

Area of Surfaces, Normal
Math 473
Introduction to Differential Geometry
Lecture 20

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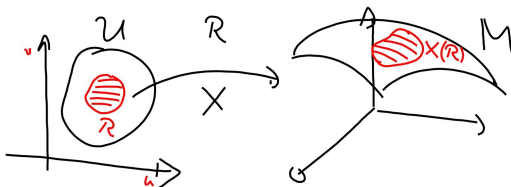
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Area of Surfaces

Definition (1):

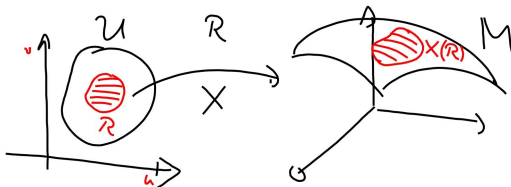
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Area of Surfaces

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Then, the **area of the domain $X(R)$ on the surface X** is

$$\text{Area}(X(R)) = \int_R \sqrt{(EG - F^2)(u, v)} \, du \, dv = \int_R \sqrt{\det I(u, v)} \, du \, dv.$$

Remark:

We have seen that

$$|X_u \times X_v| = \sqrt{EG - F^2}.$$

Example (1):

Let $U \subset \mathbb{R}^2$. Let $X : U \rightarrow \mathbb{R}^3$ be a surface given by $X(u, v) = (u, v, uv)$. Show that X is a regular surface. Compute the first fundamental form. Let $R = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 \leq 1\}$ be the unit disc. Then find area of the domain $X(R)$ on the surface X .

Proposition (1):

A surface patch $X : U \rightarrow \mathbb{R}^3$ is regular at $(u, v) \in U$ if and only if the determinant $\det I(u, v) = (EG - F^2)(u, v)$ is not equal to zero.

Proof

Definition (2): Normal Vector

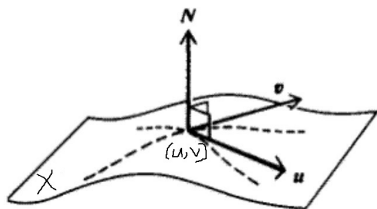
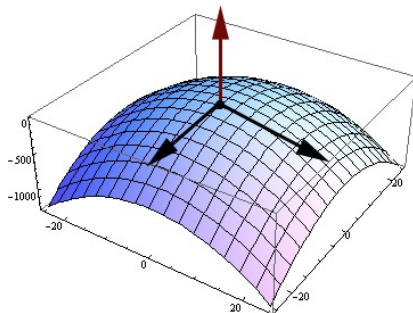
A **normal vector** to the surface X at a regular point $X(u, v)$ is a vector orthogonal to the tangent plane to X at the point $X(u, v)$, i.e. a multiple of the vector $X_u(u, v) \times X_v(u, v)$.

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Definition (3): Unit Normal Vector

A **unit normal vector** to the surface X at a regular point $X(u, v)$ is a normal vector to the surface X at the point $X(u, v)$ of length one.



Remark: At each regular point $X(u, v)$ there are two unit normal vectors. To choose a unit normal vector $N(u, v)$ at each point $X(u, v)$ in such a way that $N(u, v)$ depends continuously on (u, v) we can take

$$N(u, v) = \frac{(X_u \times X_v)(u, v)}{|(X_u \times X_v)(u, v)|}.$$

At a regular point the three vectors X_u , X_v and N give a (not orthonormal) basis.

Example (2):

Let $U \subset \mathbb{R}^2$. Let $X : U \rightarrow \mathbb{R}^3$ be a surface given by $X(u, v) = (u, v, uv)$. Find a unit normal vector to the surface X at a point $X(u, v)$.

Example (3): (Surface of Revolution)

Let $\alpha_1, \alpha_2 : \mathbb{R} \rightarrow \mathbb{R}$ be functions. Let $X : \mathbb{R} \times (-\pi, \pi) \rightarrow \mathbb{R}^3$ be a parametrization of the surface of revolution obtained by rotation of the regular curve $\alpha(u) = (\alpha_1(u), 0, \alpha_2(u))$

$$X(u, v) = (\alpha_1(u) \cos v, \alpha_1(u) \sin v, \alpha_2(u)).$$

We shall assume that $\alpha_1(u)$ is never zero and that X is injective.

- i) Prove that the surface patch X is regular.
- ii) Calculate the coefficients of the first fundamental form of X .
- iii) Find a unit normal vector to the surface X at a point $X(u, v)$.

Thanks for listening.