# Surfaces
# Math 473
# Introduction to Differential Geometry
# Lecture 18

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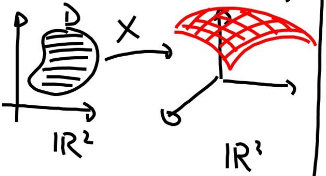
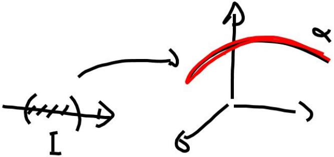
## Surface

$$X : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto X(u,v)$$




## Curve

$$\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \alpha(t)$$



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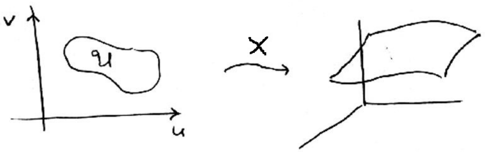

**Examples of surfaces**

Sphere, plane, helicoid, ellipsoid, torus, cylinder, cone.

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We can **describe** surfaces by their equations:

1. The horizontal $x$-$y$-plane $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$,
2. The unit sphere $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$,
3. The ellipsoid $\left\{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1\right\}$.

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We discussed that plane curves resp. space curves can be described by their equations or as a trajectory of a moving point, i.e. as a map from an interval $I$ in $\mathbb{R}$ into $\mathbb{R}^2$ or $\mathbb{R}^3$ resp. Similarly, we can describe surfaces by their equations or as a map from a subset of $\mathbb{R}^2$ into $\mathbb{R}^3$.

**Definition (1):** A **surface patch** is a map $X : U \to \mathbb{R}^3$, where $U$ is a subset of $\mathbb{R}^2$.

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**Remark (1):** A surface patch $X : U \to \mathbb{R}^3$ can be described by the **coordinate functions** $X_1, X_2, X_3 : U \to \mathbb{R}$:

$$X(u, v) = (X_1(u, v), X_2(u, v), X_3(u, v)).$$

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**Remark(2):** We will consider the partial derivatives of $X$:

$$\frac{\partial X}{\partial u}(u,v) = \left(\frac{\partial X_1}{\partial u}(u,v), \frac{\partial X_2}{\partial u}(u,v), \frac{\partial X_3}{\partial u}(u,v)\right),$$
$$\frac{\partial X}{\partial v}(u,v) = \left(\frac{\partial X_1}{\partial v}(u,v), \frac{\partial X_2}{\partial v}(u,v), \frac{\partial X_3}{\partial v}(u,v)\right).$$

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**Notation:**

$$X_u = \frac{\partial X}{\partial u}, \quad X_v = \frac{\partial X}{\partial v},$$

i.e. $X_u(u,v) = \frac{\partial X}{\partial u}(u,v)$ and $X_v(u,v) = \frac{\partial X}{\partial v}(u,v)$ for any $(u,v) \in U$.

**Definition (2):** A surface patch $X : U \to \mathbb{R}^3$ is **smooth** if the partial derivatives $X_u$, $X_v$ exist and are differentiable.

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**Definition (3):** A surface patch $X : U \to \mathbb{R}^3$ is **injective** if $X(u_1, v_1) \neq X(u_2, v_2)$ for any points $(u_1, v_1)$ and $(u_2, v_2)$ in $U$ with $(u_1, v_1) \neq (u_2, v_2)$.

**Definition (4):** A surface patch $X : U \to \mathbb{R}^3$ is **regular** at a point $(u, v) \in U$ if the vectors $X_u(u, v)$ and $X_v(u, v)$ are linearly independent, i.e. are not multiples of each other.
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**Proposition (1):** A surface patch $X : U \to \mathbb{R}^3$ is regular at $(u, v) \in U$ if and only if

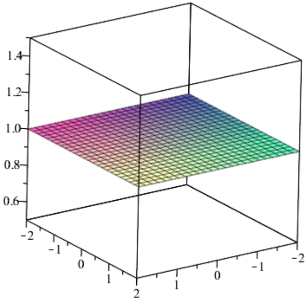
$$X_u(u, v) \times X_v(u, v) \neq 0.$$

**Example(1)**

Let $U = \mathbb{R}^2$ and consider the surface patch $X : U \to \mathbb{R}^3$ given by

$$X(u, v) = (u, v, 1).$$

Show that this determines a regular surface patch. Can you describe this surface geometrically?

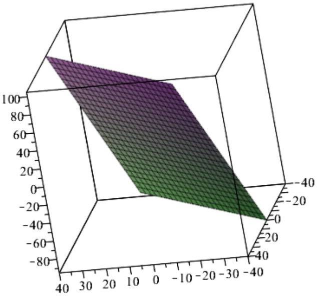


**Example(2)**

Let $U = \mathbb{R}^2$ and consider the surface patch $X : U \to \mathbb{R}^3$ given by

$$X(u,v) = (u+v, u-v, 2u+3v+5).$$

Show that this determines a regular surface patch. Can you describe this surface geometrically?

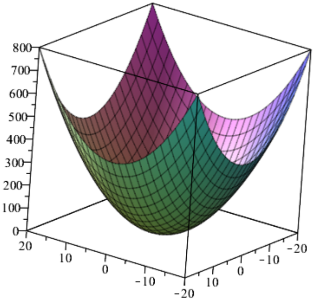


**Example(3)**

Let $U = \mathbb{R}^2$ and consider the surface patch $X : U \to \mathbb{R}^3$ given by

$$X(u, v) = (u, v, u^2 + v^2).$$

Show that this determines a regular surface patch. Can you describe this surface geometrically?



*Thanks for listening.*