Numerical Methods

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Aims

In this lecture, we will . . .

- Introduce the concept of Newton's General Interpolating Formula
- Introduce the Interpolation with Spline Functions

Newton's General Interpolating Formula

Since we noted in the previous section that for a small number of data point one can easily use the Lagrange formula of the interpolating polynomial. However, for a large number of data points there will be many multiplication and more significantly, whenever a new data point is added to an existing set, the interpolating polynomial has to be completely recalculated. Here, we describe an efficient way of organizing the calculations so as to overcome these disadvantages.

Let us consider the *nth*-degree polynomial $p_n(x)$ that agrees with the function $f(x)$ at the distinct numbers $x_0, x_1, \ldots, x_n$. The **divided differences** of $f(x)$ with respect to $x_0, x_1, \ldots, x_n$ are derived to express $p_n(x)$ in the form

$$\begin{aligned} f(x) = p_n(x) = a_0 \quad &+ \quad a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\ &+ \quad a_n(x - x_0)(x - x_1) \cdots (x - x_{n-1}), \end{aligned} \tag{1}$$

for appropriate constants $a_0, a_1, \ldots, a_n$.
Now to determine the constants, firstly, by evaluating $p_n(x)$ at x_0, we have

$$p_n(x_0) = a_0 = f(x_0) \tag{2}$$

Similarly, when $p_n(x)$ is evaluated at x_1, then

$$p_n(x_1) = a_0 + a_1(x_1 - x_0) = f(x_1),$$

which implies that

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \tag{3}$$

divided differences

Now we express the interpolating polynomial in terms of **divided difference**.
Firstly, we define the *Zeroth divided difference* at the point x_i by

$$f[x_i] = f(x_i), \tag{4}$$

which is simply the value of the function $f(x)$ at x_i.
The *first-order* or *first divided difference* at the points x_i and x_{i+1} can be defined by

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}. \tag{5}$$

In general, the *nth divided difference* $f[x_i, x_{i+1}, \ldots, x_{i+n}]$ is defined by

$$f[x_i, x_{i+1}, \ldots, x_{i+n}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+n}] - f[x_i, x_{i+1}, \ldots, x_{i+n-1}]}{x_{i+n} - x_i}. \quad (6)$$

By using this definition, (2) and (3) can be written as

$$a_0 = f[x_0]; \qquad a_1 = f[x_0, x_1],$$

respectively. Similarly, one can have the values of other constants involving in (1) such as

$$\begin{aligned} a_2 &= f[x_0, x_1, x_2], \\ a_3 &= f[x_0, x_1, x_2, x_3], \\ \cdots &= \cdots \\ \cdots &= \cdots \\ a_n &= f[x_0, x_1, \ldots, x_n]. \end{aligned}$$

Table: Divided difference table for a function $y = f(x)$

k	x_k	Zero Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	x_0	$f[x_0]$			
1	x_1	$f[x_1]$	$f[x_0, x_1]$		
2	x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
3	x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

Putting the values of these constants in (1), we get

$$\begin{aligned} f(x) = p_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}), \end{aligned} \tag{7}$$

which can also be written as

$$f(x) = p_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1) \cdots (x - x_{k-1}). \tag{8}$$

This type of polynomial is known as the *Newton's interpolatory divided difference polynomial*. Table 1 shows the divided difference for a function $f(x)$.

Example 0.1

Construct the fourth divided differences table for $f(x) = 4x^4 + 3x^3 + 2x^2 + 10$ for the values $x = 3, 4, 5, 6, 7, 8$.

Solution. The result are listed in Table 2.

From the results in Table 2, one can note that the *nth* divided difference for the *nth* polynomial equation is always constant and the *(n+1)th* divided difference is always zero for the *nth* polynomial equation. •

Table: Divided differences table for $f(x) = e^x$ at given points

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference	Fifth Divided difference
0	3	433					
1	4	1258	825				
2	5	2935	1677	426			
3	6	5914	2979	651	75		
4	7	10741	4827	924	91	4	
5	8	18058	7317	1245	107	4	0

Linear Newton's Interpolating Polynomial

The linear Newton's interpolating polynomial passing through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ can be written as

$$f(x) = p_1(x) = f[x_0] + (x - x_0)f[x_0, x_1].$$

The quadratic Newton's interpolating polynomial passing through the points $(x_0, f(x_0))$,
$(x_1, f(x_1))$ and $(x_2, f(x_2))$ can be written in terms of divided differences as

$$f(x) = p_2(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2].$$

This polynomial can also be written as

$$f(x) = p_2(x) = p_1(x) + (x - x_0)(x - x_1)f[x_0, x_1, x_2],$$

that is, the interpolating polynomial of degree 2 makes full use of the polynomial of degree 1, simply adding one extra term to $p_1(x)$. This is one of the advantages of the Newton's polynomial over Lagrange polynomial.

Cubic Newton's Interpolating Polynomial

Similarly, the cubic Newton's interpolating polynomial passing through the points $(x_0, f(x_0))$,
$(x_1, f(x_1)), (x_2, f(x_2))$ and $(x_3, f(x_3))$ can be written in terms of divided differences as

$$f(x) = p_3(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2).$$

This polynomial can also be written as

$$p_3(x) = p_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3],$$

that is, the interpolating polynomial of degree 3 makes full use of the polynomial of degree 2, simply adding one extra term to $p_2(x)$. Note that using linear polynomial in quadratic polynomial, the starting point x_0 for both polynomials should be same.

Nth Degree Newton's Interpolating Polynomial

Repeating this entire process again, $p_3(x), p_4(x)$ and higher degree interpolating polynomials can be consecutively obtained in the same way. In general, the interpolating polynomial $p_n(x)$ passing through the points $(x_i, f(x_i))(i = 0, 1, \ldots, n)$, can be written in terms of divided differences as

$$\begin{aligned} f(x) = p_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1}), \end{aligned} \tag{9}$$

which can also be written as

$$f(x) = p_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0)(x - x_1) \cdots (x - x_{k-1}), \tag{10}$$

or

$$f(x) = p_n(x) = f[x_0] + \sum_{k=0}^{n} f[x_0, x_1, \cdots, x_k] \prod_{i=0}^{k-1} (x - x_i). \tag{11}$$

This type of polynomial is known as the *Newton's interpolatory divided difference polynomial.*

Theorem 1

(Newton's Interpolating Polynomial)

Suppose that $x_0, x_1, \dots, x_n$ *are* $(n+1)$ *distinct points in the interval* $[a, b]$. *There exists a unique polynomial* $p_n(x)$ *of degree at most* n *with the property that*

$$f(x_i) = p_n(x_i), \quad \textit{for} \quad i = 0, 1, \dots, n.$$

The Newton's form of this polynomial is

$$f(x) = p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}),$$

where

$$a_k = f[x_0, x_1, x_2, \cdots, x_k], \quad \textit{for} \quad k = 0, 1, 2, \dots, n.$$

•

Example 0.2

Show that the Newton's interpolating polynomial $p_2(x)$ of degree 2 satisfies the interpolation conditions

$$p_2(x_i) = f(x_i), \qquad i = 0, 1, 2.$$

Solution. Since the Newton's interpolating polynomial of degree 2 is

$$f(x) = p_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1).$$

First take $x = x_0$, we have

$$p_2(x_0) = f[x_0] + 0 + 0 = f(x_0).$$

Now take $x = x_1$, we have

$$p_2(x_1) = f[x_0] + f[x_0, x_1](x_1 - x_0) + 0 = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_1 - x_0),$$

it gives

$$p_2(x_1) = f(x_0) + f(x_1) - f(x_0) = f(x_1).$$

Finally, take $x = x_2$, we have

$$p_2(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_0) + f[x_0, x_1, x_2](x_2 - x_0)(x_2 - x_1),$$

which can be written as

$$p_2(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_0) + \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}(x_2 - x_0)(x_2 - x_1).$$

It gives

$$p_2(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_1 + x_1 - x_0) + f[x_1, x_2](x_2 - x_1) - f[x_0, x_1](x_2 - x_1),$$

or

$$p_2(x_2) = f[x_0] + f[x_0, x_1](x_1 - x_0) + f[x_1, x_2](x_2 - x_1).$$

From (5), we have

$$p_2(x_2) = f[x_0] + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_1 - x_0) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x_2 - x_1),$$

which gives

$$p_2(x_2) = f(x_0) + f(x_1) - f(x_0) + f(x_2) - f(x_1) = f(x_2).$$

Example 0.3

The cubic Newton's polynomial $p_3(x) = 2 - (x+1) + x(x+1) - 2x(x+1)(x-1)$ interpolates the first four points in the following table:

x	-1	0	1	2	3
$f(x)$	2	1	2	-7	10

By adding one additional term $(3, 10)$ to $p_3(x)$, find Newton's polynomial $p_4(x)$ that interpolates the whole table and then use it to find the approximation of $f(0.5)$.

Solution. Since the Newton's polynomial for the whole table data points is the four degree Newton's interpolating polynomial and it can be written as

$$f(x) = p_4(x) = p_3(x) + x(x+1)(x-1)(x-2)f[x_0, x_1, x_2, x_3].$$

Now to find fourth divided difference $f[x_0, x_1, x_2, x_3]$, we have to construct the required divided differences table. The result are listed in Table 3.

Table: Divided differences table for $f(x) = e^x$ at given points

k	x_k	Zeroth Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided difference
0	-1	2				
1	0	1	-1			
2	1	2	1	1		
3	2	-7	-9	-5	-2	
4	3	10	17	13	6	2

Thus the Newton's interpolating polynomial passing through all the given data points is

$$f(x) = p_4(x) = 2 - (x+1) + x(x+1) - 2x(x+1)(x-1) + 2x(x+1)(x-1)(x-2).$$

Thus at $x = 0.5$, we get

$$f(0.5) \approx p_4(0.5) = 3.1250,$$

the required approximation of the function. •

Example 0.4

Consider the following table of date points

x	3	1	5	6
$f(x)$	1	-3	2	4

Find the third divided difference $f[3, 1, 5, 6]$ and use it to find the Newton's form of the interpolating polynomial. Find approximation of $f(2)$.

Solution. The third divided differences for the given data points are listed in Table 4.

Table: Divided difference table for a function $y = f(x)$

k	x_k	Zero Divided Difference	First Divided Difference	Second Divided Difference	Third Divided Difference
0	$x_0 = 3$	$f[x_0] = 1$			
1	$x_1 = 1$	$f[x_1] = -3$	$f[x_0, x_1] = 2$		
2	$x_2 = 5$	$f[x_2] = 2$	$f[x_1, x_2] = 5/4$	$f[x_0, x_1, x_2] = -3/8$	
3	$x_3 = 6$	$f[x_3] = 4$	$f[x_2, x_3] = 2$	$f[x_1, x_2, x_3] = 3/20$	A

where $f[x_0, x_1, x_2, x_3] = 7/40$.

The cubic Newton's interpolating polynomial passing through the given points can be written as

$$p_3(x) = f[x_0]+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1$$

so using Table 4, we have

$$f(x) = p_3(x) = 1 + 2(x - x_0) - \frac{3}{8}(x - x_0)(x - x_1) + \frac{7}{40}(x - x_0)(x - x_1)(x - x_2),$$

or

$$f(x) = p_3(x) = \frac{1}{40}[7x^3 - 78x^2 + 301x - 350].$$

Thus at $x = 2$, we get

$$f(2) \approx p_3(2) = \frac{1}{40}[7(2)^3 - 78(2)^2 + 301(2) - 350] = -\frac{1}{10},$$

the required approximation of the function at $x = 2$. •

Interpolation with Spline Functions

In the previous sections we studied the use of interpolation polynomials for approximating the values of the functions on closed intervals. An alternative approach is divide the interval into a collection of subintervals and construct a different approximating polynomial on each subinterval. Approximation by polynomial of this type is called *piecewise polynomial approximation*. Here, we will discuss some of the examples of a **piecewise curve fitting** techniques; the use of the *piecewise linear interpolation*.

Definition 2

(Spline Function)
Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$. A function $s : [a, b] \rightarrow \mathbf{R}$ is a *spline* or *spline function* of degree m with points $x_0, x_1, \ldots, x_n$ if:
1. A function s is a piecewise polynomial such that, on each subinterval $[x_k, x_{k+1}]$, s has degree at most m.
2. A function s is $m - 1$ times differentiable everywhere. •

A **spline** is a flexible drafting device that can be constrained to pass smoothly through a set of plotted data points. Spline functions are a mathematical tool which is an adaptation of this idea.

Piecewise Linear Interpolation

It is the one of the simplest **piecewise polynomial** interpolation for the approximation of the function, called *linear spline*. The *linear spline* is continuous function and the basic of it is simply connect consecutive points with straight lines. Consider the set of seven data points (x_0, y_0), (x_1, y_1), (x_2, y_2), $(x_3, y_3), (x_4, y_4), (x_5, y_5)$ and (x_6, y_6) which define six subintervals. These intervals are denoted as $[x_0, x_1], [x_1, x_2], [x_2, x_3]$, $[x_3, x_4], [x_4, x_5]$ and $[x_5, x_6]$, where $x_0, x_1, x_2, x_3, x_4, x_5$, and x_6 are distinct x-values. If we use a straight line on each subinterval (see Figure 1) then we can interpolate the data with a piecewise linear function, where

$$s_k(x) = p_k(x) = \frac{(x - x_{k+1})}{(x_k - x_{k+1})} y_k + \frac{(x - x_k)}{(x_{k+1} - x_k)} y_{k+1},$$

or

$$s_k(x) = y_k + \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)} (x - x_k).$$

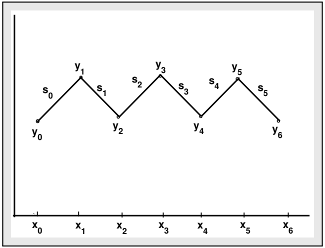

Figure: Linear spline.

It gives us

$$s_k(x) = A_k + B_k(x - x_k), \tag{12}$$

where the values of the coefficients A_k and B_k are given as

$$A_k = y_k \quad \text{and} \quad B_k = \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)}. \tag{13}$$

Note that the linear spline must be continuous at given points $x_0, x_1, \ldots, x_n$ and

$$s(x_k) = f(x_k) = y_k, \qquad \text{for} \quad k = 0, 1, \ldots, n.$$

Example 0.5

Find the linear splines which interpolates the following data

x_k	1	2	3	4
y_k	1.0	0.67	0.50	0.40

Find the approximation of the function $y(x) = \dfrac{2}{x+1}$ at $x = 2.9$. Compute absolute error.

Solution. Given $x_0 = 1.0, x_1 = 2.0, x_2 = 3.0, x_3 = 4.0$, then using (13), we have

$$A_0 = y_0 = 1.0,\ A_1 = y_1 = 0.67,\ A_2 = y_2 = 0.50,\ A_3 = y_3 = 0.4,$$

and

$$\begin{aligned} B_0 &= \frac{(y_1 - y_0)}{(x_1 - x_0)} = \frac{(0.67 - 1.0)}{(2.0 - 1.0)} = -0.33, \\ B_1 &= \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(0.50 - 0.67)}{(3.0 - 2.0)} = -0.17, \\ B_2 &= \frac{(y_3 - y_2)}{(x_3 - x_2)} = \frac{(0.40 - 0.50)}{(4.0 - 3.0)} = -0.10. \end{aligned}$$

Now using (12), the linear splines for three subintervals are define as

$$s(x) = \begin{cases} s_0(x) & = & 1.0 - 0.33(x - 1.0) = 1.33 - 0.33x, & 1 \leq x \leq 2, \\ s_1(x) & = & 0.67 - 0.17(x - 2.0) = 1.01 - 0.17x, & 2 \leq x \leq 3, \\ s_2(x) & = & 0.50 - 0.10(x - 3.0) = 0.80 - 0.10x, & 3 \leq x \leq 4. \end{cases}$$

The value $x = 2.9$ lies in the interval $[2, 3]$, so

$$f(2.9) \approx s_1(2.9) = 1.01 - 0.17(2.9) = 0.517.$$

Also,

$$|f(2.9) - s_1(2.9)| = |0.513 - 0.517| = 0.004,$$

the required absolute error. •

Summary

In this lecture, we ...

- Introduced the concept of Newton's General Interpolating Formula
- Introduced the Interpolation with Spline Functions