Curvatures of Curves on Surfaces
Math 473
Introduction to Differential Geometry
Lecture 22

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To get some useful quantities describing the position of the curve on the surface, we will split γ'' into a sum of two vectors, one normal vector and one tangent vector.

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For example, we can choose

$$N(u, v) = \frac{X_u(u, v) \times X_v(u, v)}{|X_u(u, v) \times X_v(u, v)|}.$$

Let $\gamma : I \to \mathbb{R}^3$ be a curve on the surface X, i.e. $\gamma(t) = X(u(t), v(t))$.

Definition (1): Darboux basis

The **Darboux basis** of γ consists of the **unit tangent** T, the **normal in the tangent plane** U and the **normal** N to the surface patch X along the curve γ given by

$$T(t) = \frac{\gamma'(t)}{|\gamma'(t)|}, \quad N(t) = N(u(t), v(t)), \quad U(t) = N(t) \times T(t).$$

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Remark:

The Darboux basis is an orthonormal basis and therefore satisfies the following equations

$$\begin{aligned}
&T \bullet T = U \bullet U = N \bullet N = 1, \\
&T \bullet U = U \bullet T = U \bullet N = N \bullet U = N \bullet T = T \bullet N = 0, \\
&T \times U = N, \quad U \times N = T, \quad N \times T = U, \\
&U \times T = -N, \quad N \times U = -T, \quad T \times N = -U.
\end{aligned}$$

Definition (2): The **geodesic curvature** κ_g, the **normal curvature** κ_n and the **geodesic torsion** κ_t of the curve γ on X are defined by the following **Darboux equations**:

$$\begin{pmatrix} T' \\ U' \\ N' \end{pmatrix} = |\gamma'| \cdot \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \kappa_t \\ -\kappa_n & -\kappa_t & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ U \\ N \end{pmatrix},$$

i.e.

$$\begin{aligned} T' &= |\gamma'| \cdot \kappa_g \cdot U + |\gamma'| \cdot \kappa_n \cdot N, \\ U' &= -|\gamma'| \cdot \kappa_g \cdot T + |\gamma'| \cdot \kappa_t \cdot N, \\ N' &= -|\gamma'| \cdot \kappa_n \cdot T - |\gamma'| \cdot \kappa_t \cdot U. \end{aligned}$$

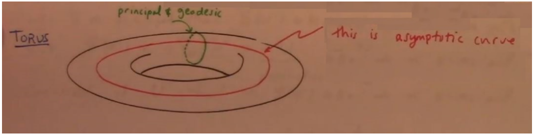
Definition (3):

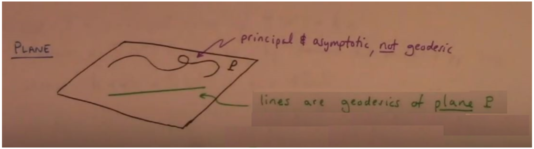
We can express the geodesic curvature κ_g, the normal curvature κ_n and the geodesic torsion κ_t as dot-products

$$\kappa_g = \frac{T' \bullet U}{|\gamma'|} = -\frac{T \bullet U'}{|\gamma'|},$$
$$\kappa_n = \frac{T' \bullet N}{|\gamma'|} = -\frac{T \bullet N'}{|\gamma'|},$$
$$\kappa_t = \frac{U' \bullet N}{|\gamma'|} = -\frac{U \bullet N'}{|\gamma'|}.$$

Definition (4):

- The curve γ is a **geodesic curve** (or a **geodesic**) if $\kappa_g(t) = 0$ for all $t \in I$.
- The curve γ is an **asymptotic curve** if $\kappa_n(t) = 0$ for all $t \in I$.
- The curve γ is a **principal curve** (or a **line of curvature**) if $\kappa_t(t) = 0$ for all $t \in I$.





Remark:

Darboux equations imply that for a geodesic curve T' is a multiple of N, i.e. T only changes in the normal direction, not in tangent directions. One could say that a geodesic curve "drives straight ahead". **Indeed, geodesics are (locally) the shortest way on the surface to get from one point to another.**

Example (1):
For the surface $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by $X(u,v) = (u, v, \sqrt{1-u^2-v^2})$, $(u,v) \in U = \{(u,v) \in \mathbb{R}^2, u^2+v^2 < 1\}$. Let $M = x(U)$ is the upper half of the unit sphere. Find the geodesic curvature κ_g, normal curvature κ_n and geodesic torsion κ_t of the curve $\gamma(t) = (\sin t, 0, \cos t)$ lies on M? Is $\gamma(t)$ geodesic curve? **Why**? Is $\gamma(t)$ principal curve? **Why**?

Example (2):
Consider the cylinder surface $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by $X(u, v) = (\cos u, \sin u, v)$, . The curve

$$\gamma(t) = X(t, 2t) = (\cos t, \sin t, 2t)$$

on the cylinder X is a helix. Find the geodesic curvature κ_g, normal curvature κ_n and geodesic torsion κ_t? Is $\gamma(t)$ geodesic curve? **Why**?

Exercise (1):

For the surface $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by $X(u, v) = (u, v, u^2 + v^2)$ find the normal curvature κ_n of the curve $\gamma(t) = X(t^2, t)$ at $t = 1$.

Thanks for listening.