APPROXIMATION AND CONVEX DECOMPOSITION BY EXTREMALS AND THE $\lambda$-FUNCTION IN JBW$^*$-TRIPLES

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[Received 15 September 2014]

Abstract

We establish new estimates to compute the $\lambda$-function of Aron and Lohman on the unit ball of a JB$^*$-triple. It is established that for every Brown–Pedersen quasi-invertible element $a$ in a JB$^*$-triple $E$ we have

$$\text{dist}(a,E(E_1)) = \max\{1 - m_q(a), \|a\| - 1\},$$

where $E(E_1)$ denotes the set of extreme points of the closed unit ball $E_1$ of $E$. It is proved that $\lambda(a) = (1 + m_q(a))/2$, for every Brown–Pedersen quasi-invertible element $a$ in $E_1$, where $m_q(a)$ is the square root of the quadratic conorm of $a$. For an element $a$ in $E_1$ which is not Brown–Pedersen quasi-invertible, we can only estimate that $\lambda(a) \leq \frac{1}{2}(1 - \alpha_q(a))$. A complete description of the $\lambda$-function on the closed unit ball of every JBW$^*$-triple is also provided, and as a consequence, we prove that every JBW$^*$-triple satisfies the uniform $\lambda$-property.

1. Introduction

In [3], Aron and Lohman, defined a function on the closed unit ball, $X_1$, of an arbitrary Banach space $X$, which is determined by the geometric structure of the set $E(X_1)$ of extreme points of the closed unit ball of $X$. The mentioned function is called the $\lambda$-function of the space $X$. The concrete definition reads as follows: Let us assume that $E(X_1) \neq \emptyset$, let $x$ and $y$ be elements in $X_1$ and let $e$ be an element in $E(X_1)$. For each $0 < \lambda \leq 1$, the ordered triplet $(e, y, \lambda)$ is said to be amenable to $x$ when $x = \lambda e + (1 - \lambda)y$. The $\lambda$-function is defined by

$$\lambda(x) := \sup S(x),$$

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where $S(x) := \{\lambda : (e, y, \lambda) \text{ is a triplet amenable to } x\}$. The space $X$ satisfies the $\lambda$-property if $\lambda(x) > 0$, for every $x \in X$. The Banach space $X$ has the uniform $\lambda$-property when $\inf\{\lambda(x) : x \in X\} > 0$. Aron, Lohman and Suárez explored the first properties of the $\lambda$-function and gave explicitly the form of this function for certain classical function and sequence spaces in $[3, 4]$. In $[3$, Question 4.1], Aron and Lohman posed the following challenge: 'What spaces of operators have the $\lambda$-property and what does the $\lambda$-function look like for these spaces?'. This question motivated a whole series of papers, in which Brown and Pedersen determined the exact form of the $\lambda$-function for every von Neumann algebra and for every unital C$^*$-algebra (cf. $[7, 8, 37]$). In their study of the $\lambda$-function, Brown and Pedersen introduce the set $A_q^{-1}$ of quasi-invertible elements in a unital C$^*$-algebra $A$, and study the geometric properties of $A_1$ in relation to the set $A_q^{-1}$. The following explicit formulae to compute the distance from an element in $A_1$ to the set of quasi-invertible elements or to $\mathcal{E}(A_1)$ are established by Brown and Pedersen:

$$
\text{dist}(a, \mathcal{E}(A_1)) = \begin{cases} 
\max\{1 - m_q(a), \|a\| - 1\} & \text{if } a \in A_q^{-1}, \\
\max\{1 + \alpha_q(a), \|a\| - 1\} & \text{if } a \notin A_q^{-1},
\end{cases}
$$

where $\alpha_q(a) = \text{dist}(a, A_q^{-1})$ and $m_q(a) = \text{dist}(a, A \cap A_q^{-1})$ (cf. $[8$, Theorem 2.3$]$). The $\lambda$-function is given by

$$
\lambda(a) = \begin{cases} 
\frac{1 + m_q(a)}{2} & \text{if } a \in A_1 \cap A_q^{-1}, \\
\frac{1}{2} (1 - \alpha_q(a)) & \text{if } a \in A_1 \setminus A_q^{-1},
\end{cases}
$$

(cf. $[8$, Theorem 3.7$]$). Furthermore, every von Neumann algebra (i.e. a C$^*$-algebra which is also a dual Banach space) satisfies the uniform $\lambda$-property, actually the expression $\lambda(a) = (1 + m_q(a))/2$ holds for every element $a$ in the closed unit ball of a von Neumann algebra (cf. $[37$, Theorem 4.2$]$).

There exists a class of complex Banach spaces defined by certain holomorphic properties of their open unit balls, we refer to the class of JB$^*$-triples. Harris shows in $[22]$ that the open unit ball of every C$^*$-algebra $A$ is a bounded symmetric domain, and the same conclusion holds for the open unit ball of every closed linear subspace $U \subseteq A$ invariant under the Jordan triple product

$$
\{x, y, z\} := \frac{1}{2} (xy^*z + yz^*x).
$$

In $[30]$, Kaup introduces the concept of a JB$^*$-triple, and shows that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB$^*$-triple and, in this way, the category of all bounded symmetric domains with base point is equivalent to the category of JB$^*$-triples. Actually, every C$^*$-algebra is a JB$^*$-triple with respect to (1.1), however, the class of JB$^*$-triples is strictly wider than the class of C$^*$-algebras (see next section for definitions and examples).

For each complex Banach space $E$ in the class of JB$^*$-triples, the open unit ball of $E$ enjoys similar geometric properties to those exhibited by the closed unit ball of a C$^*$-algebra. Many geometric properties studied in the setting of C$^*$-algebras have been studied in the wider class of JB$^*$-triples. For example, in recent papers, the first, third and fourth authors of this note extend the notion of quasi-invertible elements from the setting of C$^*$-algebras to the wider class of JB$^*$-triples introducing the concept of Brown–Pedersen quasi-invertible elements (see $[25, 26, 42]$). Once the class $E_q^{-1}$
of Brown–Pedersen quasi-invertible elements in a JB$^*$-triple $E$ has been introduced, the following question seems the natural problem to be studied.

**Problem 1.1** What JB$^*$-triples have the $\lambda$-property and what does the $\lambda$-function look like in the case of a JB$^*$-triple?

Only partial answers to the above problem are known. Accordingly to the terminology employed by Brown and Pedersen, for each element $x$ in a JB$^*$-triple $E$, the symbol $\alpha_q(x)$ will denote the distance from $x$ to the set $E_q^{-1}$ of Brown–Pedersen quasi-invertible elements in $E$, that is, $\alpha_q(x) = \text{dist}(x, E_q^{-1})$. The known estimates for the $\lambda$-function in the setting of JB$^*$-triples are the following: for each (complete tripotent) $v \in E(E_1)$, and each element $x$ in the closed unit ball of the Peirce-2 subspace $E_2(v)$ which is not Brown–Pedersen quasi-invertible in $E$ we have:

$$\lambda(x) \leq \frac{1}{2}(1 - \alpha_q(x));$$  \hspace{1cm} (1.2)

consequently, $\lambda(x) = 0$ whenever $\alpha_q(x) = 1$ (cf. [26, Theorem 3.7]).

In this paper, we continue with the study of the $\lambda$ function in the general setting of JB$^*$-triples. In Section 2, we introduce the basic facts and definitions needed in the paper, and we revisit the concept of Brown–Pedersen quasi-invertibility by finding new characterizations of this notion in terms of the triple spectrum and the orthogonal complement of an element.

We begin Section 3 proving that, for each element $x$ in a JB$^*$-triple $E$, the square root of the quadratic conorm, $\gamma_q(x)$, introduced in [11], measures the distance from $x$ to the set $E \setminus E_q^{-1}$ (see Theorem 3.1), where by convention $\gamma_q(x) = 0$ for every $x \in E \setminus E_q^{-1}$. It is established that for every Brown–Pedersen quasi-invertible element $a$ in $E$ we have:

$$\text{dist}(a, E(E_1)) = \max\{1 - m_q(a), \|a\| - 1\}$$

(see Proposition 3.2). This formula is complemented with Theorem 3.4 where we prove that $\lambda(a) = (1 + m_q(a))/2$, for every Brown–Pedersen quasi-invertible element $a$ in $E_1$.

For elements in the closed unit ball of a JB$^*$-triple which are not Brown–Pedersen quasi-invertible, we improve the estimates in (1.2) (see [26]) by proving that for every JB$^*$-triple $E$ with $E(E_1) \neq \emptyset$, the inequalities

$$1 + \|a\| \geq \text{dist}(a, E(E_1)) \geq \max\{1 + \alpha_q(a), \|a\| - 1\},$$

hold for every $a$ in $E \setminus E_q^{-1}$ (Theorem 3.6). Consequently, the inequality

$$\lambda(a) \leq \frac{1}{2}(1 - \alpha_q(a)),$$

holds for every $a \in E_1 \setminus E_q^{-1}$ without assuming that $a$ lies in the Peirce-2 subspace associated with a complete tripotent $v$ in $E$ (see Corollary 3.7).

A JBW$^*$-triple is a JB$^*$-triple which is also a dual Banach space. In the setting of JB$^*$-triples, JBW$^*$-triples play an analogue role to that played by von Neumann algebras in the class of C$^*$-algebras. In Section 4, we prove that every JBW$^*$-triple satisfies the uniform $\lambda$-property (see Corollary 4.3), a result which extends [37, Theorem 4.2] to the context of JBW$^*$-triples. This result will follow from
Theorem 4.2, where it is established that for every JBW$^*$-triple $W$ the $\lambda$-function on $W_1$ is given by the expression:

$$\lambda(a) = \begin{cases} 
\frac{1 + m_q(a)}{2} & \text{if } a \in W_1 \cap W^{-1}, \\
\frac{1}{2}(1 - \alpha_q(a)) = \frac{1}{2} & \text{if } a \in W_1 \setminus W^{-1}.
\end{cases}$$

The paper finishes with a result establishing that, for every element $a$ in the closed unit ball of a JB$^*$-triple $E$ which is not Brown–Pedersen quasi-invertible, if $\mathcal{E}(E_1) \neq \emptyset$, then the distance from $a$ to the latter set is given by the formula

$$\text{dist}(a, \mathcal{E}(E_1)) = 1 + \alpha_q(a)$$

(see Theorem 4.5).

2. von Neumann regularity and Brown–Pedersen invertibility

From a purely algebraic point of view, a complex Jordan triple system is a complex linear space $E$ equipped with a triple product

$\{., ., .\} : E \times E \times E \to E,$

$$(x, y, z) \mapsto \{x, y, z\},$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies the Jordan identity

$$L(x, y) \{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all $x, y, a, b, c \in E$, where $L(x, y) : E \to E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$.

Given an element $a$ in a complex Jordan triple system $E$, the symbol $Q(a)$ will denote the conjugate linear operator on $E$ given by $Q(a)(x) := \{a, x, a\}$. It is known that the fundamental identity

$$Q(x)Q(y)Q(x) = Q(Q(x)y)$$

(2.1)

holds for every $x, y$ in a complex Jordan triple system $E$ (cf. [13, Lemma 1.2.4]).

The studies on von Neumann regular elements in Jordan triple systems began with the contributions of Loos [35] and Fernández-López et al. [16]. We recall that an element $a$ in a Jordan triple system $E$ is called von Neumann regular if $a \in Q(a)(E)$ and strongly von Neumann regular when $a \in Q(a)^3(E)$.

Enriching the geometrical structure of a complex Jordan triple system, we find the class of complex Banach spaces called JB$^*$-triples, introduced by Kaup to classify bounded symmetric domains in arbitrary complex Banach spaces (cf. [30]). More concretely, a JB$^*$-triple is a complex Jordan triple system $E$ which is a Banach space satisfying the additional geometric axioms:

(a) For each $x \in E$, the map $L(x, x)$ is a hermitian operator with non-negative spectrum;
(b) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$.

The basic bibliography on JB$^*$-triples can be found in [13, 43].
Examples of JB*-triples include all C*-algebras with the triple product given in (1.1), all JB*-algebras with triple product

$$\{a, b, c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*,$$

and the Banach space $L(H, K)$ of all bounded linear operators between two complex Hilbert spaces $H, K$ with respect to (1.1).

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique isometric predual [5]). The triple product of every JBW*-triple is separately weak* continuous (cf. [5]), and the second dual, $E^{**}$, of a JB*-triple $E$ is a JBW*-triple (cf. [14]).

An element $a$ in a JB*-triple $E$ is von Neumann regular if, and only if, it is strongly von Neumann regular if, and only if, there exists $b \in E$ such that $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)Q(b) - Q(b)Q(a) = 0$ (cf. [16, Theorem 1; 31, Lemma 4.1]). Although for a von Neumann regular element $a$ in a JB*-triple $E$, there exist many elements $c$ in $E$ such that $Q(a)(c) = a$, there exists a unique element $b \in E$ satisfying $Q(a)(b) = a$, $Q(b)(a) = b$ and $[Q(a), Q(b)] := Q(a)Q(b) - Q(b)Q(a) = 0$, this unique element $b$ is called the generalized inverse of $a$ in $E$ and it is denoted by $a^\dagger$.

The simplest examples of von Neumann regular elements, probably, are tripotents. We recall that a von Neumann triple $E$ is full of complete tripotents.

A tripotent in $E$ is called tripotent when $\{e, e, e\} = e$. Each tripotent $e$ in $E$ induces a decomposition of $E$ (called the Peirce decomposition) in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where for $i = 0, 1, 2$, $E_i(e)$ is the $i/2$ eigenspace of $L(e, e)$. The Peirce rules affirm that

$$\{E_1(e), E_2(e), E_k(e)\} \text{ is contained in } E_{i-j+k}(e) \text{ if } i-j+k \in \{0, 1, 2\} \text{ and is zero otherwise.}$$

In addition,

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.$$

The projection $P_i(e)$ of $E$ onto $E_i(e)$ is called the Peirce $k$-projection. It is known that Peirce projections are contractive (cf. [20]) and satisfy that $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$ and $P_0(e) = \text{Id}_E - 2L(e, e) + Q(e)^2$. A tripotent $e$ in $E$ is said to be unitary if $L(e, e)$ coincides with the identity map on $E$, that is, $E_2(e) = E$. We shall say that $e$ is complete when $E_0(e) = \{0\}$.

The Peirce space $E_2(e)$ is a unital JB*-algebra with unit $e$, product $x \circ_e y := \{x, e, y\}$ and involution $x^{e*} := \{e, x, e\}$, respectively. Furthermore, the triple product in $E_2(e)$ is given by

$$\{a, b, c\} = (a \circ_e b^{e*}) \circ_e c + (c \circ_e b^{e*}) \circ_e a - (a \circ_e c) \circ_e b^{e*} (a, b, c \in E_2(e)).$$

When a C*-algebra $A$ is regarded as a JB*-triple with the product given in (1.1), tripotent elements in $A$ are precisely partial isometries of $A$. A JB*-triple might not contain a single tripotent element (consider, for example, $C_0[0, 1]$ the C*-algebra of all complex-valued continuous functions on $[0, 1]$ vanishing at 0). However, since the complete tripotents of a JB*-triple $E$ coincide with the complex and the real extreme points of its closed unit ball (cf. [6, Lemma 4.1; 33, Proposition 3.5] or [13, Theorem 3.2.3]), every JBW*-triple is full of complete tripotents.

As shown by Kaup in [31], the triple spectrum is one of the most appropriate tools to study and determine von Neumann regular elements. The triple spectrum of an element $a$ in a JB*-triple $E$ is
the set
\[ \text{Sp}(a) := \{ t \in \mathbb{C} : a \notin (L(a, a) - t^2 \text{Id}_E)(E) \} \].

The extended spectrum of \( a \) is the set \( \text{Sp}'(a) := \text{Sp}(a) \cup \{ 0 \} \). As usually, the smallest closed complex subtriple of \( E \) containing \( a \) will be denoted by \( E_a \). The set
\[ \Sigma(a) := \{ s \in \mathbb{C} : (L(a, a) - s \text{Id}_E)\vert_{E_a} \text{ is not invertible in } L(E_a) \} \]

stands for the usual spectrum of the restricted operator \( L(a, a)\vert_{E_a} \) in \( L(E_a) \). Following standard notation, we assume that \( \Sigma(a) = \emptyset \) whenever \( a = 0 \) (this is actually an equivalence, compare [31, Lemma 3.2]). The following properties were established in [31].

(i) \( \Sigma(a) \) is a compact subset of \( \mathbb{R} \) with \( \Sigma(a) \geq 0 \) and the origin cannot be an isolated point of \( \Sigma(a) \). The origin cannot be an isolated point of \( \text{Sp}(a) \) and \( \text{Sp}(a) = -\text{Sp}(a) \).

(ii) \( \text{Sp}(a) = \{ t \in \mathbb{C} : t^2 \in \Sigma(a) \} \) and \( \text{Sp}(a) \neq \emptyset \), whenever \( a \neq 0 \).

(iii) \( S_a := \text{Sp}(a) \cap [0, \infty) \) is a compact subset of \( \mathbb{R} \), \( \|a\| \in S_a \subseteq [0, \|a\|] \), and there exists a unique triple isomorphism \( \Psi : E_a \rightarrow C_0(S_a \cup \{0\}) \) such that \( \Psi(a)(s) = s \) for every \( s \in S_a \), where \( C_0(S_a \cup \{0\}) \) denotes the space of all complex-valued, continuous functions on \( S_a \cup \{0\} \) vanishing at zero. If \( 0 \in S_a \), then it is not isolated in \( S_a \).

(iv) The spectrum \( \text{Sp}(a) \) does not change when computed with respect to any closed complex subtriple \( F \subseteq E \) containing \( a \).

(v) The element \( a \) is von Neumann regular if, and only if, \( 0 \notin \text{Sp}(a) \).

The basic properties of the triple spectrum lead us to the continuous triple functional calculus. Given an element \( a \) in a JB*-triple \( E \) and a function \( f \in C_0(S_a \cup \{0\}) \), \( f_t(a) \) will denote the unique element in \( E_a \) which is mapped to \( f \) when \( E_a \) is identified as JB*-triple with \( C_0(S_a \cup \{0\}) \). Consequently, for each natural \( n \), the element \( a^{1/(2n-1)} \) coincides with \( f_t(a) \), where \( f_t(\lambda) := \lambda^{1/(2n-1)} \). When \( a \) is an element in a JBW*-triple \( W \), the sequence \( (a^{1/(2n-1)}) \) converges in the weak*-topology of \( W \) to a tripotent, denoted by \( r(a) \), and called the range tripotent of \( a \). The tripotent \( r(a) \) is the smallest tripotent \( e \in W \) satisfying that \( a \) is positive in the JBW*-algebra \( W_2(e) \) (see, for example, [15, comments before Lemma 3.1] or [9, Section 2]).

We shall habitually regard a Banach space \( X \) as being contained in its bidual, \( X^{**} \), and we identify the weak*-closure, in \( X^{**} \), of a closed subspace \( Y \) of \( X \) with \( Y^{**} \). For an element \( a \) in a JB*-triple \( E \), the range tripotent \( r(a) \) is defined in \( E^{**} \). Having this in mind, the range tripotent of an element \( a \) in a JB*-triple is the element in \( E_a^{**} \equiv (C_0(S_a \cup \{0\}))^{**} \) corresponding with the characteristic function of the set \( S_a \).

We recall that an element \( a \) in a unital Jordan Banach algebra \( J \) is called invertible whenever there exists \( b \in J \) satisfying \( a \circ b = 1 \) and \( a^2 \circ b = a \). The element \( b \) is unique and it will be denoted by \( a^{-1} \). The set \( J^{-1} = \text{inv}(J) \) of all invertible elements in \( J \) is open in the norm topology and \( a \in J^{-1} \) whenever \( \|a - 1\| < 1 \). It is well known that \( a \) is invertible if, and only if, the mapping \( x \mapsto U_a(x) := 2(x \circ a) \circ a - a^2 \circ x \) is invertible, and in that case \( U_a^{-1} = U_{a^{-1}} \) (see, for example [13, p. 107]).

The reduced minimum modulus was introduced in [11] to study the quadratic conorm of an element in a JB*-triple. The reduced minimum modulus of a non-zero bounded linear or conjugate linear operator \( T \) between two normed spaces \( X \) and \( Y \) is defined by
\[ \gamma(T) := \inf \{ \|T(x)\| : \text{dist}(x, \text{ker}(T)) \geq 1 \}. \] (2.2)
Following [29], we set $\gamma(0) = \infty$ (reader should be awarded that in [2] $\gamma(0) = 0$). When $X$ and $Y$ are Banach spaces, we have

$$\gamma(T) > 0 \iff T(X) \text{ is norm closed}$$

(cf. [29, Theorem IV.5.2]). The quadratic conorm of an element $a$ in a JB*-triple $E$ is defined as the reduced minimum modulus of $Q(a)$ and it will be denoted by $\gamma^q(a)$, that is, $\gamma^q(a) = \gamma(Q(a))$. The main results in [11] show, among many other things, that:

(S. vi) An element $a$ is von Neumann regular if, and only if, $Q(a)$ has norm-closed image if, and only if, the range tripotent $r(a)$ of $a$ lies in $E$ and $a$ is positive and invertible element of the JB*-algebra $E_2(r(a))$. Furthermore, when $a$ is von Neumann regular we have:

$$Q(a)Q(a^\dagger) = P_2(r(a)) = Q(a^\dagger)Q(a)$$

and

$$L(a, a^\dagger) = L(a^\dagger, a) = L(r(a), r(a))$$

(cf. [32, comments after Lemma 3.2] or [11, p. 192]).

(S. vii) For each element $a$ in $E$, $\gamma^q(a) = \inf\{\Sigma(a)\} = \inf\{t^2 : t \in \text{Sp}(a)\}$.

Let us recall that the Bergmann operator associated with a couple of elements $x, y$ in a JB*-triple $E$ is the mapping $B(x, y) : E \to E$ defined by $B(x, y) = \text{Id} - 2L(x, y) + Q(x)Q(y)$ (cf. [36] or [43, p. 305]).

Inspired by the definition of quasi-invertible elements in a C*-algebra developed by Brown and Pedersen in [7, 8], Tahlawi, Siddiqui and Jamjoom introduced and developed, in [25, 26, 42], the notion of Brown–Pedersen quasi-invertible elements in a JB*-triple $E$. An element $a$ in $E$ is Brown–Pedersen quasi-invertible (BP-quasi-invertible for short) if there exists $b \in E$ such that $B(a, b) = 0$. It was established in [25, 42] that an element $a$ in $E$ is BP-quasi-invertible if, and only if, one of the following equivalent statements holds:

(a) $a$ is von Neumann regular and its range tripotent $r(a)$ is an extreme point of the closed unit ball of $E$ (i.e. $r(a)$ is a complete tripotent of $E$);

(b) there exists a complete tripotent $e \in E$ such that $a$ is positive and invertible in the JB*-algebra $E_2(e)$.

The set of all BP-quasi-invertible elements in $E$ is denoted by $E_q^{-1}$. Let us observe that, in principle, the notion of invertibility makes no sense in a general JB*-triple. By [25, Theorem 8], $E_q^{-1}$ is open in the norm topology (the reason being that, for each complete tripotent $e$, the set of invertible elements in the JB*-algebra $E_2(e)$ is open and the Peirce projections are contractive).

Let us observe that when a C*-algebra $A$ is regarded as a JB*-triple with product given by (1.1), the BP-quasi-invertible elements in $A$, as JB*-triple, are exactly the quasi-invertible elements of $A$ in the terminology of Brown and Pedersen in [7, 8].

We shall also need a characterization of BP-quasi-invertible elements in terms of the orthogonal complement. First, we recall that elements $a, b$ in a JB*-triple $E$ are said to be orthogonal (denoted by $a \perp b$) when $L(a, b) = 0$. By [10, Lemma 1], we know that $a \perp b$ if, and only if, one of the
following statements holds:
\[
{a, a, b} = 0; \quad a \perp r(b); \quad r(a) \perp r(b);
E_2^{**}(r(a)) \perp E_2^{**}(r(b)); \quad E_a \perp E_b; \quad \{b, b, a\} = 0.
\] (2.3)

For each subset \(M \subseteq E\), we write \(M^\perp_E\) for the (orthogonal) annihilator of \(M\) defined by
\[
M^\perp_E := \{y \in E : y \perp x, \forall x \in M\}.
\]

It is known that, for each tripotent \(e\) in \(E\), \(\{e\}^\perp = E_0(e)\). Furthermore, the identity \(\{a\}^\perp = (E^{**})_0(r(a)) \cap E\) holds for every \(a \in E\) (cf. [12, Lemma 3.2]). We therefore have the following lemma.

**Lemma 2.1** Let \(a\) be an element in a JB*-triple \(E\). Then \(a\) is BP-quasi-invertible if, and only if, \(a\) is von Neumann regular and \(\{a\}^\perp = \{0\}\).

We initiate the novelties with a series of technical lemmas.

**Lemma 2.2** Let \(e\) be a complete tripotent in a JB*-triple \(E\) and let \(z\) be an element in \(E\). Suppose that \(P_2(e)(z)\) is invertible in the JB*-algebra \(E_2(e)\). Then \(z\) is BP-quasi-invertible.

**Proof.** By hypothesis, \(z_2 = P_2(e)(z)\) is invertible in the JB*-algebra \(E_2(e)\) with inverse \(z_2^{-1}\), and since \(e\) is complete, \(z = z_2 + z_1\) where \(z_1 = P_1(e)(z)\). Let us observe that \(z_2\) is von Neumann regular in \(E\) and \(z_2^\perp = z_2^{-1}\).

We claim that the invertibility of \(z_2\) in \(E_2(e)\) also implies that \(r(z_2)\) is a unitary tripotent in the JB*-triple \(E_2(e)\). Indeed, since for each \(x \in E_2(e)\),
\[
x = P_2(r(z_2))(x) = Q(z_2)Q(z_2^{-1})(x) = U_{z_2}U_{z_2^{-1}}(x),
\]
we deduce that \(P_2(r(z_2))|_{E_2(e)} = \text{Id}_{E_2(e)}\), proving the claim.

Clearly, \(E_2(e) = E_2(r(z_2))\). Given \(x \in E\), the condition
\[
\{r(z_2), x, r(z_2)\} = 0
\]
implies \(0 = Q(r(z_2))^2(x) = P_2(r(z_2))(x) = P_2(e)(x)\), and hence \(x = P_1(e)(x)\) lies in \(E_1(e)\). Thus, \(E_1(r(z_2)) \oplus E_0(r(z_2)) \subseteq E_1(e)\). Taking \(x \in E_0(r(z_2))\), having in mind that \(e \in E_2(r(z_2))\), it follows from Peirce arithmetic that \(\{e, x, e\} = 0\), which shows that \(E_0(r(z_2)) \subseteq E_0(e) = \{0\}\). Therefore, \(r(z_2)\) is a complete tripotent in \(E\) and \(E_j(r(z_2)) = E_j(e)\), for every \(j = 0, 1, 2\).
Now, by Peirce arithmetic we have:

\[ Q(z)(z_2^2) = Q(z_2)(z_2^1) + 2Q(z_2, z_1)(z_2^1) + Q(z_1)(z_2^1) = z_2 + 2L(z_2, z_2^1)(z_1) + 0 \]
\[ = z_2 + 2L(r(z_2), r(z_2))(z_1) = z_2 + 2L(e, e)(z_1) = z_2 + z_1 = z \]

and

\[ Q(z_2^1)(z) = Q(z_2^1)(z_2) + Q(z_2^1)(z_1) = z_2^1. \]

This shows that \( z \) is von Neumann regular. Take \( a \in \{ z \}^{-1} \). Since

\[ 0 = \{ z_2^{-1}, z, a \} = \{ z_2^{-1}, z_2, a \} + \{ z_2^{-1}, z_1, a \} \]
\[ = P_2(e)(a) + \frac{1}{2}P_1(e)(a) + \{ z_2^{-1}, z_1, P_2(e)a \} + \{ z_2^{-1}, z_1, P_1(e)(a) \} \]
\[ = (\text{by Peirce arithmetic}) = P_2(e)(a) + \frac{1}{2}P_1(e)(a) + \{ z_2^{-1}, z_1, P_1(e)(a) \}, \]

which shows that \( P_1(e)(a) = 0, P_2(e)(a) = 0 \), and hence \( a = 0 \). Lemma 2.1 concludes the proof. □

**Remark 2.3** We would like to isolate the following fact, which has been established in the proof of Lemma 2.2: For each invertible element \( b \) in a unital JB*-algebra, \( J \), its range tripotent \( r(b) \) is a unitary element belonging to \( J \).

**Corollary 2.4** Let \( e \) be a complete tripotent in a JB*-triple \( E \). Suppose that \( a \) is an element in \( E \) satisfying \( \|a - e\| < 1 \), then \( a \) is BP-quasi-invertible.

**Proof.** Having in mind that \( P_2(e) \) is a contractive projection, we get

\[ \| P_2(e)(a) - e \| = \| P_2(e)(a - e) \| \leq \| a - e \| < 1. \]

Since \( E_2(e) \) is a unital JB*-algebra with unit \( e \), it follows that \( P_2(e)(a) \) is an invertible element in \( E_2(e) \). The conclusion of the corollary follows from Lemma 2.2. □

Let \( u, v \) be tripotents in a JB*-triple \( E \). We recall [36, Section 5] that \( u \leq v \) if \( v - u \) is a tripotent with \( u \perp v - u \). It is known that \( u \leq v \) if, and only if, \( P_2(u)(v) = u \), or equivalently, \( L(u, u)(v) = u \) (cf. [20, Lemma 1.6 and subsequent remarks]). In particular, \( u \leq v \) if, and only if, \( u \) is a projection in the JB*-algebra \( E_2(v) \). Let us observe that the condition \( u \geq v \) implies \( L(v, v)(u) = u \). However, the condition \( L(v, v)(u) = u \) need not imply, in general, the inequality \( v \geq u \) (cf. Remark 2.6).

The following technical lemma will be repeatedly used later.

**Lemma 2.5** Let \( e \) be a complete tripotent in a JB*-triple \( E \). Suppose that \( u \) is a tripotent in \( E_2(e) \) satisfying that \( L(u, u)(e) = e \). Then \( u \) is a complete tripotent of \( E \).

**Proof.** Since \( L(u, u)e = e \), we deduce that \( e \in E_2(u) \). By Peirce arithmetic, for each \( x \in E \),

\[ Q(e)(x) = Q(e)P_2(u)(x) \in E_2(u), \]

which implies \( E_2(e) = P_2(e)(E) = Q(e)^2(E) \subseteq E_2(u) \). Since, we also have \( L(e, e)(u) = u \), we get \( E_2(e) = E_2(u) \). Therefore, the mapping \( T = Q(u)|_{E_2(e)} : E_2(e) \rightarrow E_2(2) \) satisfies that \( T^2 = P_2(u)|_{E_2(e)} = P_2(e)|_{E_2(e)} \) is the identity on \( E_2(e) \).
Since the triple product of $E_2(e)$ is given by $[a, b, c] = (a \circ_e b^e) \circ_e c + (c \circ_e b^e) \circ_e a - (a \circ_e c) \circ_e b^e$ $(a, b, c \in E_2(e))$, we can easily see that $T(x) = U_a(x^e)$ and hence $U_a$ is an invertible operator in $L(E_2(e))$. We have therefore proved that $u$ is an invertible element in $E_2(e)$. Lemma 2.2 gives the desired statement. □

Lemma 4 in [40] proves that for every complete tripotent $e$ in a JB*-triple $E$, every unitary element in the JB*-algebra $E_2(e)$ is an extreme point of the closed unit ball of $E$ (i.e. a complete tripotent of $E$). This statement follows as a direct consequence of the above Lemma 2.5. Concretely, let $u$ be a unitary element in the JB*-algebra $E_2(e)$ (i.e. $u$ is invertible in $E_2(e)$ with $u^{-1} = u^e$). Since the triple product on $E_2(e)$ is given by

$$[a, b, c] = (a \circ_e b^e) \circ_e c + (c \circ_e b^e) \circ_e a - (a \circ_e c) \circ_e b^e$$

$(a, b, c \in E_2(e))$, we can easily see that $[u, u, e] = (u \circ_e u^e) \circ_e u + (e \circ_e u^e) \circ_e u - (u \circ_e e) \circ_e u^e = e$, and Lemma 2.5 gives the statement.

The following remark clarifies the connections between Lemmas 2.2, 2.5, Corollary 2.4 and [40, Lemma 4].

**Remark 2.6** Let $e$ be a complete tripotent in a JB*-triple $E$ and let $v$ be a tripotent in $E_2(e)$. Then the following statements are equivalent:

(a) $v$ is invertible in the JB*-algebra $E_2(e)$;
(b) $v$ is a unitary element in the JB*-algebra $E_2(e)$;
(c) $v$ is a unitary element in the JB*-triple $E_2(e)$;
(d) $L(v, v)(e) = e$.

The implication (a) ⇒ (b) is established in Remark 2.3. The implication (c) ⇒ (d) is clear. To see (b) ⇒ (c), we recall that the triple product in $E_2(e)$ is given by

$$[a, b, c] = (a \circ_e b^e) \circ_e c + (c \circ_e b^e) \circ_e a - (a \circ_e c) \circ_e b^e (a, b, c \in E_2(e)).$$

Since for each $a \in E_2(e)$, we have $U_v(a^e) = Q(v)(a)$ (where $U_b(c) := 2(b \circ_e c^e) \circ_e b - (b \circ_e b) \circ_e c$, for all $b, c \in E_2(e)$), we can deduce that

$$P_2(v)(a) = Q(v)^2(a) = U_v(U_v(a^e)^e) = U_vU_{v^e}(a) = a,$$

for every $a \in E_2(e)$, which shows that $P_2(v)|_{E_2(e)} = \text{Id}_{E_2(e)}$, and hence $v$ is a unitary tripotent in $E_2(e)$. To prove (d) ⇒ (a), we recall that $L(v, v)(e) = e$ implies that $e \in E_2(v)$, and hence $E_2(e) = E_2(v)$ because $v \in E_2(e)$, which proves (d) ⇒ (c). Furthermore, recalling that $\text{Id}_{E_2(e)} = P_2(v)|_{E_2(e)} = U_vU_{v^e}$, we obtain (a).

Consider now the statements:

(e) $v$ is an extreme point of $(E_2(e))_1$, or equivalently, $v$ is a complete tripotent in $E_2(e)$;
(f) $v$ is a complete tripotent in $E$.

It should be noted that (e) ⇒ (f) ⇒ (e), while (f) do not necessarily imply any of the above statements (a)–(d). We consider, for example, an infinite-dimensional complex Hilbert space $H$, a complete tripotent $e \in L(H)$ such that $e^e = 1$ and $p = e^e e \neq 1$. In this case, $L(H)_2(e) = L(H)e^e$. The element $p$ is a complete tripotent in $L(H)_2(e)$, and since $0 \neq 1 - p \perp p$ it follows that $p$ is
not complete in $L(H)$ (this shows that $(e) \not\Rightarrow (f)$). To see the second claim, pick a complete partial isometry $v \in L(e^*e(H))$ such that $vv^* \neq e^*e$ and $v^*v = e^*e$. It is easy to see that $v$ is a complete tripotent in $L(H)_2(e)$ and $L(v, v)(e) = \frac{1}{4}(vv^*e + ev^*v) = \frac{1}{4}(vv^* + e) \neq e$.

For more information on extreme points and unitary elements in $C^*$-algebras, JB$^*$-triples and JB-algebras, the reader is referred to [1, 17, Section 2, 27, 34, 39].

3. Distance to the extremals and the $\lambda$-function

In this section, we shall give some formulas to compute the distance from an element in a JB$^*$-triple $E$ to the set $\mathcal{E}(E_1)$ of extreme points of the closed unit ball of $E$. Since, in some cases, $\mathcal{E}(E_1)$ may be an empty set, we shall assume that $\mathcal{E}(E_1) \neq \emptyset$.

Let $E$ be a JB$^*$-triple. According to the terminology employed in [7, 8, 25, 26, 42], we define $\alpha_q : E \to \mathbb{R}_+^+$, by $\alpha_q(x) = \text{dist}(x, E_q^{-1})$. Inspired by the studies of Brown and Pedersen, we also introduce a mapping $m_q : E \to \mathbb{R}_0^+$ defined by

$$m_q(x) := \begin{cases} 0 & \text{if } x \notin E_q^{-1}, \\ (\gamma^q(x))^{1/2} & \text{if } x \in E_q^{-1}. \end{cases}$$

Let us note that, for each $x \in E_q^{-1}$,

$$m_q(x) = \inf \{ t : t \in \text{Sp}(x) \cap [0, \infty) \} = \max \{ \varepsilon > 0 : -\varepsilon, \varepsilon \notin \text{Sp}(x) \},$$

and $m_q(x) > 0$ if, and only if, $x \in E_q^{-1}$.

We claim that

$$m_q(\lambda x) = \lambda m_q(x), \quad (3.1)$$

for every $\lambda \in \mathbb{C}\setminus\{0\}$, $x \in E$. Indeed, since

$$\left(\mathbb{C}\setminus\{0\}\right)E_q^{-1} = E_q^{-1} \quad \text{and} \quad \mathbb{C}(E \setminus E_q^{-1}) = E \setminus E_q^{-1},$$

we may reduce to the case $a \in E_q^{-1}$ (cf. $(\Sigma.v)$ and $(\Sigma.iii)$). Since $L(\lambda a, \lambda a) = |\lambda|^2 L(a, a)$, it follows that $\Sigma(\lambda a) = |\lambda|^2 \Sigma(a)$, which gives $m_q(\lambda a) = \inf\{ \sqrt{t} : t \in \Sigma(\lambda a) \} = |\lambda|m_q(a)$.

As in the $C^*$-algebra setting, our next result shows that $m_q$ is actually a distance (cf. [7, Proposition 1.5] for the result in the setting of $C^*$-algebras).

**Theorem 3.1** Let $E$ be a JB$^*$-triple, then

$$m_q(a) = \text{dist}(a, E \setminus E_q^{-1}),$$

for every $a \in E$. In particular, $m_q(a) = \text{dist}(a, E \setminus E_q^{-1}) = (\gamma^q(a))^{1/2}$, for every $a \in E_q^{-1}$.

**Proof.** We can assume that $a \in E_q^{-1}$. By $(\Sigma.iii)$ and $(\Sigma.v)$, $S_a := \text{Sp}(a) \cap [0, \infty)$ is a compact subset of $\mathbb{R}$, $S_a \subseteq [0, ||a||]$, $||a|| = \max(S_a)$, $0 < m_q(a) = (\gamma^q(a))^{1/2} = \min(S_a)$, and there exists a unique triple isomorphism $\Psi : E_a \to C_0(S_a \cup \{0\}) = C(S_a)$ such that $\Psi(a)(s) = s$ for every $s \in S_a$. The range tripotent $r(a)$ coincides with the unit element in $C(S_a)$. Clearly, $\gamma_0 = a - m_q(a)r(a)$.
lies in $E_a \subseteq E$ and contains zero in its triple spectrum, therefore $y_0 \in E_0 \setminus E_q^{-1}$. Since $\|a - y_0\| = \|m_q(a)(a)\| = m_q(a)$, we get $m_q(a) \geq \text{dist}(a, E_0 \setminus E_q^{-1})$.

To prove the reverse inequality, we first assume that $\|a\| \leq 1$. Arguing by reduction to the absurd, we suppose that $m_q(a) > \text{dist}(a, E_0 \setminus E_q^{-1})$, then there exists $z \in E_0 \setminus E_q^{-1}$ with $\|a - z\| < m_q(a) = (\gamma_q(a))^{1/2}$. Since $a \in E_q^{-1}$, its range tripotent, $r(a)$, is a complete tripotent in $E$, and $a$ is a positive, invertible element in $E_2(r(a))$. The contractivity of $P_2(r(a))$, assures that

$$\|a - P_2(r(a))(z)\| = \|P_2(r(a))(a - z)\| \leq \|a - z\| < m_q(a).$$

Now, we compute the distance

$$\|P_2(r(a))(z) - r(a)\| = \|P_2(r(a))(z) - a + a - r(a)\| < m_q(a) + \max\{1 - m_q(a), \|a\| - 1\} = 1.$$ 

The general theory of invertible elements in JB$^*$-algebras shows that the element $P_2(r(a))(z)$ is invertible in $E_2(r(a))$, because $r(a)$ is the unit element in the latter JB$^*$-algebra. Lemma 2.2 implies that $z \in E_q^{-1}$, which contradicts that $z \in E_0 \setminus E_q^{-1}$. We have therefore proved that $m_q(a) = \text{dist}(a, E_0 \setminus E_q^{-1})$, for every $a \in E_q^{-1}$ with $\|a\| \leq 1$.

Finally, given $a \in E_q^{-1}$, we have

$$m_q \left( \frac{a}{\|a\|} \right) = \text{dist} \left( \frac{a}{\|a\|}, E_q^{-1} \right),$$

and $\|a\| m_q(a/\|a\|) = m_q(a)$. Therefore,

$$m_q(a) = \|a\| m_q \left( \frac{a}{\|a\|} \right) \leq \|a\| \left\| \frac{a}{\|a\|} - c \right\| = \|a\| - \|a\| c$$

for every $c \in E_q^{-1}$, which shows that

$$m_q(a) \leq \text{dist}(a, \|a\|(E_q^{-1})) = \text{dist}(a, E_q^{-1}).$$

It was already noted in [25, Lemma 25] that

$$\alpha_q(\lambda x) = |\lambda| \alpha_q(x); \quad \alpha_q(x) \leq \|x\|$$

and

$$|\alpha_q(x) - \alpha_q(y)| \leq \|x - y\|$$

for every $x, y \in E, \lambda \in \mathbb{C}$. Theorem 3.1 implies that

$$|m_q(x) - m_q(y)| \leq \|x - y\|$$

for every $x, y \in E$. 

\[\square\]
Our next goal is an extension of [8, Theorem 2.3] to the more general setting of JB*-triples, and determines the distance from a BP-quasi-invertible element in a JB*-triple $E$ to the set of extreme points in $E_1$.

**Proposition 3.2** Let $a$ be a BP-quasi-invertible element in a JB*-triple $E$. Then

$$\text{dist}(a, \mathcal{E}(E_1)) = \max\{1 - m_q(a), \|a\| - 1\}.$$ 

**Proof.** Again, by (Σ.iii) and (Σ.v), the set $S_a := \text{Sp}(a) \cap [0, \infty)$ is a compact subset of $\mathbb{R}$, $S_a \subseteq [0, \|a\|]$, $\|a\| = \max(S_a)$, $0 < m_q(a) = \min(S_a)$, and there exists a unique triple isomorphism $\Psi : E_a \to C(S_a)$ such that $\Psi(a)(s) = s$ for every $s \in S_a$, and the range tripotent $r(a)$ coincides with the unit element in $C(S_a)$. Since $r(a) \in \mathcal{E}(E_1)$ and

$$\text{dist}(a, \mathcal{E}(E_1)) \leq \|a - r(a)\| = \max\{1 - m_q(a), \|a\| - 1\}.$$ 

Given $e \in \mathcal{E}(E_1)$, we always have $\|a - e\| \geq \|a\| - 1$. Since

$$m_q(a) = m_q(e - (e - a)) \geq m_q(e) - \|e - a\| = 1 - \|e - a\|,$$

we also have $\text{dist}(a, \mathcal{E}(E_1)) \geq \max\{1 - m_q(a), \|a\| - 1\}$. □

**Corollary 3.3** Let $E$ be a JB*-triple. Then

$$\{a \in E_q^{-1} : \|a\| = m_q(a) = (\gamma^q(a))^{1/2}\} = \{0, \infty\} \cap \mathcal{E}(E_1).$$

Our next result is a first estimate for the $\lambda$-function, it can be regarded as an appropriate triple version of [8, Theorem 3.1; 41, Lemma 2.4].

**Theorem 3.4** Let $a$ be a BP-quasi-invertible element in the closed unit ball of a JB*-triple $E$. Then for every $\lambda \in [\frac{1}{2}, (1 + m_q(a))/2]$ there exist $e, u$ in $\mathcal{E}(E_1)$ satisfying

$$a = \lambda e + (1 - \lambda)u.$$ 

When $1 \geq \lambda > (1 + m_q(a))/2$, such a convex decomposition cannot be obtained. Consequently, $\lambda(a) = (1 + m_q(a))/2$, for every $a \in E_q^{-1} \cap E_1$.

**Proof.** The range tripotent $r(a) \in \mathcal{E}(E_1)$ is the unit element of subtriple $E_a \equiv C(S_a)$, where $S_a := \text{Sp}(a) \cap [0, \infty)$ is a compact subset of $\mathbb{R}$, $S_a \subseteq [0, \|a\|]$, $\|a\| = \max(S_a)$, $0 < m_q(a) = \min(S_a)$ and there exists a triple isomorphism $\Psi : E_a \to C(S_a)$ such that $\Psi(a)(s) = s$ for every $s \in S_a$. It is part of the folklore in C*-algebra theory that for every $\lambda \in [\frac{1}{2}, (1 + m_q(a))/2]$, the function $\Psi(a) : s \mapsto s$ can be written in the form

$$\Psi(a) = \lambda v_1 + (1 - \lambda)v_2,$$

where $v_1, v_2$ are two unitary elements in $C(S_a)$ (see [28, Lemma 6] or [41, Lemma 2.4] for a proof in a more general setting). Since $v_1, v_2$ are unitary elements in $E_a \equiv C(S_a)$ and $r(a)$ is an extreme point of the closed unit ball of $E$, the tripotents $e = \Psi^{-1}(v_1)$ and $u = \Psi^{-1}(v_2)$ belong to $\mathcal{E}(E_1)$ (cf. Lemma 2.5) and $a = \lambda e + (1 - \lambda)u.$
Given $1 \geq \lambda > (1 + m_q(a))/2$, if we assume that $a = \lambda e + (1 - \lambda)y$, where $e \in \mathcal{E}(E_1)$ and $y \in E_1$, we have

$$\|a - e\| = (1 - \lambda)\|y - e\| \leq 2(1 - \lambda),$$

which shows that $\text{dist}(a, \mathcal{E}(E_1)) \leq 2(1 - \lambda)$. However, by Proposition 3.2, $1 - m_q(a) = \text{dist}(a, \mathcal{E}(E_1))$, and hence $\lambda \leq (1 + m_q(a))/2$, which is impossible.

Our next result was in [26, Theorem 3.5]. We can give now an alternative proof from the above results.

**Corollary 3.5** Let $E$ be a JB*-triple. Let $a$ be an element in $E_1$. Then $a \in E_q^{-1}$ if, and only if, $a = \alpha v_1 + (1 - \alpha)v_2$ for some extreme points $v_1, v_2$ in $\mathcal{E}(E_1)$ and $0 \leq \alpha < \frac{1}{2}$.

**Proof.** ($\Rightarrow$) Since $a \in E_q^{-1}\setminus \mathcal{E}(E_1)$, the distance $m_q$, satisfies $0 < m_q(a) < 1$, and hence $(\frac{1}{2}, (1 + m_q(a))/2] \neq \emptyset$. Take $\lambda \in (\frac{1}{2}, (1 + m_q(a))/2]$. Theorem 3.4 implies the existence of $v_1, v_2$ in $\mathcal{E}(E_1)$ satisfying $a = \lambda v_2 + (1 - \lambda)v_1$. The statement follows for $\alpha = 1 - \lambda$.

($\Leftarrow$) Note that $\|a - v_2\| = \alpha \|v_1 - v_2\| < 1$. Corollary 2.4 implies that $a \in (J)_q^{-1}$. \hfill \Box

In [25, Theorem 26], the authors show that, given a complete tripotent $e$ in a JB*-triple $E$ (i.e., $e \in \mathcal{E}(E_1)$), then for each element $a \in E_2(e)\setminus E_q^{-1}$ we have:

$$\text{dist}(a, \mathcal{E}(E_1)) \geq \max\{1 + \alpha_q(a), \|a\| - 1\}.$$ 

Our next result shows that there is no need to assume that the element $a$ lies in the Peirce-2 subspace of a complete tripotent to prove the same inequality.

**Theorem 3.6** Let $E$ be a JB*-triple satisfying $\mathcal{E}(E_1) \neq \emptyset$. Then the inequalities

$$1 + \|a\| \geq \text{dist}(a, \mathcal{E}(E_1)) \geq \max\{1 + \alpha_q(a), \|a\| - 1\}$$

hold for every $a \in E \setminus E_q^{-1}$.

**Proof.** Let us fix $a \in E \setminus E_q^{-1}$. Clearly, for each $e \in \mathcal{E}(E_1)$, $\|e - a\| > \|a\| - 1$, and hence

$$\text{dist}(a, \mathcal{E}(E_1)) \geq \|a\| - 1.$$ 

Fix an arbitrary $e \in \mathcal{E}(E_1)$. If $\|e - a\| < \beta$, then $\beta > 1$, otherwise $\|e - a\| < 1$ and Corollary 2.4 implies that $a \in E_q^{-1}$, which is impossible. Now, the inequality

$$m_q((\beta - 1)e + e) = m_q(\beta e + e - e) \geq m_q(\beta e) - \|e - a\| = \beta - \|e - a\| > 0,$$

shows that $(\beta - 1)e + e$ lies in $E_q^{-1}$. Then

$$\alpha_q(a) \leq \|a - ((\beta - 1)e + e)\| = \beta - 1.$$ 

This proves that

$$\alpha_q(a) + 1 \leq \beta,$$

for every $e \in \mathcal{E}(E_1)$ and $\beta > \|a - e\|$, witnessing that $\text{dist}(a, \mathcal{E}(E_1)) \geq 1 + \alpha_q(a)$.

\hfill \Box
COROLLARY 3.7 Let $E$ be a JB*-triple satisfying $\mathcal{E}(E_1) \neq \emptyset$. Then
\[ \lambda(a) \leq \frac{1}{2}(1 - \alpha_q(a)), \]
for every $a \in E_1 \setminus E_q^{-1}$.

Proof. Let us fix $a \in E_1 \setminus E_q^{-1}$. By Theorem 3.6, we have
\[ \text{dist}(a, \mathcal{E}(E_1)) \geq \max\{\alpha_q(a) + 1, \|a\| - 1\}. \]
Thus, if $a$ writes in the form $a = \lambda e + (1 - \lambda)y$, where $e \in \mathcal{E}(E_1)$, $y \in E_1$ and $0 \leq \lambda \leq 1$ we have $a - e = (\lambda - 1)e + (1 - \lambda)y$, which gives
\[ \alpha_q(a) + 1 \leq \text{dist}(a, \mathcal{E}(E_1)) \leq \|a - e\| = |1 - \lambda|\|y - e\| \leq 2(1 - \lambda), \]
which proves $\lambda \leq \frac{1}{2}(1 - \alpha_q(a))$. \qed

4. The $\lambda$-function of a JBW*-triple

We can present now a precise description of the $\lambda$-function in the case of a JBW*-triple. The main goal of this section is to prove that every JBW*-triple satisfies the uniform $\lambda$-property, extending the result established by Pedersen in [37, Theorem 4.2] in the context of von Neumann algebras.

First, we observe that whenever we replace JB*-triples with JBW*-triples, the $\alpha_q$ function is much more simpler to compute on the closed unit ball.

PROPOSITION 4.1 Let $W$ be a JBW*-triple. Then, for each $a$ in $W_1$ we have
\[ \text{dist}(a, \mathcal{E}(W_1)) = 1 - m_q(a). \]
In particular, $\alpha_q(a) = 0$, for every $a \in W_1 \setminus W_q$.

Proof. When $a \in W_1 \setminus W_q$, the statement follows from Proposition 3.2. Let us assume that $a \notin W_q$, then $0$ is not an isolated point in $S_a$ (cf. (Σ.i)). One more time, we shall identify $W_a$ (the (norm-closed) JB*-subtriple of $W$ generated by $a$) with $C_0(S_a \cup \{0\})$. Therefore, for each $\delta > 0$ the sets $]0, \|a\| \cap S_a$ and $]0, \delta] \cap S_a$ are non-empty. The characteristic function $r_3 = \chi_{]0,\|a\|]}(W_a)$ is a range tripotent of an element in $W_a$, and hence $r_3$ is a tripotent in $W$.

By [24, Lemma 3.12], there exists $e \in \mathcal{E}(W_1)$ such that $Q(e)(r_3) = r_3$, that is, $e = r_3 + (e - r_3)$ and $r_3 \perp (e - r_3)$. Since $P_1(r_3)(a - e) = 0$, we can write
\[ a - e = P_2(r_3)(a - e) + P_0(r_3)(a - e) = P_2(r_3)(a - r_3) + P_0(r_3)(a - e). \]
Clearly,
\[ \|P_2(r_3)(a - r_3)\| = \max\{1 - \delta, \|a\| - 1\} = 1 - \delta, \]
while $\|P_0(r_3)(a - e)\| \leq \|P_0(r_3)(a)\| + \|P_0(r_3)(e)\| \leq 1 + \delta$. Now, observing that $P_2(r_3)(a - r_3) \perp P_0(r_3)(a - e)$, we deduce from [20, Lemma 1.3(a)] that
\[ \text{dist}(a, \mathcal{E}(W_1)) \leq \|a - e\| \leq \max\{1 + \delta, 1 - \delta\} = 1 + \delta. \]
The arbitrariness of $\delta > 0$ implies that $\text{dist}(a, \mathcal{E}(W_1)) \leq 1$. 


Finally, the equality \( \text{dist}(a, \mathcal{E}(W_1)) = 1 \) and the final statement follows from Theorem 3.6. \( \square \)

The detailed description of the \( \lambda \)-function in the case of a JBW*-triple reads as follows.

**Theorem 4.2** Let \( W \) be a JBW*-triple. Then the \( \lambda \)-function on \( W_1 \) is given by the expression:

\[
\lambda(a) = \begin{cases} 
\frac{1 + m_q(a)}{2} & \text{if } a \in W_1 \cap W_q^{-1}, \\
\frac{1}{2} (1 - \alpha_q(a)) & \text{if } a \in W_1 \setminus W_q^{-1}.
\end{cases}
\]

**Proof.** The case \( a \in W_1 \cap W_q^{-1} \) follows from Theorem 3.4. Suppose \( a \in W_1 \setminus W_q^{-1} \). Corollary 3.7 and Proposition 4.1 imply that \( \lambda(a) \leq \frac{1}{2} (1 - \alpha_q(a)) = \frac{1}{2} \).

Let \( r = r(a) \) denote the range tripotent of \( a \) in \( W \). Let us observe that, by [24, Lemma 3.12], there exists a complete tripotent \( e \in \mathcal{E}(W_1) \) such that \( e = r_3 + (e - r_3) \) and \( r_3 \perp (e - r_3) \). This implies that \( a \) is a positive element in the closed unit ball of the JBW*-algebra \( W_2(e) \). Since \( a \notin W_q^{-1} \), 0 lies in the triple spectrum of \( a \) (cf. \((\Sigma, v)\)). Furthermore, the triple spectrum of \( a \) does not change when computed as an element in \( W_2(e) \) (see \((\Sigma, iv)\)), thus \( a \) is not BP-quasi-invertible in \( W_2(e) \). Let \( J_{a,e} \) denote the JBW*-algebra of \( W_2(e) \) generated by \( e \) and \( a \). It is known that \( J_{a,e} \) is isometrically isomorphic, as JBW*-algebra, to an abelian von Neumann algebra with unit \( e \) (cf. [21, Lemma 4.1.11]). Since, in the terminology of [7, 8], \( a \) neither is quasi-invertible in the abelian von Neumann algebra \( J_{a,e} \), we deduce, via [37, Theorem 4.2], that there exist unitary elements \( e_1 \) and \( e_2 \) in \( J_{a,e} \) satisfying \( a = \frac{1}{2} e_1 + \frac{1}{2} e_2 \). Since \( e \in \mathcal{E}(W_1) \) is the unit element in \( J_{a,e} \) and \( e_1, e_2 \) are unitary element in the latter von Neumann algebra, we conclude that \( e_1, e_2 \in \mathcal{E}(W_1) \) (cf. Lemma 2.5 or [40, Lemma 4]), which shows that \( \frac{1}{2} \leq \lambda(a) \). \( \square \)

As in the C*-setting, an element \( a \) in the closed unit ball of a JBW*-triple is BP-quasi-invertible if, and only if, \( \lambda(a) > \frac{1}{2} \).

**Corollary 4.3** Every JBW*-triple satisfies the uniform \( \lambda \)-property.

In [26, Section 4] (see also [42, Section 5.3]), the authors introduce the \( \Lambda \)-condition in the setting of JB*-triples in the following sense: a JB*-triple \( E \) satisfies the \( \Lambda \)-condition if for each complete tripotent \( e \in \mathcal{E}(E) \) and every \( a \in (E_2(e))_1 \mathcal{E}_q^{-1} \), the condition \( \lambda(a) = 0 \) implies \( \alpha_q(a) = 1 \). We can affirm now that every JBW*-triple actually satisfies a stronger property, because, by Theorem 4.2 (see also Proposition 4.1), the minimum value of the \( \lambda \)-function on the closed unit ball of a JBW*-triple is \( \frac{1}{2} \) (cf. [37, Theorem 4.2] for the appropriate result in von Neumann algebras).

Our next goal is to complete the statement of Theorem 3.6 in the case of a general JB*-triple.

**Proposition 4.4** Let \( a \) and \( b \) be elements in a JB*-triple \( E \). Suppose \( \| b - a \| < \beta \) and \( b \in E_q^{-1} \). Then \( a + \beta r(b) \in E_q^{-1} \) and the inequality

\[
m_q(a + \beta r(b)) \geq \beta - \| b - a \|,
\]

holds. Furthermore, under the above conditions, the element \( P_2(r(b))(a) + \beta r(b) \) is invertible in the JB*-algebra \( E_2(r(b)) \).
Proof. Let us write $a + \beta r(b) = a - b + b + \beta r(b)$. Considering the JB*-subtriple $E_b$ generated by $b$, we can easily see that $m_q(b + \beta r(b)) = \beta + m_q(b)$. Therefore, by (3.2),

$$m_q(a + \beta r(b)) \geq m_q(b + \beta r(b)) - \|a - b\| = \beta + m_q(b) - \|a - b\| > \beta - \|b - a\| > 0,$$

which proves the first statement.

Now, set $c = P_2(r(b))(a - b)$. Clearly, $\|c\| \leq \|a - b\| < \beta$. We write

$$P_2(r(b))(a) + \beta r(b) = c + P_2(r(b))(b) + \beta r(b) = c + b + \beta r(b).$$

Since $b$ is invertible and positive in the JB*-algebra $E_2(r(b))$, we deduce that $d = b + \beta r(b)$ is a positive invertible element in $E_2(r(b))$, with inverse $d^{-1} \in E_2(r(b))$. It is easy to see that $\|d^{-1}\|^{-1} = \|(b + \beta r(b))^{-1}\|^{-1} \geq \beta + m_q(b) > \beta$, and hence

$$\|U_{d^{-1/2}}(a - b)\| \leq \|d^{-1/2}\|^2 \|a - b\| < \frac{1}{\beta} \beta = 1,$$

which implies that $r(b) + U_{d^{-1/2}}(a - b)$ is invertible in the JB*-algebra $E_2(r(b))$. Finally, the identity

$$P_2(r(b))(a) + \beta r(b) = P_2(r(b))(a - b) + P_2(r(b))(b) + \beta r(b)$$

$$= U_{d^{-1/2}}(U_{d^{-1/2}}(a - b) + U_{d^{-1/2}}(b + \beta r(b)))$$

$$= U_{d^{-1/2}}(U_{d^{-1/2}}(a - b) + r(b)),$$

gives the final statement and concludes the proof. \qed

We can now extend Proposition 4.1 to the setting of JB*-triples.

**Theorem 4.5** Let $E$ be a JB*-triple satisfying $\mathcal{E}(E_1) \neq \emptyset$. Then the formula

$$\text{dist}(a, \mathcal{E}(E_1)) = 1 + \alpha_q(a)$$

holds for every $a$ in $E_1 \setminus E_q^{-1}$.

**Proof.** Fix $a$ in $E_1 \setminus E_q^{-1}$. Theorem 3.6 proves that $2 \geq \text{dist}(a, \mathcal{E}(E_1)) \geq 1 + \alpha_q(a)$. In particular, $0 \leq \alpha_q(a) \leq 1$. When $\alpha_q(a) = 1$, we have $2 \geq \text{dist}(a, \mathcal{E}(E_1)) \geq 1 + \alpha_q(a) = 2$, we may therefore assume that $\alpha_q(a) < 1$.

We shall prove now that for each pair $(\delta, \beta)$ with $1 > \delta > \beta > \alpha_q(a)$ there exists $e \in \mathcal{E}(E_1)$ with $\|a - e\| < \max(1 + \beta, 2\delta)$. Indeed, by definition there exists $b \in E_q^{-1}$ such that $\|a - b\| < \beta < \delta$. By Proposition 4.4, the element $z = a + \delta r(b) \in E_q^{-1}$ and $m_q(z) = m_q(a + \delta r(b)) \geq \delta - \|b - a\| > \delta - \beta$. 

Clearly, $\|a - z\| = \|\delta r(b)\| = \delta$. Since $z \in E_q^{-1}$, its range tripotent $e = r(z) \in \mathcal{E}(E_1)$. It is known that

$$\|z - r(z)\| = \max\{1 - m_q(z), \|z\| - 1\} < \max\{1 - \delta + \beta, \|a\| + \delta - 1\}.$$ 

Therefore,

$$\|a - r(z)\| \leq \|a - z\| + \|z - r(z)\| < \delta + \max\{1 - \delta + \beta, \|a\| + \delta - 1\} \leq \max\{1 + \beta, 2\delta\}.$$ 

This proves that for each pair $(\delta, \beta)$ with $1 > \delta > \beta > \alpha_q(a)$ we have

$$\text{dist}(a, \mathcal{E}(E_1)) \leq \max\{1 + \beta, 2\delta\},$$

letting $\beta, \delta \to \alpha_q(a)$ we get

$$\text{dist}(a, \mathcal{E}(E_1)) \leq \max\{1 + \alpha_q(a), 2\alpha_q(a)\} = 1 + \alpha_q(a),$$

which concludes the proof. \[\square\]

The set of extreme points of the closed unit ball of a unital C*-algebra is always non-empty. Since every C*-algebra is a JB*-triple, [8, Theorem 2.3] derives as a direct consequence of our Theorem 4.5. Actually, the proof above provides a simpler argument to obtain the result in [8]. Let us observe that the introduction of JB*-triple techniques makes the proofs easier because the set of extreme points is not directly linked to the order structure of a C*-algebra.

**Remark 4.6** In order to determine the $\lambda$ function on $E_1 \setminus E_1^{-1}$, it would be very interesting to know if the distance formula established in Theorem 4.5 can be improved to show that, under the same hypothesis, the equality

$$\text{dist}(a, \mathcal{E}(E_1)) = \max\{1 + \alpha_q(a), \|a\| - 1\}$$

(4.1)

holds for every $a$ in $E \setminus E_q^{-1}$.

We recall that a JB*-triple $E$ is said to be **commutative** or **abelian** if the identity

$$\{\{x, y, z\}, a, b\} = \{\{x, y, z, a, b\}\} = \{x, \{y, z, a, b\}\}$$

holds for all $x, y, z, a, b \in E$, equivalently, $L(a, b)L(c, d) = L(c, d)L(a, b)$, for every $a, b, c, d \in E$. Suppose that $E$ is a commutative JB*-triple with $\mathcal{E}(E_1) \neq \emptyset$. It is known (cf. [18, Theorems 2 and 4] or [23, Lemma 6.2]) that for each $e \in \mathcal{E}(E_1)$, the JB*-triple $E$ is a commutative C*-algebra with unit $e$, product and involution given by $a \circ b := [a, e, b]$ and $a^* := [e, a, e]$ $(a, b \in E)$, respectively, and the same norm. We have already observed that when a C*-algebra $A$ is regarded as a JB*-triple with the triple product given in (1.1), the BP-quasi-invertible elements in $A$, as JB*-triple, are exactly the quasi-invertible elements of the C*-algebra $A$ introduced and studied by Brown and Pedersen in [7, 8].
Since the Banach space underlying $E$ has not been changed, we can deduce from [8, Theorem 2.3] that
\[
\text{dist}(a, \mathcal{E}(E_1)) = \max\{1 + \alpha_q(a), \|a\| - 1\},
\]
for every $a \in E \setminus E_q^{-1}$, that is, (4.1) holds for every commutative JB$^*$-triple $E$ with $\mathcal{E}(E_1) \neq \emptyset$. It can be also shown that in this case,
\[
\lambda(a) \leq \frac{1}{2}(1 - \alpha_q(a)),
\]
for every $a \in E_1 \setminus E_q^{-1}$. Since commutative JB$^*$-triples are also example of function spaces (cf. [19, 30, Section 1]), the last result complements the study developed in [3, Theorem 1.9].

**Funding**

The authors extend their appreciation to the Deanship of Scientific Research at King Saud University for funding this work through research group no. RG-1435-20. A.M.P. was also partially supported by the Spanish Ministry of Science and Innovation, D.G.I. project no. MTM2011-23843.

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