item. Suppose the item has reached the age of t. Let X_t denote its remaining life and define

$$\bar{F}_t(a) = P\{X_t > a\}$$

In words, $\bar{F}_t(a)$ is the probability that a t-year-old item survives an additional time a. Show that

(a) $\bar{F}_t(a) = \bar{F}(t + a)/\bar{F}(t)$ where F is the distribution function of X.
(b) Another definition of IFR is to say that F is IFR if $\bar{F}_t(a)$ decreases in t, for all a. Show that this definition is equivalent to the one given in the text when F has a density.

23. Show that if each (independent) component of a series system has an IFR distribution, then the system lifetime is itself IFR by

(a) showing that

$$\lambda_F(t) = \sum_i \lambda_i(t)$$

where $\lambda_F(t)$ is the failure rate function of the system; and $\lambda_i(t)$ the failure rate function of the lifetime of component i.

(b) using the definition of IFR given in Exercise 22.

24. Show that if F is IFR, then it is also IFRA, and show by counterexample that the reverse is not true.

*25. We say that ζ is a p-percentile of the distribution F if $F(\zeta) = p$. Show that if ζ is a p-percentile of the IFRA distribution F, then

$$\bar{F}(x) \leqslant e^{-\theta x}, \quad x \geqslant \zeta$$
$$\bar{F}(x) \geqslant e^{-\theta x}, \quad x \leqslant \zeta$$

where

$$\theta = \frac{-\log(1-p)}{\zeta}$$

26. Prove Lemma 9.3.

Hint: Let $x = y + \delta$. Note that $f(t) = t^\alpha$ is a concave function when $0 \leqslant \alpha \leqslant 1$, and use the fact that for a concave function $f(t+h) - f(t)$ is decreasing in t.

27. Let $r(p) = r(p, p, \ldots, p)$. Show that if $r(p_0) = p_0$, then

$$r(p) \geqslant p \quad \text{for } p \geqslant p_0$$
$$r(p) \leqslant p \quad \text{for } p \leqslant p_0$$

Hint: Use Proposition 9.2.

28. Find the mean lifetime of a series system of two components when the component lifetimes are respectively uniform on $(0, 1)$ and uniform on $(0, 2)$. Repeat for a parallel system.

29. Show that the mean lifetime of a parallel system of two components is

$$\frac{1}{\mu_1 + \mu_2} + \frac{\mu_1}{(\mu_1 + \mu_2)\mu_2} + \frac{\mu_2}{(\mu_1 + \mu_2)\mu_1}$$

when the first component is exponentially distributed with mean $1/\mu_1$ and the second is exponential with mean $1/\mu_2$.

*30. Compute the expected system lifetime of a three-out-of-four system when the first two component lifetimes are uniform on $(0, 1)$ and the second two are uniform on $(0, 2)$.

31. Show that the variance of the lifetime of a k-out-of-n system of components, each of whose lifetimes is exponential with mean θ, is given by

$$\theta^2 \sum_{i=k}^{n} \frac{1}{i^2}$$

32. In Section 9.6.1 show that the expected number of X_i that exceed c^* is equal to 1.

33. Let X_i be an exponential random variable with mean $8 + 2i$, for $i = 1, 2, 3$. Use the results of Section 9.6.1 to obtain an upper bound on $E[\max X_i]$, and then compare this with the exact result when the X_i are independent.

34. For the model of Section 9.7, compute for a k-out-of-n structure (i) the average up time, (ii) the average down time, and (iii) the system failure rate.

35. Prove the combinatorial identity

$$\binom{n-1}{i-1} = \binom{n}{i} - \binom{n}{i+1} + \cdots \pm \binom{n}{n}, \quad i \leqslant n$$

(a) by induction on i

(b) by a backwards induction argument on i—that is, prove it first for $i = n$, then assume it for $i = k$ and show that this implies that it is true for $i = k - 1$.

36. Verify Equation (9.36).

(b) It is clearly true when $i = n$, so assume it for i. We must show that

$$\begin{bmatrix} n-1 \\ i-2 \end{bmatrix} = \begin{bmatrix} n \\ i-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + \cdots \pm \begin{bmatrix} n \\ n \end{bmatrix}$$

which, using the induction hypothesis, reduces to

$$\begin{bmatrix} n-1 \\ i-2 \end{bmatrix} = \begin{bmatrix} n \\ i-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}$$

which is true.

Chapter 9

1. If $x_i = 0$, $\phi(x) = \phi(0_i, x)$.
   If $x_i = 1$, $\phi(x) = \phi(1_i, x)$.

2. (a) If $\min_i\ x_i = 1$, then $\underline{x} = (1, 1, \ldots, 1)$ and so $\phi(\underline{x}) = 1$.
   If $\max_i\ x_i = 0$, then $\underline{x} = (0, 0, \ldots, 0)$ and so $\phi(\underline{x}) = 0$.
   (b) $\max(x, y) \geq x \Rightarrow \phi(\max(x, y)) \geq \phi(x)$
   $\max(x, y) \geq y \Rightarrow \phi(\max(x, y)) \geq \phi(y)$
   $\therefore\ \phi(\max(x, y)) \geq \max(\phi(x), \phi(y))$.
   (c) Similar to (b).

3. (a) If ϕ is series, then $\phi(x) = \min_i x_i$ and so $\phi^D(\underline{x}) = 1 - \min_i\ (1 - x_i) = \max x_i$, and vice versa.
   (b) $$\begin{aligned}\phi^{D,D}(x) &= 1 - \phi^D(1 - x)\\ &= 1 - [1 - \phi(1 - (1 - x))]\\ &= \phi(x)\end{aligned}$$
   (c) An $n - k + 1$ of n.
   (d) Say $\{1, 2, \ldots, r\}$ is a minimal path set. Then $\phi(\underbrace{1, 1, \ldots,}_{r} 1, 0, 0, \ldots 0) = 1$, and so
   $\phi^D(\underbrace{0,\ 0, \ldots,}_{r} 0, 1, 1, \ldots, 1) = 1 - \phi(1, 1, \ldots,$ $1, 0, 0, \ldots, 0) = 0$, implying that $\{1, 2, \ldots, r\}$ is a cut set. We can easily show it to be minimal. For instance,
   $$\begin{aligned}&\phi^D(\underbrace{0, 0, \ldots,}_{r-1} 0, 1, 1, \ldots, 1)\\ &\quad = 1 - \phi(\underbrace{1, 1, \ldots,}_{r-1} 1, 0, 0, \ldots, 0) = 1,\end{aligned}$$
   since $\phi(\underbrace{1, 1, \ldots,}_{r-1} 1, 0, 0, \ldots, 0) = 0$ since $\{1, 2, \ldots, r - 1\}$ is not a path set.

4. (a) $\phi(x) = x_1 \max(x_2, x_3, x_4) x_5$
   (b) $\phi(x) = x_1 \max(x_2 x_4, x_3 x_5) x_6$
   (c) $\phi(x) = \max(x_1, x_2 x_3) x_4$

5. (a) Minimal path sets are
   $\{1, 8\}$, $\{1, 7, 9\}$, $\{1, 3, 4, 7, 8\}$, $\{1, 3, 4, 9\}$, $\{1, 3, 5, 6, 9\}$, $\{1, 3, 5, 6, 7, 8\}$, $\{2, 5, 6, 9\}$, $\{2, 5, 6, 7, 8\}$, $\{2, 4, 9\}$, $\{2, 4, 7, 8\}$, $\{2, 3, 7, 9\}$, $\{2, 3, 8\}$.
   Minimal cut sets are
   $\{1, 2\}$, $\{2, 3, 7, 8\}$, $\{1, 3, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{1, 3, 7, 9\}$, $\{4, 5, 7, 8\}$, $\{4, 6, 7, 8\}$, $\{8, 9\}$.

6. A minimal cut set has to contain at least one component of each minimal path set. There are 6 minimal cut sets:
   $\{1, 5\}$, $\{1, 6\}$, $\{2, 5\}$, $\{2, 3, 6\}$, $\{3, 4, 6\}$, $\{4, 5\}$.

7. $\{1, 4, 5\}$, $\{3\}$, $\{2, 5\}$.

8. The minimal path sets are $\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{2, 4, 5\}$, $\{2, 4, 6\}$. The minimal cut sets are $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{1, 4\}$, $\{2, 3\}$.

9. (a) A component is irrelevant if its functioning or not functioning can never make a difference as to whether or not the system functions.
   (b) Use the representation (2.1.1).
   (c) Use the representation (2.1.2).

10. The system fails the first time at least one component of each minimal path set is down—thus the left side of the identity. The right side follows by noting that the system fails the first time all of the components of at least one minimal cut set are failed.

11. $$\begin{aligned}r(p) &= P\{\text{either } x_1 x_3 = 1 \text{ or } x_2 x_4 = 1\}\\ &\quad P\{\text{either of 5 or 6 work}\}\\ &= (p_1 p_3 + p_2 p_4 - p_1 p_3 p_2 p_4)\\ &\quad (p_5 + p_6 - p_5 p_5)\end{aligned}$$

12. The minimal path sets are

$\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{3,4\}$, $\{3,5\}$.

With $q_i = 1 - P_i$, the structure function is

$$r(p) = P\{\text{either of } 1, 2, \text{ or } 3 \text{ works}\}$$
$$P\{\text{either of 4 or 5 works}\}$$
$$= (1 - q_1 q_2 q_3)(1 - q_4 q_5)$$

13. Taking expectations of the identity

$$\phi(X) = X_i \phi(1_i, X) + (1 - X_i)\phi(0_i, X)$$

noting the independence of X_i and $\phi(1_i, X)$ and of $\phi(0_i, X)$.

14. $$r(p) = p_3 P\{\max(X_1, X_2) = 1 = \max(X_4, X_5)\}$$
$$+ (1 - p_3)P\{\max(X_1 X_4, X_2 X_5) = 1\}$$
$$= p_3(p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5)$$
$$+ (1 - p_3)(p_1 p_4 + p_2 p_5 - p_1 p_4 p_2 p_5)$$

15. (a) $\frac{7}{32} \le \mathrm{r}\left[\frac{1}{2}\right] \le 1 - \left[\frac{7}{8}\right]^3 = \frac{169}{512}$

The exact value is $r(1/2) = 7/32$, which agrees with the minimal cut lower bound since the minimal cut sets $\{1\}$, $\{5\}$, $\{2,3,4\}$ do not overlap.

17. $$E[N^2] = E[N^2 | N > 0]P\{N > 0\}$$
$$\ge (E[N|N > 0])^2 P\{N > 0\}$$

since $E[X^2] \ge (E[X])^2$.

Thus,

$$E[N^2]P\{N > 0\} \ge (E[N|N > 0]P\{N > 0\})^2$$
$$= (E[N])^2$$

Let N denote the number of minimal path sets having all of its components functioning. Then $r(p) = P\{N > 0\}$.

Similarly, if we define N as the number of minimal cut sets having all of its components failed, then $1 - r(p) = P\{N > 0\}$.

In both cases we can compute expressions for $E[N]$ and $E[N^2]$ by writing N as the sum of indicator (i.e., Bernoulli) random variables. Then we can use the inequality to derive bounds on $r(p)$.

18. (a) $\{3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$.

(b) $P\left\{\text{system life} > \frac{1}{2}\right\} = r\left[\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right]$

Now $r(p) = p_1 p_2 p_3 + p_3 p_4 p_5 - p_1 p_2 p_3 p_4 p_5$

and so

$$P\left\{\text{system life} < \frac{1}{2}\right\} = 1 - \frac{1}{8} - \frac{1}{8} + \frac{1}{32}$$
$$= \frac{25}{32}$$

19. $X_{(i)}$ is the system life of an $n - i + 1$ of n system each having the life distribution F. Hence, the result follows from Example 5e.

20. The densities are related as follows.

$$g(t) = a[\bar{F}(t)]^{a-1} f(t)$$

Therefore,

$$\lambda_C(t) = a[\bar{F}(t)]^{a-1} f(t)/[\bar{F}(t)]^a$$
$$= a\, f(t)/\bar{F}(t)$$
$$= a\, \lambda_F(t)$$

21. (a) (i), (ii), (iv) – (iv) because it is two-of-three.

(b) (i) because it is series, (ii) because it can be thought of as being a series arrangement of 1 and the parallel system of 2 and 3, which as $F_2 = F_3$ is IFR.

(c) (i) because it is series.

22. (a) $$F_t(a) = P\{X > t + a \mid X > t\}$$
$$= \frac{P\{X > t + a\}}{P\{X > t\}} = \frac{\bar{F}(t + a)}{\bar{F}(t)}$$

(b) Suppose $\lambda(t)$ is increasing. Recall that

$$\bar{F}(t) = e^{-\int_0^t \lambda(s)ds}$$

Hence,

$\frac{\bar{F}(t + a)}{\bar{F}(t)} = e^{-\int_0^{t+a} \lambda(s)ds}$, which decreases in t since $\lambda(t)$ is increasing. To go the other way, suppose $\bar{F}(t + a)/\bar{F}(t)$ decreases in t. Now for a small

$$\bar{F}(t + a)/\bar{F}(t) = e^{-a\lambda(t)}$$

Hence, $e^{-a\lambda(t)}$ must decrease in t and thus $\lambda(t)$ increases.

23. (a) $\bar{F}(t) = \prod_{i=1}^{n} F_i(t)$

$$\lambda_F(t) = \frac{\frac{d}{dt}\bar{F}(t)}{\bar{F}(t)} = \frac{\sum_{j=1}^{n} F_j'(t) \prod_{i \neq j} F_j(t)}{\prod_{i=1}^{n} F_i(t)}$$

$$= \frac{\sum_{j=1}^{n} F_j'(t)}{F_j(t)}$$

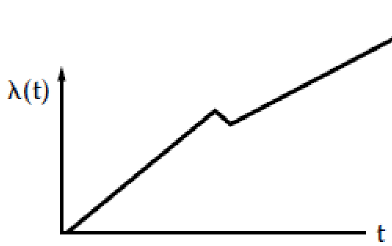
$$= \sum_{j=1}^{n} \lambda_j(t)$$

(b) $F_t(a) = P\{\text{additional life of } t\text{-year-old} > a\}$

$$= \frac{\prod_{1}^{n} F_i(t+a)}{F_i(t)}$$

where F_i is the life distribution for component i. The point being that as the system is series, it follows that knowing that it is alive at time t is equivalent to knowing that all components are alive at t.

24. It is easy to show that $\lambda(t)$ increasing implies that $\int_0^t \lambda(s)\,ds/t$ also increases. For instance, if we differentiate, we get $t\lambda(t) - \int_0^t \lambda(s)\,ds/t^2$, which is nonnegative since $\int_0^t \lambda(s)\,ds \le \int_0^t \lambda(t)\,dt = t\lambda(t)$. A counterexample is



25. For $x \ge \xi$,

$1 - p = 1 - F(\xi) = 1 - F(x(\xi/x)) \ge [1 - F(x)]^{\xi/x}$

since IFRA.

Hence,

$1 - F(x) \le (1-p)^{x/\xi} = e^{-\theta x}$

For $x \le \xi$,

$1 - F(x) = 1 - F(\xi(x/\xi)) \ge [1 - F(\xi)]^{x/\xi}$

since IFRA.

Hence,

$1 - F(x) \ge (1-p)^{x/\xi} = e^{-\theta x}$

26. Either use the hint in the text or the following, which does not assume a knowledge of concave functions.

To show: $h(y) \equiv \lambda^\alpha x^\alpha + (1-\lambda^\alpha)y^\alpha$
$\quad - (\lambda x + (1-\lambda)y)^\alpha \ge 0,$
$\quad 0 \le y \le x,$
$\quad$ where $0 \le \lambda \le 1,\ 0 \le \alpha \le 1.$

Note: $h(0) = 0$, assume $y > 0$, and let $g(y) = h(y)/y^a$

$$g(y) = \left[\frac{\lambda x}{y}\right]^\alpha + 1 - \lambda^\alpha - \left[\frac{\lambda x}{y} + 1 - \lambda\right]^\alpha$$

Let $z = x/y$. Now $g(y) \ge 0\ \forall\ 0 < y < x \Leftrightarrow f(z) \ge 0\ \forall\ z \ge 1$

where $f(z) = (\lambda z)^\alpha + 1 - \lambda^\alpha - (\lambda z + 1 - \lambda)^\alpha$.

Now $f(1) = 0$ and we prove the result by showing that $f'(z) \ge 0$ whenever $z > 1$. This follows since

$$f'(z) = \alpha\lambda(\lambda z)^{\alpha-1} - \alpha\lambda(\lambda z + 1 - \lambda)^{\alpha-1}$$

$$f'(z) \ge 0 \Leftrightarrow (\lambda z)^{\alpha-1} \ge (\lambda z + 1 - \lambda)^{\alpha-1}$$

$$\Leftrightarrow (\lambda z)^{1-\alpha} \le (\lambda z + 1 - \lambda)^{1-\alpha}$$

$$\Leftrightarrow \lambda z \le \lambda z + 1 - \lambda$$

$$\Leftrightarrow \lambda \le 1$$

27. If $p > p_0$, then $p = p_0{}^\alpha$ for some $a \in (0,1)$. Hence,

$r(p) = r(p_0{}^\alpha) \ge [r(p_0)]^\alpha = p_0{}^\alpha = p$

If $p < p_0$, then $p_0 = p^\alpha$ for some $a \in (0,1)$. Hence,

$p^\alpha = p_0 = r(p_0) = r(p^\alpha) \ge [r(p)]^\alpha$

28. (a) $\bar{F}(t) = (1-t)\left[\frac{2-t}{2}\right], \quad 0 \le t \le 1$

$$E[\text{lifetime}] = \frac{1}{2}\int_0^1 (1-t)(2-t)\,dt = \frac{5}{12}$$

(b) $\bar{F}(t) = \begin{cases} 1 - t^2/2, & 0 \le t \le 1 \\ 1 - t/2, & 1 \le t \le 2 \end{cases}$

$$E[\text{lifetime}] = \frac{1}{2}\int_0^1 (2-t^2)\,dt + \frac{1}{2}\int_1^2 (2-t)\,dt$$

$$= \frac{13}{12}$$

29. Let X denote the time until the first failure and let Y denote the time between the first and second failure. Hence, the desired result is

$$EX + EY = \frac{1}{\mu_1 + \mu_2} + EY$$

Now,

$$E[Y] = E[Y|\mu_1 \text{ component fails first}]\frac{\mu_1}{\mu_1 + \mu_2}$$
$$+ E[Y|\mu_2 \text{ component fails first}]\frac{\mu_2}{\mu_1 + \mu_2}$$
$$= \frac{1}{\mu_2}\frac{\mu_1}{\mu_1 + \mu_2} + \frac{1}{\mu_1}\frac{\mu_2}{\mu_1 + \mu_2}$$

30. $r(p) = p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4$
$-3p_1p_2p_3p_4$

$r(1 - \bar{F}(t))$

$$= \begin{cases} 2(1-t)^2(1-t/2) + 2(1-t)(1-t/2)^2 & \\ -3(1-t)^2(1-t/2)^2, & 0 \le t \le 1 \\ 0, & 1 \le t \le 2 \end{cases}$$

$$E[\text{lifetime}] = \int_0^1 \Big[2(1-t)^2(1-t/2)$$
$$+ 2(1-t)(1-t/2)^2$$
$$- 3(1-t)^2(1-t/2)^2\Big]dt$$
$$= \frac{31}{60}$$

31. Use the remark following Equation (6.3).

32. Let I_i equal 1 if $X_i > c^\alpha$ and let it be 0 otherwise. Then,

$$E\left[\sum_{i=1}^n I_i\right] = \sum_{i=1}^n E[I_i] = \sum_{i=1}^n P\{X_i > c^\infty\}$$

33. The exact value can be obtained by conditioning on the ordering of the random variables. Let M denote the maximum, then with $A_{i,j,k}$ being the even that $X_i < X_j < X_k$, we have that

$$E[M] = \sum E[M|A_{i,j,k}]P(A_{i,j,k})$$

where the preceding sum is over all 6 possible permutations of $1, 2, 3$. This can now be evaluated by using

$$P(A_{i,j,k}) = \frac{\lambda_i}{\lambda_i + \lambda_j + \lambda_k}\frac{\lambda_j}{\lambda_j + \lambda_k}$$
$$E[M|A_{i,j,k}] = \frac{1}{\lambda_i + \lambda_j + \lambda_k} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k}$$

35. (a) It follows when $i = 1$ since $0 = (1-1)^n = 1 - \left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right] \cdots \pm \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]$. So assume it true for i and consider $i + 1$. We must show that

$$\begin{bmatrix} n-1 \\ i \end{bmatrix} = \begin{bmatrix} n \\ i+1 \end{bmatrix} - \begin{bmatrix} n \\ i+2 \end{bmatrix} + \cdots \pm \begin{bmatrix} n \\ n \end{bmatrix}$$

which, using the induction hypothesis, is equivalent to

$$\begin{bmatrix} n-1 \\ i \end{bmatrix} = \begin{bmatrix} n \\ i \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}$$

which is easily seen to be true.

(b) It is clearly true when $i = n$, so assume it for i. We must show that

$$\begin{bmatrix} n-1 \\ i-2 \end{bmatrix} = \begin{bmatrix} n \\ i-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + \cdots \pm \begin{bmatrix} n \\ n \end{bmatrix}$$

which, using the induction hypothesis, reduces to

$$\begin{bmatrix} n-1 \\ i-2 \end{bmatrix} = \begin{bmatrix} n \\ i-1 \end{bmatrix} - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}$$

which is true.

Brownian Motion and Stationary Processes

Exercises

In the following exercises $\{B(t),\ t \geqslant 0\}$ is a standard Brownian motion process and T_a denotes the time it takes this process to hit a.

*1. What is the distribution of $B(s) + B(t)$, $s \leqslant t$?

2. Compute the conditional distribution of $B(s)$ given that $B(t_1) = A$ and $B(t_2) = B$, where $0 < t_1 < s < t_2$.

*3. Compute $E[B(t_1)B(t_2)B(t_3)]$ for $t_1 < t_2 < t_3$.

4. Show that

$$P\{T_a < \infty\} = 1,$$
$$E[T_a] = \infty, \qquad a \neq 0$$

*5. What is $P\{T_1 < T_{-1} < T_2\}$?

6. Suppose you own one share of a stock whose price changes according to a standard Brownian motion process. Suppose that you purchased the stock at a price $b + c$,

$c > 0$, and the present price is b. You have decided to sell the stock either when it reaches the price $b + c$ or when an additional time t goes by (whichever occurs first). What is the probability that you do not recover your purchase price?

7. Compute an expression for

$$P\left\{\max_{t_1 \leqslant s \leqslant t_2} B(s) > x\right\}$$

8. Consider the random walk that in each Δt time unit either goes up or down the amount $\sqrt{\Delta t}$ with respective probabilities p and $1 - p$, where $p = \frac{1}{2}(1 + \mu\sqrt{\Delta t})$.
   (a) Argue that as $\Delta t \to 0$ the resulting limiting process is a Brownian motion process with drift rate μ.
   (b) Using part (a) and the results of the gambler's ruin problem (Section 4.5.1), compute the probability that a Brownian motion process with drift rate μ goes up A before going down B, $A > 0$, $B > 0$.
9. Let $\{X(t), t \geqslant 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. What is the joint density function of $X(s)$ and $X(t)$, $s < t$?

*10. Let $\{X(t), t \geqslant 0\}$ be a Brownian motion process with drift coefficient μ and variance parameter σ^2. What is the conditional distribution of $X(t)$ given that $X(s) = c$ when
   (a) $s < t$?
   (b) $t < s$?

11. Consider a process whose value changes every h time units; its new value being its old value multiplied either by the factor $e^{\sigma\sqrt{h}}$ with probability $p = \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{h})$, or by the factor $e^{-\sigma\sqrt{h}}$ with probability $1 - p$. As h goes to zero, show that this process converges to geometric Brownian motion with drift coefficient μ and variance parameter σ^2.
12. A stock is presently selling at a price of \$50 per share. After one time period, its selling price will (in present value dollars) be either \$150 or \$25. An option to purchase y units of the stock at time 1 can be purchased at cost cy.
   (a) What should c be in order for there to be no sure win?
   (b) If $c = 4$, explain how you could guarantee a sure win.
   (c) If $c = 10$, explain how you could guarantee a sure win.
   (d) Use the arbitrage theorem to verify your answer to part (a).
13. Verify the statement made in the remark following Example 10.2.
14. The present price of a stock is 100. The price at time 1 will be either 50, 100, or 200. An option to purchase y shares of the stock at time 1 for the (present value) price ky costs cy.
   (a) If $k = 120$, show that an arbitrage opportunity occurs if and only if $c > 80/3$.
   (b) If $k = 80$, show that there is not an arbitrage opportunity if and only if $20 \leqslant c \leqslant 40$.
15. The current price of a stock is 100. Suppose that the logarithm of the price of the stock changes according to a Brownian motion process with drift coefficient $\mu = 2$ and variance parameter $\sigma^2 = 1$. Give the Black-Scholes cost of an option to buy the stock at time 10 for a cost of

(a) 100 per unit.
(b) 120 per unit.
(c) 80 per unit.
Assume that the continuously compounded interest rate is 5 percent.

A stochastic process $\{Y(t),\ t \geqslant 0\}$ is said to be a *Martingale* process if, for $s < t$,

$$E[Y(t)|Y(u),\ 0 \leqslant u \leqslant s] = Y(s)$$

16. If $\{Y(t),\ t \geqslant 0\}$ is a Martingale, show that

$$E[Y(t)] = E[Y(0)]$$

17. Show that standard Brownian motion is a Martingale.
18. Show that $\{Y(t),\ t \geqslant 0\}$ is a Martingale when

$$Y(t) = B^2(t) - t$$

What is $E[Y(t)]$?

Hint: First compute $E[Y(t)|B(u),\ 0 \leqslant u \leqslant s]$.

*19. Show that $\{Y(t), t \geqslant 0\}$ is a Martingale when

$$Y(t) = \exp\{cB(t) - c^2 t/2\}$$

where c is an arbitrary constant. What is $E[Y(t)]$?

An important property of a Martingale is that if you continually observe the process and then stop at some time T, then, subject to some technical conditions (which will hold in the problems to be considered),

$$E[Y(T)] = E[Y(0)]$$

The time T usually depends on the values of the process and is known as a *stopping time* for the Martingale. This result, that the expected value of the stopped Martingale is equal to its fixed time expectation, is known as the *Martingale stopping theorem*.

*20. Let

$$T = \text{Min}\{t\colon B(t) = 2 - 4t\}$$

That is, T is the first time that standard Brownian motion hits the line $2 - 4t$. Use the Martingale stopping theorem to find $E[T]$.

21. Let $\{X(t),\ t \geqslant 0\}$ be Brownian motion with drift coefficient μ and variance parameter σ^2. That is,

$$X(t) = \sigma B(t) + \mu t$$

Let $\mu > 0$, and for a positive constant x let

$$\begin{aligned} T &= \text{Min}\{t\colon X(t) = x\} \\ &= \text{Min}\left\{t\colon B(t) = \frac{x - \mu t}{\sigma}\right\} \end{aligned}$$

That is, T is the first time the process $\{X(t),\ t \geqslant 0\}$ hits x. Use the Martingale stopping theorem to show that

$$E[T] = x/\mu$$

22. Let $X(t) = \sigma B(t) + \mu t$, and for given positive constants A and B, let p denote the probability that $\{X(t),\ t \geqslant 0\}$ hits A before it hits $-B$.
    (a) Define the stopping time T to be the first time the process hits either A or $-B$. Use this stopping time and the Martingale defined in Exercise 19 to show that

    $$E[\exp\{c(X(T) - \mu T)/\sigma - c^2 T/2\}] = 1$$

    (b) Let $c = -2\mu/\sigma$, and show that

    $$E[\exp\{-2\mu X(T)/\sigma\}] = 1$$

    (c) Use part (b) and the definition of T to find p.

    Hint: What are the possible values of $\exp\{-2\mu X(T)/\sigma^2\}$?
23. Let $X(t) = \sigma B(t) + \mu t$, and define T to be the first time the process $\{X(t),\ t \geqslant 0\}$ hits either A or $-B$, where A and B are given positive numbers. Use the Martingale stopping theorem and part (c) of Exercise 22 to find $E[T]$.

*24. Let $\{X(t),\ t \geqslant 0\}$ be Brownian motion with drift coefficient μ and variance parameter σ^2. Suppose that $\mu > 0$. Let $x > 0$ and define the stopping time T (as in Exercise 21) by

$$T = \text{Min}\{t\colon X(t) = x\}$$

Use the Martingale defined in Exercise 18, along with the result of Exercise 21, to show that

$$\text{Var}(T) = x\sigma^2/\mu^3$$

25. Compute the mean and variance of
    (a) $\int_0^1 t\, dB(t)$
    (b) $\int_0^1 t^2\, dB(t)$
26. Let $Y(t) = tB(1/t)$, $t > 0$ and $Y(0) = 0$.
    (a) What is the distribution of $Y(t)$?
    (b) Compare $\text{Cov}(Y(s), Y(t))$.
    (c) Argue that $\{Y(t),\ t \geqslant 0\}$ is a standard Brownian motion process.

*27. Let $Y(t) = B(a^2 t)/a$ for $a > 0$. Argue that $\{Y(t)\}$ is a standard Brownian motion process.

28. For $s < t$, argue that $B(s) - \frac{s}{t}B(t)$ and $B(t)$ are independent.
29. Let $\{Z(t),\ t \geqslant 0\}$ denote a Brownian bridge process. Show that if

$$Y(t) = (t+1)Z(t/(t+1))$$

then $\{Y(t),\ t \geqslant 0\}$ is a standard Brownian motion process.

30. Let $X(t) = N(t+1) - N(t)$ where $\{N(t),\ t \geqslant 0\}$ is a Poisson process with rate λ. Compute

$$\mathrm{Cov}[X(t), X(t+s)]$$

*31. Let $\{N(t), t \geqslant 0\}$ denote a Poisson process with rate λ and define $Y(t)$ to be the time from t until the next Poisson event.

(a) Argue that $\{Y(t),\ t \geqslant 0\}$ is a stationary process.

(b) Compute $\mathrm{Cov}[Y(t), Y(t+s)]$.

32. Let $\{X(t), -\infty < t < \infty\}$ be a weakly stationary process having covariance function $R_X(s) = \mathrm{Cov}[X(t), X(t+s)]$.

(a) Show that

$$\mathrm{Var}(X(t+s) - X(t)) = 2R_X(0) - 2R_X(t)$$

(b) If $Y(t) = X(t+1) - X(t)$ show that $\{Y(t),\ -\infty < t < \infty\}$ is also weakly stationary having a covariance function $R_Y(s) = \mathrm{Cov}[Y(t), Y(t+s)]$ that satisfies

$$R_Y(s) = 2R_X(s) - R_X(s-1) - R_X(s+1)$$

33. Let Y_1 and Y_2 be independent unit normal random variables and for some constant w set

$$X(t) = Y_1 \cos wt + Y_2 \sin wt, \qquad -\infty < t < \infty$$

(a) Show that $\{X(t)\}$ is a weakly stationary process.

(b) Argue that $\{X(t)\}$ is a stationary process.

34. Let $\{X(t),\ -\infty < t < \infty\}$ be weakly stationary with covariance function $R(s) = \mathrm{Cov}(X(t), X(t+s))$ and let $\tilde{R}(w)$ denote the power spectral density of the process.

(i) Show that $\tilde{R}(w) = \tilde{R}(-w)$. It can be shown that

$$R(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{R}(w)e^{iws}\,dw$$

(ii) Use the preceding to show that

$$\int_{-\infty}^{\infty} \tilde{R}(w)\,dw = 2\pi E[X^2(t)]$$

Chapter 10

1. $X(s) + X(t) = 2X(s) + X(t) - X(s)$.

   Now $2X(s)$ is normal with mean 0 and variance $4s$ and $X(t) - X(s)$ is normal with mean 0 and variance $t - s$. As $X(s)$ and $X(t) - X(s)$ are independent, it follows that $X(s) + X(t)$ is normal with mean 0 and variance $4s + t - s = 3s + t$.

2. The conditional distribution $X(s) - A$ given that $X(t_1) = A$ and $X(t_2) = B$ is the same as the conditional distribution of $X(s - t_1)$ given that $X(0) = 0$ and $X(t_2 - t_1) = B - A$, which by Equation (10.4) is normal with mean $\frac{s - t_1}{t_2 - t_1}(B - A)$ and variance $\frac{(s - t_1)}{t_2 - t_1}(t_2 - s)$. Hence the desired conditional distribution is normal with mean $A + \frac{(s - t_1)(B - A)}{t_2 - t_1}$ and variance $\frac{(s - t_1)(t_2 - s)}{t_2 - t_1}$.

3. $E[X(t_1)X(t_2)X(t_3)]$

   $$\begin{aligned}
   &= E[E[X(t_1)X(t_2)X(t_3) \mid X(t_1), X(t_2)]] \\
   &= E[X(t_1)X(t_2)E[X(t_3) \mid X(t_1), X(t_2)]] \\
   &= E[X(t_1)X(t_2)X(t_2)] \\
   &= E[E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \\
   &= E[X(t_1)E[X^2(t_2) \mid X(t_1)]] \qquad (*) \\
   &= E[X(t_1)\{(t_2 - t_1) + X^2(t_1)\}] \\
   &= E[X^3(t_1)] + (t_2 - t_1)E[X(t_1)] \\
   &= 0
   \end{aligned}$$

   where the equality $(*)$ follows since given $X(t_1)$, $X(t_2)$ is normal with mean $X(t_1)$ and variance $t_2 - t_1$. Also, $E[X^3(t)] = 0$ since $X(t)$ is normal with mean 0.

4. (a) $P\{T_a < \infty\} = \lim_{t\to\infty} P\{T_a \le t\}$

   $$= \frac{2}{\sqrt{2r}}\int_0^\infty e^{-y^2/2}dy \quad \text{by (10.6)}$$

   $$= 2P\{N(0,1) > 0\} = 1$$

   Part (b) can be proven by using

   $$E[T_a] = \int_0^\infty P\{T_a > t\}dt$$

   in conjunction with Equation (10.7).

5. $P\{T_1 < T_{-1} < T_2\} = P\{\text{hit 1 before } -1 \text{ before } 2\}$

   $$\begin{aligned}
   &= P\{\text{hit 1 before } -1\} \\
   &\quad \times P\{\text{hit } -1 \text{ before } 2 \mid \text{hit 1 before } -1\} \\
   &= \frac{1}{2}P\{\text{down 2 before up 1}\} \\
   &= \frac{1}{2}\frac{1}{3} = \frac{1}{6}
   \end{aligned}$$

   The next to last equality follows by looking at the Brownian motion when it first hits 1.

6. The probability of recovering your purchase price is the probability that a Brownian motion goes up c by time t. Hence the desired probability is

   $$1 - P\{\max_{0\le s\le t} X(s) \ge c\} = 1 - \frac{2}{\sqrt{2\pi t}}\int_{c/\sqrt{t}}^\infty e^{-y^2/2}dy$$

7. Let $M = \{\max_{t_1\le s\le t_2} X(s) > x\}$. Condition on $X(t_1)$ to obtain

   $$P(M) = \int_{-\infty}^\infty P(M|X(t_1) = y)\frac{1}{\sqrt{2\pi t_1}}e^{-y^2/2t_1}dy$$

   Now, use that

   $$P(M|X(t_1) = y) = 1, \quad y \ge x$$

   and, for $y < x$

   $$\begin{aligned}
   P(M|X(t_1) = y) &= P\{\max_{0<s<t_2-t_1} X(s) > x - y\} \\
   &= 2P\{X(t_2 - t_1) > x - y\}
   \end{aligned}$$

8. (a) Let $X(t)$ denote the position at time t. Then

   $$X(t) = \sqrt{\Delta t}\sum_{i=1}^{[t/\Delta t]} X_i$$

where

$$X_i = \begin{cases} +1, & \text{if } i^{th} \text{ step is up} \\ -1, & \text{if } i^{th} \text{ step is down} \end{cases}$$

As

$$E[X_1] = p - 1(1-p)$$
$$= 2p - 1$$
$$= \mu\sqrt{\Delta t}$$

and

$$Var(X_i) = E\left[X_i^2\right] - (E[X_i])^2$$
$$= 1 - \mu^2\Delta t \quad \text{since } X_i^2 = 1$$

we obtain

$$E[X(t)] = \sqrt{\Delta t}\left[\frac{t}{\Delta t}\right]\mu\sqrt{\Delta t}$$
$$\to \mu t \text{ as } \Delta t \to 0$$

$$Var(X(t)) = \Delta t\left[\frac{t}{\Delta t}\right](1 - \mu^2\Delta t)$$
$$\to t \text{ as } \Delta t \to 0.$$

(b) By the gambler's ruin problem the probability of going up A before going down B is

$$\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

when each step is either up 1 or down 1 with probabilities p and $q = 1-p$. (This is the probability that a gambler starting with B will reach his goal of $A + B$ before going broke.) Now, when $p = \frac{1}{2}(1 + \mu\sqrt{\Delta t}), q = 1 - p = \frac{1}{2}(1 - \mu\sqrt{\Delta t})$ and so $q/p = \frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}$. Hence, in this case the probability of going up $A/\sqrt{\Delta t}$ before going down $B/\sqrt{\Delta t}$ (we divide by $\sqrt{\Delta t}$ since each step is now of this size) is

$$(*) \qquad \frac{1 - \left[\dfrac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}\right]^{B/\sqrt{\Delta t}}}{1 - \left[\dfrac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}\right]^{(A+B/\sqrt{\Delta t})}}$$

Now

$$\lim_{\Delta t \to 0}\left[\frac{1-\mu\sqrt{\Delta t}}{1+\mu\sqrt{\Delta t}}\right]^{1/\sqrt{\Delta t}} = \lim_{h\to 0}\left[\frac{1-\mu h}{1+\mu h}\right]^{1/h}$$
$$= \lim_{n\to\infty}\left[\frac{1-\frac{\mu}{n}}{1+\frac{\mu}{n}}\right]^n$$

by $n = 1/h$

$$= \frac{e^{-\mu}}{e^{\mu}} = e^{-2\mu}$$

where the last equality follows from

$$\lim_{n\to\infty}\left[1+\frac{x}{n}\right]^n = e^x$$

Hence the limiting value of $(*)$ as $\Delta t \to 0$ is

$$\frac{1-e^{-2\mu B}}{1-e^{-2\mu(A+B)}}$$

11. Let $X(t)$ denote the value of the process at time $t = nh$. Let $X_i = 1$ if the i^{th} change results in the state value becoming larger, and let $X_i = 0$ otherwise. Then, with $u = e^{\sigma\sqrt{h}}$, $d = e^{-\sigma\sqrt{h}}$

$$X(t) = X(0)u^{\sum_{i=1}^n X_i}d^{n-\sum_{i=1}^n X_i}$$
$$= X(0)d^n\left(\frac{u}{d}\right)^{\sum_{i=1}^n X_i}$$

Therefore,

$$\log\left(\frac{X(t)}{X(0)}\right) = n\log(d) + \sum_{i=1}^n X_i\log(u/d)$$
$$= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\sum_{i=1}^{t/h} X_i$$

By the central limit theorem, the preceding becomes a normal random variable as $h \to 0$. Moreover, because the X_i are independent, it is easy to see that the process has independent increments. Also,

$$E\left[\log\left(\frac{X(t)}{X(0)}\right)\right]$$
$$= -\frac{t}{h}\sigma\sqrt{h} + 2\sigma\sqrt{h}\frac{t}{h}\frac{1}{2}(1+\frac{\mu}{\sigma}\sqrt{h})$$
$$= \mu t$$

and

$$Var\left[\log\left(\frac{X(t)}{X(0)}\right)\right] = 4\sigma^2 h\frac{t}{h}p(1-p)$$
$$\to \sigma^2 t$$

where the preceding used that $p \to 1/2$ as $h \to 0$.

12. If we purchase x units of the stock and y of the option then the value of our holdings at time 1 is

$$\text{value} = \begin{cases} 150x + 25y & \text{if price is } 150 \\ 25x & \text{if price is } 25 \end{cases}$$

So if

$$150x + 25y = 25x, \text{ or } y = -5x$$

then the value of our holdings is $25x$ no matter what the price is at time 1. Since the cost of purchasing x units of the stock and $-5x$ units of options is $50x - 5xc$ it follows that our profit from such a purchase is

$$25x - 50x + 5xc = x(5c - 25)$$

(a) If $c = 5$ then there is no sure win.

(b) Selling $|x|$ units of the stock and buying $-5|x|$ units of options will realize a profit of $5|x|$ no matter what the price of the stock is at time 1. (That is, buy x units of the stock and $-5x$ units of the options for $x < 0$.)

(c) Buying x units of the stock and $-5x$ units of options will realize a positive profit of $25x$ when $x > 0$.

(d) Any probability vector $(p, 1-p)$ on $(150, 25)$, the possible prices at time 1, under which buying the stock is a fair bet satisfies the following:

$$50 = p(150) + (1-p)(25)$$

or

$$p = 1/5$$

That is, $(1/5, 4/5)$ is the only probability vector that makes buying the stock a fair bet. Thus, in order for there to be no arbitrage possibility, the price of an option must be a fair bet under this probability vector. This means that the cost c must satisfy

$$c = 25(1/5) = 5$$

13. If the outcome is i then our total winnings are

$$\begin{aligned} x_i o_i - \sum_{j \neq i} x_j &= \frac{o_i(1+o_i)^{-1} - \sum_{j\neq i}(1+o_j)^{-1}}{1 - \sum_k (1+o_k)^{-1}} \\ &= \frac{(1+o_i)(1+o_i)^{-1} - \sum_j (1+o_j)^{-1}}{1 - \sum_k (1+o_k)^{-1}} \\ &= 1 \end{aligned}$$

14. Purchasing the stock will be a fair bet under probabilities $(p_1,\ p_2,\ 1-p_1-p_2)$ on $(50, 100, 200)$, the set of possible prices at time 1, if

$$100 = 50p_1 + 100p_2 + 200(1-p_1-p_2)$$

or equivalently, if

$$3p_1 + 2p_2 = 2$$

(a) The option bet is also fair if the probabilities also satisfy

$$c = 80(1-p_1-p_2)$$

Solving this and the equation $3p_1 + 2p_2 = 2$ for p_1 and p_2 gives the solution

$$p_1 = c/40,\ p_2 = (80-3c)/80$$

$$1 - p_1 - p_2 = c/80$$

Hence, no arbitrage is possible as long as these p_i all lie between 0 and 1. However, this will be the case if and only if

$$80 \geq 3c$$

(b) In this case, the option bet is also fair if

$$c = 20p_2 + 120(1-p_1-p_2)$$

Solving in conjunction with the equation $3p_1 + 2p_2 = 2$ gives the solution

$$p_1 = (c-20)/30,\ p_2 = (40-c)/20$$

$$1 - p_1 - p_2 = (c-20)/60$$

These will all be between 0 and 1 if and only if $20 \leq c \leq 40$.

15. The parameters of this problem are

$$\sigma = .05, \quad \sigma = 1, \quad x_o = 100, \quad t = 10.$$

(a) If $K = 100$ then from Equation (4.4)

$$\begin{aligned} b &= [.5 - 5 - \log(100/100)]/\sqrt{10} \\ &= -4.5\sqrt{10} = -1.423 \end{aligned}$$

and

$$\begin{aligned} c &= 100\phi(\sqrt{10} - 1.423) - 100e^{-.5}\phi(-1.423) \\ &= 100\phi(1.739) - 100e^{-.5}[1 - \phi(1.423)] \\ &= 91.2 \end{aligned}$$

The other parts follow similarly.

16. Taking expectations of the defining equation of a Martingale yields

$$E[Y(s)] = E[E[Y(t)/Y(u),\ 0 \le u \le s]] = E[Y(t)]$$

That is, $E[Y(t)]$ is constant and so is equal to $E[Y(0)]$.

17. $E\,[B(t)|B(u),\ 0 \le u \le s]$

$$= E[B(s) + B(t) - B(s)|B(u),\ 0 \le u \le s]$$

$$= E[B(s)|B(u),\ 0 \le u \le s]$$

$$+\ E[B(t) - B(s)|B(u), 0 \le u \le s]$$

$= B(s) + E[B(t) - B(s)]$ by independent increments

$$= B(s)$$

18. $E[B^2(t)|B(u),\ 0 \le u \le s] = E[B^2(t)|B(s)]$

where the above follows by using independent increments as was done in Problem 17. Since the conditional distribution of $B(t)$ given $B(s)$ is normal with mean $B(s)$ and variance $t - s$ it follows that

$$E[B^2(t)|B(s)] = B^2(s) + t - s$$

Hence,

$$E[B^2(t) - t|B(u),\ 0 \le u \le s] = B^2(s) - s$$

Therefore, the conditional expected value of $B^2(t) - t$, given all the values of $B(u)$, $0 \le u \le s$, depends only on the value of $B^2(s)$. From this it intuitively follows that the conditional expectation given the squares of the values up to time s is also $B^2(s) - s$. A formal argument is obtained by conditioning on the values $B(u)$, $0 \le u \le s$ and using the above. This gives

$$E[B^2(t) - t|B^2(u),\ 0 \le u \le s]$$

$$= E\,[E[B^2(t) - t|B(u),\ 0 \le u \le s]|B^2(u),$$

$$0 \le u \le s]$$

$$= E[B^2(s) - s|B^2(u),\ 0 \le u \le s]$$

$$= B^2(s) - s$$

which proves that $\{B^2(t) - t,\ t \ge 0\}$ is a Martingale. By letting $t = 0$, we see that

$$E[B^2(t) - t] = E[B^2(0)] = 0$$

19. Since knowing the value of $Y(t)$ is equivalent to knowing $B(t)$ we have

$$E[Y(t)|Y(u),\ 0 \le u \le s]$$

$$= e^{-c^2t/2}E[e^{cB(t)}|B(u),\ 0 \le u \le s]$$

$$= e^{-c^2t/2}E[e^{cB(t)}|B(s)]$$

Now, given $B(s)$, the conditional distribution of $B(t)$ is normal with mean $B(s)$ and variance $t - s$. Using the formula for the moment generating function of a normal random variable we see that

$$e^{-c^2t/2}E[e^{cB(t)}|B(s)]$$

$$= e^{-c^2t/2}e^{cB(s)+(t-s)c^2/2}$$

$$= e^{-c^2s/2}e^{cB(s)}$$

$$= Y(s)$$

Thus, $\{Y(t)\}$ is a Martingale.

$$E[Y(t)] = E[Y(0)] = 1$$

20. By the Martingale stopping theorem

$$E[B(T)] = E[B(0)] = 0$$

However, $B(T) = 2 - 4T$ and so

$$2 - 4E[T] = 0$$

or, $E[T] = 1/2$

21. By the Martingale stopping theorem

$$E[B(T)] = E[B(0)] = 0$$

But, $B(T) = (x - \mu T)/\sigma$ and so

$$E[(x - \mu T)/\sigma] = 0$$

or

$$E[T] = x/\mu$$

22. (a) It follows from the results of Problem 19 and the Martingale stopping theorem that

$$E[\exp\{cB(T) - c^2T/2\}]$$

$$= E[\exp\{cB(0)\}] = 1$$

Since $B(T) = [X(T) - \mu T]/\sigma$ part (a) follows.

(b) This follows from part (a) since

$$-2\mu[X(T) - \mu T]/\sigma^2 - (2\mu/\sigma)^2T/2$$

$$= -2\mu X(T)/\sigma^2$$

(c) Since T is the first time the process hits A or $-B$ it follows that

$$X(T) = \begin{cases} A, & \text{with probability } p \\ -B, & \text{with probability } 1-p \end{cases}$$

Hence, we see that

$$1 = E[e^{-2\mu X(T)/\sigma^2}] = pe^{-2\mu A/\sigma^2} + (1-p)e^{2\mu B/\sigma^2}$$

and so

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}$$

23. By the Martingale stopping theorem we have

$$E[B(T)] = E[B(0)] = 0$$

Since $B(T) = [X(T) - \mu T]/\sigma$ this gives the equality

$$E[X(T) - \mu T] = 0$$

or

$$E[X(T)] = \mu E[T]$$

Now

$$E[X(T)] = pA - (1-p)B$$

where, from part (c) of Problem 22,

$$p = \frac{1 - e^{2\mu B/\sigma^2}}{e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2}}$$

Hence,

$$E[T] = \frac{A(1 - e^{2\mu B/\sigma^2}) - B(e^{-2\mu A/\sigma^2} - 1)}{\mu(e^{-2\mu A/\sigma^2} - e^{2\mu B/\sigma^2})}$$

24. It follows from the Martingale stopping theorem and the result of Problem 18 that

$$E[B^2(T) - T] = 0$$

where T is the stopping time given in this problem and $B(t) = [X(t) - \mu t]/\sigma$. Therefore,

$$E[(X(T) - \mu T)^2/\sigma^2 - T] = 0$$

However, $X(T) = x$ and so the above gives that

$$E[(x - \mu T)^2] = \sigma^2 E[T]$$

But, from Problem 21, $E[T] = x/\mu$ and so the above is equivalent to

$$Var(\mu T) = \sigma^2 x/\mu$$

or

$$Var(T) = \sigma^2 x/\mu^3$$

25. The means equal 0.

$$Var\left[\int_0^1 t\,dX(t)\right] = \int_0^1 t^2 dt = \frac{1}{3}$$

$$Var\left[\int_0^1 t^2 dX(t)\right] = \int_0^1 t^4 dt = \frac{1}{5}$$

26. (a) Normal with mean and variance given by

$$E[Y(t)] = tE[X(1/t)] = 0$$

$$Var(Y(t)) = t^2 Var[X(1/t)] = t^2/t = t$$

(b)
$$\begin{aligned} Cov(Y(s), Y(t)) &= Cov(sX(1/s), tX(1/t)) \\ &= st\, Cov(X(1/s), X(1/t)) \\ &= st\frac{1}{t}, \quad \text{when } s \le t \\ &= s, \quad \text{when } s \le t \end{aligned}$$

(c) Clearly $\{Y(t)\}$ is Gaussian. As it has the same mean and covariance function as the Brownian motion process (which is also Gaussian) it follows that it is also Brownian motion.

27. $E[X(a^2t)/a] = \dfrac{1}{a}E[X(a^2t)] = 0$

For $s < t$,

$$\begin{aligned} Cov(Y(s), Y(t)) &= \frac{1}{a^2}\, Cov(X(a^2s), X(a^2t)) \\ &= \frac{1}{a^2}\, a^2 s = s \end{aligned}$$

As $\{Y(t)\}$ is clearly Gaussian, the result follows.

28.
$$\begin{aligned} Cov(B(s) - \tfrac{s}{t}B(t),\ B(t)) &= Cov(B(s),\ B(t)) \\ &\quad - \frac{s}{t}Cov(B(t),\ B(t)) \\ &= s - \frac{s}{t}t = 0 \end{aligned}$$

29. $\{Y(t)\}$ is Gaussian with

$$E[Y(t)] = (t+1)E(Z[t/(t+1)]) = 0$$

and for $s \le t$

$$\begin{aligned} &Cov(Y(s), Y(t)) \\ &= (s+1)(t+1)\,Cov\left[Z\left[\frac{s}{s+1}\right],\ Z\left[\frac{t}{t+1}\right]\right] \\ &= (s+1)(t+1)\frac{s}{s+1}\left[1 - \frac{t}{t+1}\right] \quad (*) \\ &= s \end{aligned}$$

where (∗) follows since $Cov(Z(s), Z(t)) = s(1-t)$. Hence, $\{Y(t)\}$ is Brownian motion since it is also Gaussian and has the same mean and covariance function (which uniquely determines the distribution of a Gaussian process).

30. For $s < 1$

$Cov[X(t), X(t+s)]$

$$= Cov[N(t+1) - N(t), N(t+s+1) - N(t+s)]$$

$$= Cov(N(t+1), N(t+s+1) - N(t+s))$$

$$-Cov(N(t), N(t+s+1) - N(t+s))$$

$$= Cov(N(t+1), N(t+s+1) - N(t+s)) \quad (*)$$

where the equality (∗) follows since $N(t)$ is independent of $N(t+s+1) - N(t+s)$. Now, for $s \le t$,

$$Cov(N(s), N(t)) = Cov(N(s), N(s) + N(t) - N(s))$$

$$= Cov(N(s), N(s))$$

$$= \lambda s$$

Hence, from (∗) we obtain that, when $s < 1$,

$$Cov(X(t), X(t+s)) = Cov(N(t+1), N(t+s+1))$$

$$-Cov(N(t+1), N(t+s))$$

$$= \lambda(t+1) - \lambda(t+s)$$

$$= \lambda(1-s)$$

When $s \ge 1, N(t+1) - N(t)$ and $N(t+s+1) - N(t+s)$ are, by the independent increments property, independent and so their covariance is 0.

31. (a) Starting at any time t the continuation of the Poisson process remains a Poisson process with rate λ.

(b) $E[Y(t)Y(t+s)]$

$$= \int_0^\infty E[Y(t)Y(t+s) \mid Y(t) = y]\lambda e^{-\lambda y}dy$$

$$= \int_0^\infty yE[Y(t+s) \mid Y(t) = y]\lambda e^{-\lambda y}dy$$

$$+ \int_s^\infty y(y-s)\lambda e^{-\lambda y}dy$$

$$= \int_0^s y\frac{1}{\lambda}\lambda e^{-\lambda y}dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y}dy$$

where the above used that

$E[Y(t)Y(t+s)|Y(t) = y]$

$$= \begin{cases} yE(Y(t+s)) = \dfrac{y}{\lambda}, & \text{if } y < s \\ y(y-s), & \text{if } y > s \end{cases}$$

Hence,

$Cov(Y(t), Y(t+s))$

$$= \int_0^s ye^{-y\lambda}dy + \int_s^\infty y(y-s)\lambda e^{-\lambda y}dy - \frac{1}{\lambda^2}$$

32. (a) $Var(X(t+s) - X(t))$

$$= Cov(X(t+s) - X(t), X(t+s) - X(t))$$

$$= R(0) - R(s) - R(s) + R(0)$$

$$= 2R(0) - 2R(s)$$

(b) $Cov(Y(t), Y(t+s))$

$$= Cov(X(t+1) - X(t), X(t+s+1)$$

$$- X(t+s))$$

$$= R_x(s) - R_x(s-1) - R_x(s+1) + R_x(s)$$

$$= 2R_x(s) - R_x(s-1) - R_x(s+1), \quad s \ge 1$$

33. $Cov(X(t), X(t+s))$

$$= Cov(Y_1 \cos wt + Y_2 \sin wt,$$

$$Y_1 \cos w(t+s) + Y_2 \sin w(t+s))$$

$$= \cos wt \cos w(t+s) + \sin wt \sin w(t+s)$$

$$= \cos(w(t+s) - wt)$$

$$= \cos ws$$

Simulation

Exercises

*1. Suppose it is relatively easy to simulate from the distributions F_i, $i = 1, 2, \ldots, n$. If n is small, how can we simulate from

$$F(x) = \sum_{i=1}^{n} P_i F_i(x), \quad P_i \geqslant 0, \qquad \sum_i P_i = 1?$$

Give a method for simulating from

$$F(x) = \begin{cases} \dfrac{1 - e^{-2x} + 2x}{3}, & 0 < x < 1 \\ \dfrac{3 - e^{-2x}}{3}, & 1 < x < \infty \end{cases}$$

2. Give a method for simulating a negative binomial random variable.

*3. Give a method for simulating a hypergeometric random variable.

4. Suppose we want to simulate a point located at random in a circle of radius r centered at the origin. That is, we want to simulate X, Y having joint density

$$f(x, y) = \frac{1}{\pi r^2}, \quad x^2 + y^2 \leqslant r^2$$

(a) Let $R = \sqrt{X^2 + Y^2}$, $\theta = \tan^{-1} Y/X$ denote the polar coordinates. Compute the joint density of R, θ and use this to give a simulation method. Another method for simulating X, Y is as follows:

Step 1: Generate independent random numbers U_1, U_2 and set $Z_1 = 2rU_1 - r$, $Z_2 = 2rU_2 - r$. Then Z_1, Z_2 is uniform in the square whose sides are of length $2r$ and which encloses, the circle of radius r (see Figure 11.6).

Step 2: If (Z_1, Z_2) lies in the circle of radius r—that is, if $Z_1^2 + Z_2^2 \leqslant r^2$—set $(X, Y) = (Z_1, Z_2)$. Otherwise return to step 1.

(b) Prove that this method works, and compute the distribution of the number of random numbers it requires.

5. Suppose it is relatively easy to simulate from F_i for each $i = 1, \ldots, n$. How can we simulate from

(a) $F(x) = \prod_{i=1}^{n} F_i(x)$?

(b) $F(x) = 1 - \prod_{i=1}^{n}(1 - F_i(x))$?

(c) Give two methods for simulating from the distribution $F(x) = x^n$, $0 < x < 1$.

*6. In Example 11.4 we simulated the absolute value of a standard normal by using the Von Neumann rejection procedure on exponential random variables with rate 1. This raises the question of whether we could obtain a more efficient algorithm by using a different exponential density—that is, we could use the density $g(x) = \lambda e^{-\lambda x}$. Show that the mean number of iterations needed in the rejection scheme is minimized when $\lambda = 1$.

7. Give an algorithm for simulating a random variable having density function

$$f(x) = 30(x^2 - 2x^3 + x^4), \quad 0 < x < 1$$

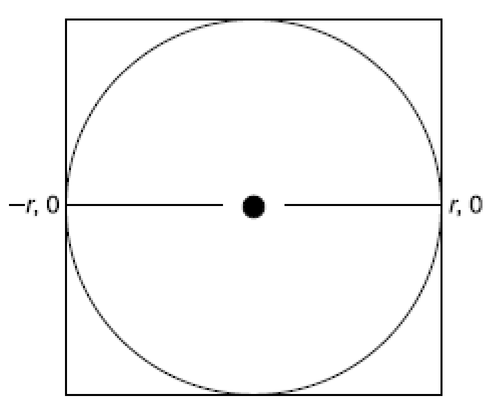


Figure 11.6

8. Consider the technique of simulating a gamma (n, λ) random variable by using the rejection method with g being an exponential density with rate λ/n.
   (a) Show that the average number of iterations of the algorithm needed to generate a gamma is $n^n e^{1-n}/(n-1)!$.
   (b) Use Stirling's approximation to show that for large n the answer to part (a) is approximately equal to $e[(n-1)/(2\pi)]^{1/2}$.
   (c) Show that the procedure is equivalent to the following:

   Step 1: Generate Y_1 and Y_2, independent exponentials with rate 1.

   Step 2: If $Y_1 < (n-1)[Y_2 - \log(Y_2) - 1]$, return to step 1.

   Step 3: Set $X = nY_2/\lambda$.

   (d) Explain how to obtain an independent exponential along with a gamma from the preceding algorithm.
9. Set up the alias method for simulating from a binomial random variable with parameters $n = 6, p = 0.4$.
10. Explain how we can number the $\mathbf{Q}^{(k)}$ in the alias method so that k is one of the two points that $\mathbf{Q}^{(k)}$ gives weight.

    Hint: Rather than giving the initial $\mathbf{Q}$ the name $\mathbf{Q}^{(1)}$, what else could we call it?
11. Complete the details of Example 11.10.
12. Let $X_1, \ldots, X_k$ be independent with

$$P\{X_i = j\} = \frac{1}{n}, \quad j = 1, \ldots, n, \; i = 1, \ldots, k$$

    If D is the number of distinct values among $X_1, \ldots, X_k$ show that

$$E[D] = n\left[1 - \left(\frac{n-1}{n}\right)^k\right]$$

$$\approx k - \frac{k^2}{2n} \quad \text{when } \frac{k^2}{n} \text{ is small}$$

13. *The Discrete Rejection Method:* Suppose we want to simulate X having probability mass function $P\{X = i\} = P_i, i = 1, \ldots, n$ and suppose we can easily simulate from the probability mass function $Q_i, \sum_i Q_i = 1, Q_i \geqslant 0$. Let C be such that $P_i \leqslant CQ_i, i = 1, \ldots, n$. Show that the following algorithm generates the desired random variable:

    Step 1: Generate Y having mass function Q and U an independent random number.

    Step 2: If $U \leqslant P_Y/CQ_Y$, set $X = Y$. Otherwise return to step 1.
14. *The Discrete Hazard Rate Method:* Let X denote a nonnegative integer valued random variable. The function $\lambda(n) = P\{X = n \mid X \geqslant n\}, n \geqslant 0$, is called the *discrete hazard rate function*.
    (a) Show that $P\{X = n\} = \lambda(n) \prod_{i=0}^{n-1}(1 - \lambda(i))$.
    (b) Show that we can simulate X by generating random numbers $U_1, U_2, \ldots$ stopping at

$$X = \min\{n : U_n \leqslant \lambda(n)\}$$

(c) Apply this method to simulating a geometric random variable. Explain, intuitively, why it works.

(d) Suppose that $\lambda(n) \leqslant p < 1$ for all n. Consider the following algorithm for simulating X and explain why it works: Simulate $X_i, U_i, i \geqslant 1$ where X_i is geometric with mean $1/p$ and U_i is a random number. Set $S_k = X_1 + \cdots + X_k$ and let

$$X = \min\{S_k : U_k \leqslant \lambda(S_k)/p\}$$

15. Suppose you have just simulated a normal random variable X with mean μ and variance σ^2. Give an easy way to generate a second normal variable with the same mean and variance that is negatively correlated with X.

16. Suppose n balls having weights $w_1, w_2, \ldots, w_n$ are in an urn. These balls are sequentially removed in the following manner: At each selection, a given ball in the urn is chosen with a probability equal to its weight divided by the sum of the weights of the other balls that are still in the urn. Let $I_1, I_2, \ldots, I_n$ denote the order in which the balls are removed—thus $I_1, \ldots, I_n$ is a random permutation with weights.
(a) Give a method for simulating $I_1, \ldots, I_n$.
(b) Let X_i be independent exponentials with rates $w_i, i = 1, \ldots, n$. Explain how X_i can be utilized to simulate $I_1, \ldots, I_n$.

17. *Order Statistics:* Let $X_1, \ldots, X_n$ be i.i.d. from a continuous distribution F, and let $X_{(i)}$ denote the ith smallest of $X_1, \ldots, X_n, i = 1, \ldots, n$. Suppose we want to simulate $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$. One approach is to simulate n values from F, and then order these values. However, this ordering, or *sorting*, can be time consuming when n is large.
(a) Suppose that $\lambda(t)$, the hazard rate function of F, is bounded. Show how the hazard rate method can be applied to generate the n variables in such a manner that no sorting is necessary.

Suppose now that F^{-1} is easily computed.
(b) Argue that $X_{(1)}, \ldots, X_{(n)}$ can be generated by simulating $U_{(1)} < U_{(2)} < \cdots < U_{(n)}$—the ordered values of n independent random numbers—and then setting $X_{(i)} = F^{-1}(U_{(i)})$. Explain why this means that $X_{(i)}$ can be generated from $F^{-1}(\beta_i)$ where β_i is beta with parameters $i, n + i + 1$.
(c) Argue that $U_{(1)}, \ldots, U_{(n)}$ can be generated, without any need for sorting, by simulating i.i.d. exponentials $Y_1, \ldots, Y_{n+1}$ and then setting

$$U_{(i)} = \frac{Y_1 + \cdots + Y_i}{Y_1 + \cdots + Y_{n+1}}, \quad i = 1, \ldots, n$$

Hint: Given the time of the $(n + 1)$st event of a Poisson process, what can be said about the set of times of the first n events?
(d) Show that if $U_{(n)} = y$ then $U_{(1)}, \ldots, U_{(n-1)}$ has the same joint distribution as the order statistics of a set of $n - 1$ uniform $(0, y)$ random variables.
(e) Use part (d) to show that $U_{(1)}, \ldots, U_{(n)}$ can be generated as follows:
Step 1: Generate random numbers $U_1, \ldots, U_n$.

Step 2: Set

$$U_{(n)} = U_1^{1/n}, \qquad U_{(n-1)} = U_{(n)}(U_2)^{1/(n-1)},$$
$$U_{(j-1)} = U_{(j)}(U_{n-j+2})^{1/(j-1)}, \quad j = 2, \ldots, n-1$$

18. Let $X_1, \ldots, X_n$ be independent exponential random variables each having rate 1. Set

$$W_1 = X_1/n,$$
$$W_i = W_{i-1} + \frac{X_i}{n-i+1}, \quad i = 2, \ldots, n$$

Explain why $W_1, \ldots, W_n$ has the same joint distribution as the order statistics of a sample of n exponentials each having rate 1.

19. Suppose we want to simulate a large number n of independent exponentials with rate 1—call them $X_1, X_2, \ldots, X_n$. If we were to employ the inverse transform technique we would require one logarithmic computation for each exponential generated. One way to avoid this is to first simulate S_n, a gamma random variable with parameters $(n, 1)$ (say, by the method of Section 11.3.3). Now interpret S_n as the time of the nth event of a Poisson process with rate 1 and use the result that given S_n the set of the first $n-1$ event times is distributed as the set of $n-1$ independent uniform $(0, S_n)$ random variables. Based on this, explain why the following algorithm simulates n independent exponentials:

Step 1: Generate S_n, a gamma random variable with parameters $(n, 1)$.
Step 2: Generate $n-1$ random numbers $U_1, U_2, \ldots, U_{n-1}$.
Step 3: Order the U_i, $i = 1, \ldots, n-1$ to obtain $U_{(1)} < U_{(2)} < \cdots < U_{(n-1)}$.
Step 4: Let $U_{(0)} = 0, U_{(n)} = 1$, and set $X_i = S_n(U_{(i)} - U_{(i-1)}), i = 1, \ldots, n$.

When the ordering (step 3) is performed according to the algorithm described in Section 11.5, the preceding is an efficient method for simulating n exponentials when all n are simultaneously required. If memory space is limited, however, and the exponentials can be employed sequentially, discarding each exponential from memory once it has been used, then the preceding may not be appropriate.

20. Consider the following procedure for randomly choosing a subset of size k from the numbers $1, 2, \ldots, n$: Fix p and generate the first n time units of a renewal process whose interarrival distribution is geometric with mean $1/p$—that is, $P\{\text{interarrival time} = k\} = p(1-p)^{k-1}, k = 1, 2, \ldots$. Suppose events occur at times $i_1 < i_2 < \cdots < i_m \leqslant n$. If $m = k$, stop; $i_1, \ldots, i_m$ is the desired set. If $m > k$, then randomly choose (by some method) a subset of size k from $i_1, \ldots, i_m$ and then stop. If $m < k$, take $i_1, \ldots, i_m$ as part of the subset of size k and then select (by some method) a random subset of size $k-m$ from the set $\{1, 2, \ldots, n\} - \{i_1, \ldots, i_m\}$. Explain why this algorithm works. As $E[N(n)] = np$ a reasonable choice of p is to take $p \approx k/n$. (This approach is due to Dieter.)

21. Consider the following algorithm for generating a random permutation of the elements $1, 2, \ldots, n$. In this algorithm, $P(i)$ can be interpreted as the element in position i.

Step 1: Set $k = 1$.

Step 2: Set $P(1) = 1$.

Step 3: If $k = n$, stop. Otherwise, let $k = k + 1$.

Step 4: Generate a random number U, and let

$$P(k) = P([kU] + 1),$$
$$P([kU] + 1) = k.$$

Go to step 3.

(a) Explain in words what the algorithm is doing.

(b) Show that at iteration k—that is, when the value of $P(k)$ is initially set—that $P(1), P(2), \ldots, P(k)$ is a random permutation of $1, 2, \ldots, k$.

Hint: Use induction and argue that

$$\begin{aligned}
&P_k\{i_1, i_2, \ldots, i_{j-1}, k, i_j, \ldots, i_{k-2}, i\} \\
&\quad = P_{k-1}\{i_1, i_2, \ldots, i_{j-1}, i, i_j, \ldots, i_{k-2}\}\frac{1}{k} \\
&\quad = \frac{1}{k!} \quad \text{by the induction hypothesis}
\end{aligned}$$

The preceding algorithm can be used even if n is not initially known.

22. Verify that if we use the hazard rate approach to simulate the event times of a nonhomogeneous Poisson process whose intensity function $\lambda(t)$ is such that $\lambda(t) \leqslant \lambda$, then we end up with the approach given in method 1 of Section 11.5.

*23. For a nonhomogeneous Poisson process with intensity function $\lambda(t)$, $t \geqslant 0$, where $\int_0^\infty \lambda(t)\, dt = \infty$, let $X_1, X_2, \ldots$ denote the sequence of times at which events occur.

(a) Show that $\int_0^{X_1} \lambda(t)\, dt$ is exponential with rate 1.

(b) Show that $\int_{X_{i-1}}^{X_i} \lambda(t)\, dt, i \geqslant 1$, are independent exponentials with rate 1, where $X_0 = 0$.

In words, independent of the past, the additional amount of hazard that must be experienced until an event occurs is exponential with rate 1.

24. Give an efficient method for simulating a nonhomogeneous Poisson process with intensity function

$$\lambda(t) = b + \frac{1}{t + a}, \quad t \geqslant 0$$

25. Let (X, Y) be uniformly distributed in a circle of radius r about the origin. That is, their joint density is given by

$$f(x, y) = \frac{1}{\pi r^2}, \quad 0 \leqslant x^2 + y^2 \leqslant r^2$$

Let $R = \sqrt{X^2 + Y^2}$ and $\theta = \arctan Y/X$ denote their polar coordinates. Show that R and θ are independent with θ being uniform on $(0, 2\pi)$ and $P\{R < a\} = a^2/r^2$, $0 < a < r$.

26. Let R denote a region in the two-dimensional plane. Show that for a two-dimensional Poisson process, given that there are n points located in R, the points are independently and uniformly distributed in R—that is, their density is $f(x, y) = c, (x, y) \in R$ where c is the inverse of the area of R.

27. Let $X_1, \ldots, X_n$ be independent random variables with $E[X_i] = \theta$, $\mathrm{Var}(X_i) = \sigma_i^2$ $i = 1, \ldots, n$, and consider estimates of θ of the form $\sum_{i=1}^n \lambda_i X_i$ where $\sum_{i=1}^n \lambda_i = 1$. Show that $\mathrm{Var}\left(\sum_{i=1}^n \lambda_i X_i\right)$ is minimized when

$$\lambda_i = (1/\sigma_i^2) \Big/ \left(\sum_{j=1}^n 1/\sigma_j^2\right), \quad i = 1, \ldots, n.$$

Possible Hint: If you cannot do this for general n, try it first when $n = 2$.

The following two problems are concerned with the estimation of $\int_0^1 g(x)\,dx = E[g(U)]$ where U is uniform $(0, 1)$.

28. *The Hit–Miss Method:* Suppose g is bounded in $[0, 1]$—for instance, suppose $0 \leqslant g(x) \leqslant b$ for $x \in [0, 1]$. Let U_1, U_2 be independent random numbers and set $X = U_1, Y = bU_2$—so the point (X, Y) is uniformly distributed in a rectangle of length 1 and height b. Now set

$$I = \begin{cases} 1, & \text{if } Y < g(X) \\ 0, & \text{otherwise} \end{cases}$$

That is, accept (X, Y) if it falls in the shaded area of Figure 11.7.

(a) Show that $E[bI] = \int_0^1 g(x)\,dx$.

(b) Show that $\mathrm{Var}(bI) \geqslant \mathrm{Var}(g(U))$, and so hit–miss has larger variance than simply computing g of a random number.

29. *Stratified Sampling:* Let $U_1, \ldots, U_n$ be independent random numbers and set $\bar{U}_i = (U_i + i - 1)/n$, $i = 1, \ldots, n$. Hence, $\bar{U}_i, i \geqslant 1$, is uniform on $((i-1)/n, i/n)$. $\sum_{i=1}^n g(\bar{U}_i)/n$ is called the stratified sampling estimator of $\int_0^1 g(x)\,dx$.

(a) Show that $E\left[\sum_{i=1}^n g(\bar{U}_i)/n\right] = \int_0^1 g(x)\,dx$.

(b) Show that $\mathrm{Var}\left[\sum_{i=1}^n g(\bar{U}_i)/n\right] \leqslant \mathrm{Var}\left[\sum_{i=1}^n g(U_i)/n\right]$.

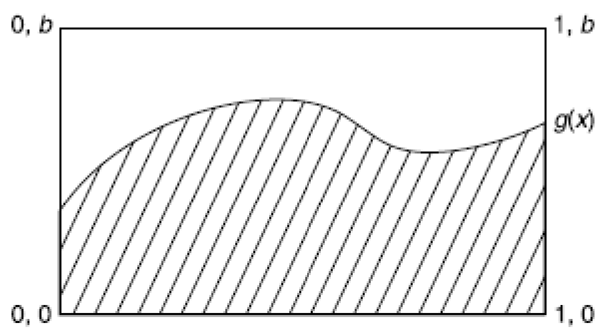


Figure 11.7

Hint: Let U be uniform $(0, 1)$ and define N by $N = i$ if $(i - 1)/n < U < i/n$, $i = 1, \ldots, n$. Now use the conditional variance formula to obtain

$$\begin{aligned}
\mathrm{Var}(g(U)) &= E[\mathrm{Var}(g(U)|N)] + \mathrm{Var}(E[g(U)|N]) \\
&\geqslant E[\mathrm{Var}(g(U)|N)] \\
&= \sum_{i=1}^{n} \frac{\mathrm{Var}(g(U)|N = i)}{n} = \sum_{i=1}^{n} \frac{\mathrm{Var}[g(\bar{U}_i)]}{n}
\end{aligned}$$

30. If f is the density function of a normal random variable with mean μ and variance σ^2, show that the tilted density f_t is the density of a normal random variable with mean $\mu + \sigma^2 t$ and variance σ^2.

31. Consider a queueing system in which each service time, independent of the past, has mean μ. Let W_n and D_n denote, respectively, the amounts of time customer n spends in the system and in queue. Hence, $D_n = W_n - S_n$ where S_n is the service time of customer n. Therefore,

$$E[D_n] = E[W_n] - \mu$$

If we use simulation to estimate $E[D_n]$, should we
(a) use the simulated data to determine D_n, which is then used as an estimate of $E[D_n]$; or
(b) use the simulated data to determine W_n and then use this quantity minus μ as an estimate of $E[D_n]$?
Repeat for when we want to estimate $E[W_n]$.

*32. Show that if X and Y have the same distribution then

$$\mathrm{Var}((X + Y)/2) \leqslant \mathrm{Var}(X)$$

Hence, conclude that the use of antithetic variables can never increase variance (though it need not be as efficient as generating an independent set of random numbers).

33. If $0 \leqslant X \leqslant a$, show that
(a) $E[X^2] \leqslant aE[X]$,
(b) $\mathrm{Var}(X) \leqslant E[X](a - E[X])$,
(c) $\mathrm{Var}(X) \leqslant a^2/4$.

34. Suppose in Example 11.19 that no new customers are allowed in the system after time t_0. Give an efficient simulation estimator of the expected additional time after t_0 until the system becomes empty.

35. Suppose we are able to simulate independent random variables X and Y. If we simulate $2k$ independent random variables $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$, where the X_i have the same distribution as does X, and the Y_j have the same distribution as does Y, how would you use them to estimate $P(X < Y)$?

36. If U_1, U_2, U_3 are independent uniform $(0, 1)$ random variables, find $P(\prod_{i=1}^{3} U_i > 0.1)$.

Hint: Relate the desired probability to one about a Poisson process.

Chapter 11

1. (a) Let u be a random number. If $\sum_{j=1}^{i-1} P_j < u \le \sum_{j=1}^{i} P_j$ then simulate from F_i.

$\left(\text{In the above } \sum_{j=1}^{i-1} P_j \equiv 0 \text{ when } i = 1.\right)$

(b) Note that

$$F(x) = \frac{1}{3}F_1(X) + \frac{2}{3}F_2(x)$$

where

$$F_1(x) = 1 - e^{-2x}, \quad 0 < x < \infty$$

$$F_2(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x \end{cases}$$

Hence, using (a), let U_1, U_2, U_3 be random numbers and set

$$X = \begin{cases} \frac{-\log U_2}{2}, & \text{if } U_1 < 1/3 \\ U_3, & \text{if } U_1 > 1/3 \end{cases}$$

The above uses the fact that $\frac{-\log U_2}{2}$ is exponential with rate 2.

2. Simulate the appropriate number of geometrics and sum them.

3. If a random sample of size n is chosen from a set of $N + M$ items of which N are acceptable then X, the number of acceptable items in the sample, is such that

$$P\{X = k\} = \begin{bmatrix} N \\ k \end{bmatrix} \begin{bmatrix} M \\ n-k \end{bmatrix} \bigg/ \begin{bmatrix} N+M \\ k \end{bmatrix}$$

To simulate X note that if

$$I_j = \begin{cases} 1, & \text{if the } j^{th} \text{ section is acceptable} \\ 0, & \text{otherwise} \end{cases}$$

then

$$P\{I_j = 1 | I_1, \ldots, I_{j-1}\} = \frac{N - \sum_1^{j-1} I_i}{N + M - (j-1)}.$$ Hence, we can simulate $I_1, \ldots, I_n$ by generating random numbers $U_1, \ldots, U_n$ and then setting

$$I_j = \begin{cases} 1, & \text{if } U_j < \dfrac{N - \sum_1^{j-1} I_i}{N + M - (j-1)} \\ 0, & \text{otherwise} \end{cases}$$

$X = \sum_{j=1}^{n} I_j$ has the desired distribution.

Another way is to let

$$X_j = \begin{cases} 1, & \text{the } j^{th} \text{ acceptable item is in the sample} \\ 0, & \text{otherwise} \end{cases}$$

and then simulate $X_1, \ldots, X_N$ by generating random numbers $U_1, \ldots, U_N$ and then setting

$$X_j = \begin{cases} 1, & \text{if } U_j < \dfrac{N - \sum_{i=1}^{j-1} I_i}{N + M - (j-1)} \\ 0, & \text{otherwise} \end{cases}$$

$X = \sum_{j=1}^{N} X_j$ then has the desired distribution.

The former method is preferable when $n \le N$ and the latter when $N \le n$.

4. $$\frac{\partial R}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \frac{\partial R}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left[\frac{y}{x}\right]^2} \left[\frac{-y}{x^2}\right] = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \left[\frac{y}{x}\right]^2} \left[\frac{1}{x}\right] = \frac{x}{x^2 + y^2}$$

Hence, the Jacobian of the transformation is

$$J = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}}$$

The joint density of R, θ is thus

$$\begin{aligned} f_{R,\theta}(s,\theta) &= sf_{X,Y}\left[\sqrt{x^2+y^2}, \tan^{-1} y/x\right] \\ &= \frac{s}{\pi r^2} \\ &= \frac{1}{2\pi} \cdot \frac{2s}{r^2}, \quad 0 < \theta < 2\pi, \quad 0 < s < r \end{aligned}$$

Hence, R and θ are independent with

$$f_R(s) = \frac{2s}{r^2}, \quad 0 < s < r$$

$$f_\theta(\theta) = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi$$

As $F_R(s) = \dfrac{2s}{r^2}$ and so $F_R^{-1}(U) = \sqrt{r^2 U} = r\sqrt{U}$, it follows that we can generate R, θ by letting U_1 and U_2 be random numbers and then setting $R = r\sqrt{U_1}$ and $\theta = 2rU_2$.

(b) It is clear that the accepted point is uniformly distributed in the desired circle. Since

$$P\left\{Z_1^2 + Z_2^2 \le r^2\right\} = \frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi r^2}{4r^2} = \frac{\pi}{4}$$

it follows that the number of iterations needed (or equivalently that one-half the number of random numbers needed) is geometric with mean $\pi/4$.

7. Use the rejection method with $g(x) = 1$. Differentiating $f(x)/g(x)$ and equating to 0 gives the two roots 1/2 and 1. As $f(.5) = 30/16 > f(1) = 0$, we see that $c = 30/16$, and so the algorithm is

Step 1: Generate random numbers U_1 and U_2.

Step 2: If $U_2 \le 16(U_1^2 - 2U_1^3 + U_1^4)$, set $X = U_1$. Otherwise return to step 1.

8. (a) With $f(x) = \dfrac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}$

and $g(x) = \dfrac{\lambda e^{-\lambda x/n}}{n}$

$$f(x)/g(x) = \frac{n(\lambda x)^{n-1} e^{-\lambda x(1-1/n)}}{(n-1)!}$$

Differentiating this ratio and equating to 0 yields the equation

$$(n-1)x^{n-2} = x^{n-1}\lambda(1-1/n)$$

or $\quad x = n/\lambda. \quad$ Therefore,

$$c = \max[f(x)/g(x)] = \frac{n^n e^{-(n-1)}}{(n-1)!}$$

(b) By Stirling's approximation

$$(n-1)! \approx (n-1)^{n-1/2} e^{-(n-1)} (2\pi)^{1/2}$$

and so

$$\begin{aligned} &n^n e^{-(n-1)}/(n-1) \\ &\quad \approx (2\pi)^{-1/2}\left[\frac{n}{n-1}\right]^n (n-1)^{1/2} \\ &\quad = \frac{[(n-1)/2\pi]^{1/2}}{(1-1/n)^n} \\ &\quad \approx e[(n-1)/2\pi]^{1/2} \end{aligned}$$

since $(1-1/n)^n \approx e^{-1}$.

(c) Since

$$f(x)/cg(x) = e^{-\lambda x(1-1/n)}(\lambda x)^{n-1}\frac{e^{n-1}}{n^{n-1}}$$

the procedure is

Step 1: Generate Y, an exponential with rate λ/n and a random number U.

Step 2: If $U \le f(Y)/cg(Y)$, set $X = Y$. Otherwise return to step 1.

The inequality in step 2 is equivalent, upon taking logs, to

$$\begin{aligned} \log U \le\ & n - 1 - \lambda Y(1-1/n) \\ & + (n-1)\log(\lambda Y) - (n-1)\log n \end{aligned}$$

or

$$\begin{aligned} -\log U \ge\ & (n-1)\lambda Y/n + 1 - n \\ & -(n-1)\log(\lambda Y/n) \end{aligned}$$

Now, $Y_1 = -\log U$ is exponential with rate 1, and $Y_2 = \lambda Y/n$ is also exponential with rate 1. Hence, the algorithm can be written as given in part (c).

(d) Upon acceptance, the amount by which Y_1 exceeds $(n-1)\{Y_2 - \log(Y_2) - 1\}$ is exponential with rate 1.

10. Whenever i is the chosen value that satisfies Lemma 11.1 name the resultant $\underline{Q}$ as $Q^{(i)}$.

12. Let

$$I_j = \begin{cases} 1, & \text{if } X_i = j \text{ for some } i \\ 0, & \text{otherwise} \end{cases}$$

then

$$D = \sum_{j=1}^{n} I_j$$

and so

$$E[D] = \sum_j = 1^n E[I_j] = \sum_{j=1}^{n} \left[1 - \left[\frac{n-1}{n}\right]^k\right]$$

$$= n\left[1 - \left[\frac{n-1}{n}\right]^k\right]$$

$$\approx n\left[1 - 1 + \frac{k}{n} - \frac{k(k-1)}{2n^2}\right]$$

13. $P\{X = i\} = P\{Y = i | U \le P_Y / CQ_Y\}$

$$= \frac{P\{Y = i, U \le P_Y/CQ_Y\}}{K}$$

$$= \frac{Q_i P\{U \le P_Y/CQ_Y | Y = i\}}{K}$$

$$= \frac{Q_i P_i/CQ_i}{K}$$

$$= \frac{P_i}{CK}$$

where $K = P\{U \le P_Y/CQ_Y\}$. Since the above is a probability mass function it follows that $KC = 1$.

14. (a) By induction we show that

$$(*) P\{X > k\} = (1 - \lambda(1)) \cdots (1 - \lambda(k))$$

The above is obvious for $k = 1$ and so assume it true. Now

$$P\{X > k + 1\}$$

$$= P\{X > k + 1 | X > k\} P\{X > k\}$$

$$= (1 - \lambda(k + 1)) P\{X > k\}$$

which proves (*). Now

$$P\{X = n\}$$

$$= P\{X = n | X > n - 1\} P\{X > n - 1\}$$

$$= \lambda(n) P\{X > n - 1\}$$

and the result follows from (*).

(b) $P\{X = n\}$

$$= P\{U_1 > \lambda(1), U_2 > \lambda(2), \ldots, U_{n-1} > \lambda(n-1), U_n \le \lambda(n)\}$$

$$= (1 - \lambda(1))(1 - \lambda(2)) \cdots (1 - \lambda(n-1))\lambda(n)$$

(c) Since $\lambda(n) \equiv p$ it sets

$$X = \min\{n : U \le p\}$$

That is, if each trial is a success with probability p then it stops at the first success.

(d) Given that $X \ge n$, then

$$P\{X = n | X > n\} = P\frac{\lambda(n)}{p} = \lambda(n)$$

15. Use $2\mu = X$.

16. (b) Let I_j denote the index of the j^{th} smallest X_i.

17. (a) Generate the $X_{(i)}$ sequentially using that given $X_{(1)}, \ldots, X_{(i-1)}$ the conditional distribution of $X_{(i)}$ will have failure rate function $\lambda_i(t)$ given by

$$\lambda_i(t) = \begin{cases} 0, & t < X_{(i-1)} \\ & , X_{(0)} \equiv 0. \\ (n - i + 1)\lambda(t), & t > X_{(i-1)} \end{cases}$$

(b) This follows since as F is an increasing function the density of $U_{(i)}$ is

$$f_{(i)}(t) = \frac{n!}{(i-1)!(n-i)} (F(t))^{i-1} \times (F(t))^{n-i} f(t)$$

$$= \frac{n!}{(i-1)!(n-i)} t^{i-1}(1-t)^{n-i}, \quad 0 < t < 1$$

which shows that $U_{(i)}$ is beta.

(c) Interpret Y_i as the i^{th} interarrival time of a Poisson process. Now given $Y_1 + \cdots + Y_{n+1} = t$, the time of the $(n + 1)^{st}$ event, it follows that the first n event times are distributed as the ordered values of n uniform $(0, t)$ random variables. Hence,

$$\frac{Y_1 + \cdots + Y_i}{Y_1 + \cdots + Y_{n+i}}, \quad i = 1, \ldots, n$$

will have the same distribution as $U_{(1)}, \ldots, U_{(n)}$.

(d) $f_{U_{(1)},\dots|U_{(n)}}(y_1,\dots,y_{n-1}|y_n)$

$$= \frac{f(y_1,\dots,y_n)}{f_{U_{(n)}}(y_n)}$$

$$= \frac{n!}{ny^{n-1}}$$

$$= \frac{(n-1)!}{y^{n-1}}, 0 < y_1 < \cdots < y_{n-1} < y$$

where the above used that

$$F_{U_{(n)}}(y) = P\{\max U_i \le y\} = y^n$$

and so

$$F_{U_{(n)}}(y) = ny^{n-1}$$

(e) Follows from (d) and the fact that if $F(y) = y^n$ then $F^{-1}(U) = U^{1/n}$.

18. Consider a set of n machines each of which independently functions for an exponential time with rate 1. Then W_1, the time of the first failure, is exponential with rate n. Also given W_{i-1}, the time of the i^{th} failure, the additional time until the next failure is exponential with rate $n-(i-1)$.

20. Since the interarrival distribution is geometric, it follows that independent of when renewals prior to k occurred there will be a renewal at k with probability p. Hence, by symmetry, all subsets of k points are equally likely to be chosen.

21. $P_{m+1}\{i_1,\dots,i_{k-1},m+1\}$

$$= \sum_{\substack{j\le m \\ j\ne i_1,\dots,i_{k-1}}} P_m\{i_1,\dots,i_{k-1},j\}\frac{k}{m+1}\frac{1}{k}$$

$$= (m-(k-1))\frac{1}{\begin{bmatrix} m \\ k\end{bmatrix}}\frac{1}{m+1}\frac{1}{\begin{bmatrix} m+1 \\ k\end{bmatrix}}$$

25. See Problem 4.

27. First suppose $n = 2$.

$$Var(\lambda X_1 + (1-\lambda)X_2) = \lambda^2\sigma_1^2 + (1-\lambda)^2\sigma_2^2.$$

The derivative of the above is $2\lambda\sigma_1^2 - 2(1-\lambda)\sigma_2^2$ and equating to 0 yields

$$\lambda = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \frac{1/\sigma_1^2}{1/\sigma_1^2 + 1/\sigma_2^2}$$

Now suppose the result is true for $n-1$. Then

$$Var\left[\sum_{i=1}^{n}\lambda_i X_i\right] = Var\left[\sum_{i=1}^{n-1}\lambda_i X_i\right] + Var(\lambda_n X_n)$$

$$= (1-\lambda_n)^2\, Var\left[\sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}X_i\right]$$

$$+ \lambda_n^2\, Var\, X_n$$

Now by the inductive hypothesis for fixed λ_n the above is minimized when

$$(*)\ \frac{\lambda_i}{1-\lambda_n} = \frac{1/\sigma_i^2}{\sum_{j=1}^{n-1} 1/\sigma_j^2}, \quad i = 1,\dots,n-1$$

Hence, we now need choose λ_n so as to minimize

$$(1-\lambda_n)^2\frac{1}{\sum_{j=1}^{n-1}1/\sigma_j^2} + \lambda_n^2\,\sigma_n^2$$

Calculus yields that this occurs when

$$\lambda_n = \frac{1}{1+\sigma_n^2\sum_{j=1}^{n-1}1/\sigma_j^2} = \frac{1/\sigma_n^2}{\sum_{j=1}^{n}1/\sigma_j^2}$$

Substitution into (*) now gives the result.

28. (a) $E[I] = P\{Y < g(X)\}$

$$= \int_0^1 P\{Y < g(X)|X = x\}dx$$

since $X = U_1$

$$= \int_0^1 \frac{g(x)}{b}dx$$

since Y is uniform $(0, b)$.

(b) $Var(bI) = b^2 Var(I)$

$$= b^2(E[I] - E^2[I]) \text{ since } I \text{ is Bernoulli}$$

$$= b\int_0^1 g(x)dx - \left[\int_0^1 g(x)dx\right]^2$$

On the other hand
$Var\, g(U) = E[g^2(U)] - E^2[g(U)]$

$$= \int_0^1 g^2(x)dx - \left[\int_0^1 g(x)dx\right]^2$$

$$\le \int_0^1 bg(x)dx - \left[\int_0^1 g(x)dx\right]^2$$

since $g(x) \le b$
$$= Var(bI)$$

29. Use Hint.

30. In the following, the quantities C_i do not depend on x.

$$f_t(x) = C_1 e^{tx} e^{-(x-\mu)^2/(2\sigma)}$$
$$= C_2 \exp\{-(x^2 - (2\mu + 2t\sigma^2)x)/(2\sigma)\}$$
$$= C_3 \exp\{-(x - (\mu + t\sigma^2))^2/(2\sigma)\}$$

31. Since $E[W_n|D_n] = D_n + \mu$, it follows that to estimate $E[W_n]$ we should use $D_n + \mu$. Since $E[D_n|W_n] \ne W_n - \mu$, the reverse is not true and so we should use the simulated data to determine D_n and then use this as an estimate of $E[D_n]$.

32. $Var[(X+Y)/2]$

$$= \frac{1}{4}[Var(X) + Var(Y) + 2Cov(X, Y)]$$
$$= \frac{Var(X) + Cov(X, Y)}{2}$$

Now it is always true that

$$\frac{Cov(V, W)}{\sqrt{Var(V)Var(W)}} \le 1$$

and so when X and Y have the same distribution

$$Cov(X, Y) \le Var(X)$$

33. (a) $E[X^2] \le E[aX] = aE[X]$

(b) $Var(X) = E[X^2] - E^2[X] \le aE[X] - E^2[X]$

(c) From (b) we have that

$$Var(X) \le a^2 \left(\frac{E[X]}{a}\right)$$
$$\left(1 - \frac{E[X]}{a}\right) \le a^2 \max_{0<p<1} p(1-p) = a^2/4$$

34. Use the estimator $R + X_Q E[S]$. Let A be the amount of time the person in service at time t_0 has already spent in service. If $E[R|A]$ is easily computed, an even better estimator is $E[R|A] + X_Q E[S]$.

35. Use the estimator $\sum_{i=1}^{k} N_i/k^2$ where N_i = number of $j = 1, \ldots, k : X_i < Y_j$.

36.
$$P\left(\prod_{i=1}^{3} U_i > .1\right) = P\left(\sum_{i=1}^{3} \log(U_i) > -\log(10)\right)$$
$$= P\left(\sum_{i=1}^{3} -\log(U_i) < \log(10)\right)$$
$$= P(N(\log(10)) \ge 3)$$

where $N(t)$ is the number of events by time t of a Poisson process with rate 1. Hence,

$$P\left(\prod_{i=1}^{3} U_i > .1\right) = 1 - \frac{1}{10}\sum_{i=0}^{2} (\log(10))^i/i!$$