Principal Curvature
Math 473
Introduction to Differential Geometry
Lecture 25

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Corollary (1):

The normal curvature of a curve on a surface only depends on the velocity of the curve, i.e. if γ_1, γ_2 are two curves through a point on the surface with the same velocity at this point then their normal curvatures at this point coincide(equal).

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Remark:

The sign of the normal curvature depends on the choice of the unit normal. If N is a unit normal on the surface then $-N$ is also a unit normal. The change of sign $N \mapsto -N$ for the unit normal causes further sign changes: $(T, U, N) \mapsto (T, -U, -N)$, $\kappa_g \mapsto -\kappa_g$, $\kappa_n \mapsto -\kappa_n$, $\kappa_t \mapsto \kappa_t$, $\mathrm{I} \mapsto \mathrm{I}$, $\mathrm{II} \mapsto -\mathrm{II}$.

Definition (1):
The **sectional curvature** of the surface X at the point $X(u_0, v_0)$ in the direction of the tangent vector $x \cdot X_u(u_0, v_0) + y \cdot X_v(u_0, v_0)$, $x, y \in \mathbb{R}$, is given by

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$$\kappa(x \cdot X_u + y \cdot X_v) = \frac{x^2 \cdot e + 2xy \cdot f + y^2 \cdot g}{x^2 \cdot E + 2xy \cdot F + y^2 \cdot G} = \frac{\begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} e & f \\ f & g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}{\begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}.$$

Remark:
For a curve $\gamma(t) = X(u(t), v(t))$ on the surface X the normal curvature κ_n of γ is equal to the sectional curvature of X in the direction of the tangent vector $\gamma' = u' \cdot X_u + v' \cdot X_v$.

Example (1):
We consider the cylinder $X(u,v) = (\cos u, \sin u, v)$.
The first and the second fundamental forms are

$$\mathrm{I}(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{II}(u,v) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

What is the range of possible values of the sectional curvature

$$\kappa(x,y) := \kappa(x \cdot X_u + y \cdot X_v) = \frac{x^2 \cdot e + 2xy \cdot f + y^2 \cdot g}{x^2 \cdot E + 2xy \cdot F + y^2 \cdot G},$$

what are the maxima and the minima?

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Proof

Remark:

The matrix $\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is invertible since

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Note:

The sectional curvature is the same along any line $\{r \cdot (x, y),\ r \neq 0\}$ since

$$\begin{aligned} p(rx, ry) =& (rx)^2 \cdot e + 2(rx)(ry) \cdot f + (ry)^2 \cdot g \\ =& r^2 \cdot (x^2 \cdot e + 2xy \cdot f + y^2 \cdot g) = r^2 \cdot p(x, y) \end{aligned}$$

and similarly $q(rx, ry) = r^2 \cdot q(x, y)$, therefore

$$\begin{aligned} \kappa(rx, ry) =& \frac{p(rx, ry)}{q(rx, ry)} \\ =& \frac{r^2 \cdot p(x, y)}{r^2 \cdot q(x, y)} = \frac{p(x, y)}{q(x, y)} = \kappa(x, y). \end{aligned}$$

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Theorem (1):

Let I resp. II be the matrices of the first resp. second fundamental form at a point p on the surface X. Then the matrix $\mathrm{I}^{-1} \cdot \mathrm{II}$ has real eigenvalues κ_1 and κ_2 which coincide with the global maximum and the global minimum of the sectional curvature over all non-zero tangent vectors. The following two cases are possible:

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1. If $\kappa_1 \neq \kappa_2$ then if (a, b) is a non-zero eigenvector of the matrix $\mathrm{I}^{-1} \cdot \mathrm{II}$ to the eigenvalue κ_i then $\kappa(a \cdot X_u + b \cdot X_v) = \kappa_i$.
2. If $\kappa_1 = \kappa_2$ then the sectional curvature is constant and equal to $\kappa_1 = \kappa_2$ and is achieved in any direction. In this case the point is called **umbilic**.

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Example (2): All points on a sphere are umbilic. On the paraboloid $X(u, v) = (u, v, u^2 + v^2)$ the point $X(0, 0) = (0, 0, 0)$ is umbilic.

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Proof

Definition (2):
The **principal curvatures** of the surface X at a point p, denoted by κ_1 and κ_2, are the global maximum and the global minimum of the sectional curvature at the point p.

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Note
We have shown that principal curvatures correspond to eigenvalues and that principal directions $x \cdot X_u + y \cdot X_v$ correspond to eigenvectors (x, y) of the matrix $I^{-1} \cdot II$.

Recall:

The sectional curvature in a direction is the normal curvature of the curves in this direction. The normal curvature is given by $\kappa_n = \frac{1}{|\gamma'|} \cdot (T' \bullet N)$, hence $\kappa_n > 0$ if the curve is bending in the direction of N and $\kappa_n < 0$ if the curve is bending away from N.

Let κ_1 and κ_2 be the principal curvatures.

- If $\kappa_1, \kappa_2 > 0$, then the sectional curvature κ is positive in all directions. All sections of X at p are bending in the direction of N. In a neighbourhood of the point p the surface X is on the same side of its tangent plane as N.

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- If $\kappa_1, \kappa_2 < 0$, then the sectional curvature κ is negative in all directions. All sections of X at p are bending away from N. In a neighbourhood of the point p the surface X is on the other side of its tangent plane than N.

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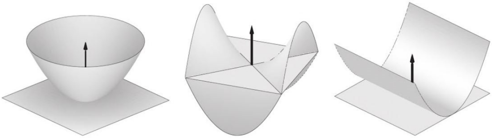
- If $\kappa_1, \kappa_2 > 0$, then the sectional curvature κ is positive in all directions. All sections of X at p are bending in the direction of N. In a neighbourhood of the point p the surface X is on the same side of its tangent plane as N.
- If $\kappa_1, \kappa_2 < 0$, then the sectional curvature κ is negative in all directions. All sections of X at p are bending away from N. In a neighbourhood of the point p the surface X is on the other side of its tangent plane than N.
- If $\kappa_1 < 0, \kappa_2 > 0$ or $\kappa_1 > 0, \kappa_2 < 0$, then the sectional curvature is positive in some directions, negative in some directions and equal to zero in some directions. In a neighbourhood of the point p the points of the surface X are on both sides of its tangent plane and the surface looks like the saddle surface.

Elliptic, Hyperbolic and Parabolic Points

Definition (3): A point on a surface

- is **elliptic** if the principal curvatures are non-zero and of the same sign.
- is **hyperbolic** if the principal curvatures are non-zero and of different signs.
- is **parabolic** if at least one of the principal curvatures is equal to zero.



(a) Elliptic point (b) Hyperbolic point (c) Parabolic point

Examples (3):

All points on a sphere are elliptic. All points on the cylinder are parabolic. The origin on the saddle surface is hyperbolic.

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Remark:
In general, a surface can be subdivided into regions consisting of elliptic and hyperbolic points respectively, the boundaries between the regions are curves that consist of parabolic points.

Example (4):
Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $X(u, v) = (u, v, 1)$. Compute the first fundamental form of the surface X. Compute the second fundamental form of the surface X. Compute the principal curvatures of this surface. Is there umbilic point.

Example (5):
Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by $X(u, v) = (\cos u, \sin u, v)$. Compute the first fundamental form of the surface X. Compute the second fundamental form of the surface X. Compute the principal curvatures of this surface. Is there umbilic point.

Exercise (1):
Let $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be given by
$X(u, v) = (\sin u \sin v, \cos u \sin v, \cos v)$. Compute the first fundamental form of the surface X. Compute the second fundamental form of the surface X. Compute the principal curvatures of this surface. Is there umbilic point.

Thanks for listening.