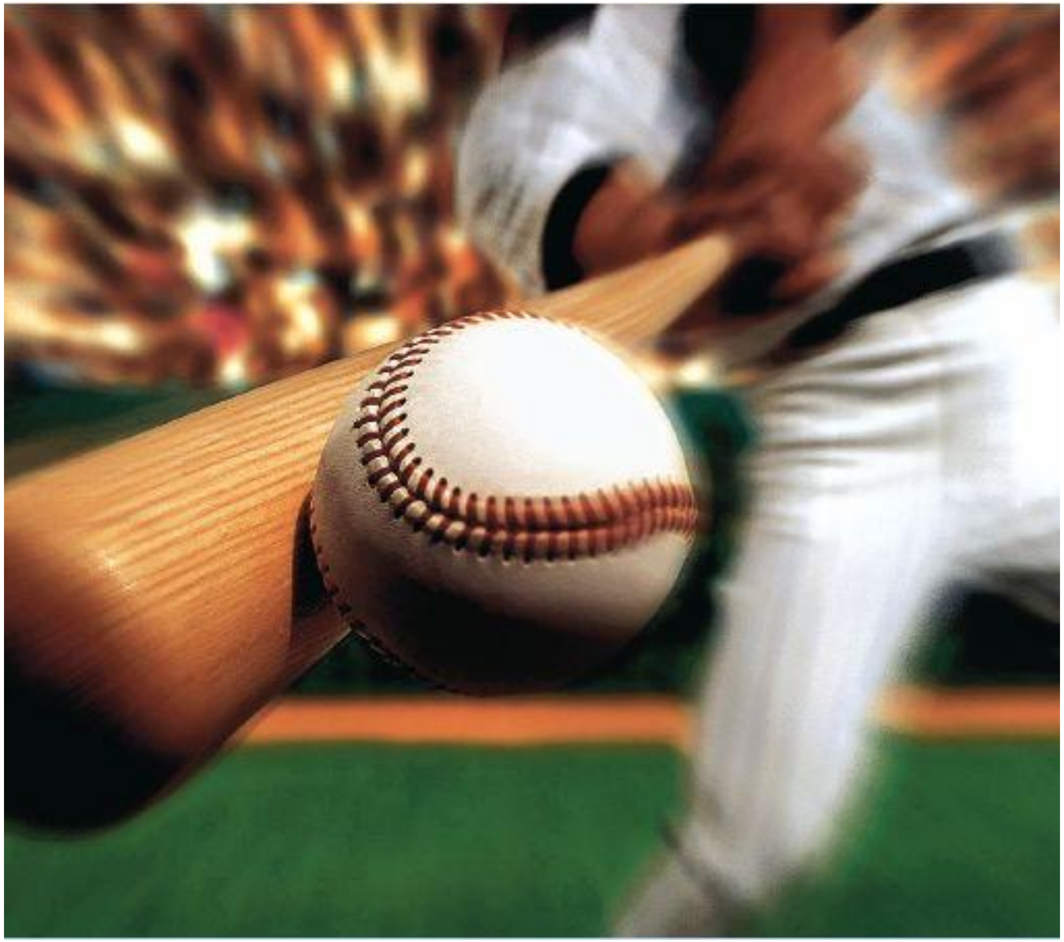
10 Conics, Parametric Equations, and Polar Coordinates



10.5

Area and Arc Length in Polar Coordinates

Objectives

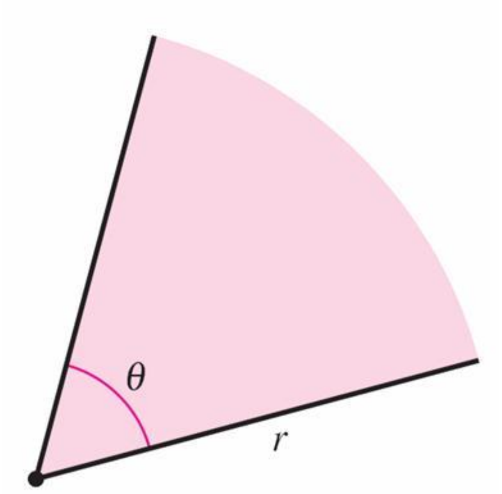
- Find the area of a region bounded by a polar graph.
- Find the points of intersection of two polar graphs.
- Find the arc length of a polar graph.
- Find the area of a surface of revolution (polar form).

Area of a Polar Region

Area of a Polar Region

The development of a formula for the area of a polar region parallels that for the area of a region on the rectangular coordinate system, but uses sectors of a circle instead of rectangles as the basic elements of area.

In Figure 10.49, note that the area of a circular sector of radius *r* is given by $\frac{1}{2}\theta r^2$, provided θ is measured in radians.



The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.

Figure 10.49

Area of a Polar Region

Consider the function given by $r = f(\theta)$, where f is continuous and nonnegative on the interval given by $\alpha \le \theta \le \beta.$

The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 10.50(a).

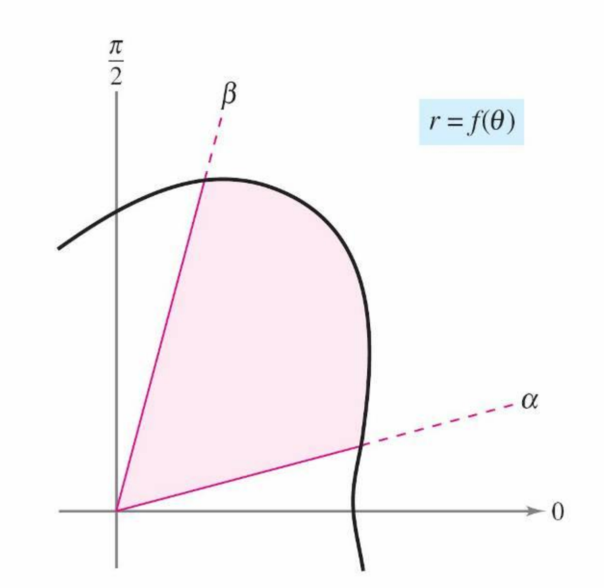


Figure 10.50(a)

To find the area of this region, partition the interval $[\alpha, \beta]$ into n equal subintervals

$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = \beta.$$

Area of a Polar Region

Then approximate the area of the region by the sum of the areas of the n sectors, as shown in Figure 10.50(b).

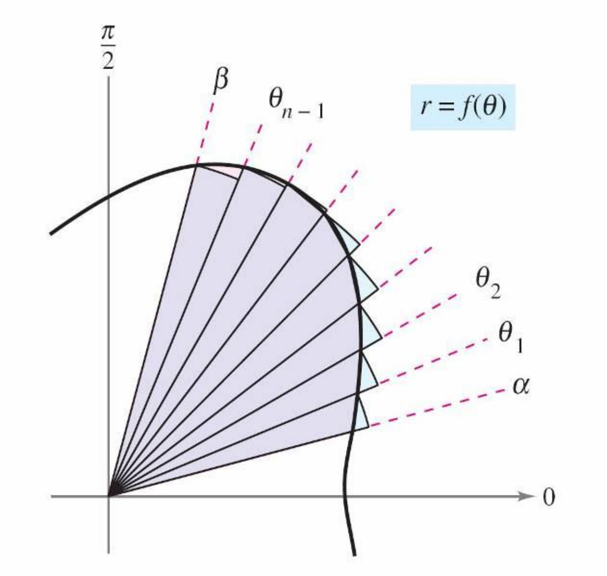


Figure 10.50(b)

Area of a Polar Region

Radius of ith sector = $f(\theta_i)$

Central angle of ith sector = $\dfrac{\beta - \alpha}{n} = \Delta\theta$

$$A \approx \sum_{i=1}^{n} \left(\frac{1}{2}\right) \Delta\theta [f(\theta_i)]^2$$

Taking the limit as $n \to \infty$ produces

$$\begin{aligned} A &= \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} [f(\theta_i)]^2 \, \Delta\theta \\ &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta \end{aligned}$$

Area of a Polar Region

THEOREM 10.13 AREA IN POLAR COORDINATES

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta$$

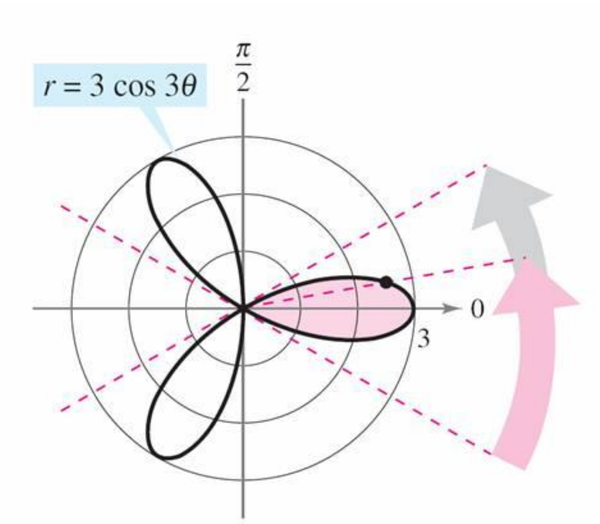
$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta. \qquad 0 < \beta - \alpha \leq 2\pi$$

Example 1 – *Finding the Area of a Polar Region*

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.

Solution:

In Figure 10.51, you can see that the petal on the right is traced as θ increases from $-\pi/6$ to $\pi/6$.



The area of one petal of the rose curve that lies between the radial lines $\theta = -\pi/6$ and $\theta = \pi/6$ is $3\pi/4$

Figure 10.51

Example 1 – *Solution*

So, the area is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2 \, d\theta$$

Formula for area in polar coordinates

$$= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1 + \cos 6\theta}{2} \, d\theta$$

Trigonometric identity

$$= \frac{9}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6}$$

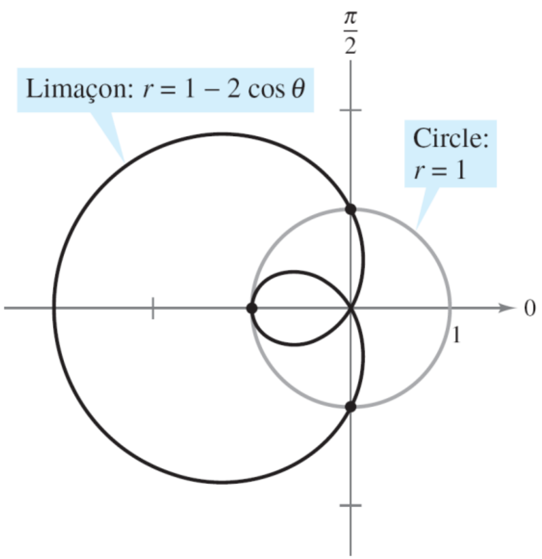
$$= \frac{9}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = \frac{3\pi}{4}.$$

Points of Intersection of Polar Graphs

Points of Intersection of Polar Graphs

Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs.

For example, consider the points of intersection of the graphs of
$r = 1 - 2\cos\theta$ and $r = 1$
as shown in Figure 10.53.



Three points of intersection:
$(1, \pi/2)$, $(-1, 0)$, $(1, 3\pi/2)$

Figure 10.53

Points of Intersection of Polar Graphs

If, as with rectangular equations, you attempted to find the points of intersection by solving the two equations simultaneously, you would obtain

$r = 1 - 2\cos\theta$	First equation
$1 = 1 - 2\cos\theta$	Substitute $r = 1$ from 2nd equation into 1st equation.
$\cos\theta = 0$	Simplify.
$\theta = \frac{\pi}{2}, \quad \frac{3\pi}{2}.$	Solve for θ.

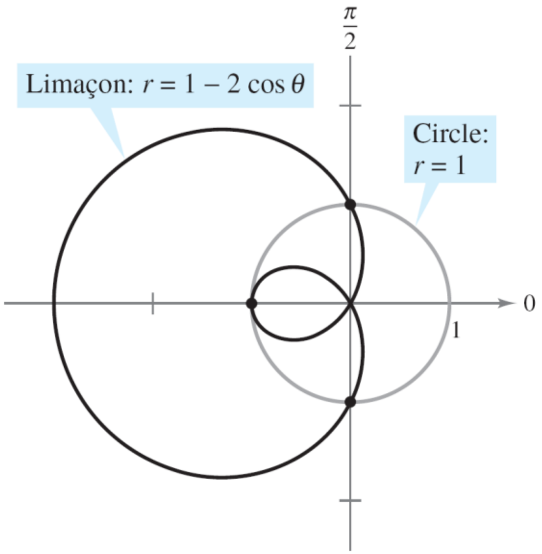
The corresponding points of intersection are $(1, \pi/2)$ and $(1, 3\pi/2)$.

Points of Intersection of Polar Graphs

However, from Figure 10.53 you can see that there is a *third* point of intersection that did not show up when the two polar equations were solved simultaneously.

The reason the third point was not found is that it does not occur with the same coordinates in the two graphs.

On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of $r = 1 - 2\cos\theta$, the point occurs with coordinates $(-1, 0)$.



Three points of intersection:
$(1, \pi/2)$, $(-1, 0)$, $(1, 3\pi/2)$

Figure 10.53

Points of Intersection of Polar Graphs

You can compare the problem of finding points of intersection of two polar graphs with that of finding collision points of two satellites in intersecting orbits about Earth, as shown in Figure 10.54.

The satellites will not collide as long as they reach the points of intersection at different times (θ-values). Collisions will occur only at the points of intersection that are "simultaneous points"—those reached at the same time (θ-value).

The paths of satellites can cross without causing a collision.

Figure 10.54.

Example 3 – *Finding the Area of a Region Between Two Curves*

Find the area of the region common to the two regions bounded by the following curves.

$r = -6 \cos \theta$ Circle

$r = 2 - 2 \cos \theta$ Cardioid

Solution:

Because both curves are symmetric with respect to the x-axis, you can work with the upper half-plane, as shown in Figure 10.55.

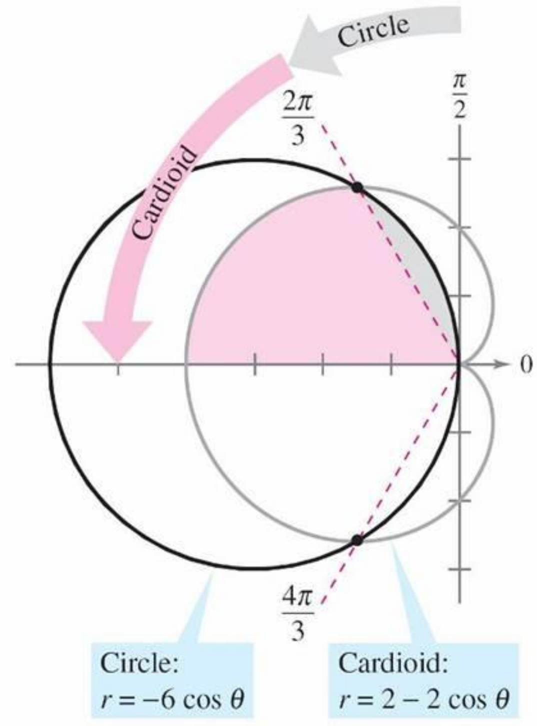


Figure 10.55

Example 3 – *Solution*

The gray shaded region lies between the circle and the radial line $\theta = 2\pi/3$.

Because the circle has coordinates $(0, \pi/2)$ at the pole, you can integrate between $\pi/2$ and $2\pi/3$ to obtain the area of this region.

The region that is shaded red is bounded by the radial lines $\theta = 2\pi/3$ and $\theta = \pi$ and the cardioid.

So, you can find the area of this second region by integrating between $2\pi/3$ and π.

Example 3 – *Solution*

cont'd

The sum of these two integrals gives the area of the common region lying *above* the radial line $\theta = \pi$.

$$\frac{A}{2} = \underbrace{\frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2\,d\theta}_{\text{Region between circle and radial line } \theta = 2\pi/3} + \underbrace{\frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2\,d\theta}_{\text{Region between cardioid and radial lines } \theta = 2\pi/3 \text{ and } \theta = \pi}$$

$$= 18\int_{\pi/2}^{2\pi/3} \cos^2\theta\,d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (4 - 8\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= 9\int_{\pi/2}^{2\pi/3} (1 + \cos 2\theta)\,d\theta + \int_{2\pi/3}^{\pi} (3 - 4\cos\theta + \cos 2\theta)\,d\theta$$

Example 3 – *Solution*

cont'd

$$= 9\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{2\pi/3} + \left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_{2\pi/3}^{\pi}$$

$$= 9\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{2}\right) + \left(3\pi - 2\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}\right)$$

$$= \frac{5\pi}{2}$$

$$\approx 7.85$$

Finally, multiplying by 2, you can conclude that the total area is 5π.

Arc Length in Polar Form

Arc Length in Polar Form

The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations.

THEOREM 10.14 ARC LENGTH OF A POLAR CURVE

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example 4 – *Finding the Length of a Polar Curve*

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid $r = f(\theta) = 2 - 2\cos\theta$ as shown in Figure 10.56.

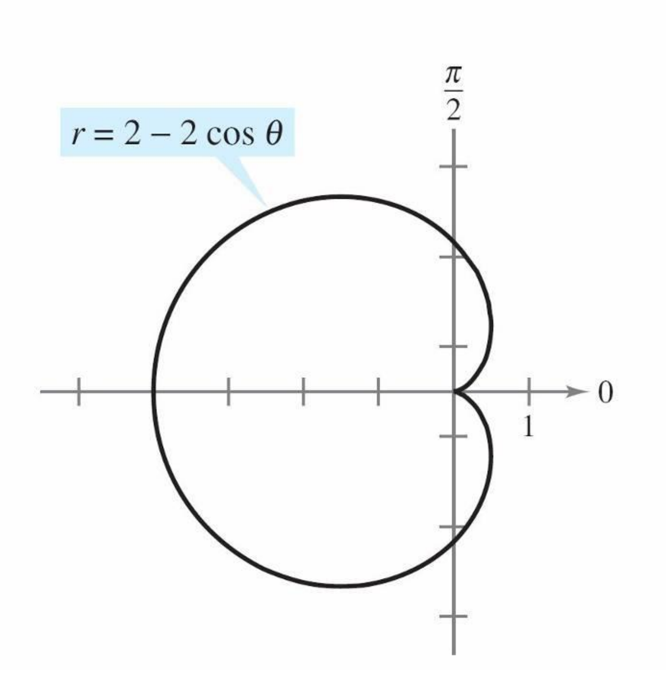


Figure 10.56

Example 4 – *Solution*

Because $f'(\theta) = 2 \sin \theta$, you can find the arc length as follows.

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

Formula for arc length of a polar curve

$$= \int_{0}^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta$$

$$= 2\sqrt{2} \int_{0}^{2\pi} \sqrt{1 - \cos\theta}\, d\theta$$

Simplify.

$$= 2\sqrt{2} \int_{0}^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\, d\theta$$

Trigonometric identity

Example 4 – *Solution*

cont'd

$$= 4\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta$$

sin $\frac{\theta}{2}$ ≥ 0 for 0 ≤ θ ≤ 2π

$$= 8\left[-\cos\frac{\theta}{2}\right]_0^{2\pi}$$

$$= 8(1 + 1)$$

$$= 16$$

Example 4 – *Solution*

cont'd

In the fifth step of the solution, it is legitimate to write

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\,\sin(\theta/2)$$

rather than

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\,|\sin(\theta/2)|$$

because $\sin(\theta/2) \geq 0$ for $0 \leq \theta \leq 2\pi$.

Area of a Surface of Revolution

Area of a Surface of Revolution

The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

THEOREM 10.15 AREA OF A SURFACE OF REVOLUTION

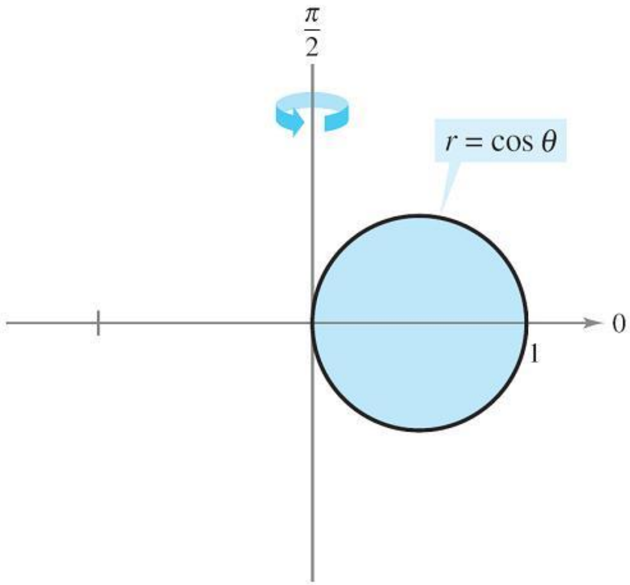
Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

1. $S = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ — About the polar axis

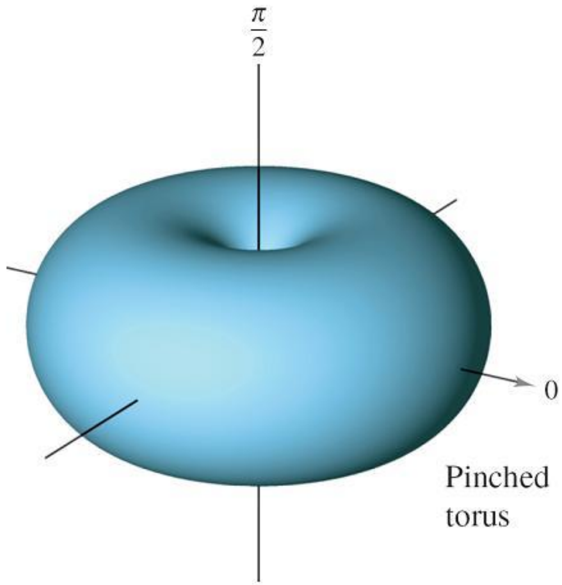
2. $S = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ — About the line $\theta = \frac{\pi}{2}$

Example 5 – *Finding the Area of Surface of Revolution*

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown in Figure 10.57.



(a)



(b)

Figure 10.57

Example 5 – *Solution*

You can use the second formula given in Theorem 10.15 with $f'(\theta) = -\sin\theta$. Because the circle is traced once as θ increases from 0 to π, you have

$$S = 2\pi\int_{\alpha}^{\beta} f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$$

Formula for area of a surface of revolution

$$= 2\pi\int_{0}^{\pi} \cos\theta(\cos\theta)\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 2\pi\int_{0}^{\pi} \cos^2\theta\,d\theta$$

Trigonometric identity

$$= \pi\int_{0}^{\pi} (1 + \cos 2\theta)\,d\theta$$

Trigonometric identity

$$= \pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = \pi^2.$$