Area of Surfaces, Normal
Math 473
Introduction to Differential Geometry
Lecture 22

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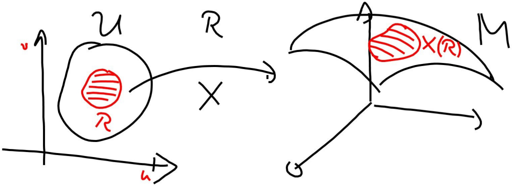
November 10, 2018

Definition (1):

Let $X : U \to \mathbb{R}^3$ be an injective regular surface patch. Let R be a subset of U and $X(R)$ the corresponding domain on the surface X.



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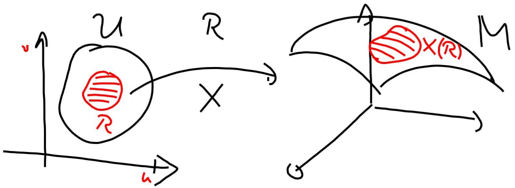

Then, the **area of the domain** $X(R)$ **on the surface** X is

$$\text{Area}(X(R)) = \int_R \sqrt{(EG - F^2)(u, v)}\, du\, dv = \int_R \sqrt{\det I(u, v)}\, du\, dv.$$

Remark:
We have seen that

$$|X_u \times X_v| = \sqrt{EG - F^2}.$$

Example (1):
Let $U \subset \mathbb{R}^2$. Let $X : U \to \mathbb{R}^3$ be a surface given by $X(u, v) = (u, v, uv)$. Show that X is a regular surface. Compute the first fundamental form. Let $R = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 \leq 1\}$ be the unit disc. Then find area of the domain $X(R)$ on the surface X.

Proposition (1):
A surface patch $X : U \to \mathbb{R}^3$ is regular at $(u, v) \in U$ if and only if the determinant $\det I(u, v) = (EG - F^2)(u, v)$ is not equal to zero.
Proof

Definition (2): Normal Vector

A **normal vector** to the surface X at a regular point $X(u, v)$ is a vector orthogonal to the tangent plane to X at the point $X(u, v)$, i.e. a multiple of the vector $X_u(u, v) \times X_v(u, v)$.

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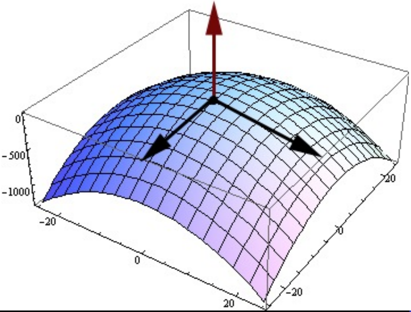
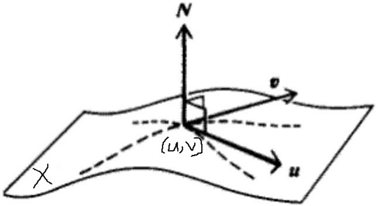
Definition (3): Unit Normal Vector

A **unit normal vector** to the surface X at a regular point $X(u,v)$ is a normal vector to the surface X at the point $X(u,v)$ of length one.





Remark: At each regular point $X(u,v)$ there are two unit normal vectors. To choose a unit normal vector $N(u,v)$ at each point $X(u,v)$ in such a way that $N(u,v)$ depends continuously on (u,v) we can take

$$N(u,v) = \frac{(X_u \times X_v)(u,v)}{|(X_u \times X_v)(u,v)|}.$$

At a regular point the three vectors X_u, X_v and N give a (not orthonormal) basis.

Example (2):
Let $U \subset \mathbb{R}^2$. Let $X : U \to \mathbb{R}^3$ be a surface given by $X(u,v) = (u, v, uv)$. Find a unit normal vector to the surface X at a point $X(u,v)$.

Example (3): (Surface of Revolution)
Let $\alpha_1, \alpha_2 : \mathbb{R} \to \mathbb{R}$ be functions. Let $X : \mathbb{R} \times (-\pi, \pi) \to \mathbb{R}^3$ be a parametrization of the surface of revolution obtained by rotation of the regular curve $\alpha(u) = (\alpha_1(u), 0, \alpha_2(u))$

$$X(u,v) = (\alpha_1(u)\cos v, \alpha_1(u)\sin v, \alpha_2(u)).$$

We shall assume that $\alpha_1(u)$ is never zero and that X is injective.

(i) Prove that the surface patch X is regular.
(ii) Calculate the coefficients of the first fundamental form of X.
(iii) Find a unit normal vector to the surface X at a point $X(u,v)$.

Thanks for listening.