



**Faculty of Engineering**  
**Mechanical Engineering Department**

# **CALCULUS FOR ENGINEERS**

## **MATH 1110**

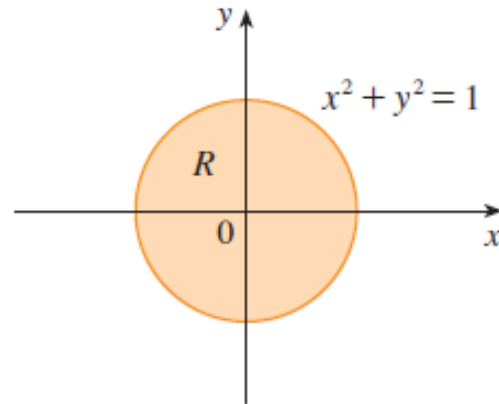
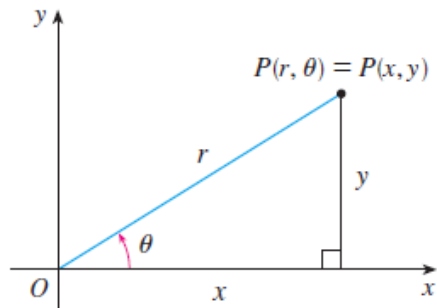
**: Instructor**

**Dr. O. Philips Agboola**

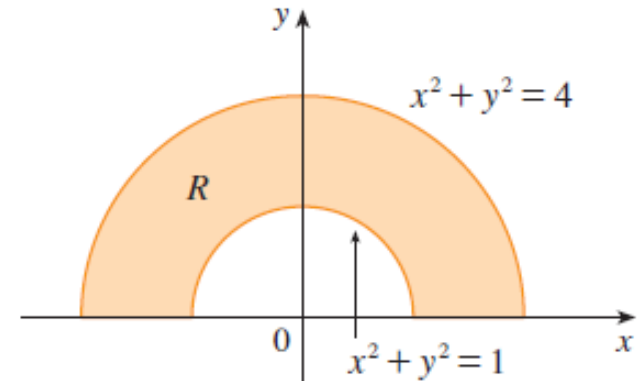
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# Double INTEGRAL IN POLAR COORDINATES



(a)  $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$



(b)  $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

The rectangular coordinates  $(x, y)$  and polar coordinates  $(r, \theta)$  of a point  $P$  are related as follows:

(i)  $x = r \cos \theta, \quad y = r \sin \theta$

(ii)  $\tan \theta = \frac{y}{x}, \quad r^2 = x^2 + y^2$

# Double INTEGRAL IN POLAR COORDINATES

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

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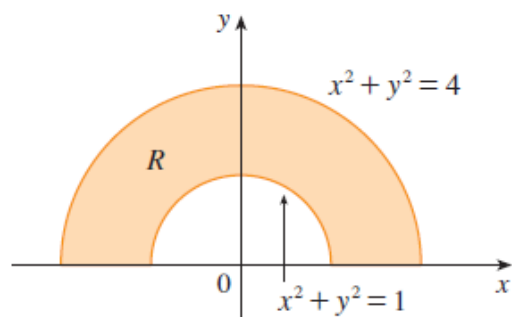
**EXAMPLE 1** Evaluate  $\iint_R (3x + 4y^2) dA$ , where  $R$  is the region in the upper half-plane bounded by the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .

**SOLUTION** The region  $R$  can be described as

$$R = \{(x, y) \mid y \geq 0, 1 \leq x^2 + y^2 \leq 4\}$$

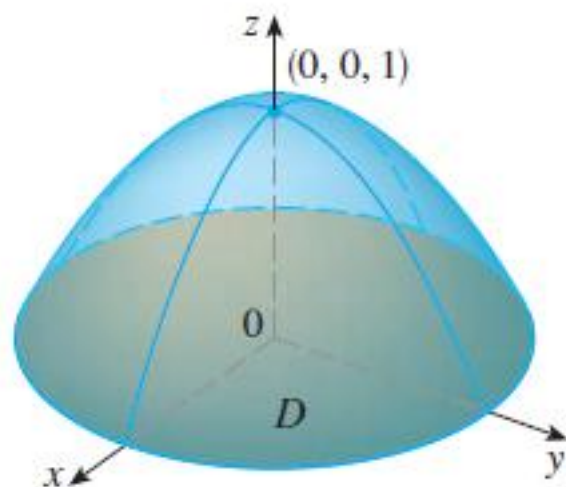
It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by  $1 \leq r \leq 2$ ,  $0 \leq \theta \leq \pi$ . Therefore, by Formula 2,

$$\begin{aligned} \iint_R (3x + 4y^2) dA &= \int_0^\pi \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta \\ &= \int_0^\pi \int_1^2 (3r^2 \cos \theta + 4r^3 \sin^2 \theta) dr d\theta \\ &= \int_0^\pi [r^3 \cos \theta + r^4 \sin^2 \theta]_{r=1}^{r=2} d\theta = \int_0^\pi (7 \cos \theta + 15 \sin^2 \theta) d\theta \\ &= \int_0^\pi \left[ 7 \cos \theta + \frac{15}{2}(1 - \cos 2\theta) \right] d\theta \\ &= 7 \sin \theta + \frac{15\theta}{2} - \frac{15}{4} \sin 2\theta \Big|_0^\pi = \frac{15\pi}{2} \end{aligned}$$



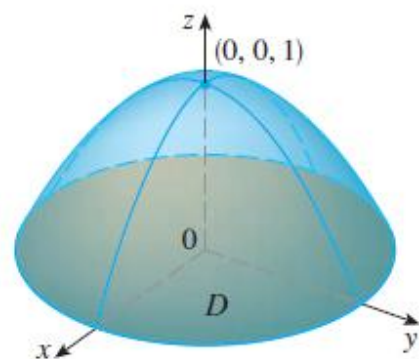
(b)  $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

**EXAMPLE 2** Find the volume of the solid bounded by the plane  $z = 0$  and the paraboloid  $z = 1 - x^2 - y^2$ .

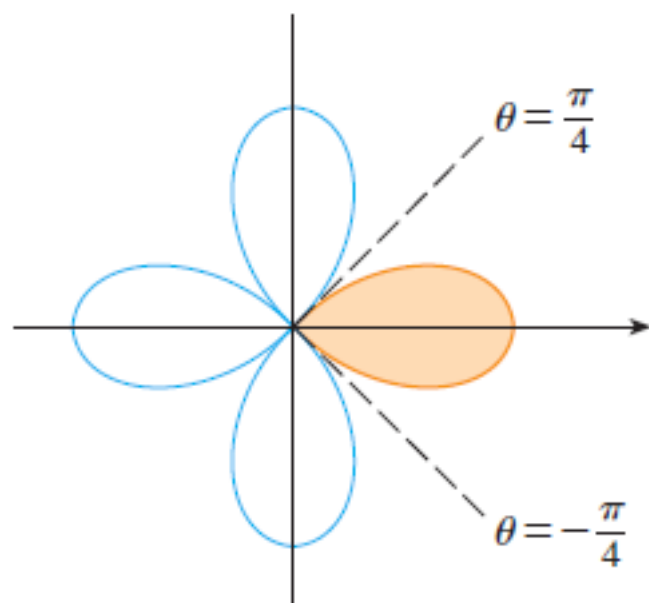


**SOLUTION** If we put  $z = 0$  in the equation of the paraboloid, we get  $x^2 + y^2 = 1$ . This means that the plane intersects the paraboloid in the circle  $x^2 + y^2 = 1$ , so the solid lies under the paraboloid and above the circular disk  $D$  given by  $x^2 + y^2 \leq 1$  [see Figures 6 and 1(a)]. In polar coordinates  $D$  is given by  $0 \leq r \leq 1$ ,  $0 \leq \theta \leq 2\pi$ . Since  $1 - x^2 - y^2 = 1 - r^2$ , the volume is

$$\begin{aligned} V &= \iint_D (1 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \\ &= \int_0^{2\pi} d\theta \int_0^1 (r - r^3) dr = 2\pi \left[ \frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2} \end{aligned}$$



**EXAMPLE 3** Use a double integral to find the area enclosed by one loop of the four-leaved rose  $r = \cos 2\theta$ .



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**SOLUTION** From the sketch of the curve in Figure 8, we see that a loop is given by the region

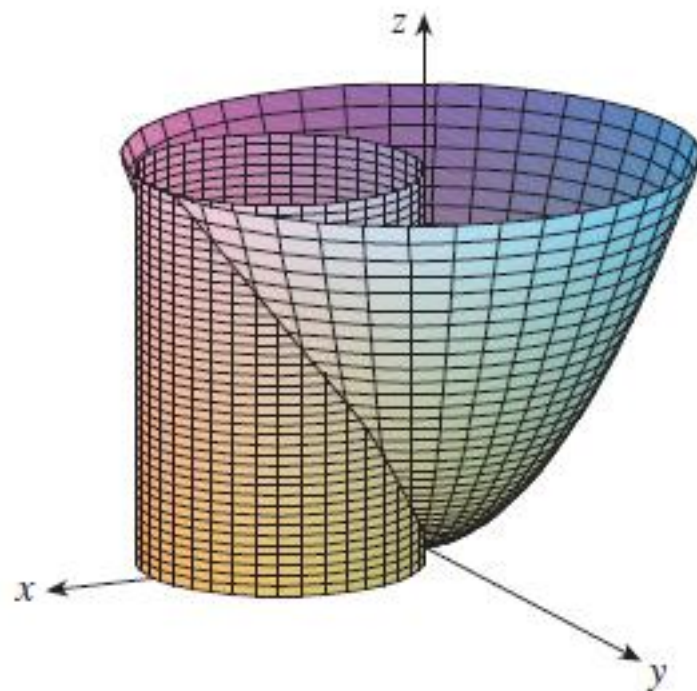
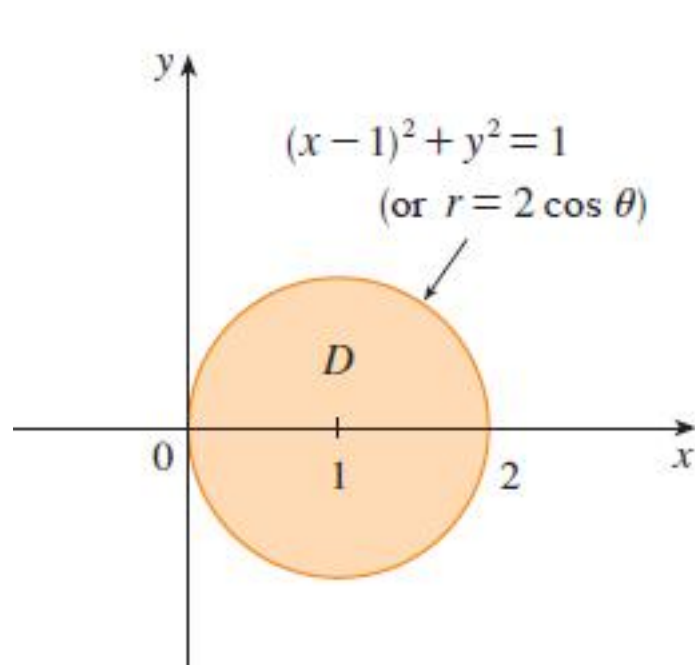
$$D = \{(r, \theta) \mid -\pi/4 \leq \theta \leq \pi/4, 0 \leq r \leq \cos 2\theta\}$$

So the area is

$$\begin{aligned} A(D) &= \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r \, dr \, d\theta \\ &= \int_{-\pi/4}^{\pi/4} \left[ \frac{1}{2} r^2 \right]_0^{\cos 2\theta} d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta \\ &= \frac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta) \, d\theta = \frac{1}{4} \left[ \theta + \frac{1}{4} \sin 4\theta \right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

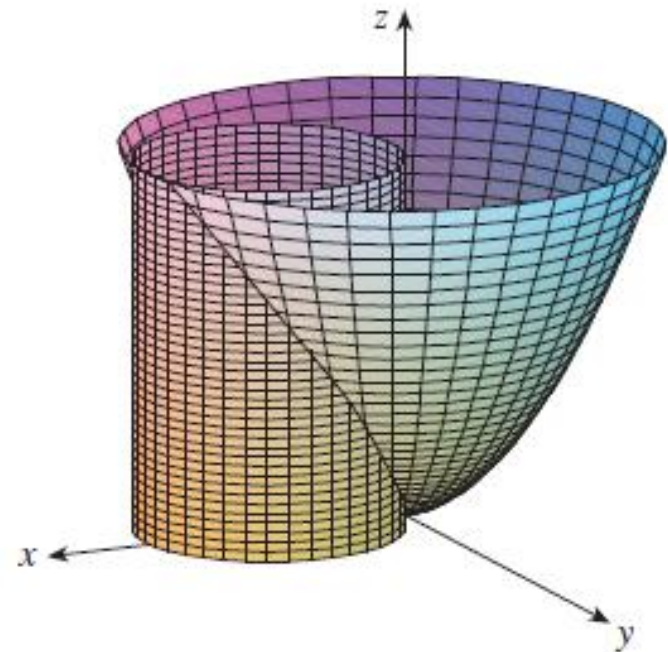
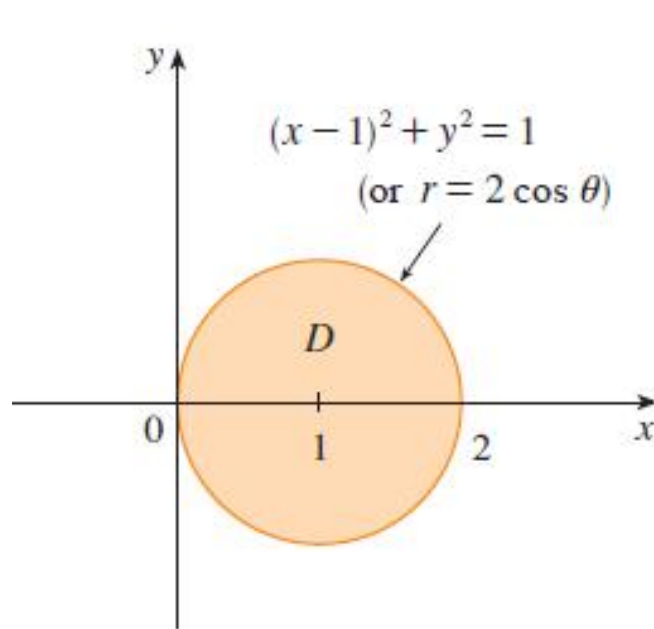


**EXAMPLE 4** Find the volume of the solid that lies under the paraboloid  $z = x^2 + y^2$ , above the  $xy$ -plane, and inside the cylinder  $x^2 + y^2 = 2x$ .



**SOLUTION** The solid lies above the disk  $D$  whose boundary circle has equation  $x^2 + y^2 = 2x$  or, after completing the square,

$$(x - 1)^2 + y^2 = 1$$



In polar coordinates we have  $x^2 + y^2 = r^2$  and  $x = r \cos \theta$ , so the boundary circle becomes  $r^2 = 2r \cos \theta$ , or  $r = 2 \cos \theta$ . Thus the disk  $D$  is given by

$$D = \{(r, \theta) \mid -\pi/2 \leq \theta \leq \pi/2, 0 \leq r \leq 2 \cos \theta\}$$

and, by Formula 3, we have

$$\begin{aligned} V &= \iint_D (x^2 + y^2) dA = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^2 r dr d\theta = \int_{-\pi/2}^{\pi/2} \left[ \frac{r^4}{4} \right]_0^{2 \cos \theta} d\theta \\ &= 4 \int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta = 8 \int_0^{\pi/2} \cos^4 \theta d\theta = 8 \int_0^{\pi/2} \left( \frac{1 + \cos 2\theta}{2} \right)^2 d\theta \\ &= 2 \int_0^{\pi/2} \left[ 1 + 2 \cos 2\theta + \frac{1}{2}(1 + \cos 4\theta) \right] d\theta \\ &= 2 \left[ \frac{3}{2} \theta + \sin 2\theta + \frac{1}{8} \sin 4\theta \right]_0^{\pi/2} = 2 \left( \frac{3}{2} \right) \left( \frac{\pi}{2} \right) = \frac{3\pi}{2} \end{aligned}$$

