Curvatures of Curves on Surfaces
Math 473
Introduction to Differential Geometry
Lecture 25

Dr. Nasser Bin Turki

King Saud University
Department of Mathematics

November 18, 2018

We have seen in the computations for the helix on the cylinder (**Example 2, Lecture 22**) were feasible because the helix is a curve of constant speed. For a general curve the expressions for the Darboux vectors T, U and N are much more complicated. The following proposition will simplify computations in the general case.

Proposition (1): Let $\tilde{T}$, $\tilde{U}$, $\tilde{N}$ be positive multiples of T, U, N, i.e.

$$\tilde{T}(t) = \lambda(t) \cdot T(t), \quad \tilde{U}(t) = \mu(t) \cdot U(t), \quad \tilde{N}(t) = \nu(t) \cdot N(t)$$

for some functions $\lambda, \mu, \nu : I \to \mathbb{R}$ with $\lambda(t), \mu(t), \nu(t) > 0$ for all $t \in I$. Then

$$\kappa_g = \frac{\tilde{T}' \bullet \tilde{U}}{|\gamma'| \cdot |\tilde{T}| \cdot |\tilde{U}|} = -\frac{\tilde{T} \bullet \tilde{U}'}{|\gamma'| \cdot |\tilde{T}| \cdot |\tilde{U}|},$$

$$\kappa_n = \frac{\tilde{T}' \bullet \tilde{N}}{|\gamma'| \cdot |\tilde{T}| \cdot |\tilde{N}|} = -\frac{\tilde{T} \bullet \tilde{N}'}{|\gamma'| \cdot |\tilde{T}| \cdot |\tilde{N}|},$$

$$\kappa_t = \frac{\tilde{U}' \bullet \tilde{N}}{|\gamma'| \cdot |\tilde{U}| \cdot |\tilde{N}|} = -\frac{\tilde{U} \bullet \tilde{N}'}{|\gamma'| \cdot |\tilde{U}| \cdot |\tilde{N}|}.$$

Remark:

For example we could consider

$$\tilde{T} = \gamma' = |\gamma'| \cdot T, \quad \lambda = |\gamma'|,$$
$$\tilde{N} = X_u \times X_v = |X_u \times X_v| \cdot N, \quad \mu = |X_u \times X_v|,$$
$$\tilde{U} = \tilde{N} \times \tilde{T} = |\gamma'| \cdot |X_u \times X_v| \cdot (N \times T) = |\gamma'| \cdot |X_u \times X_v| \cdot U,$$
$$\nu = \lambda \cdot \mu = |\gamma'| \cdot |X_u \times X_v|.$$

Proof of Proposition (1):

Proposition (2):

We have

$$\kappa_g^2 + \kappa_n^2 = \kappa^2$$

, where κ_g is the geodesic curvature, κ_n is the normal curvature and κ is the curvature of the curve γ.

Example (1):*

Let X be the surface patch $X(u,v) = (u, v, uv)$.

a. Check that the non-unit speed curve $\gamma(t) = X(t, -t) = (t, -t, -t^2)$ is a geodesic.
b. Let $\gamma(t) = X(u(t), v(t)) = (u(t), v(t), u(t)v(t))$ be a curve on the surface X.
   i. Show that a (non-unit) normal to X along the curve γ is given by $\tilde{N} = (-v, -u, 1)$.
   ii. Show that a (non-unit) tangent of the curve γ is given by $\tilde{T} = (u', v', u'v + uv')$.
   iii. Find a condition on u and v for the curve γ to be a geodesic.
   iv. Use the condition you have found to find as many geodesics on X as you can.

Thanks for listening.