Reparametrisation and Arc-length
Math 473
Introducation to Differential Geometry
Lecture 3

Dr. Nasser Bin Turki

King Saud University
Department of Mathematics

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Defnation (1):
Let $\alpha : [a, b] \mapsto \mathbb{R}^3$ be a parametrised regular curve. We define the **Length of** α as

$$L = \int_a^b |\alpha'(t)| dt.$$

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$$S(t) = \int_{t_0}^{t} |\alpha'(u)| du \qquad (t \in I) \ i.e. \quad S : I \mapsto \mathbb{R}.$$

Example(1): The Length of the curve $\alpha : \mathbb{R} \mapsto \mathbb{R}^3, \alpha(t) = (\cos t, \sin t, 6)$ is

Example(2): Show that for the curve $\alpha(t) = \frac{1}{2}(t, \frac{1}{t}, \sqrt{2}\ln t)$, where $t \in (0, \infty)$, arc-length measured from $t = 1$ to $t = t_0$ is

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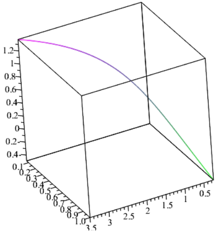

Defnation (2):
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Example(3) Show that the curve $\alpha(t) = (\cos t, \sin t, 2)$ is a unit speed curve.

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Unit speed curves are often easier to work with. But does every curve have a unit speed parametrisation?

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α is a unit speed curve if and only if α is parametrised by its arc-length.

Defnation (3):
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The parametrised curve $\beta = \alpha \circ h : J \mapsto \mathbb{R}^3$ is called a **reparametrisation** of the curve α.

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Let $\alpha : I \mapsto \mathbb{R}^3$ be a regular parametrised curve. Then there exists a parameter transformation $h : J \mapsto I$ for α such that the reparametrisation $\beta = \alpha \circ h : J \mapsto \mathbb{R}^3$ is a unit speed curve. Moreover, the parameter transformation h has the property $h'(n) > 0$ for all $n \in J$.

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Let $\alpha : I \mapsto \mathbb{R}^3$ be a regular parametrised curve. Then there exists a parameter transformation $h : J \mapsto I$ for α such that the reparametrisation $\beta = \alpha \circ h : J \mapsto \mathbb{R}^3$ is a unit speed curve. Moreover, the parameter transformation h has the property $h'(n) > 0$ for all $n \in J$.
Proof

Remark

Remark The reparametrisation of the curve α in the previous Theorem (1) is called **normal Reparametrisation** of α.

Example(4) Reparametrisation the curve $\alpha(t) = \frac{1}{2}(t, \frac{1}{t}, \sqrt{2}\ln t)$, where $t \in (0, \infty)$, (Taken in Example 2) using the arc-length?(Find the normal reparametrisation of α)

Example(5) For the regular curve $\alpha(t) = (a\cos t, a\sin t, bt)$ where $a, b \in \mathbb{R}^*$.

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Example(6) Show that the curve $\alpha(t) = (e^t - 1, \frac{2\sqrt{2}}{3} e^{\frac{3t}{2}}, \frac{1}{2} e^{2t})$ is regular.

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Thanks for listening.