

Bertrand Curves
Math 473
Introduction to Differential Geometry
Lecture 12

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Definition (1): Let $\alpha : I \mapsto \mathbb{R}^3$ be unit speed curve. Let the curvature $\kappa(t) > 0$ and the torsion $\tau(t) \neq 0$ for all $t \in I$.

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$$N_{\alpha}(t) = \pm N_{\beta}(t).$$

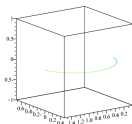
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$$N_{\alpha}(t) = \pm N_{\beta}(t).$$

In this case β is called a Bertrand mate of α .

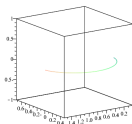
Example

Let $\alpha(t) = \frac{1}{2}(\cos^{-1} t - t\sqrt{1-t^2}, 1-t^2, 0)$ be unit speed curve

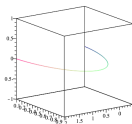


Example

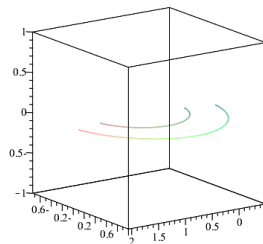
Let $\alpha(t) = \frac{1}{2}(\cos^{-1} t - t\sqrt{1-t^2}, 1-t^2, 0)$ be unit speed curve



and $\beta(t) = \frac{1}{2}(\cos^{-1} t - t\sqrt{1-t^2}, 1-t^2 + \sqrt{1-t^2}, 0)$.



The curve α and β are Bertrand curves and we say that β is a Bertrand mate of α .



Theorem(1):

Let $\alpha : I \mapsto \mathbb{R}^3$ be a Bertrand curve and $\beta : I \mapsto \mathbb{R}^3$ be the Bertrand mate of α . Then, the distance between corresponding points of $\alpha(t)$ and $\beta(t)$ is constant.

Theorem(1):

Let $\alpha : I \mapsto \mathbb{R}^3$ be a Bertrand curve and $\beta : I \mapsto \mathbb{R}^3$ be the Bertrand mate of α . Then, the distance between corresponding points of $\alpha(t)$ and $\beta(t)$ is constant.

(Hint: if $\alpha(t)$ is unit speed parametrisation curve, then there is a function $\lambda(t)$ such that $\alpha(t) + \lambda(t)N_\alpha(t) = \beta(t)$ gives the other curve.)

Proof:

Theorem(2):

Let $\alpha : I \mapsto \mathbb{R}^3$ be a Bertrand curve and $\beta : I \mapsto \mathbb{R}^3$ be the Bertrand mate of α . Then

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Let $\alpha : I \mapsto \mathbb{R}^3$ be a Bertrand curve and $\beta : I \mapsto \mathbb{R}^3$ be the Bertrand mate of α . Then

$$(i) \quad T_\alpha(t) \bullet T_\beta(t) = c_1$$

$$(ii) \quad B_\alpha(t) \bullet T_\beta(t) = c_2,$$

where c_1, c_2 are constant.

Proof:

Thanks for listening.