14 Multiple Integration

14.3

Change of Variables: Polar Coordinates

Objective

- Write and evaluate double integrals in polar coordinates.

Double Integrals in Polar Coordinates

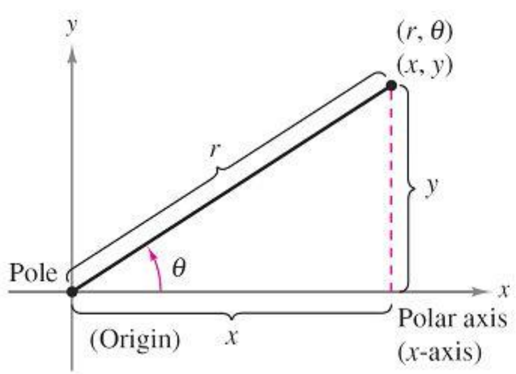
Some double integrals are easier to evaluate in polar form than in rectangular form.
It is particularly true for regions with circular symmetry and integrands involving $x^2+y^2 = r^2$

Double Integrals in Polar Coordinates

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

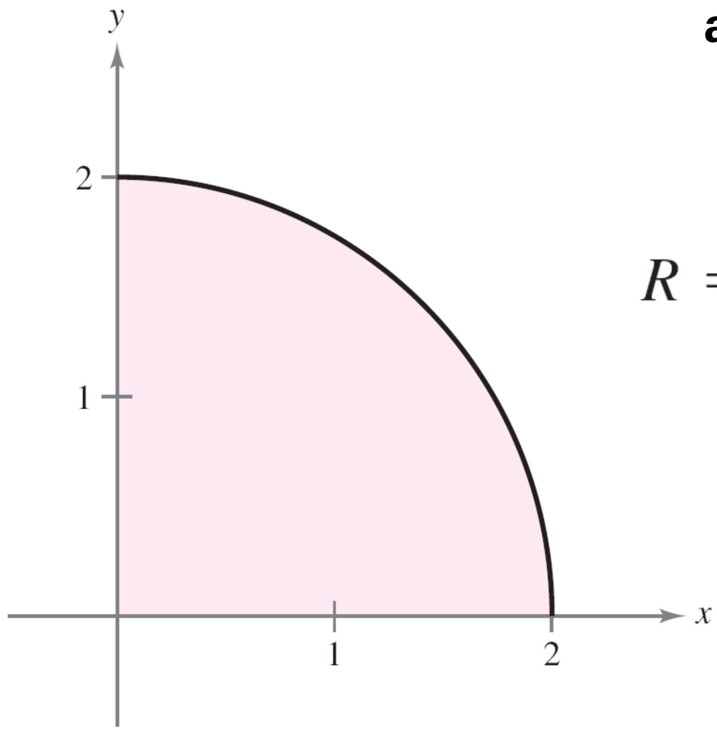
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$



Relating polar and rectangular coordinates

Example 1 – *Using Polar Coordinates to Describe a Region*

Use polar coordinates to describe each region shown in Figure 14.24.



(a)

a. The region R is a quarter circle of radius 2.

It can be described in polar coordinates as

$$R = \{(r, \theta): 0 \leq r \leq 2, \quad 0 \leq \theta \leq \pi/2\}.$$

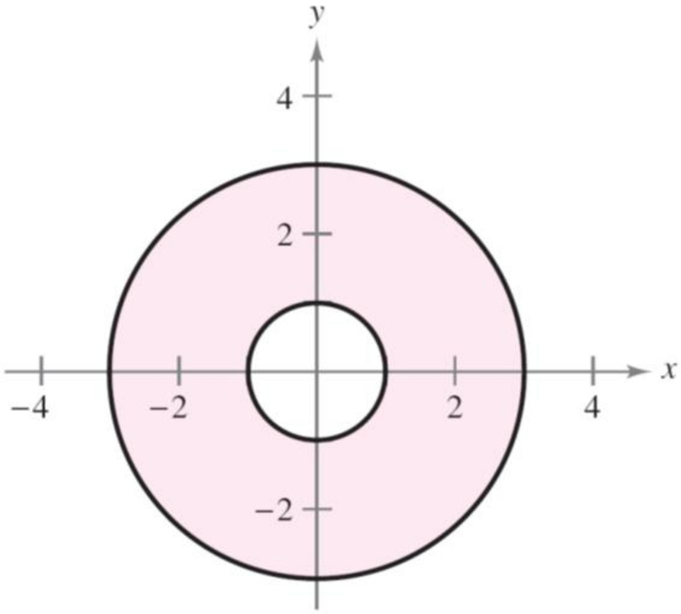
Figure 14.24

Example 1 – *Solution*

b. The region R consists of all points between concentric circles of radii 1 and 3.

It can be described in polar coordinates as

$$R = \{(r, \theta): 1 \le r \le 3, \quad 0 \le \theta \le 2\pi\}.$$



(b)

Double Integrals in Polar Coordinates

The regions in Example 1 are special cases of **polar sectors**

$$R = \{(r, \theta): r_1 \leq r \leq r_2, \quad \theta_1 \leq \theta \leq \theta_2\}$$

Polar sector

as shown in Figure 14.25.

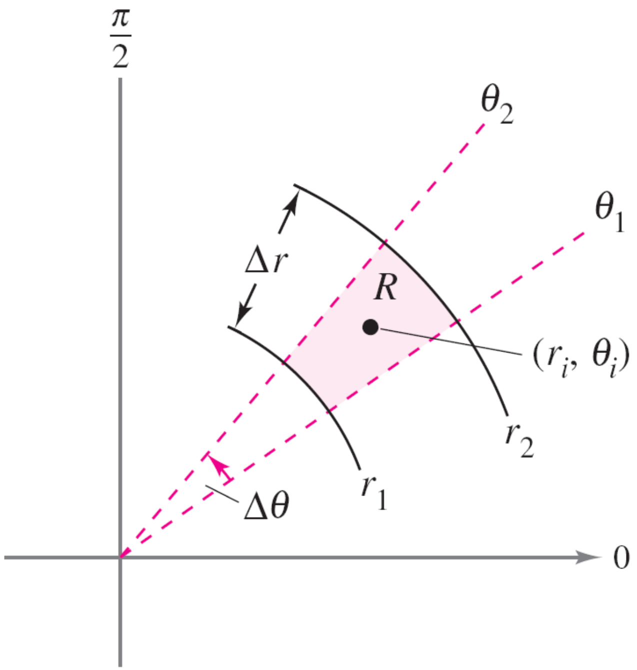
Area of this polar sector R :

1st let's recall formula for area of any sector with

angle $\Delta\theta = \dfrac{\Delta\theta}{2\pi}\pi r^2 = \dfrac{\Delta\theta\, r^2}{2}$

$$Area(R_i) = \frac{\Delta\theta\, r_2^2}{2} - \frac{\Delta\theta\, r_1^2}{2} = \frac{\Delta\theta\,(r_2^2 - r_1^2)}{2} =$$

$$\frac{\Delta\theta\,(r_2 - r_1)(r_2 + r_1)}{2} \approx \frac{\Delta\theta\,\Delta r\, 2r_{av}}{2} = r_{av}\Delta r\Delta\theta$$



Polar sector

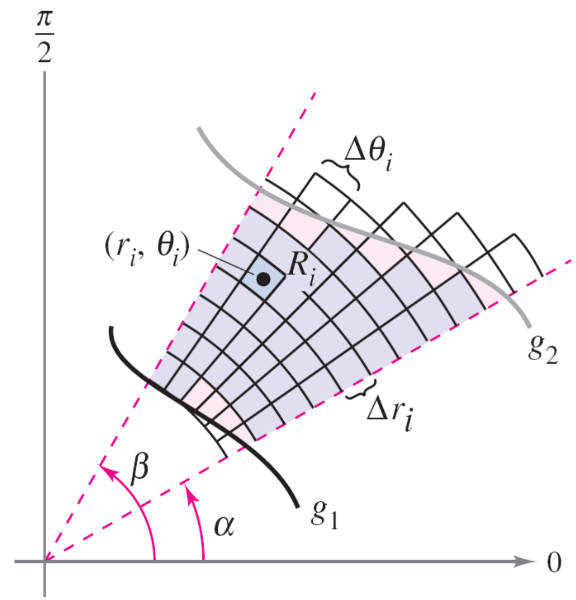
Figure 14.25

Double Integrals in Polar Coordinates

To define a double integral of a continuous function $z = f(x, y)$ in polar coordinates, consider a region R bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$.

Instead of partitioning R into small rectangles, use a partition of small polar sectors.

On R, superimpose a polar grid made of rays and circular arcs, as shown in Figure 14.26.



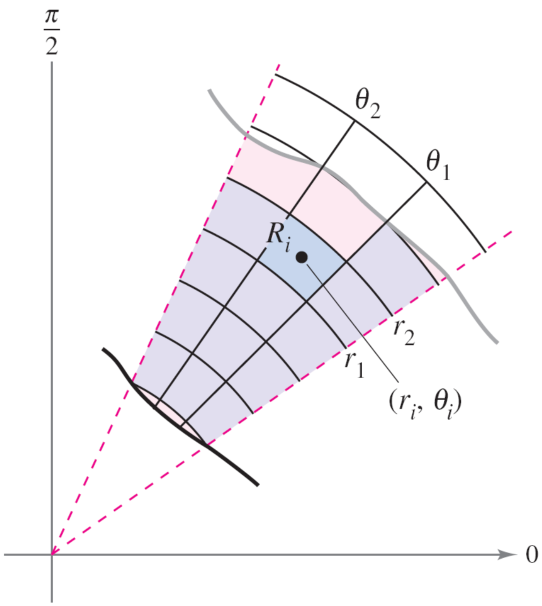
Polar grid superimposed over region R

Figure 14.26

Double Integrals in Polar Coordinates

The polar sectors R_i lying entirely within R form an **inner polar partition Δ**, whose **norm ||Δ||** is the length of the longest diagonal of the n polar sectors.

Consider a specific polar sector R_i, as shown in Figure 14.27.



The polar sector R_i is the set of all points (r, θ) such that $r_1 \le r \le r_2$ and $\theta_1 \le \theta \le \theta_2$.

Figure 14.27

Double Integrals in Polar Coordinates

It can be shown that the area of R_i is

$$\Delta A_i = r_i \Delta r_i \Delta\theta_i$$

Area of R_i

Consider specific polar sector R_i :

$$Area(R_i) = \frac{\Delta\theta\, r_2^2}{2} - \frac{\Delta\theta\, r_1^2}{2} = \frac{\Delta\theta\,(r_2^2 - r_1^2)}{2} = \frac{\Delta\theta\,(r_2 - r_1)(r_2 + r_1)}{2} \approx \frac{\Delta\theta_i\, \Delta r_i\, 2r_i}{2} = r_i \Delta r_i \Delta\theta_i$$

where $\Delta r_i = r_2 - r_1$ and $\Delta\theta_i = \theta_2 - \theta_1$.

This implies that the volume of the solid of height

$$f(r_i \cos\theta_i, r_i \sin\theta_i)$$

above R_i is approximately

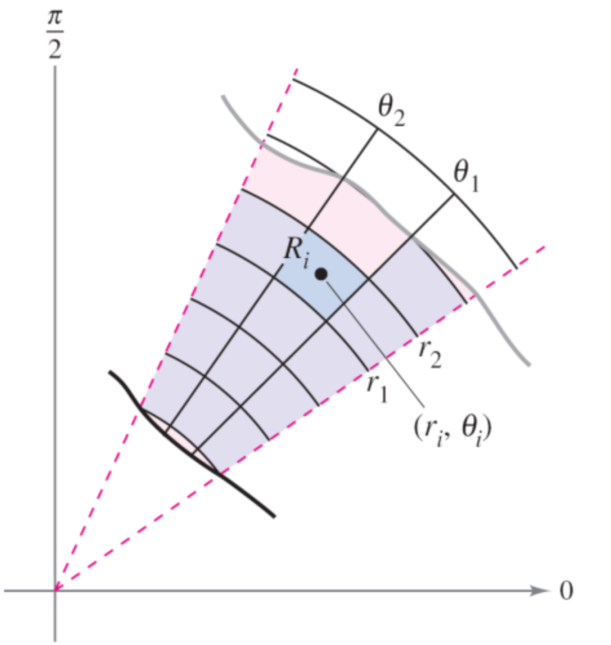
$$f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i$$

and you have

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{n} f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i.$$

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

$\boxed{dA = r dr d\theta}$



The polar sector R_i is the set of all points (r, θ) such that $r_1 \le r \le r_2$ and $\theta_1 \le \theta \le \theta_2$.

skip

Double Integrals in Polar Coordinates

$$\int_R\int f(x, y)\, dA \approx \sum_{i=1}^{n} f(r_i \cos \theta_i, r_i \sin \theta_i) r_i\, \Delta r_i\, \Delta\theta_i.$$

The sum on the right can be interpreted as a Riemann sum for $f(r \cos \theta, r \sin \theta)r.$

The region R corresponds to a *horizontally simple* region S in the $r\theta$-plane, as shown in Figure 14.28.

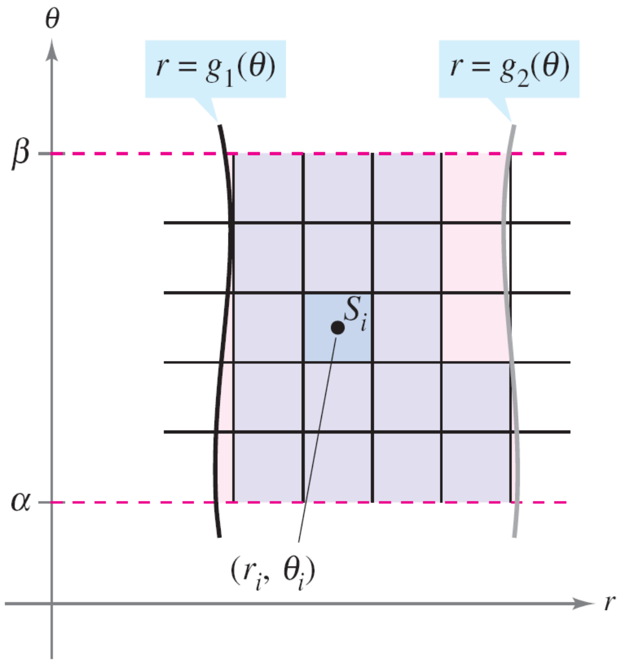


Figure 14.28

Horizontally simple region S

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Double Integrals in Polar Coordinates

The polar sectors R_i correspond to rectangles S_i, and the area ΔA_i of S_i is $\Delta r_i \Delta \theta_i$.

So, the right-hand side of the equation corresponds to the double integral

$$\iint_S f(r\cos\theta, r\sin\theta) r\, dA.$$

From this, you can write

$$\iint_R f(x, y)\, dA = \iint_S f(r\cos\theta, r\sin\theta) r\, dA$$
$$= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Double Integrals in Polar Coordinates

This suggests the following theorem 14.3

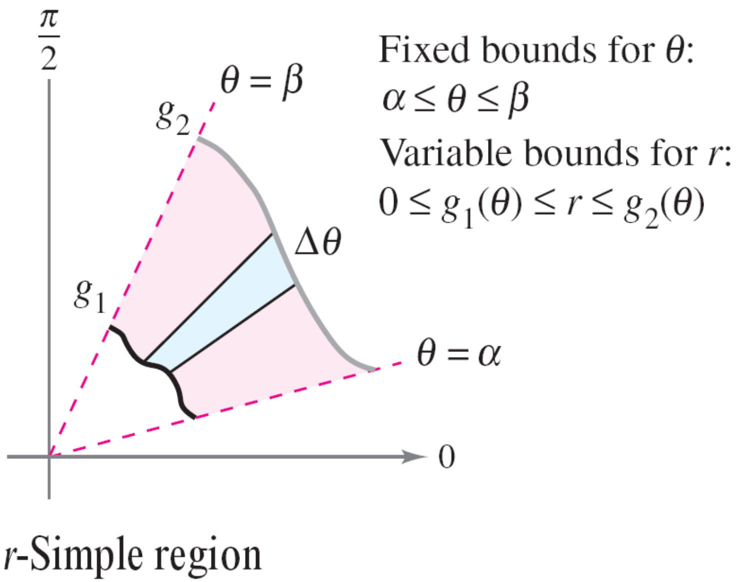
THEOREM 14.3 CHANGE OF VARIABLES TO POLAR FORM

Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\int_R\int f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Double Integrals in Polar Coordinates

The region R is restricted to two basic types, ***r*-simple** regions and ***θ*-simple** regions, as shown in Figure 14.29.



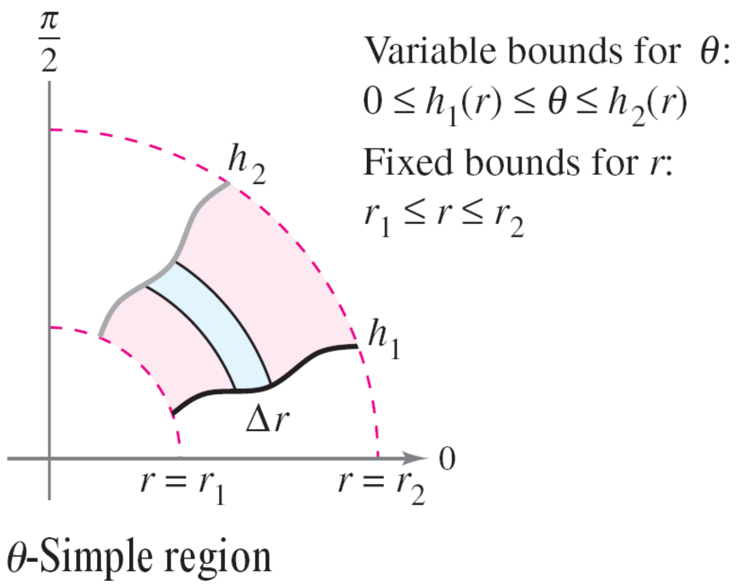


Figure 14.29

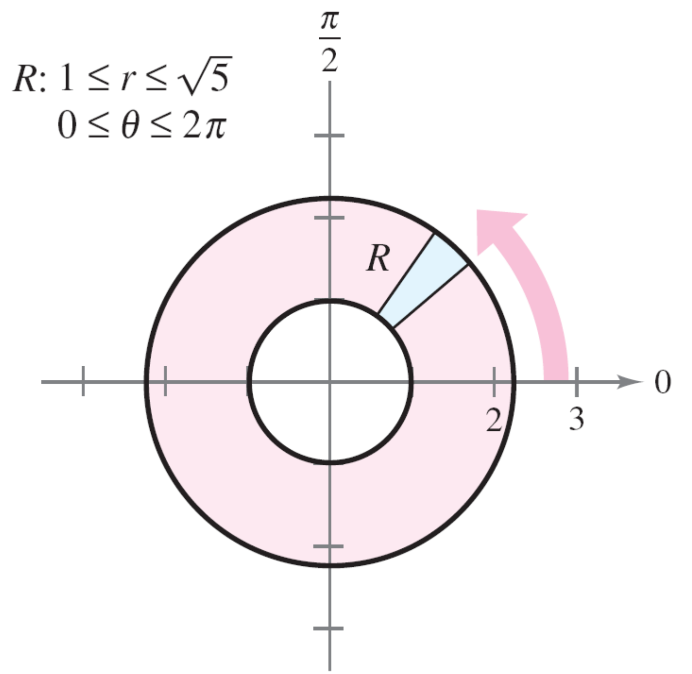
Example 2 – *Evaluating a Double Polar Integral*

Let R be the annular region lying between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$.

Evaluate the integral $\int_R\int (x^2 + y)\, dA$.

Solution:

The polar boundaries are $1 \le r \le \sqrt{5}$ and $0 \le \theta \le 2\pi$, as shown in Figure 14.30.



r-Simple region

Figure 14.30

Example 2 – *Solution*

cont'd

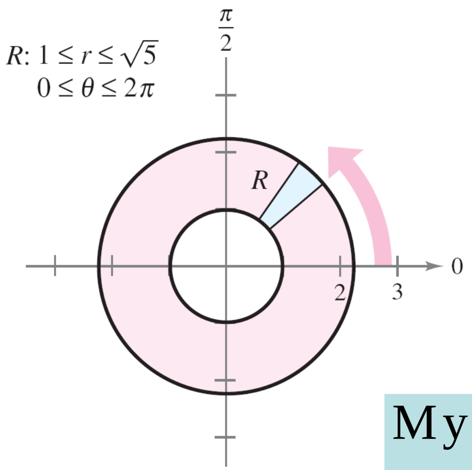
Furthermore, $x^2 = (r\cos\theta)^2$ and $y = r\sin\theta$.

So, you have

$$\iint_R (x^2 + y)\,dA = \int_0^{2\pi}\int_1^{\sqrt{5}} (r^2\cos^2\theta + r\sin\theta)r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^3\cos^2\theta + r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta\right)\Big]_1^{\sqrt{5}}\,d\theta$$



r-Simple region

My solution - 1st integrate in angle θ:

$$\int_1^{\sqrt{5}}\int_0^{2\pi}(r^2\cos^2(\theta) + r\sin(\theta))d\theta rdr = \int_1^{\sqrt{5}}\int_0^{2\pi}\left(r^2\frac{1+\cos(2\theta)}{2} + r\sin(\theta)\right)d\theta rdr =$$

$$\int_1^{\sqrt{5}}\left(r^2\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right) - r\cos(\theta)\right)_0^{2\pi} rdr = \pi\int_1^{\sqrt{5}} r^2 rdr = \pi\frac{r^4}{4}\Big|_1^{\sqrt{5}} = 6\pi$$

Example 2 – *Solution*

cont'd

$$= \int_0^{2\pi} \left(6\cos^2\theta + \frac{5\sqrt{5} - 1}{3}\sin\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(3 + 3\cos 2\theta + \frac{5\sqrt{5} - 1}{3}\sin\theta\right) d\theta$$

$$= \left(3\theta + \frac{3\sin 2\theta}{2} - \frac{5\sqrt{5} - 1}{3}\cos\theta\right)\Bigg]_0^{2\pi}$$

$$= 6\pi.$$

My example:

Find volume between surfaces for positive $x > 0$:

$$z = f_1(x, y) = 3(x^2 + y^2) = 3r^2$$

$$z = f_2(x, y) = 4 - (x^2 + y^2) = 4 - r^2$$

Intersection: $3r^2 = 4 - r^2 => r^2 = 1 => r = 1$

$$Volume = \int_{-\pi/2}^{\pi/2} \int_0^1 ((4 - r^2) - 3r^2) r dr d\theta = 4\pi \int_0^1 (r - r^3) dr =$$

$$4\pi \left(\frac{r^2}{2} - \frac{r^4}{4} \right)_0^1 = \pi$$

```
kill(all)$  load("draw")$

f1 : 3*(x^2 + y^2);    f2 : 4 - (x^2 + y^2) ;
```

$3(y^2+x^2)$

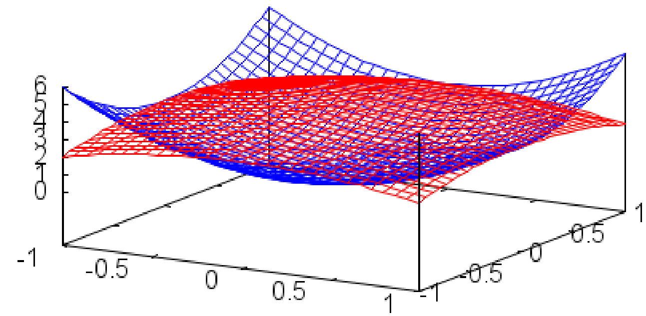
$-y^2-x^2+4$

```
f1r : trigsimp(at(f1,[x=r*cos(theta),y=r*sin(theta)]));
f2r : trigsimp(at(f2,[x=r*cos(theta),y=r*sin(theta)]));
```

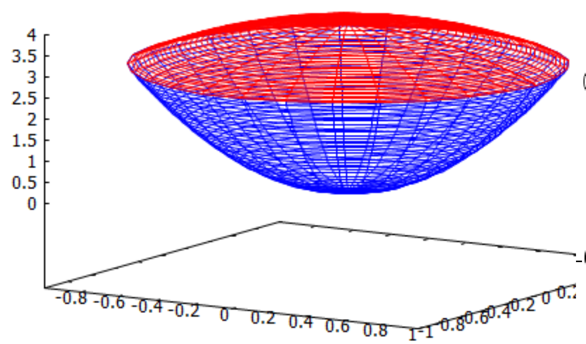
$3r^2$

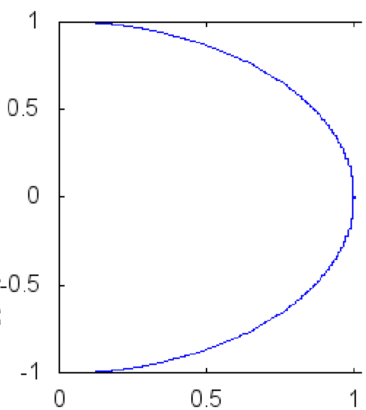
$4-r^2$

```
wxdraw3d(color=blue,explicit(f1,x,-1,1,y,-1,1),
        color=red,explicit(f2,x,-1,1,y,-1,1));
```



```
draw3d(color=blue,cylindrical(sqrt(z/3),z,0,3,theta,0,2*%pi),
       color=red,cylindrical(sqrt(4-z),z,3,4,theta,0,2*%pi));
```





```
wxdraw2d(implicit(f1=f2,x,0,2,y,-1,1));
```

```
solve(f1r=f2r,r);
```

$[r=-1, r=1]$

```
integrate((f2r-f1r)*r,r);
Int1 : integrate((f2r-f1r)*r,r,0,1);
```

$-\frac{(4-4r^2)^2}{16}$

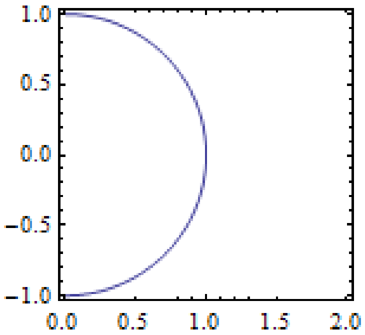
1

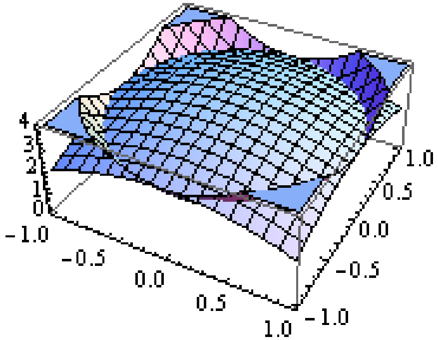
```
integrate(Int1,theta) ;
Vol : integrate(Int1,theta,-%pi/2,%pi/2);
```

θ

π

```
f1 = 3 * (x^2 + y^2);   f2 = 4 - (x^2 + y^2);
Plot3D[{f1, f2}, {x, -1, 1}, {y, -1, 1}, PlotRange -> {0, 4}]
 ContourPlot[f1 == f2, {x, 0, 2}, {y, -1, 1}]
f1 = Simplify[f1 /. {x → r * Cos[θ], y → r * Sin[θ]}]
f2 = Simplify[f2 /. {x → r * Cos[θ], y → r * Sin[θ]}]
```





$3 r^2$

$4 - r^2$

```
Solve[f1 == f2, r]
```

{{r → -1}, {r → 1}}

```
Integrate[(f2 - f1) * r, r]
Int1 = Simplify[Integrate[(f2 - f1) * r, {r, 0, 1}]]
```

$2 r^2 - r^4$

1

```
Integrate[Int1, θ]
Volume = Integrate[Int1, {θ, -Pi / 2, Pi / 2}]
```

θ

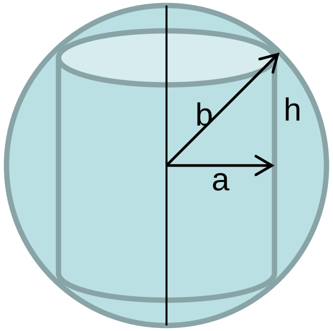
π

My example:

Find volume remaining in a sphere of radius b after a cylindrical hole of radius a has been drilled along its center.

$$x^2 + y^2 + z^2 = b^2 \Rightarrow r^2 + z^2 = b^2 \Rightarrow z = \pm\sqrt{b^2 - r^2}$$

Note that xy projection is annulus of inner radius a and outer radius b.



$$Volume = 2\int_0^{2\pi}\int_a^b \sqrt{b^2 - r^2}\, r\,dr\,d\theta = 4\pi\int_a^b \sqrt{b^2 - r^2}\, r\,dr =$$

$$[u = b^2 - r^2 \Rightarrow du = -2r\,dr] = 4\pi \int_{b^2-a^2}^{0} \sqrt{u}\left(-\frac{du}{2}\right) =$$

$$2\pi \int_0^{b^2-a^2} \sqrt{u}\,du = 2\pi \left.\frac{u^{3/2}}{3/2}\right|_0^{b^2-a^2} = \frac{4\pi}{3}(b^2 - a^2)^{3/2} = \frac{4\pi}{3} h^{3/2}$$

where h is half-length of the drilled sphere.

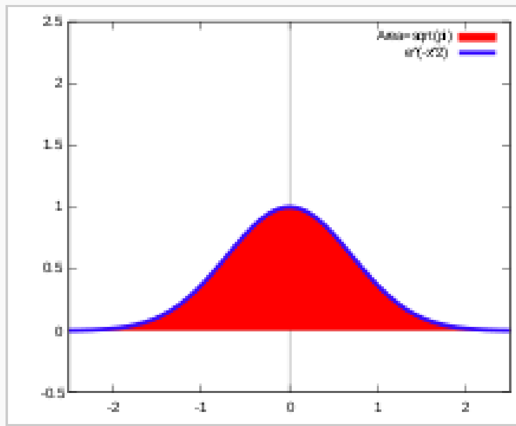
The **Gaussian integral**, also known as the **Euler-Poisson integral**[1] is the integral of the Gaussian function e^{-x^2} over the entire real line. It is named after the German mathematician and physicist Carl Friedrich Gauss. The integral is:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

This integral has wide applications. When normalized so that its value is 1, it is the cumulative distribution function of the normal distribution. It is closely related to the error function, which is the same integral with finite limits.

Although no elementary function exists for the error function, as can be proven by the Risch algorithm, the Gaussian integral can be solved analytically through the tools of calculus. That is, there is no elementary *indefinite integral* for $\int e^{-x^2}\,dx$, but the definite integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ can be evaluated.

The Gaussian integral is encountered very often in physics and numerous generalizations of the integral are encountered in quantum field theory.



A graph of $f(x) = e^{-x^2}$ and the area between the function and the x-axis, which is equal to $\sqrt{\pi}$.

$$\text{Consider}\quad \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}dydx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dydx = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta =$$

$$2\pi\int_0^{\infty} e^{-r^2}rdr = [u = r^2 \Rightarrow du = 2rdr] = 2\pi\int_0^{\infty} e^{-u}\frac{du}{2} = \pi\left.\frac{e^{-u}}{-1}\right|_0^{\infty} = \pi \Rightarrow$$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = [u = \sqrt{a}x \quad (a>0) \Rightarrow du = \sqrt{a}dx] = \int_{-\infty}^{\infty} e^{-u^2}\frac{du}{\sqrt{a}} = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\frac{\pi}{a}}$$

Just as with rectangular coordinates, the double integral

$$\iint_R dA$$

can be used to find the area of a region in the plane.

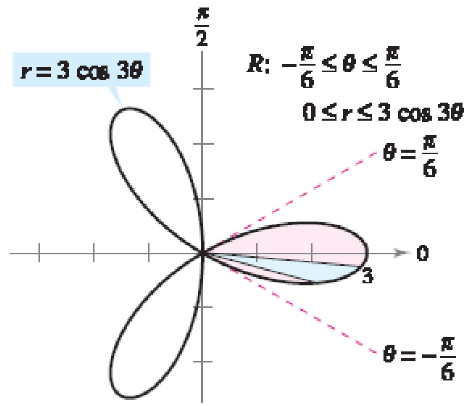
EXAMPLE 4 Finding Areas of Polar Regions

Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.

Solution Let R be one petal of the curve shown in Figure 14.32. This region is r-simple, and the boundaries are as follows.

$$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \qquad \text{Fixed bounds on } \theta$$

$$0 \le r \le 3\cos 3\theta \qquad \text{Variable bounds on } r$$



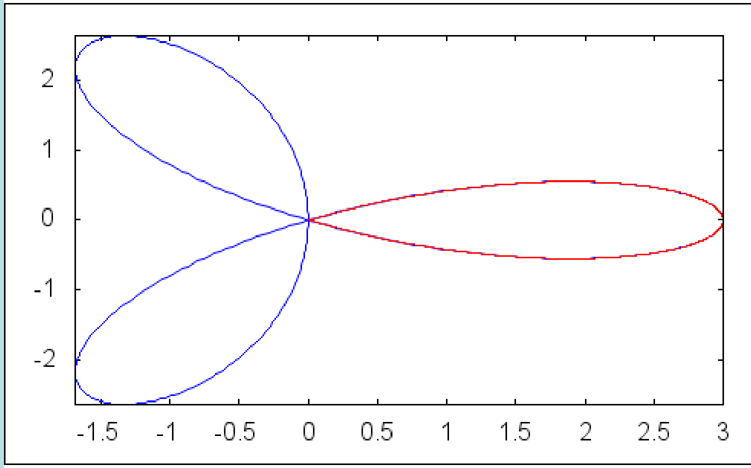
So, the area of one petal is

$$\begin{aligned}\frac{1}{3}A = \iint_R dA &= \int_{-\pi/6}^{\pi/6}\int_0^{3\cos 3\theta} r\,dr\,d\theta \\ &= \int_{-\pi/6}^{\pi/6}\left[\frac{r^2}{2}\right]_0^{3\cos 3\theta} d\theta \\ &= \frac{9}{2}\int_{-\pi/6}^{\pi/6}\cos^2 3\theta\,d\theta \\ &= \frac{9}{4}\int_{-\pi/6}^{\pi/6}(1+\cos 6\theta)\,d\theta = \frac{9}{4}\left[\theta + \frac{1}{6}\sin 6\theta\right]_{-\pi/6}^{\pi/6} = \frac{3\pi}{4}.\end{aligned}$$

So, the total area is $A = 9\pi/4$.

```
r1 : 3*cos(3*theta);   f : 1;
3 cos(3 θ)
1

wxdraw2d(nticks=500,polar(r1,theta,0,%pi),
         color=red,polar(r1,theta,-%pi/6,%pi/6));
```




```
integrate(f*r,r);
Int1 : integrate(f*r,r,0,r1);
```

$$\frac{r^2}{2}$$

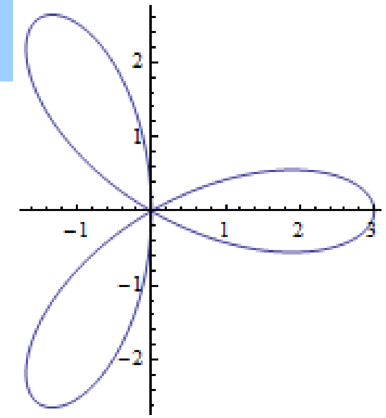
$$\frac{9\cos(3\theta)^2}{2}$$

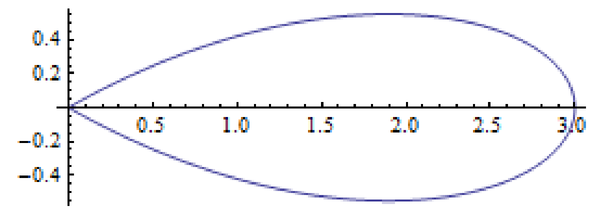
```
integrate(Int1,theta) ;
Int : 3*integrate(Int1,theta,-%pi/6,%pi/6);
```

$$\frac{3\left(\frac{\sin(6\theta)}{2}+3\theta\right)}{4}$$

$$\frac{9\pi}{4}$$

```
r1 = 3*Cos[3*θ];  f = 1;
PolarPlot[r1, {θ, 0, π}]
PolarPlot[r1, {θ, -π/6, π/6}]
```





```
Integrate[f*r, r]
Int1 = Simplify[Integrate[f*r, {r, 0, r1}]]
```

$$\frac{r^2}{2}$$

$$\frac{9}{2}\,\mathrm{Cos}[3\,\theta]^2$$

```
Integrate[Int1, θ]
Area = 3*Integrate[Int1, {θ, -Pi/6, Pi/6}]
```

$$\frac{9}{2}\left(\frac{\theta}{2}+\frac{1}{12}\,\mathrm{Sin}[6\,\theta]\right)$$

$$\frac{9\,\pi}{4}$$

As illustrated in Example 4, the area of a region in the plane can be represented by

$$A = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} r\,dr\,d\theta.$$

If $g_1(\theta) = 0$, you obtain

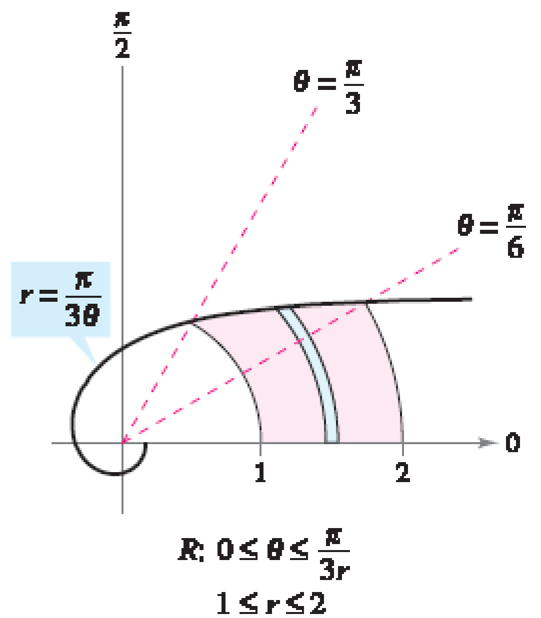
$$A = \int_{\alpha}^{\beta} \int_{0}^{g_2(\theta)} r\,dr\,d\theta = \int_{\alpha}^{\beta} \frac{r^2}{2}\bigg]_{0}^{g_2(\theta)} d\theta = \int_{\alpha}^{\beta} \frac{1}{2}(g_2(\theta))^2\,d\theta$$

which agrees with Theorem 10.13.

So far in this section, all of the examples of iterated integrals in polar form have been of the form

$$\int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

in which the order of integration is with respect to r first. Sometimes you can obtain a simpler integration problem by switching the order of integration, as illustrated in the next example.



EXAMPLE 5 Changing the Order of Integration

Find the area of the region bounded above by the spiral $r = \pi/(3\theta)$ and below by the polar axis, between $r = 1$ and $r = 2$.

Solution The region is shown in Figure 14.33. The polar boundaries for the region are

$$1 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le \frac{\pi}{3r}.$$

So, the area of the region can be evaluated as follows.

$$A = \int_{1}^{2} \int_{0}^{\pi/(3r)} r\,d\theta\,dr = \int_{1}^{2} r\theta\bigg]_{0}^{\pi/(3r)} dr = \int_{1}^{2} \frac{\pi}{3}\,dr = \frac{\pi r}{3}\bigg]_{1}^{2} = \frac{\pi}{3}$$