King Saud University
College of Since
Mathematical Department

Mid-Term 1/ 2017-2018
Full Marke:25. Time 1H30min
12/12/2017

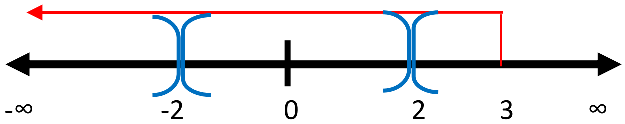
Question 1[4,4] a) Discuss the existence of unique solution of the following initial value problem

$$\begin{cases} (x-2)y'' + \frac{x}{\sqrt{3-x}}y' + \frac{1}{x^2-4}y = \cos x \\ y(1) = 0, \; y'(1) = 1. \end{cases}$$

$(x-2)y''$ and $\cos(x)$ are continues on $\mathcal{R}$

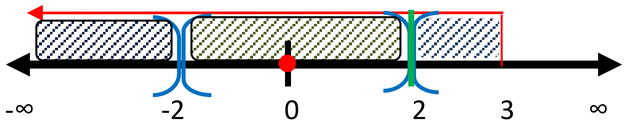
$\frac{x}{\sqrt{3-x}}$ are continues on $x < 3$

$\frac{1}{x^2-4}$ are continues on $\mathcal{R} - \{-2.2\}$



$a(x) \neq 0 \quad . \; x \neq 2$

$x_o = 0 \in (-2.2)$



The initial value problem has a unique solution on $I = (-2.2)$

b) Solve the nonhomogeneous differential equation

$$y'' - 2y' - 3y = e^{2x} + 5\cos 2x$$

1# Find y_c

$y_c \rightarrow \quad y'' - 2y' - 3y = 0$

Let $y = e^{mx}$; $\quad y' = me^{mx}$; $\quad y'' = m^2e^{mx}$

$$m^2e^{mx} - 2me^{mx} - 3e^{mx} = 0 \qquad \text{Divided by } e^{mx}$$

Characteristic equation $\qquad m^2 - 2m - 3 = 0$

$$m = 3 \quad \rightarrow \quad y_1 = e^{3x}$$

$$m = -1 \quad \rightarrow \quad y_2 = e^{-x}$$

$$y_c = c_1e^{3x} + c_2e^{-x}$$

2# Find y_p

$$f(x) = e^{2x} + 5\cos(2x)$$

$$f(x)1 = 5\cos(2x) \quad \rightarrow \quad m = \pm 2i \quad \rightarrow \quad y_p = \mathrm{a}\cos(2x) + bsin(2x)$$

$$f(x)2 = e^{2x} \quad \rightarrow \quad m = 2 \quad \rightarrow \quad y_p = de^{2x}$$

$$y_p = de^{2x} + \mathrm{a}\cos(2x) + bsin(2x)$$

$$y'_p = 2de^{2x} - 2a\sin(2x) + 2b\cos(2x)$$

$$y''_p = 4de^{2x} - 4a\cos(2x) - 4b\sin(2x)$$

$$y'' - 2y' - 3y = e^{2x} + 5\cos(2x)$$

$$\left(4de^{2x} - 4a\cos(2x) - 4b\sin(2x)\right) - 2\left(2de^{2x} - 2a\sin(2x) + 2b\cos(2x)\right)$$

$$-3\left(de^{2x} + \mathrm{a}\cos(2x) + bsin(2x)\right) = e^{2x} + 5\cos(2x)$$

$$-3de^{2x} - 7a\cos(2x) - 4\,\mathrm{b}\cos(2x) - 7b\sin(2x) + 4a\sin(2x) = e^{2x} + 5\cos(2x)$$

$$-3de^{2x} + (-7a - 4b)\cos(2x) + (-7b + 4a)\sin(2x) = e^{2x} + 5\cos(2x)$$

Divided by e^{2x} $\quad -3de^{2x} = e^{2x}$

$$-3d = 1 \quad \rightarrow \quad d = -\frac{1}{3}$$

$$-7a - 4b = 5$$

Multiply by $\frac{7}{4}$ $\quad 4a - 7b = 0$

$$\cancel{-7}a - 4b = 5$$

$$\cancel{7a} - \frac{49}{4}b = 0$$

$$-\frac{65}{4}b = 5$$

$$b = -\frac{4}{13}$$

$$a = -\frac{7}{13}$$

$$y = y_c + y_p$$

$$y = -\frac{1}{3}e^{2x} - \frac{7}{13}\cos(2x) - \frac{4}{13}sin(2x) + c_1e^{3x} + c_2e^{-x}$$

Question 2 [4,3]. a) If $y_1 = x^3 e^x$ is a solution of the differential equation

$$xy'' - 2(x+1)y' + (x+2)y = 0, \quad x \neq 0,$$

then use reduction of order method to obtain its general solution.

$$xy'' - 2(x+1)y' + (x+2)y = 0$$

$$y = y_1 u = x^3 e^x u$$

$$y' = 3x^2 e^x u + x^3 e^x u + x^3 e^x u'$$

$$y'' = 6xe^x u + 3x^2 e^x u + 3x^2 e^x u' + 3x^2 e^x u + x^3 e^x u + x^3 e^x u'$$

$$+3x^2 e^x u' + x^3 e^x u' + x^3 e^x u''$$

$$y'' = 6xe^x u + 6x^2 e^x u + 6x^2 e^x u' + 2x^3 e^x u' + x^3 e^x u + x^3 e^x u''$$

$$xy'' - 2y'x - 2y' + xy + 2y = 0$$

$$6x^2 e^x u + 6x^3 e^x u + 6x^3 e^x u' + 2x^4 e^x u' + x^4 e^x u + x^4 e^x u'' - 6x^3 e^x u - 2x^4 e^x u$$

$$- 2x^4 e^x u' - 6x^2 e^x u - 2x^3 e^x u - 2x^3 e^x u' + x^4 e^x u + 2x^3 e^x u = 0$$

$$x^4 e^x u'' + 4x^3 e^x u' = 0$$ Divided by $x^3 e^x$

$$xu'' + 4u' = 0$$

Let $w = u'$ $. w' = u''$

$$x\frac{dw}{dx} + 4w = 0$$

$$x\frac{dw}{dx} + 4w = 0 \quad \rightarrow \quad xdw + 4wdx = 0 \quad \rightarrow \quad \frac{dw}{w} + 4\frac{dx}{x} = 0$$

$$\int \frac{dw}{w} + 4\int \frac{dx}{x} = 0$$

$$\ln(w) + 4\ln(x) = \mathrm{c}$$

$$\ln(wx^4) = c$$

$$wx^4 = \pm c$$

$$w = c_1 x^{-4}$$

$$w = u' = c_1 x^{-4}$$

$$u = c_1 \int x^{-4} dx$$

$$u = -\frac{c_1}{3} x^{-3} + c_2$$

$$y = x^3 e^x \left(-\frac{c_1}{3} x^{-3} + c_2\right)$$

$$y = -\frac{c_1}{3} e^x + c_2 x^3 e^x$$

b) Determine either the functions

$$f_1(x) = e^{2x}, \quad f_2(x) = e^{-2x}, \quad f_3(x) = \cosh 2x,$$

are linearly independent or linearly dependent on $(-\infty, \infty)$.

$f_1(x) = e^{2x}$. $f_2(x) = e^{-2x}$. $f_3(x) = \cosh(2x)$

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$$

$$\cosh(2x) = \frac{e^{-2x}}{2} + \frac{e^{2x}}{2}$$

$$\cosh(2x) - \frac{e^{-2x}}{2} - \frac{e^{2x}}{2} = 0$$

$$. \quad f_3(x) - \frac{1}{2} f_2(x) - \frac{1}{2} f_1(x) = 0$$

$c_1 = -\frac{1}{2}$. $c_2 = -\frac{1}{2}$. $c_3 = 1$

All are not zeros

$f_1 . f_2 . f_3$ are linearly dependent on $\mathcal{R}$

Question 3 [5] Find the general solution of the differential equation

$$x^2y'' - 3xy' + 3y = x^4e^x; \quad x > 0.$$

$$x^2y'' - 3xy' + 3y = x^4e^x$$

Let $y = x^m$. $y' = mx^{m-1}$. $y'' = m(m-1)x^{m-2}$

$$a_2m(m-1) + a_1m + a_0 = 0$$

$$a_2m^2 - a_2m + a_1m + a_0 = 0$$

$$a_2m^2 + m(a_1 - a_2) + a_0 = 0$$

Where

$a_2 = 1$

$a_1 = -3$

$a_0 = 3$

$$y_c \rightarrow \quad m^2 - 4m + 3 = 0$$

Roots

$$m_1 = 3 \quad \rightarrow \quad y_1 = x$$

$$m_2 = 1 \quad \rightarrow \quad y_2 = x^3$$

$$y_c = c_1x + c_2x^3$$

$$y_p = y_1u_1 + y_2u_2$$

$g(x) = x^4e^x$

$a_n = x^2$

$$f(x) = \frac{g(x)}{a_n} = \frac{x^4e^x}{x^2}$$

$$f(x) = x^2e^x$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} \rightarrow w = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3$$

$$w = 2x^3$$

$$w_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix} \rightarrow w_1 = \begin{vmatrix} 0 & x^3 \\ x^2e^x & 3x^2 \end{vmatrix} = -x^5e^x$$

$$w_2 = \begin{vmatrix} y_1 & 0 \\ y'_1 & f(x) \end{vmatrix} \rightarrow w_1 = \begin{vmatrix} x & 0 \\ 1 & x^2e^x \end{vmatrix} = x^3e^x$$

$w = 2x^3$. $w_1 = -x^5e^x$, $w_2 = x^3e^x$

$$u_1' = \frac{w_1}{w} = \frac{-x^5e^x}{2x^3} = -\frac{1}{2}x^2e^x$$

$$u_2' = \frac{w_2}{w} = \frac{x^3e^x}{2x^3} = \frac{1}{2}xe^x$$

$$u_1 = \int \frac{w_1}{w}dx = -\frac{1}{2}\int x^2e^x\,dx = -\frac{1}{2}(x^2e^x - 2xe^x + 2e^x)$$

$$u_1 = (-\frac{x^2}{2} + x - 1)e^x$$

$$u_2 = \int \frac{w_2}{w}dx = \frac{1}{2}\int e^x dx = \frac{1}{2}e^x$$

$$y_p = (-\frac{x^2}{2} + x - 1)e^x * x + \frac{1}{2}e^x * x^3$$

$$y_p = (-\frac{x^3}{2} + x^2 - x)e^x + \frac{1}{2}e^x x^3$$

$$y_p = x(x-1)e^x$$

$$y = y_c + y_p$$

$$y = c_1x + c_2x^3 + x(x-1)e^x$$

Question 4 [5] Solve the following linear system of differential equations.

$$\begin{cases} x' = x - y + t \\ y' = x + 3y - 3t \end{cases}$$

$$x' - x + y = t$$

$$y' - x - 3y = -3t$$

$$Dx - x + y = t$$

$$Dy - 3y - x = -3t$$

$$-(D-3)* \qquad (D-1)x + y = t$$

$$(D-3)y - x = -3t$$

$$-(D-3)(D-1)x - \cancel{(D-3)y} = -(D-3)t$$

$$\cancel{(D-3)y} - x = -3t$$

$$-(D^2 - 4D + 3)x - x = -(Dt - 3t) - 3t$$

$$x'' - 4x' + 4x = 1$$

$$x = e^{mt} \quad . \quad x' = me^{mt} \quad . \quad x'' = m^2 e^{mt}$$

$$x_c \quad \rightarrow \quad m^2 - 4m + 4 = 0$$

$$m_1 = 2 \quad \rightarrow \quad x_1 = e^{2t}$$

$$m_2 = 2 \quad \rightarrow \quad x_2 = te^{2t}$$

$$x_c = c_1 e^{2t} + c_2 t e^{2t}$$

$$f(t) = 1 \quad \rightarrow \quad m = 0 \quad \rightarrow x_p = A$$

$$x' = 0 \quad x'' = 0$$

$$x'' - 4x' + 4x = 1$$

$$4A = 1$$

$$A = \frac{1}{4}$$

$$x_p = \frac{1}{4}$$

$$x = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{4}$$

$$x' = 2c_1 e^{2t} + 2c_2 t e^{2t} + c_2 e^{2t}$$

$$y = x - x' + t$$

$$y = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{4} - (2c_1 e^{2t} + 2c_2 t e^{2t} + c_2 e^{2t}) + t$$

$$y = -(c_1 + c_2) e^{2t} - c_2 t e^{2t} + t + \frac{1}{4}$$