

Curvatures of Curves on Surfaces  
Math 473  
Introduction to Differential Geometry  
Lecture 23

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December 10, 2017

We have seen in the computations for the helix on the cylinder (**Example 2, Lecture 22**) were feasible because the helix is a curve of constant speed. For a general curve the expressions for the Darboux vectors  $T$ ,  $U$  and  $N$  are much more complicated. The following proposition will simplify computations in the general case.

**Proposition (1):** Let  $\tilde{T}$ ,  $\tilde{U}$ ,  $\tilde{N}$  be positive multiples of  $T$ ,  $U$ ,  $N$ , i.e.

$$\tilde{T}(t) = \lambda(t) \cdot T(t), \quad \tilde{U}(t) = \mu(t) \cdot U(t), \quad \tilde{N}(t) = \nu(t) \cdot N(t)$$

for some functions  $\lambda, \mu, \nu : I \rightarrow \mathbb{R}$  with  $\lambda(t), \mu(t), \nu(t) > 0$  for all  $t \in I$ . Then

$$\begin{aligned} \kappa_g &= \frac{\tilde{T}' \bullet \tilde{U}}{|\gamma'| \cdot |\tilde{T}| \cdot |\tilde{U}|} = -\frac{\tilde{T} \bullet \tilde{U}'}{|\gamma'| \cdot |\tilde{T}| \cdot |\tilde{U}|}, \\ \kappa_n &= \frac{\tilde{T}' \bullet \tilde{N}}{|\gamma'| \cdot |\tilde{T}| \cdot |\tilde{N}|} = -\frac{\tilde{T} \bullet \tilde{N}'}{|\gamma'| \cdot |\tilde{T}| \cdot |\tilde{N}|}, \\ \kappa_t &= \frac{\tilde{U}' \bullet \tilde{N}}{|\gamma'| \cdot |\tilde{U}| \cdot |\tilde{N}|} = -\frac{\tilde{U} \bullet \tilde{N}'}{|\gamma'| \cdot |\tilde{U}| \cdot |\tilde{N}|}. \end{aligned}$$

### Remark:

For example we could consider

$$\tilde{T} = \gamma' = |\gamma'| \cdot T, \quad \lambda = |\gamma'|,$$

$$\tilde{N} = X_u \times X_v = |X_u \times X_v| \cdot N, \quad \mu = |X_u \times X_v|,$$

$$\tilde{U} = \tilde{N} \times \tilde{T} = |\gamma'| \cdot |X_u \times X_v| \cdot (N \times T) = |\gamma'| \cdot |X_u \times X_v| \cdot U,$$

$$\nu = \lambda \cdot \mu = |\gamma'| \cdot |X_u \times X_v|.$$

## Proof of Proposition (1):

## Proposition (2):

We have

$$\kappa_g^2 + \kappa_n^2 = \kappa^2$$

, where  $\kappa_g$  is the geodesic curvature,  $\kappa_n$  is the normal curvature and  $\kappa$  is the curvature of the curve  $\gamma$ .

## Example (1):\*

Let  $X$  be the surface patch  $X(u, v) = (u, v, uv)$ .

- Ⓐ Check that the non-unit speed curve  $\gamma(t) = X(t, -t) = (t, -t, -t^2)$  is a geodesic.
- Ⓑ Let  $\gamma(t) = X(u(t), v(t)) = (u(t), v(t), u(t)v(t))$  be a curve on the surface  $X$ .
  - Ⓐ Show that a (non-unit) normal to  $X$  along the curve  $\gamma$  is given by  $\tilde{N} = (-v, -u, 1)$ .
  - Ⓑ Show that a (non-unit) tangent of the curve  $\gamma$  is given by  $\tilde{T} = (u', v', u'v + uv')$ .
  - Ⓒ Find a condition on  $u$  and  $v$  for the curve  $\gamma$  to be a geodesic.
  - Ⓓ Use the condition you have found to find as many geodesics on  $X$  as you can.

*Thanks for listening.*