First Fundamental Form
Math 473
Introduction to Differential Geometry
Lecture 21

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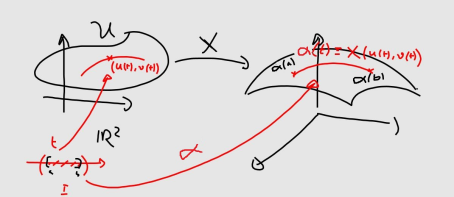
First Fundamental Form

Let $X : U \to \mathbb{R}^3$ be a regular surface patch. Let $(u(t), v(t)) \in U$. Let $\alpha : I \subset \mathbb{R} \to U \subset \mathbb{R}^2$, $\alpha(t) == X(u(t), v(t))$ be a regular curve on U.



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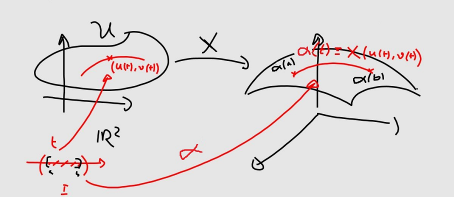

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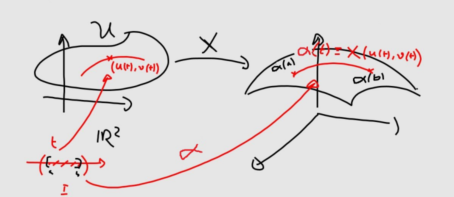

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$$
\begin{aligned}
|\alpha'|^2 &= \alpha' \bullet \alpha' = (u'X_u + v'X_v) \bullet (u'X_u + v'X_v) \\
&= (u')^2(X_u \bullet X_u) + u'v'(X_u \bullet X_v) + v'u'(X_v \bullet X_u) + (v')^2(X_v \bullet X_v) \\
&= (u')^2(X_u \bullet X_u) + 2u'v'(X_u \bullet X_v) + (v')^2(X_v \bullet X_v).
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$$F(u,v) = X_u(u,v) \bullet X_v(u,v) = X_v(u,v) \bullet X_u(u,v),$$
$$G(u,v) = X_v(u,v) \bullet X_v(u,v),$$

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The **first fundamental form** of X is

$$I = E(u')^2 + 2Fu'v' + G(v')^2,$$

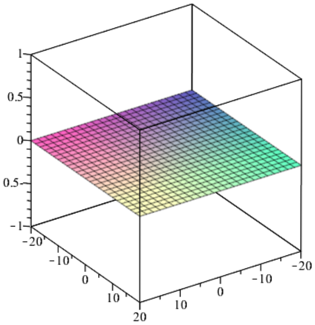
or

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

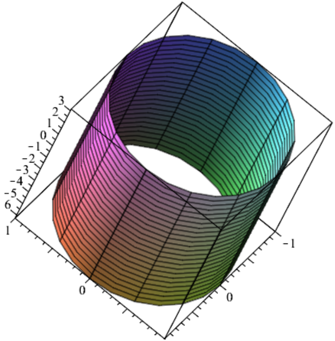
Note:

The first fundamental form is an impartment tools which allows us to make measurements on the surface (lengths of curves, angles of tangent vectors, areas of regions).

Example (1):
Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $X(u, v) = (u, v, 0)$. Compute the first fundamental form of the surface X.



Example (2):
Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $X(u, v) = (\cos u, \sin u, v)$. Compute the first fundamental form of the surface X.



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We can also rewrite this formula using the matrix notation

$$|\alpha'|^2 = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Let $X : U \to \mathbb{R}^3$ be a surface patch. Let $(u_0, v_0) \in U$.

Proposition (2):

For a curve $\alpha(t) = X(u(t), v(t))$ on the surface X we have

$$|\alpha'| = \sqrt{u'^2 E + 2u'v'F + v'^2 G} = \sqrt{\begin{pmatrix} u' & v' \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix}},$$

where $u' = u'(t)$, $v' = v'(t)$, $E = E(u(t), v(t))$, $F = F(u(t), v(t))$, $G = G(u(t), v(t))$.

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where $u' = u'(t)$, $v' = v'(t)$, $E = E(u(t), v(t))$, $F = F(u(t), v(t))$, $G = G(u(t), v(t))$. The length of the curve α from $t = t_1$ to $t = t_2$ is

$$\int_{t_1}^{t_2} |\alpha'|\, dt = \int_{t_1}^{t_2} \sqrt{u'^2 E + 2u'v'F + v'^2 G}\, dt$$

$$= \int_{t_1}^{t_2} \sqrt{\begin{pmatrix} u' & v' \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix} \cdot \begin{pmatrix} u' \\ v' \end{pmatrix}}\, dt.$$

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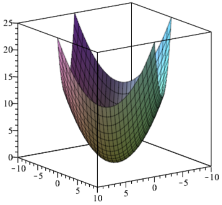
Proof:

Example (3):

Let

$$X(u,v) = \left(u - v, u + v, \frac{u^2 + v^2}{2}\right).$$

Show that X defines a regular surface patch. Calculate the coefficients E, F, G of the first fundamental form for this surface. Write down an integral which gives the length of the curve $\gamma_1(t) = X(t, 1)$ on this surface from $t = 1$ to $t = 2$.



Angles between Curves on Surfaces

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Proposition (3):
Let $X : U \to \mathbb{R}^3$ be an injective regular surface patch and let $\alpha_1(t) = X(u_1(t), v_1(t))$, $\alpha_2(t) = X(u_2(t), v_2(t))$ be curves on the surface X that intersect at a point $\alpha_1(t_1) = \alpha_2(t_2)$.

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angle between the curves α_1 and α_2 at the point of their intersection $\alpha_1(t_1) = \alpha_2(t_2)$. Then

$$\cos\theta = \frac{u_1' u_2' E + (u_1' v_2' + v_1' u_2')F + v_1' v_2' G}{|\alpha_1'| \cdot |\alpha_2'|} = \frac{\begin{pmatrix} u_1' & v_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u_2' \\ v_2' \end{pmatrix}}{|\alpha_1'| \cdot |\alpha_2'|},$$

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where u_i', v_i' are taken at t_i for $i = 1, 2$ and E, F, G are taken at the point of intersection of the curves. Note that $|\alpha_i'|$ can be computed using the first fundamental form as explained in the previous slides.

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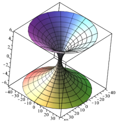
Proof:

Example (4):

Let

$$X(u,v) = \left((2+u^2)\cos v, (2+u^2)\sin v, u\right).$$

1. (i) Compute X_u and X_v.
2. (ii) Show that X defines a regular surface.
3. (iii) Compute the first fundamental form of the surface X.
4. (iv) Write down, but do not evaluate, an integral which gives the length of the curve $\delta(t) = X(t,0)$ on X from $t=-1$ to $t=2$.
5. (v) Calculate the cosine of the angle between the curves $\gamma_1(t) = X(0,t)$ and $\gamma_2(t) = X(2t, t+\pi)$ on X at the point $X(0,\pi) = (-2,0,0)$ where they meet.



Thanks for listening.