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APPROXIMATION AND CONVEX DECOMPOSITION BY EXTREMALS AND THE λ-FUNCTION IN JBW*-TRIPLES

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Abstract

We establish new estimates to compute the λ-function of Aron and Lohman on the unit ball of a JB*-triple. It is established that for every Brown–Pedersen quasi-invertible element a in a JB*-triple E we have

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) = \max\{1 - m_q(a), \|a\| - 1\},$$

where $\mathfrak{E}(E_1)$ denotes the set of extreme points of the closed unit ball E_1 of E. It is proved that $\lambda(a) = (1 + m_q(a))/2$, for every Brown–Pedersen quasi-invertible element a in E_1, where $m_q(a)$ is the square root of the quadratic conorm of a. For an element a in E_1 which is not Brown–Pedersen quasi-invertible, we can only estimate that $\lambda(a) \leq \frac{1}{2}(1 - \alpha_q(a))$. A complete description of the λ-function on the closed unit ball of every JBW*-triple is also provided, and as a consequence, we prove that every JBW*-triple satisfies the uniform λ-property.

1. Introduction

In [3], Aron and Lohman, defined a function on the closed unit ball, X_1, of an arbitrary Banach space X, which is determined by the geometric structure of the set $\mathfrak{E}(X_1)$ of extreme points of the closed unit ball of X. The mentioned function is called the λ-*function* of the space X. The concrete definition reads as follows: Let us assume that $\mathfrak{E}(X_1) \neq \emptyset$, let x and y be elements in X_1 and let e be an element in $\mathfrak{E}(X_1)$. For each $0 < \lambda \leq 1$, the ordered triplet (e, y, λ) is said to be *amenable* to x when $x = \lambda e + (1 - \lambda) y$. The λ-function is defined by

$$\lambda(x) := \sup \mathcal{S}(x),$$

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where $\mathcal{S}(x) := \{\lambda : (e, y, \lambda)$ is a triplet amenable to $x\}$. The space X satisfies the *λ-property* if $\lambda(x) > 0$, for every $x \in X_1$. The Banach space X has *the uniform λ-property* when $\inf\{\lambda(x) : x \in X_1\} > 0$. Aron, Lohman and Suárez explored the first properties of the λ-function and gave explicitly the form of this function for certain classical function and sequence spaces in [3, 4].

In [3, Question 4.1], Aron and Lohman posed the following challenge: 'What spaces of operators have the λ-property and what does the λ-function look like for these spaces?'. This question motivated a whole series of papers, in which Brown and Pedersen determined the exact form of the λ-function for every von Neumann algebra and for every unital C*-algebra (cf. [7, 8, 37]). In their study of the λ-function, Brown and Pedersen introduce the set A_q^{-1} of *quasi-invertible elements* in a unital C*-algebra A, and study the geometric properties of A_1 in relation to the set A_q^{-1}. The following explicit formulae to compute the distance from an element in A_1 to the set of quasi-invertible elements or to $\mathfrak{E}(A_1)$ are established by Brown and Pedersen:

$$\operatorname{dist}(a, \mathfrak{E}(A_1)) = \begin{cases} \max\{1 - m_q(a), \|a\| - 1\} & \text{if } a \in A_q^{-1}, \\ \max\{1 + \alpha_q(a), \|a\| - 1\} & \text{if } a \notin A_q^{-1}, \end{cases}$$

where $\alpha_q(a) = \operatorname{dist}(a, A_q^{-1})$ and $m_q(a) = \operatorname{dist}(a, A \backslash A_q^{-1})$ (cf. [8, Theorem 2.3]). The λ-function is given by

$$\lambda(a) = \begin{cases} \dfrac{1 + m_q(a)}{2} & \text{if } a \in A_1 \cap A_q^{-1}, \\[2ex] \dfrac{1}{2}(1 - \alpha_q(a)) & \text{if } a \in A_1 \backslash A_q^{-1}, \end{cases}$$

(cf. [8, Theorem 3.7]). Furthermore, every von Neumann algebra (i.e. a C*-algebra which is also a dual Banach space) satisfies the uniform λ-property, actually the expression $\lambda(a) = (1 + m_q(a))/2$ holds for every element a in the closed unit ball of a von Neumann algebra (cf. [37, Theorem 4.2]).

There exists a class of complex Banach spaces defined by certain holomorphic properties of their open unit balls, we refer to the class of JB*-triples. Harris shows in [22] that the open unit ball of every C*-algebra A is a *bounded symmetric domain*, and the same conclusion holds for the open unit ball of every closed linear subspace $U \subseteq A$ invariant under the Jordan triple product

$$\{x, y, z\} := \tfrac{1}{2}(xy^*z + zy^*x). \tag{1.1}$$

In [30], Kaup introduces the concept of a JB*-triple, and shows that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple and, in this way, the category of all bounded symmetric domains with base point is equivalent to the category of JB*-triples. Actually, every C*-algebra is a JB*-triple with respect to (1.1), however, the class of JB*-triples is strictly wider than the class of C*-algebras (see next section for definitions and examples).

For each complex Banach space E in the class of JB*-triples, the open unit ball of E enjoys similar geometric properties to those exhibited by the closed unit ball of a C*-algebra. Many geometric properties studied in the setting of C*-algebras have been studied in the wider class of JB*-triples. For example, in recent papers, the first, third and fourth authors of this note extend the notion of quasi-invertible elements from the setting of C*-algebras to the wider class of JB*-triples introducing the concept of *Brown–Pedersen quasi-invertible* elements (see [25, 26, 42]). Once the class E_q^{-1}

of Brown–Pedersen quasi-invertible elements in a JB*-triple E has been introduced, the following question seems the natural problem to be studied.

PROBLEM 1.1 What JB*-triples have the λ-property and what does the λ-function look like in the case of a JB*-triple?

Only partial answers to the above problem are known. Accordingly to the terminology employed by Brown and Pedersen, for each element x in a JB*-triple E, the symbol $\alpha_q(x)$ will denote the distance from x to the set E_q^{-1} of Brown–Pedersen quasi-invertible elements in E, that is, $\alpha_q(x) = \mathrm{dist}(x, E_q^{-1})$. The known estimates for the λ-function in the setting of JB*-triples are the following: for each (complete tripotent) $v \in \mathfrak{E}(E_1)$, and each element x in the closed unit ball of the Peirce-2 subspace $E_2(v)$ which is not Brown–Pedersen quasi-invertible in E we have:

$$\lambda(x) \leq \tfrac{1}{2}(1 - \alpha_q(x)); \tag{1.2}$$

consequently, $\lambda(x) = 0$ whenever $\alpha_q(x) = 1$ (cf. [26, Theorem 3.7]).

In this paper, we continue with the study of the λ function in the general setting of JB*-triples. In Section 2, we introduce the basic facts and definitions needed in the paper, and we revisit the concept of Brown–Pedersen quasi-invertibility by finding new characterizations of this notion in terms of the triple spectrum and the orthogonal complement of an element.

We begin Section 3 proving that, for each element x in a JB*-triple E, the square root of the quadratic conorm, $\gamma^q(x)$, introduced in [11], measures the distance from x to the set $E\backslash E_q^{-1}$ (see Theorem 3.1), where by convention $\gamma^q(x) = 0$ for every $x \in E\backslash E_q^{-1}$. It is established that for every Brown–Pedersen quasi-invertible element a in E we have

$$\mathrm{dist}(a, \mathfrak{E}(E_1)) = \max\{1 - m_q(a), \|a\| - 1\}$$

(see Proposition 3.2). This formula is complemented with Theorem 3.4 where we prove that $\lambda(a) = (1 + m_q(a))/2$, for every Brown–Pedersen quasi-invertible element a in E_1.

For elements in the closed unit ball of a JB*-triple which are not Brown–Pedersen quasi-invertible, we improve the estimates in (1.2) (see [26]) by proving that for every JB*-triple E with $\mathfrak{E}(E_1) \neq \emptyset$, the inequalities

$$1 + \|a\| \geq \mathrm{dist}(a, \mathfrak{E}(E_1)) \geq \max\{1 + \alpha_q(a), \|a\| - 1\},$$

hold for every a in $E\backslash E_q^{-1}$ (Theorem 3.6). Consequently, the inequality

$$\lambda(a) \leq \tfrac{1}{2}(1 - \alpha_q(a)),$$

holds for every $a \in E_1\backslash E_q^{-1}$ without assuming that a lies in the Peirce-2 subspace associated with a complete tripotent v in E (see Corollary 3.7).

A JBW*-triple is a JB*-triple which is also a dual Banach space. In the setting of JB*-triples, JBW*-triples play an analogue role to that played by von Neumann algebras in the class of C*-algebras. In Section 4, we prove that every JBW*-triple satisfies the uniform λ-property (see Corollary 4.3), a result which extends [37, Theorem 4.2] to the context of JBW*-triples. This result will follow from

Theorem 4.2, where it is established that for every JBW*-triple W the λ-function on W_1 is given by the expression:

$$\lambda(a) = \begin{cases} \dfrac{1+m_q(a)}{2} & \text{if } a \in W_1 \cap W_q^{-1}, \\ \dfrac{1}{2}(1-\alpha_q(a)) = \dfrac{1}{2} & \text{if } a \in W_1 \backslash W_q^{-1}. \end{cases}$$

The paper finishes with a result establishing that, for every element a in the closed unit ball of a JB*-triple E which is not Brown–Pedersen quasi-invertible, if $\mathfrak{E}(E_1) \neq \emptyset$, then the distance from a to the latter set is given by the formula

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) = 1 + \alpha_q(a)$$

(see Theorem 4.5).

2. von Neumann regularity and Brown–Pedersen invertibility

From a purely algebraic point of view, a *complex Jordan triple system* is a complex linear space E equipped with a triple product

$$\{.,.,.\} : E \times E \times E \to E,$$
$$(x, y, z) \mapsto \{x, y, z\},$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies the *Jordan identity*:

$$L(x, y)\{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all $x, y, a, b, c \in E$, where $L(x, y) : E \to E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$.

Given an element a in a complex Jordan triple system E, the symbol $Q(a)$ will denote the conjugate linear operator on E given by $Q(a)(x) := \{a, x, a\}$. It is known that the fundamental identity

$$Q(x)Q(y)Q(x) = Q(Q(x)y) \tag{2.1}$$

holds for every x, y in a complex Jordan triple system E (cf. [**13**, Lemma 1.2.4]).

The studies on von Neumann regular elements in Jordan triple systems began with the contributions of Loos [**35**] and Fernández-López *et al.* [**16**]. We recall that an element a in a Jordan triple system E is called *von Neumann regular* if $a \in Q(a)(E)$ and *strongly von Neumann regular* when $a \in Q(a)^2(E)$.

Enriching the geometrical structure of a complex Jordan triple system, we find the class of complex Banach spaces called JB*-triples, introduced by Kaup to classify bounded symmetric domains in arbitrary complex Banach spaces (cf. [**30**]). More concretely, a *JB*-triple* is a complex Jordan triple system E which is a Banach space satisfying the additional geometric axioms:

(a) For each $x \in E$, the map $L(x, x)$ is a hermitian operator with non-negative spectrum;
(b) $\| \{x, x, x\} \| = \|x\|^3$ for all $x \in E$.

The basic bibliography on JB*-triples can be found in [**13**, **43**].

Examples of JB*-triples include all C*-algebras with the triple product given in (1.1), all JB*-algebras with triple product

$$\{a, b, c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*,$$

and the Banach space $L(H, K)$ of all bounded linear operators between two complex Hilbert spaces H, K with respect to (1.1).

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [5]). The triple product of every JBW*-triple is separately weak* continuous (cf. [5]), and the second dual, E^{**}, of a JB*-triple E is a JBW*-triple (cf. [14]).

An element a in a JB*-triple E is von Neumann regular if, and only if, it is strongly von Neumann regular if, and only if, there exists $b \in E$ such that $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)Q(b) - Q(b)Q(a) = 0$ (cf. [16, Theorem 1; 31, Lemma 4.1]). Although for a von Neumann regular element a in a JB*-triple E, there exist many elements c in E such that $Q(a)(c) = a$, there exists a unique element $b \in E$ satisfying $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)\,Q(b) - Q(b)\,Q(a) = 0$, this unique element b is called the *generalized inverse* of a in E and it is denoted by $a^\dagger$.

The simplest examples of von Neumann regular elements, probably, are tripotents. We recall that an element e in a JB*-triple E is called *tripotent* when $\{e, e, e\} = e$. Each tripotent e in E induces a decomposition of E (called the *Peirce decomposition*) in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e)$ is the $i/2$ eigenspace of $L(e, e)$. The Peirce rules affirm that $\left\{E_i(e), E_j(e), E_k(e)\right\}$ is contained in $E_{i-j+k}(e)$ if $i - j + k \in \{0, 1, 2\}$ and is zero otherwise. In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The projection $P_k(e)$ of E onto $E_k(e)$ is called the Peirce k-projection. It is known that Peirce projections are contractive (cf. [20]) and satisfy that $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$ and $P_0(e) = \mathrm{Id}_E - 2L(e, e) + Q(e)^2$. A tripotent e in E is said to be *unitary* if $L(e, e)$ coincides with the identity map on E, that is, $E_2(e) = E$. We shall say that e is *complete* when $E_0(e) = \{0\}$.

The Peirce space $E_2(e)$ is a unital JB*-algebra with unit e, product $x \circ_e y := \{x, e, y\}$ and involution $x^{*_e} := \{e, x, e\}$, respectively. Furthermore, the triple product in $E_2(e)$ is given by

$$\{a, b, c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ_e b^{*_e} \ (a, b, c \in E_2(e)).$$

When a C*-algebra A is regarded as a JB*-triple with the product given in (1.1), tripotent elements in A are precisely partial isometries of A. A JB*-triple might not contain a single tripotent element (consider, for example, $C_0(0, 1]$ the C*-algebra of all complex-valued continuous functions on $[0, 1]$ vanishing at 0). However, since the complete tripotents of a JB*-triple E coincide with the complex and the real extreme points of its closed unit ball (cf. [6, Lemma 4.1; 33, Proposition 3.5] or [13, Theorem 3.2.3]), every JBW*-triple is full of complete tripotents.

As shown by Kaup in [31], the triple spectrum is one of the most appropriate tools to study and determine von Neumann regular elements. The *triple spectrum* of an element a in a JB*-triple E is

the set

$$\mathrm{Sp}(a) := \{t \in \mathbb{C} : a \notin (L(a,a) - t^2 \mathrm{Id}_E)(E)\}.$$

The extended spectrum of a is the set $\mathrm{Sp}'(a) := \mathrm{Sp}(a) \cup \{0\}$. As usually, the smallest closed complex subtriple of E containing a will be denoted by E_a. The set

$$\Sigma(a) := \{s \in \mathbb{C} : (L(a,a) - s\mathrm{Id}_E)|_{E_a} \text{ is not invertible in } L(E_a)\}$$

stands for the usual spectrum of the restricted operator $L(a,a)|_{E_a}$ in $L(E_a)$. Following standard notation, we assume that $\Sigma(a) = \emptyset$ whenever $a = 0$ (this is actually an equivalence, compare [31, Lemma 3.2]). The following properties were established in [31].

(Σ.i) $\Sigma(a)$ is a compact subset of $\mathbb{R}$ with $\Sigma(a) \geq 0$ and the origin cannot be an isolated point of $\Sigma(a)$. The origin cannot be an isolated point of $\mathrm{Sp}(a)$ and $\mathrm{Sp}(a) = -\mathrm{Sp}(a)$.

(Σ.ii) $\mathrm{Sp}(a) = \{t \in \mathbb{C} : t^2 \in \Sigma(a)\}$ and $\mathrm{Sp}(a) \neq \emptyset$, whenever $a \neq 0$.

(Σ.iii) $S_a := \mathrm{Sp}(a) \cap [0, \infty)$ is a compact subset of $\mathbb{R}$, $\|a\| \in S_a \subseteq [0, \|a\|]$, and there exists a unique triple isomorphism $\Psi : E_a \to C_0(S_a \cup \{0\})$ such that $\Psi(a)(s) = s$ for every $s \in S_a$, where $C_0(S_a \cup \{0\})$ denotes the space of all complex-valued, continuous functions on $S_a \cup \{0\}$ vanishing at zero. If $0 \in S_a$, then it is not isolated in S_a.

(Σ.iv) The spectrum $\mathrm{Sp}(a)$ does not change when computed with respect to any closed complex subtriple $F \subseteq E$ containing a.

(Σ.v) The element a is von Neumann regular if, and only if, $0 \notin \mathrm{Sp}(a)$.

The basic properties of the triple spectrum lead us to the *continuous triple functional calculus*. Given an element a in a JB*-triple E and a function $f \in C_0(S_a \cup \{0\})$, $f_t(a)$ will denote the unique element in E_a which is mapped to f when E_a is identified as JB*-triple with $C_0(S_a \cup \{0\})$. Consequently, for each natural n, the element $a^{[1/(2n-1)]}$ coincides with $f_t(a)$, where $f(\lambda) := \lambda^{1/(2n-1)}$. When a is an element in a JBW*-triple W, the sequence $(a^{[1/(2n-1)]})$ converges in the weak*-topology of W to a tripotent, denoted by $r(a)$, and called the *range tripotent* of a. The tripotent $r(a)$ is the smallest tripotent $e \in W$ satisfying that a is positive in the JBW*-algebra $W_2(e)$ (see, for example, [15, comments before Lemma 3.1] or [9, Section 2]).

We shall habitually regard a Banach space X as being contained in its bidual, X^{**}, and we identify the weak*-closure, in X^{**}, of a closed subspace Y of X with Y^{**}. For an element a in a JB*-triple E, the range tripotent $r(a)$ is defined in E^{**}. Having this in mind, the range tripotent of an element a in a JB*-triple is the element in $E_a^{**} \equiv (C_0(S_a \cup \{0\}))^{**}$ corresponding with the characteristic function of the set S_a.

We recall that an element a in a unital Jordan Banach algebra J is called invertible whenever there exists $b \in J$ satisfying $a \circ b = 1$ and $a^2 \circ b = a$. The element b is unique and it will be denoted by a^{-1}. The set $J^{-1} = \mathrm{inv}(J)$ of all invertible elements in J is open in the norm topology and $a \in J^{-1}$ whenever $\|a - 1\| < 1$. It is well known that a is invertible if, and only if, the mapping $x \mapsto U_a(x) := 2(a \circ x) \circ a - a^2 \circ x$ is invertible, and in that case $U_a^{-1} = U_{a^{-1}}$ (see, for example [13, p. 107]).

The reduced minimum modulus was introduced in [11] to study the quadratic conorm of an element in a JB*-triple. The *reduced minimum modulus* of a non-zero bounded linear or conjugate linear operator T between two normed spaces X and Y is defined by

$$\gamma(T) := \inf\{\|T(x)\| : \mathrm{dist}(x, \ker(T)) \geq 1\}. \tag{2.2}$$

Following [29], we set $\gamma(0) = \infty$ (reader should be awarded that in [2] $\gamma(0) = 0$). When X and Y are Banach spaces, we have

$$\gamma(T) > 0 \Leftrightarrow T(X) \text{ is norm closed}$$

(cf. [29, Theorem IV.5.2]). The quadratic conorm of an element a in a JB*-triple E is defined as the reduced minimum modulus of $Q(a)$ and it will be denoted by $\gamma^q(a)$, that is, $\gamma^q(a) = \gamma(Q(a))$. The main results in [11] show, among many other things, that:

(Σ.vi) An element a is von Neumann regular if, and only if, $Q(a)$ has norm-closed image if, and only if, the range tripotent $r(a)$ of a lies in E and a is positive and invertible element of the JB*-algebra $E_2(r(a))$. Furthermore, when a is von Neumann regular we have:

$$Q(a)Q(a^\dagger) = P_2(r(a)) = Q(a^\dagger)Q(a)$$

and

$$L(a, a^\dagger) = L(a^\dagger, a) = L(r(a), r(a))$$

(cf. [32, comments after Lemma 3.2] or [11, p. 192]).

(Σ.vii) For each element a in E, $\gamma^q(a) = \inf\{\Sigma(a)\} = \inf\{t^2 : t \in \mathrm{Sp}(a)\}$.

Let us recall that the *Bergmann operator* associated with a couple of elements x, y in a JB*-triple E is the mapping $B(x, y) : E \to E$ defined by $B(x, y) = \mathrm{Id} - 2L(x, y) + Q(x)Q(y)$ (cf. [36] or [43, p. 305]).

Inspired by the definition of *quasi-invertible* elements in a C*-algebra developed by Brown and Pedersen in [7, 8], Tahlawi, Siddiqui and Jamjoom introduced and developed, in [25, 26, 42], the notion of *Brown–Pedersen quasi-invertible* elements in a JB*-triple E. An element a in E is Brown–Pedersen quasi-invertible (BP-quasi-invertible for short) if there exists $b \in E$ such that $B(a, b) = 0$. It was established in [25, 42] that an element a in E is BP-quasi-invertible if, and only if, one of the following equivalent statements holds:

(a) a is von Neumann regular and its range tripotent $r(a)$ is an extreme point of the closed unit ball of E (i.e. $r(a)$ is a complete tripotent of E);
(b) there exists a complete tripotent $e \in E$ such that a is positive and invertible in the JB*-algebras $E_2(e)$.

The set of all BP-quasi-invertible elements in E is denoted by E_q^{-1}. Let us observe that, in principle, the notion of invertibility makes no sense in a general JB*-triple. By [25, Theorem 8], E_q^{-1} is open in the norm topology (the reason being that, for each complete tripotent e, the set of invertible elements in the JB*-algebra $E_2(e)$ is open and the Peirce projections are contractive).

Let us observe that when a C*-algebra A is regarded as a JB*-triple with product given by (1.1), the BP-quasi-invertible elements in A, as JB*-triple, are exactly the quasi-invertible elements of A in the terminology of Brown and Pedersen in [7, 8].

We shall also need a characterization of BP-quasi-invertible elements in terms of the orthogonal complement. First, we recall that elements a, b in a JB*-triple E are said to be *orthogonal* (denoted by $a \perp b$) when $L(a, b) = 0$. By [10, Lemma 1], we know that $a \perp b$ if, and only if, one of the

following statements holds:

$$\begin{aligned} \{a,a,b\} = 0; \quad a \perp r(b); \quad r(a) \perp r(b); \\ E_2^{**}(r(a)) \perp E_2^{**}(r(b)); \quad E_a \perp E_b; \quad \{b,b,a\} = 0. \end{aligned} \tag{2.3}$$

For each subset $M \subseteq E$, we write $M_E^{\perp}$ for the (*orthogonal*) *annihilator of M* defined by

$$M_E^{\perp} := \{y \in E : y \perp x, \forall x \in M\}.$$

It is known that, for each tripotent e in E, $\{e\}^{\perp} = E_0(e)$. Furthermore, the identity $\{a\}^{\perp} = (E^{**})_0(r(a)) \cap E$ holds for every $a \in E$ (cf. [12, Lemma 3.2]). We therefore have the following lemma.

LEMMA 2.1 *Let a be an element in a JB*-triple E. Then a is BP-quasi-invertible if, and only if, a is von Neumann regular and $\{a\}^{\perp} = \{0\}$.*

We initiate the novelties with a series of technical lemmas.

LEMMA 2.2 *Let e be a complete tripotent in a JB*-triple E and let z be an element in E. Suppose that $P_2(e)(z)$ is invertible in the JB*-algebra $E_2(e)$. Then z is BP-quasi-invertible.*

Proof. By hypothesis, $z_2 = P_2(e)(z)$ is invertible in the JB*-algebra $E_2(e)$ with inverse z_2^{-1}, and since e is complete, $z = z_2 + z_1$ where $z_1 = P_1(e)(z)$. Let us observe that z_2 is von Neumann regular in E and $z_2^{\dagger} = z_2^{-1}$.

We claim that the invertibility of z_2 in $E_2(e)$ also implies that $r(z_2) \in E_2(z_2)$ is a unitary tripotent in the JB*-triple $E_2(e)$. Indeed, since for each $x \in E_2(e)$,

$$x = P_2(r(z_2))(x) = Q(z_2)Q(z_2^{-1})(x) = U_{z_2}U_{z_2^{-1}}(x),$$

we deduce that $P_2(r(z_2))|_{E_2(e)} = \mathrm{Id}_{E_2(e)}$, proving the claim.

Clearly, $E_2(e) = E_2(r(z_2))$. Given $x \in E$, the condition

$$\{r(z_2), x, r(z_2)\} = 0$$

implies $0 = Q(r(z_2))^2(x) = P_2(r(z_2))(x) = P_2(e)(x)$, and hence $x = P_1(e)(x)$ lies in $E_1(e)$. Thus, $E_1(r(z_2)) \oplus E_0(r(z_2)) \subseteq E_1(e)$. Taking $x \in E_0(r(z_2))$, having in mind that $e \in E_2(r(z_2))$, it follows from Peirce arithmetic that $\{e, e, x\} = 0$, which shows that $E_0(r(z_2)) \subseteq E_0(e) = \{0\}$. Therefore, $r(z_2)$ is a complete tripotent in E and $E_j(r(z_2)) = E_j(e)$, for every $j = 0, 1, 2$.

Now, by Peirce arithmetic we have:

$$\begin{aligned} Q(z)(z_2^\dagger) &= Q(z_2)(z_2^\dagger) + 2Q(z_2, z_1)(z_2^\dagger) + Q(z_1)(z_2^\dagger) = z_2 + 2L(z_2, z_2^\dagger)(z_1) + 0 \\ &= z_2 + 2L(r(z_2), r(z_2))(z_1) = z_2 + 2L(e,e)(z_1) = z_2 + z_1 = z \end{aligned}$$

and

$$Q(z_2^\dagger)(z) = Q(z_2^\dagger)(z_2) + Q(z_2^\dagger)(z_1) = z_2^\dagger.$$

This shows that z is von Neumann regular. Take $a \in \{z\}^\perp$. Since

$$\begin{aligned} 0 &= \{z_2^{-1}, z, a\} = \{z_2^{-1}, z_2, a\} + \{z_2^{-1}, z_1, a\} \\ &= P_2(e)(a) + \tfrac{1}{2} P_1(e)(a) + \{z_2^{-1}, z_1, P_2(e)a\} + \{z_2^{-1}, z_1, P_1(e)(a)\} \\ &= \text{(by Peirce arithmetic)} = P_2(e)(a) + \tfrac{1}{2} P_1(e)(a) + \{z_2^{-1}, z_1, P_1(e)(a)\}, \end{aligned}$$

which shows that $P_1(e)(a) = 0$, $P_2(e)(a) = 0$, and hence $a = 0$. Lemma 2.1 concludes the proof. □

REMARK 2.3 We would like to isolate the following fact, which has been established in the proof of Lemma 2.2: For each invertible element b in a unital JB*-algebra, J, its range tripotent $r(b)$ is a unitary element belonging to J.

COROLLARY 2.4 *Let e be a complete tripotent in a JB*-triple E. Suppose that a is an element in E satisfying $\|a - e\| < 1$, then a is BP-quasi-invertible.*

Proof. Having in mind that $P_2(e)$ is a contractive projection, we get

$$\|P_2(e)(a) - e\| = \|P_2(e)(a - e)\| \leq \|a - e\| < 1.$$

Since $E_2(e)$ is a unital JB*-algebra with unit e, it follows that $P_2(e)(a)$ is an invertible element in $E_2(e)$. The conclusion of the corollary follows from Lemma 2.2. □

Let u, v be tripotents in a JB*-triple E. We recall [36, Section 5] that $u \leq v$ if $v - u$ is a tripotent with $u \perp v - u$. It is known that $u \leq v$ if, and only if, $P_2(u)(v) = u$, or equivalently, $L(u,u)(v) = u$ (cf. [20, Lemma 1.6 and subsequent remarks]). In particular, $u \leq v$ if, and only if, u is a projection in the JB*-algebra $E_2(v)$. Let us observe that the condition $u \geq v$ implies $L(v,v)(u) = u$. However, the condition $L(v,v)(u) = u$ need not imply, in general, the inequality $v \geq u$ (cf. Remark 2.6).

The following technical lemma will be repeatedly used later.

LEMMA 2.5 *Let e be a complete tripotent in a JB*-triple E. Suppose that u is a tripotent in $E_2(e)$ satisfying that $L(u,u)(e) = e$. Then u is a complete tripotent of E.*

Proof. Since $L(u,u)e = e$, we deduce that $e \in E_2(u)$. By Peirce arithmetic, for each $x \in E$,

$$Q(e)(x) = Q(e)P_2(u)(x) \in E_2(u),$$

which implies $E_2(e) = P_2(e)(E) = Q(e)^2(E) \subseteq E_2(u)$. Since, we also have $L(e,e)(u) = u$, we get $E_2(e) = E_2(u)$. Therefore, the mapping $T = Q(u)|_{E_2(e)} : E_2(e) \to E_2(2)$ satisfies that $T^2 = P_2(u)|_{E_2(e)} = P_2(e)|_{E_2(e)}$ is the identity on $E_2(e)$.

Since the triple product of $E_2(e)$ is given by $\{a, b, c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ_e b^{*_e}$ $(a, b, c \in E_2(e))$, we can easily see that $T(x) = U_u(x^*)$ and hence U_u is an invertible operator in $L(E_2(e))$. We have therefore proved that u is an invertible element in $E_2(e)$. Lemma 2.2 gives the desired statement. □

Lemma 4 in [**40**] proves that for every complete tripotent e in a JB*-triple E, every unitary element in the JB*-algebra $E_2(e)$ is an extreme point of the closed unit ball of E (i.e. a complete tripotent of E). This statement follows as a direct consequence of the above Lemma 2.5. Concretely, let u be a unitary element in the JB*-algebra $E_2(e)$ (i.e. u is invertible in $E_2(e)$ with $u^{-1} = u^{*_e}$). Since the triple product on $E_2(e)$ is given by

$$\{a, b, c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ_e b^{*_e}$$

$(a, b, c \in E_2(e))$, we can easily see that $\{u, u, e\} = (u \circ_e u^{*_e}) \circ_e e + (e \circ_e u^{*_e}) \circ_e u - (u \circ_e e) \circ_e u^{*_e} = e$, and Lemma 2.5 gives the statement.

The following remark clarifies the connections between Lemmas 2.2, 2.5, Corollary 2.4 and [**40**, Lemma 4].

REMARK 2.6 Let e be a complete tripotent in a JB*-triple E and let v be a tripotent in $E_2(e)$. Then the following statements are equivalent:

(a) v is invertible in the JB*-algebra $E_2(e)$;
(b) v is a unitary element in the JB*-algebra $E_2(e)$;
(c) v is a unitary element in the JB*-triple $E_2(e)$;
(d) $L(v, v)(e) = e$.

The implication (a) $\Rightarrow$ (b) is established in Remark 2.3. The implication (c) $\Rightarrow$ (d) is clear. To see (b) $\Rightarrow$ (c), we recall that the triple product in $E_2(e)$ is given by

$$\{a, b, c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ_e b^{*_e} \ (a, b, c \in E_2(e)).$$

Since for each $a \in E_2(e)$, we have $U_v(a^{*_e}) = Q(v)(a)$ (where $U_b(c) := 2(b \circ_e c^{*_e}) \circ_e b - (b \circ_e b) \circ_e c^{*_e}$, for all $b, c \in E_2(e)$), we can deduce that

$$P_2(v)(a) = Q(v)^2(a) = U_v(U_v(a^{*_e})^{*_e}) = U_v U_{v^{*_e}}(a) = a,$$

for every $a \in E_2(e)$, which shows that $P_2(v)|_{E_2(e)} = \mathrm{Id}_{E_2(e)}$, and hence v is a unitary tripotent in $E_2(e)$. To prove (d) $\Rightarrow$ (a), we recall that $L(v, v)(e) = e$ implies that $e \in E_2(v)$, and hence $E_2(e) = E_2(v)$ because $v \in E_2(e)$, which proves (d) $\Rightarrow$ (c). Furthermore, recalling that $\mathrm{Id}_{E_2(e)} = P_2(v)|_{E_2(e)} = U_v U_{v^{*_e}}$, we obtain (a).

Consider now the statements:

(e) v is an extreme point of $(E_2(e))_1$, or equivalently, v is a complete tripotent in $E_2(e)$;
(f) v is a complete tripotent in E.

It should be noted that (e) $\not\Rightarrow$ (f) $\Rightarrow$ (e), while (f) do not necessarily imply any of the above statements (a)–(d). We consider, for example, an infinite-dimensional complex Hilbert space H, a complete tripotent $e \in L(H)$ such that $ee^* = 1$ and $p = e^*e \neq 1$. In this case, $L(H)_2(e) = L(H)e^*e$. The element p is a complete tripotent in $L(H)_2(e)$, and since $0 \neq 1 - p \perp p$ it follows that p is

not complete in $L(H)$ (this shows that (e) $\not\Rightarrow$ (f)). To see the second claim, pick a complete partial isometry $v \in L(e^*e(H))$ such that $vv^* \neq e^*e$ and $v^*v = e^*e$. It is easy to see that v is a complete tripotent in $L(H)_2(e)$ and $L(v, v)(e) = \frac{1}{2}(vv^*e + ev^*v) = \frac{1}{2}(vv^* + e) \neq e$.

For more information on extreme points and unitary elements in C*-algebras, JB*-triples and JB-algebras, the reader is referred to [1, 17, Section 2, 27, 34, 39].

3. Distance to the extremals and the λ-function

In this section, we shall give some formulas to compute the distance from an element in a JB*-triple E to the set $\mathfrak{E}(E_1)$ of extreme points of the closed unit ball of E. Since, in some cases, $\mathfrak{E}(E_1)$ may be an empty set, we shall assume that $\mathfrak{E}(E_1) \neq \emptyset$.

Let E be a JB*-triple. According to the terminology employed in [7, 8, 25, 26, 42], we define $\alpha_q : E \to \mathbb{R}_0^+$, by $\alpha_q(x) = \operatorname{dist}(x, E_q^{-1})$. Inspired by the studies of Brown and Pedersen, we also introduce a mapping $m_q : E \to \mathbb{R}_0^+$ defined by

$$m_q(x) := \begin{cases} 0 & \text{if } x \notin E_q^{-1}, \\ (\gamma^q(x))^{1/2} & \text{if } x \in E_q^{-1}. \end{cases}$$

Let us note that, for each $x \in E_q^{-1}$,

$$m_q(x) = \inf\{t : t \in \operatorname{Sp}(x) \cap [0, \infty)\} = \max\{\varepsilon > 0 :] - \varepsilon, \varepsilon[\cap \operatorname{Sp}(x) = \emptyset\},$$

and $m_q(x) > 0$ if, and only if, $x \in E_q^{-1}$.

We claim that

$$m_q(\lambda x) = |\lambda| m_q(x), \tag{3.1}$$

for every $\lambda \in \mathbb{C}\backslash\{0\}$, $x \in E$. Indeed, since

$$(\mathbb{C}\backslash\{0\})E_q^{-1} = E_q^{-1} \quad \text{and} \quad \mathbb{C}(E\backslash E_q^{-1}) = E\backslash E_q^{-1},$$

we may reduce to the case $a \in E_q^{-1}$ (cf. (Σ.v) and (Σ.iii)). Since $L(\lambda a, \lambda a) = |\lambda|^2 L(a, a)$, it follows that $\Sigma(\lambda a) = |\lambda|^2 \Sigma(a)$, which gives $m_q(\lambda a) = \inf\{\sqrt{t} : t \in \Sigma(\lambda a)\} = |\lambda| m_q(a)$.

As in the C*-algebra setting, our next result shows that m_q is actually a distance (cf. [7, Proposition 1.5] for the result in the setting of C*-algebras).

THEOREM 3.1 *Let E be a JB*-triple, then*

$$m_q(a) = \operatorname{dist}(a, E\backslash E_q^{-1}),$$

for every $a \in E$. In particular, $m_q(a) = \operatorname{dist}(a, E\backslash E_q^{-1}) = (\gamma^q(a))^{1/2}$, for every $a \in E_q^{-1}$.

Proof. We can assume that $a \in E_q^{-1}$. By (Σ.iii) and (Σ.v), $S_a := \operatorname{Sp}(a) \cap [0, \infty)$ is a compact subset of $\mathbb{R}$, $S_a \subseteq [0, \|a\|]$, $\|a\| = \max(S_a)$, $0 < m_q(a) = (\gamma^q(a))^{1/2} = \min(S_a)$, and there exists a unique triple isomorphism $\Psi : E_a \to C_0(S_a \cup \{0\}) = C(S_a)$ such that $\Psi(a)(s) = s$ for every $s \in S_a$. The range tripotent $r(a)$ coincides with the unit element in $C(S_a)$. Clearly, $y_0 = a - m_q(a)r(a)$

lies in $E_a \subseteq E$ and contains zero in its triple spectrum, therefore $y_0 \in E \backslash E_q^{-1}$. Since $\|a - y_0\| = \|m_q(a)r(a)\| = m_q(a)$, we get $m_q(a) \geq \operatorname{dist}(a, E \backslash E_q^{-1})$.

To prove the reverse inequality, we first assume that $\|a\| \leq 1$. Arguing by reduction to the absurd, we suppose that $m_q(a) > \operatorname{dist}(a, E \backslash E_q^{-1})$, then there exists $z \in E \backslash E_q^{-1}$ with $\|a - z\| < m_q(a) = (\gamma^q(a))^{1/2}$. Since $a \in E_q^{-1}$, its range tripotent, $r(a)$, is a complete tripotent in E, and a is a positive, invertible element in $E_2(r(a))$. The contractivity of $P_2(r(a))$, assures that

$$\|a - P_2(r(a))(z)\| = \|P_2(r(a))(a - z)\| \leq \|a - z\| < m_q(a).$$

Now, we compute the distance

$$\begin{aligned}\|P_2(r(a))(z) - r(a)\| &\leq \|P_2(r(a))(z) - a\| + \|a - r(a)\| \\ &< m_q(a) + \max\{1 - m_q(a), \|a\| - 1\} = 1.\end{aligned}$$

The general theory of invertible elements in JB*-algebras shows that the element $P_2(r(a))(z)$ is invertible in $E_2(r(a))$, because $r(a)$ is the unit element in the latter JB*-algebra. Lemma 2.2 implies that $z \in E_q^{-1}$, which contradicts that $z \in E \backslash E_q^{-1}$. We have therefore proved that $m_q(a) = \operatorname{dist}(a, E \backslash E_q^{-1})$, for every $a \in E_q^{-1}$ with $\|a\| \leq 1$.

Finally, given $a \in E_q^{-1}$, we have

$$m_q\left(\frac{a}{\|a\|}\right) = \operatorname{dist}\left(\frac{a}{\|a\|}, E \backslash E_q^{-1}\right),$$

and $\|a\| m_q(a/\|a\|) = m_q(a)$. Therefore,

$$m_q(a) = \|a\| m_q\left(\frac{a}{\|a\|}\right) \leq \|a\| \left\| \frac{a}{\|a\|} - c \right\| = \|a - \|a\|c\|$$

for every $c \in E \backslash E_q^{-1}$, which shows that

$$m_q(a) \leq \operatorname{dist}(a, \|a\|(E \backslash E_q^{-1})) = \operatorname{dist}(a, E \backslash E_q^{-1}).$$

□

It was already noted in [25, Lemma 25] that

$$\alpha_q(\lambda x) = |\lambda| \alpha_q(x); \quad \alpha_q(x) \leq \|x\|$$

and

$$|\alpha_q(x) - \alpha_q(y)| \leq \|x - y\|$$

for every $x, y \in E$, $\lambda \in \mathbb{C}$. Theorem 3.1 implies that

$$|m_q(x) - m_q(y)| \leq \|x - y\| \tag{3.2}$$

for every $x, y \in E$.

Our next goal is an extension of [8, Theorem 2.3] to the more general setting of JB*-triples, and determines the distance from a BP-quasi-invertible element in a JB*-triple E to the set of extreme points in E_1.

PROPOSITION 3.2 *Let a be a BP-quasi-invertible element in a JB*-triple E. Then*

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) = \max\{1 - m_q(a), \|a\| - 1\}.$$

Proof. Again, by (Σ.iii) and (Σ.v), the set $S_a := \operatorname{Sp}(a) \cap [0, \infty)$ is a compact subset of $\mathbb{R}$, $S_a \subseteq [0, \|a\|]$, $\|a\| = \max(S_a)$, $0 < m_q(a) = \min(S_a)$, and there exists a unique triple isomorphism $\Psi : E_a \to C(S_a)$ such that $\Psi(a)(s) = s$ for every $s \in S_a$, and the range tripotent $r(a)$ coincides with the unit element in $C(S_a)$. Since $r(a) \in \mathfrak{E}(E_1)$ and

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) \leq \|a - r(a)\| = \max\{1 - m_q(a), \|a\| - 1\}.$$

Given $e \in \mathfrak{E}(E_1)$, we always have $\|a - e\| \geq \|a\| - 1$. Since

$$m_q(a) = |m_q(e - (e - a))| \geq m_q(e) - \|e - a\| = 1 - \|e - a\|,$$

we also have $\operatorname{dist}(a, \mathfrak{E}(E_1)) \geq \max\{1 - m_q(a), \|a\| - 1\}$. □

COROLLARY 3.3 *Let E be a JB*-triple. Then*

$$\{a \in E_q^{-1} : \|a\| = m_q(a) = (\gamma^q(a))^{1/2}\} =]0, \infty[\ \mathfrak{E}(E_1).$$

Our next result is a first estimate for the λ-function, it can be regarded as an appropriate triple version of [8, Theorem 3.1; 41, Lemma 2.4].

THEOREM 3.4 *Let a be a BP-quasi-invertible element in the closed unit ball of a JB*-triple E. Then for every $\lambda \in [\frac{1}{2}, (1 + m_q(a))/2]$ there exist e, u in $\mathfrak{E}(E_1)$ satisfying*

$$a = \lambda e + (1 - \lambda)u.$$

When $1 \geq \lambda > (1 + m_q(a))/2$, such a convex decomposition cannot be obtained. Consequently, $\lambda(a) = (1 + m_q(a))/2$, for every $a \in E_q^{-1} \cap E_1$.

Proof. The range tripotent $r(a) \in \mathfrak{E}(E_1)$ is the unit element of subtriple $E_a \equiv C(S_a)$, where $S_a := \operatorname{Sp}(a) \cap [0, \infty)$ is a compact subset of $\mathbb{R}$, $S_a \subseteq [0, \|a\|]$, $\|a\| = \max(S_a)$, $0 < m_q(a) = \min(S_a)$ and there exists a triple isomorphism $\Psi : E_a \to C(S_a)$ such that $\Psi(a)(s) = s$ ($s \in S_a$). It is part of the folklore in C*-algebra theory that for every $\lambda \in [\frac{1}{2}, (1 + m_q(a))/2]$, the function $\Psi(a) : s \mapsto s$ can be written in the form

$$\Psi(a) = \lambda v_1 + (1 - \lambda)v_2,$$

where v_1, v_2 are two unitary elements in $C(S_a)$ (see [28, Lemma 6] or [41, Lemma 2.4] for a proof in a more general setting). Since v_1, v_2 are unitary elements in $E_a \equiv C(S_a)$ and $r(a)$ is an extreme point of the closed unit ball of E, the tripotents $e = \Psi^{-1}(v_1)$ and $u = \Psi^{-1}(v_2)$ belong to $\mathfrak{E}(E_1)$ (cf. Lemma 2.5) and $a = \lambda e + (1 - \lambda)u$.

Given $1 \geq \lambda > (1 + m_q(a))/2$, if we assume that $a = \lambda e + (1 - \lambda)y$, where $e \in \mathfrak{E}(E_1)$ and $y \in E_1$, we have

$$\|a - e\| = (1 - \lambda)\|y - e\| \leq 2(1 - \lambda),$$

which shows that $\operatorname{dist}(a, \mathfrak{E}(E_1)) \leq 2(1 - \lambda)$. However, by Proposition 3.2, $1 - m_q(a) = \operatorname{dist}(a, \mathfrak{E}(E_1))$, and hence $\lambda \leq (1 + m_q(a))/2$, which is impossible. $\square$

Our next result was in [26, Theorem 3.5]. We can give now an alternative proof from the above results.

COROLLARY 3.5 *Let E be a JB*-triple. Let a be an element in E_1. Then $a \in E_q^{-1}$ if, and only if, $a = \alpha v_1 + (1 - \alpha)v_2$ for some extreme points v_1, v_2 in $\mathcal{E}(E_1)$ and $0 \leq \alpha < \frac{1}{2}$.*

Proof. ($\Rightarrow$) Since $a \in E_q^{-1} \backslash \mathfrak{E}(E_1)$, the distance m_q, satisfies $0 < m_q(a) < 1$, and hence $(\frac{1}{2}, (1 + m_q(a))/2] \neq \emptyset$. Take $\lambda \in (\frac{1}{2}, (1 + m_q(a))/2]$. Theorem 3.4 implies the existence of v_1, v_2 in $\mathcal{E}(E_1)$ satisfying $a = \lambda v_2 + (1 - \lambda)v_1$. The statement follows for $\alpha = 1 - \lambda$.

($\Leftarrow$) Note that $\|a - v_2\| = \alpha\ \|v_1 - v_2\| < 1$. Corollary 2.4 implies that $a \in (\mathcal{J})_q^{-1}$. $\square$

In [25, Theorem 26], the authors show that, given a complete tripotent e in a JB*-triple E (i.e. $e \in \mathfrak{E}(E_1)$), then for each element a in $E_2(e) \backslash E_q^{-1}$ we have:

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) \geq \max\{1 + \alpha_q(a), \|a\| - 1\}.$$

Our next result shows that there is no need to assume that the element a lies in the Peirce-2 subspace of a complete tripotent to prove the same inequality.

THEOREM 3.6 *Let E be a JB*-triple satisfying $\mathfrak{E}(E_1) \neq \emptyset$. Then the inequalities*

$$1 + \|a\| \geq \operatorname{dist}(a, \mathfrak{E}(E_1)) \geq \max\{1 + \alpha_q(a), \|a\| - 1\}$$

hold for every a in $E \backslash E_q^{-1}$.

Proof. Let us fix a in $E \backslash E_q^{-1}$. Clearly, for each $e \in \mathfrak{E}(E_1)$, $\|a - e\| \geq |\|a\| - 1|$, and hence

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) \geq |\|a\| - 1|.$$

Fix an arbitrary $e \in \mathfrak{E}(E_1)$. If $\|a - e\| < \beta$, then $\beta > 1$, otherwise $\|a - e\| < 1$ and Corollary 2.4 implies that $a \in E_q^{-1}$, which is impossible. Now, the inequality

$$m_q((\beta - 1)e + a) = m_q(\beta e + a - e) \geq m_q(\beta e) - \|a - e\| = \beta - \|a - e\| > 0,$$

shows that $(\beta - 1)e + a$ lies in E_q^{-1}. Then

$$\alpha_q(a) \leq \|a - ((\beta - 1)e + a)\| = \beta - 1.$$

This proves that

$$\alpha_q(a) + 1 \leq \beta,$$

for every $e \in \mathfrak{E}(E_1)$ and $\beta > \|a - e\|$, witnessing that $\operatorname{dist}(a, \mathfrak{E}(E_1)) \geq 1 + \alpha_q(a)$. $\square$

COROLLARY 3.7 *Let E be a JB*-triple satisfying $\mathfrak{E}(E_1) \neq \emptyset$. Then*

$$\lambda(a) \leq \tfrac{1}{2}(1 - \alpha_q(a)),$$

for every $a \in E_1 \backslash E_q^{-1}$.

Proof. Let us fix $a \in E_1 \backslash E_q^{-1}$. By Theorem 3.6, we have

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) \geq \max\{\alpha_q(a) + 1, \|a\| - 1\}.$$

Thus, if a writes in the form $a = \lambda e + (1 - \lambda)y$, where $e \in \mathfrak{E}(E_1)$, $y \in E_1$ and $0 \leq \lambda \leq 1$ we have $a - e = (\lambda - 1)e + (1 - \lambda)y$, which gives

$$\alpha_q(a) + 1 \leq \operatorname{dist}(a, \mathfrak{E}(\mathcal{J})_1) \leq \|a - e\| = |1 - \lambda| \|y - e\| \leq 2(1 - \lambda),$$

which proves $\lambda \leq \frac{1}{2}(1 - \alpha_q(a))$. □

4. The λ-function of a JBW*-triple

We can present now a precise description of the λ-function in the case of a JBW*-triple. The main goal of this section is to prove that every JBW*-triple satisfies the uniform λ-property, extending the result established by Pedersen in [37, Theorem 4.2] in the context of von Neumann algebras.

First, we observe that whenever we replace JB*-triples with JBW*-triples, the α_q function is much more simpler to compute on the closed unit ball.

PROPOSITION 4.1 *Let W be a JBW*-triple. Then, for each a in W_1 we have*

$$\operatorname{dist}(a, \mathfrak{E}(W_1)) = 1 - m_q(a).$$

In particular, $\alpha_q(a) = 0$, for every $a \in W_1 \backslash W_q^{-1}$.

Proof. When $a \in W_q^{-1}$, the statement follows from Proposition 3.2. Let us assume that $a \notin W_q^{-1}$, then 0 is not an isolated point in S_a (cf. (Σ.i)). One more time, we shall identify W_a (the (norm-closed) JB*-subtriple of W generated by a) with $C_0(S_a \cup \{0\})$. Therefore, for each $\delta > 0$ the sets $]\delta, \|a\|] \cap S_a$ and $]0, \delta] \cap S_a$ are non-empty. The characteristic function $r_\delta = \chi_{]\delta, \|a\|]} \in (W_a)^{\sigma(W, W_*)}$ is a range tripotent of an element in W_a, and hence r_δ is a tripotent in W.

By [24, Lemma 3.12], there exists $e \in \mathfrak{E}(W_1)$ such that $Q(e)(r_\delta) = r_\delta$, that is, $e = r_\delta + (e - r_\delta)$ and $r_\delta \perp (e - r_\delta)$. Since $P_1(r_\delta)(a - e) = 0$, we can write

$$a - e = P_2(r_\delta)(a - e) + P_0(r_\delta)(a - e) = P_2(r_\delta)(a - r_\delta) + P_0(r_\delta)(a - e).$$

Clearly,

$$\|P_2(r_\delta)(a - r_\delta)\| = \max\{1 - \delta, \|a\| - 1\} = 1 - \delta,$$

while $\|P_0(r_\delta)(a - e)\| \leq \|P_0(r_\delta)(a)\| + \|P_0(r_\delta)(e)\| \leq 1 + \delta$. Now, observing that $P_2(r_\delta)(a - r_\delta) \perp P_0(r_\delta)(a - e)$, we deduce from [20, Lemma 1.3(a)] that

$$\operatorname{dist}(a, \mathfrak{E}(W_1)) \leq \|a - e\| \leq \max\{1 + \delta, 1 - \delta\} = 1 + \delta.$$

The arbitrariness of $\delta > 0$ implies that $\operatorname{dist}(a, \mathfrak{E}(W_1)) \leq 1$.

Finally, the equality $\operatorname{dist}(a, \mathfrak{E}(W_1)) = 1$ and the final statement follows from Theorem 3.6. $\square$

The detailed description of the λ-function in the case of a JBW*-triple reads as follows.

THEOREM 4.2 *Let W be a JBW*-triple. Then the λ-function on W_1 is given by the expression*:

$$\lambda(a) = \begin{cases} \dfrac{1 + m_q(a)}{2} & \text{if } a \in W_1 \cap W_q^{-1}, \\ \dfrac{1}{2}(1 - \alpha_q(a)) = \dfrac{1}{2} & \text{if } a \in W_1 \backslash W_q^{-1}. \end{cases}$$

Proof. The case $a \in W_1 \cap W_q^{-1}$ follows from Theorem 3.4. Suppose $a \in W_1 \backslash W_q^{-1}$. Corollary 3.7 and Proposition 4.1 imply that $\lambda(a) \leq \frac{1}{2}(1 - \alpha_q(a)) = \frac{1}{2}$.

Let $r = r(a)$ denote the range tripotent of a in W. Let us observe that, by [24, Lemma 3.12], there exists a complete tripotent $e \in \mathfrak{E}(W_1)$ such that $e = r_\delta + (e - r_\delta)$ and $r_\delta \perp (e - r_\delta)$. This implies that a is a positive element in the closed unit ball of the JBW*-algebra $W_2(e)$. Since $a \notin W_q^{-1}$, 0 lies in the triple spectrum of a (cf. (Σ.v)). Furthermore, the triple spectrum of a does not change when computed as an element in $W_2(e)$ (see (Σ.iv)), thus a is not BP-quasi-invertible in $W_2(e)$. Let $\mathcal{J}_{a,e}$ denote the JBW*-algebra of $W_2(e)$ generated by e and a. It is known that $\mathcal{J}_{a,e}$ is isometrically isomorphic, as JBW*-algebra, to an abelian von Neumann algebra with unit e (cf. [21, Lemma 4.1.11]). Since, in the terminology of [7, 8], a neither is quasi-invertible in the abelian von Neumann algebra $\mathcal{J}_{a,e}$, we deduce, via [37, Theorem 4.2], that there exist unitary elements e_1 and e_2 in $\mathcal{J}_{a,e}$ satisfying $a = \frac{1}{2}e_1 + \frac{1}{2}e_2$. Since $e \in \mathfrak{E}(W_1)$ is the unit element in $\mathcal{J}_{a,e}$ and e_1, e_2 are unitary element in the latter von Neumann algebra, we conclude that $e_1, e_2 \in \mathfrak{E}(W_1)$ (cf. Lemma 2.5 or [40, Lemma 4]), which shows that $\frac{1}{2} \leq \lambda(a)$. $\square$

As in the C*-setting, an element a in the closed unit ball of a JBW*-triple is BP-quasi-invertible if, and only if, $\lambda(a) > \frac{1}{2}$.

COROLLARY 4.3 *Every JBW*-triple satisfies the uniform λ-property.*

In [26, Section 4] (see also [42, Section 5.3]), the authors introduce the Λ-condition in the setting of JB*-triples in the following sense: a JB*-triple E satisfies the Λ-condition if for each complete tripotent $e \in \mathfrak{E}(E)$ and every $a \in (E_2(e))_1 \backslash E_q^{-1}$, the condition $\lambda(a) = 0$ implies $\alpha_q(a) = 1$. We can affirm now that every JBW*-triple actually satisfies a stronger property, because, by Theorem 4.2 (see also Proposition 4.1), the minimum value of the λ-function on the closed unit ball of a JBW*-triple is $\frac{1}{2}$ (cf. [37, Theorem 4.2] for the appropriate result in von Neumann algebras).

Our next goal is to complete the statement of Theorem 3.6 in the case of a general JB*-triple.

PROPOSITION 4.4 *Let a and b be elements in a JB*-triple E. Suppose $\|a - b\| < \beta$ and $b \in E_q^{-1}$. Then $a + \beta r(b) \in E_q^{-1}$ and the inequality*

$$m_q(a + \beta r(b)) \geq \beta - \|b - a\|,$$

holds. Furthermore, under the above conditions, the element $P_2(r(b))(a) + \beta r(b)$ is invertible in the JB-algebra $E_2(r(b))$.*

Proof. Let us write $a + \beta r(b) = a - b + b + \beta r(b)$. Considering the JB*-subtriple E_b generated by b, we can easily see that $m_q(b + \beta r(b)) = \beta + m_q(b)$. Therefore, by (3.2),

$$m_q(a + \beta r(b)) \geq m_q(b + \beta r(b)) - \|a - b\| = \beta + m_q(b) - \|a - b\| > \beta - \|b - a\| > 0,$$

which proves the first statement.

Now, set $c = P_2(r(b))(a - b)$. Clearly, $\|c\| \leq \|a - b\| < \beta$. We write

$$P_2(r(b))(a) + \beta r(b) = c + P_2(r(b))(b) + \beta r(b) = c + b + \beta r(b).$$

Since b is invertible and positive in the JB*-algebra $E_2(r(b))$, we deduce that $d = b + \beta r(b)$ is a positive invertible element in $E_2(r(b))$, with inverse $d^{-1} \in E_2(r(b))$. It is easy to see that $\|d^{-1}\|^{-1} = \|(b + \beta r(b))^{-1}\|^{-1} \geq \beta + m_q(b) > \beta$, and hence

$$\|U_{d^{-1/2}}(a - b)\| \leq \|d^{-1/2}\|^2 \, \|a - b\| < \frac{1}{\beta}\, \beta = 1,$$

which implies that $r(b) + U_{d^{-1/2}}(a - b)$ is invertible in the JB*-algebra $E_2(r(b))$. Finally, the identity

$$\begin{aligned} P_2(r(b))(a) + \beta r(b) &= P_2(r(b))(a - b) + P_2(r(b))(b) + \beta r(b) \\ &= U_{d^{1/2}}(U_{d^{-1/2}}(a - b) + U_{d^{-1/2}}(b + \beta r(b))) \\ &= U_{d^{1/2}}(U_{d^{-1/2}}(a - b) + r(b)), \end{aligned}$$

gives the final statement and concludes the proof. □

We can now extend Proposition 4.1 to the setting of JB*-triples.

THEOREM 4.5 *Let E be a JB*-triple satisfying $\mathfrak{E}(E_1) \neq \emptyset$. Then the formula*

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) = 1 + \alpha_q(a)$$

holds for every a in $E_1 \backslash E_q^{-1}$.

Proof. Fix a in $E_1 \backslash E_q^{-1}$. Theorem 3.6 proves that $2 \geq \operatorname{dist}(a, \mathfrak{E}(E_1)) \geq 1 + \alpha_q(a)$. In particular, $0 \leq \alpha_q(a) \leq 1$. When $\alpha_q(a) = 1$, we have $2 \geq \operatorname{dist}(a, \mathfrak{E}(E_1)) \geq 1 + \alpha_q(a) = 2$, we may therefore assume that $\alpha_q(a) < 1$.

We shall prove now that for each pair (δ, β) with $1 > \delta > \beta > \alpha_q(a)$ there exists $e \in \mathfrak{E}(E_1)$ with $\|a - e\| < \max\{1 + \beta, 2\delta\}$. Indeed, by definition there exists $b \in E_q^{-1}$ such that $\|a - b\| < \beta < \delta$. By Proposition 4.4, the element $z = a + \delta r(b) \in E_q^{-1}$ and $m_q(z) = m_q(a + \delta r(b)) \geq \delta - \|b - a\| > \delta - \beta$.

Clearly, $\|a - z\| = \|\delta r(b)\| = \delta$. Since $z \in E_q^{-1}$, its range tripotent $e = r(z) \in \mathfrak{E}(E_1)$. It is known that

$$\|z - r(z)\| = \max\{1 - m_q(z), \|z\| - 1\} < \max\{1 - \delta + \beta, \|a\| + \delta - 1\}.$$

Therefore,

$$\begin{aligned} \|a - r(z)\| &\le \|a - z\| + \|z - r(z)\| \\ &< \delta + \max\{1 - \delta + \beta, \|a\| + \delta - 1\} \le \max\{1 + \beta, 2\delta\}. \end{aligned}$$

This proves that for each pair (δ, β) with $1 > \delta > \beta > \alpha_q(a)$ we have

$$\mathrm{dist}(a, \mathfrak{E}(E_1)) \le \max\{1 + \beta, 2\delta\},$$

letting $\beta, \delta \to \alpha_q(a)$ we get

$$\mathrm{dist}(a, \mathfrak{E}(E_1)) \le \max\{1 + \alpha_q(a), 2\alpha_q(a)\} = 1 + \alpha_q(a),$$

which concludes the proof. □

The set of extreme points of the closed unit ball of a unital C*-algebra is always non-empty. Since every C*-algebra is a JB*-triple, [**8**, Theorem 2.3] derives as a direct consequence of our Theorem 4.5. Actually, the proof above provides a simpler argument to obtain the result in [**8**]. Let us observe that the introduction of JB*-triple techniques makes the proofs easier because the set of extreme points is not directly linked to the order structure of a C*-algebra.

REMARK 4.6 In order to determine the λ function on $E_1 \backslash E_q^{-1}$, it would be very interesting to know if the distance formula established in Theorem 4.5 can be improved to show that, under the same hypothesis, the equality

$$\mathrm{dist}(a, \mathfrak{E}(E_1)) = \max\{1 + \alpha_q(a), \|a\| - 1\} \tag{4.1}$$

holds for every a in $E \backslash E_q^{-1}$.

We recall that a JB*-triple E is said to be *commutative* or *abelian* if the identity

$$\{\{x, y, z\}, a, b\} = \{x, y, \{z, a, b\}\} = \{x, \{y, z, a\}, b\}$$

holds for all $x, y, z, a, b \in E$, equivalently, $L(a, b)L(c, d) = L(c, d)L(a, b)$, for every $a, b, c, d \in E$. Suppose that E is a commutative JB*-triple with $\mathfrak{E}(E_1) \ne \emptyset$. It is known (cf. [**18**, Theorems 2 and 4] or [**23**, Lemma 6.2]) that for each $e \in \mathfrak{E}(E_1)$, the JB*-triple E is a commutative C*-algebra with unit e, product and involution given by $a \circ_e b := \{a, e, b\}$ and $a^{*_e} := \{e, a, e\}$ $(a, b \in E)$, respectively, and the same norm. We have already observed that when a C*-algebra A is regarded as a JB*-triple with the triple product given in (1.1), the BP-quasi-invertible elements in A, as JB*-triple, are exactly the quasi-invertible elements of the C*-algebra A introduced and studied by Brown and Pedersen in [**7**, **8**].

Since the Banach space underlying E has not been changed, we can deduce from [8, Theorem 2.3] that

$$\operatorname{dist}(a, \mathfrak{E}(E_1)) = \max\{1 + \alpha_q(a), \|a\| - 1\},$$

for every a in $E \backslash E_q^{-1}$, that is, (4.1) holds for every commutative JB*-triple E with $\mathfrak{E}(E_1) \neq \emptyset$. It can be also shown that in this case,

$$\lambda(a) \leq \tfrac{1}{2}(1 - \alpha_q(a)),$$

for every $a \in E_1 \backslash E_q^{-1}$. Since commutative JB*-triples are also example of function spaces (cf. [19, 30, Section 1]), the last result complements the study developed in [3, Theorem 1.9].

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